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A NEW RESULT ON THE POMPEIU PROBLEM

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Abstract. A nonempty bounded open set $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is said to have the *Pompeiu property* if and only if the only continuous function f on $\mathbb{R}^n$ for which the integral of f over $\sigma(\Omega)$ is zero for all rigid motions σ of $\mathbb{R}^n$ is $f \equiv 0$. We consider a nonempty bounded open set $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ with Lipschitz boundary and we assume that the complement of $\overline{\Omega}$ is connected. We show that the failure of the Pompeiu property for Ω implies some geometric conditions. Using these conditions we prove that a special kind of solid tori in $\mathbb{R}^n$, $n \geq 3$, has the Pompeiu property. So far the result was proved only for solid tori in $\mathbb{R}^4$. We also examine the case of planar domains. Finally we extend the example of solid tori to domains in $\mathbb{R}^n$ bounded by hypersurfaces of revolution.

1. Introduction and main result

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a nonempty bounded open set. We denote by $M(n)$ the group of orientation preserving rigid motions of $\mathbb{R}^n$: $M(n)$ is generated by all translations and by the rotations in $SO(n)$. Ω induces a transformation T_Ω between the spaces of continuous functions on $\mathbb{R}^n$ and $M(n)$:

$$T_\Omega : C(\mathbb{R}^n) \to C(M(n)),$$

$$T_\Omega f(\sigma) = \int_{\sigma(\Omega)} f(x)\,dx, \quad \sigma \in M(n).$$

We say that Ω has the *Pompeiu property* if T_Ω is injective. A ball of any radius $R > 0$ fails to have the Pompeiu property. When $n = 2$ we may take $f(x_1, x_2) = \sin(ax_1)$ for any $a > 0$ satisfying $J_{n/2}(aR) = 0$, where J_λ denotes the Bessel function of order λ. The result extends to $n \geq 2$: see e.g. Williams [13]. The *Pompeiu problem* asks: which sets Ω have the Pompeiu property?

Now we assume that Ω is a nonempty bounded open set with Lipschitz boundary $\partial\Omega$, and that the complement of $\overline{\Omega}$ is connected. When $\overline{\Omega}$ is rotationally symmetric this implies that $\overline{\Omega} = \overline{B}(a, R)$, the closed ball of center a and radius R for some $a \in \mathbb{R}^n$, $0 < R < \infty$. There is another formulation of the Pompeiu problem. Ω fails to have the Pompeiu property if and only if the following boundary value problem

$$\Delta u + \lambda u + 1 = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

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where ν is the exterior normal to $\partial\Omega$, has a solution (see Williams [13] and Berenstein [1] when $\partial\Omega \in C^{2+\varepsilon}$). From Green's formula we deduce that $\lambda > 0$. The *Schiffer's conjecture* (cf. Yau [15, problem 80]) asserts that the existence of a solution to the above overdetermined eigenvalue problem implies that Ω is a ball. Williams in [14] proved that the existence of a solution to (1.1), (1.2) implies that $\partial\Omega$ is real analytic.

References and information about various aspects of the Pompeiu problem can be found in the surveys by Zalcman [16], [17]. Let us mention also the remarkable results obtained by Garofalo and Segala [8]–[10] and Ebenfelt [4]–[6] in the 2-dimensional case.

Very little is known about the Pompeiu problem in $\mathbb{R}^n$ for $n \geq 3$. It is proved in [13] that proper ellipsoids have the Pompeiu property (see [3] when $n = 2$ and also Johnsson [11] when $n \geq 2$). Finally Berenstein and Khavinson [2] proved that certain tori in $\mathbb{R}^4$ have the Pompeiu property. Even for very simple sets such as ellipsoids, the proofs mentioned above follow from deep and difficult results. Only the case considered in [2] is treated using elementary arguments.

Our purpose here is to give a simple result from which we can deduce using elementary calculations that ellipsoids and certain solid tori in $\mathbb{R}^n$ have the Pompeiu property. We also show that our necessary conditions below easily provide examples of domains in $\mathbb{R}^n$ $(n \geq 2)$ having the Pompeiu property.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a nonempty bounded open set such that $\partial\Omega \in C^2$. Assume that problem* (1.1), (1.2) *has a solution $u \in C^2(\overline{\Omega})$. Then, for any $y \in \mathbb{R}^n$, we have*

$$\int_{\partial\Omega} \nu_j^2(x)(x-y).\nu(x)\,ds \;=\; \int_{\partial\Omega} \nu_k^2(x)(x-y).\nu(x)\,ds, \quad j,k \in \{1,\cdots,n\}\,, \tag{1.3}$$

and

$$\int_{\partial\Omega} \nu_j(x)\nu_k(x)(x-y).\nu(x)\,ds \;=\; 0, \quad j \neq k\,, \tag{1.4}$$

where $\nu = (\nu_1,\cdots,\nu_n)$ is the exterior normal to $\partial\Omega$.

In Section 2 we prove Theorem 1. We show that ellipsoids have the Pompeiu property in Section 3. In Section 4 we prove that certain tori in $\mathbb{R}^n$ have the Pompeiu property. In Section 5 we consider the case of planar domains. We give an example of nonconvex planar domain having the Pompeiu property. Then we examine the necessary conditions (1.3), (1.4) in the particular case of convex planar domains. Finally in Section 6 we extend the example of Section 4 to domains in $\mathbb{R}^n$ bounded by hypersurfaces of revolution.

2. Proof of Theorem 1

Let $u \in C^2(\overline{\Omega})$ be a solution of the overdetermined problem (1.1), (1.2).

The first lemma below is a particular case of a result obtained by Pucci and Serrin ([12], (4), p. 683). However we provide a proof.

Lemma 1. *Let $h = (h_1,\cdots,h_n) : \overline{\Omega} \to \mathbb{R}^n$ be of class C^1. Then we have*

$$\int_\Omega \{(\frac{1}{2}|\nabla u|^2 - \lambda\frac{u^2}{2} - u)\operatorname{div} h \;-\; \sum_{i,j=1}^n \frac{\partial h_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\}\,dx \;=0\,.$$

Proof. If we multiply equation (1.1) by

$$\sum_{j=1}^{n} h_j \frac{\partial u}{\partial x_j}$$

and integrate by parts using (1.2), we obtain

$$\begin{aligned}\sum_{i,j=1}^{n} \int_\Omega h_j \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i^2}\, dx &= -\sum_{i,j=1}^{n} \int_\Omega \{\frac{\partial h_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + h_j \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_j}\}\, dx \\ &= -\sum_{j=1}^{n} \int_\Omega (\lambda u + 1) h_j \frac{\partial u}{\partial x_j}\, dx = \int_\Omega (\lambda \frac{u^2}{2} + u) \operatorname{div} h\, dx\,.\end{aligned}$$

But we have

$$\begin{aligned}\sum_{i,j=1}^{n} \int_\Omega h_j \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_j}\, dx &= -\sum_{i,j=1}^{n} \int_\Omega h_j \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_j}\, dx - \int_\Omega |\nabla u|^2 \operatorname{div} h\, dx \\ &= -\frac{1}{2}\int_\Omega |\nabla u|^2 \operatorname{div} h\, dx\,,\end{aligned}$$

and the lemma follows.

Lemma 2. *We have*

$$\int_\Omega \big(\frac{\partial u}{\partial x_i}\big)^2 dx = \int_\Omega \big(\frac{\partial u}{\partial x_j}\big)^2 dx\,, \quad \textit{for } i,j \in \{1,\cdots,n\} \tag{2.1}$$

and

$$\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, dx = 0\,, \quad \textit{for } i \neq j\,. \tag{2.2}$$

Proof. (2.1) is obtained by using Lemma 1 with $h = (h_1, \cdots, h_n)$ such that

$$h_j(x) = x_j\,, \quad h_i(x) = -x_i \quad \text{and} \quad h_k(x) = 0 \quad \text{for } k \neq i,j\,.$$

For (2.2) we take

$$h_j(x) = x_i \quad \text{and} \quad h_k(x) = 0 \quad \text{for } k \neq j\,.$$

Now let $l = (l_1, \cdots, l_n) \in \mathbb{R}^n$ be a unit vector. Define

$$w = \frac{\partial u}{\partial l} = \sum_{j=1}^{n} l_j \frac{\partial u}{\partial x_j}\,.$$

Then w is a solution of the following boundary value problem

$$\Delta w + \lambda w = 0 \quad \text{in } \Omega\,, \quad w = 0 \quad \text{on } \partial\Omega\,.$$

Lemma 3. *We have:* $\dfrac{\partial w}{\partial \nu} = -l.\nu$ *on* $\partial\Omega$.

Proof. Since $u = \partial u/\partial \nu = 0$ on $\partial\Omega$, we can write

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial \nu^2}\nu_i \nu_j \quad \text{on } \partial\Omega \quad \text{for } \ i,j \in \{1,\cdots,n\}\,.$$

Then on $\partial\Omega$ we have

$$\begin{aligned}\frac{\partial w}{\partial \nu} &= \sum_{j=1}^{n} \nu_j \frac{\partial w}{\partial x_j} = \sum_{j,k=1}^{n} \nu_j l_k \frac{\partial^2 u}{\partial x_j \partial x_k} \\ &= (\sum_{j,k=1}^{n} \nu_j^2 l_k \nu_k) \frac{\partial^2 u}{\partial \nu^2} = (l.\nu)\frac{\partial^2 u}{\partial \nu^2} = (l.\nu)\Delta u = -l.\nu\,.\end{aligned}$$

Now the Pohozaev identity gives

$$\lambda \int_\Omega w^2\,dx = \frac{1}{2}\int_{\partial\Omega} (\frac{\partial w}{\partial \nu})^2 (x-y).\nu(x)\,ds\,, \tag{2.3}$$

for any fixed $y \in \mathbb{R}^n$. Using Lemma 2 we get

$$\int_\Omega w^2\,dx = \sum_{j,k=1}^{n} l_j l_k \int_\Omega \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_k}\,dx = \sum_{j=1}^{n} l_j^2 \int_\Omega (\frac{\partial u}{\partial x_j})^2\,dx = \int_\Omega (\frac{\partial u}{\partial x_i})^2\,dx$$

for $1 \le i \le n$. Therefore using (2.3) and Lemma 3 we can write for $1 \le i \le n$:

$$\begin{aligned}\lambda \int_\Omega (\frac{\partial u}{\partial x_i})^2\,dx &= \frac{1}{2}\int_{\partial\Omega} (l.\nu(x))^2 (x-y).\nu(x)\,ds \\ &= \frac{1}{2}\sum_{j=1}^{n} l_j^2 \int_{\partial\Omega} \nu_j^2(x)(x-y).\nu(x)\,ds \\ &\quad + \sum_{1\le j<k\le n} l_j l_k \int_{\partial\Omega} \nu_j(x)\nu_k(x)(x-y).\nu(x)\,ds\,.\end{aligned}$$

Since $l \in \mathbb{R}^n$ is an arbitrary unit vector the theorem follows.

3. Ellipsoids in $\mathbb{R}^n$ $(n \ge 2)$

Theorem 2. *Let $a_j > 0$, $j = 1, \cdots, n$, and assume that $a_j \ne a_k$ for some $j \ne k$. Then the ellipsoid*

$$\Omega = \{x \in \mathbb{R}^n;\ \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} < 1\}$$

has the Pompeiu property.

Proof. Using Theorem 1 it is enough to show that (1.3) does not hold for some $y \in \mathbb{R}^n$. Let $a_r = \min\{a_j;\ j = 1, \cdots, n\}$ and $a_s = \max\{a_j;\ j = 1, \cdots, n\}$. Our assumption implies that $r \ne s$. When $n = 2$, $(r, s) = (1, 2)$ or $(r, s) = (2, 1)$. If $n \ge 3$, using a rotation we can assume that $(r, s) = (1, 2)$ or $(r, s) = (2, 1)$.

We denote by $\mu = (\mu_1, \cdots, \mu_n)$ the exterior normal to $\partial B(0, R)$. Using polar coordinates we can write

$$\begin{aligned}\mu_1 &= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-2} \cos\theta_{n-1} \\ \mu_2 &= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-2} \sin\theta_{n-1} \\ \mu_3 &= \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-3} \sin\theta_{n-2} \\ &\vdots \\ \mu_{n-1} &= \cos\theta_1 \sin\theta_2 \\ \mu_n &= \sin\theta_1\end{aligned}$$

where $-\frac{\pi}{2} \le \theta_1, \cdots, \theta_{n-2} \le \frac{\pi}{2}$ (if $n \ge 3$) and $-\pi \le \theta_{n-1} < \pi$. We can parametrize $\partial\Omega$ by

$$x_1 = a_1\mu_1, \cdots, x_n = a_n\mu_n\,,$$

with $-\frac{\pi}{2} < \theta_1, \cdots, \theta_{n-2} < \frac{\pi}{2}$ (if $n \ge 3$) and $-\pi \le \theta_{n-1} < \pi$. Then the exterior normal to $\partial\Omega$ is given by $\nu = (\nu_1, \cdots, \nu_n)$:

$$\nu_j \;=\; \frac{a_1 \cdots a_{j-1}a_{j+1} \cdots a_n\mu_j}{(a_2^2 \cdots a_n^2\mu_1^2 + \cdots + a_1^2 \cdots a_{n-1}^2\mu_n^2)^{1/2}}\,,$$

for $j = 1, \cdots, n$. Suppose that $a_1 < a_2$. Let

$$I_j = \int_{\partial\Omega} \nu_j^2(x)(x.\nu(x))\,ds\,, \quad j = 1, \cdots, n.$$

We have

$$\begin{aligned} ds &= (a_2^2\mu_1^2 + a_1^2\mu_2^2)^{1/2}d\theta_1 \quad \text{if } n = 2,\\ ds &= \cos^{n-2}\theta_1 \cdots \cos\theta_{n-2}\\ &\qquad \times (a_2^2 \cdots a_n^2\mu_1^2 + \cdots + a_1^2 \cdots a_{n-1}^2\mu_n^2)^{1/2}d\theta_1 \cdots d\theta_{n-1} \quad \text{if } n \ge 3\,. \end{aligned}$$

Then

$$I_1 = a_1a_2^3 \int_{-\pi}^{\pi} \frac{\cos^2\theta}{a_2^2\cos^2\theta + a_1^2\sin^2\theta}\,d\theta\,,$$

$$I_2 = a_1^3a_2 \int_{-\pi}^{\pi} \frac{\sin^2\theta}{a_2^2\cos^2\theta + a_1^2\sin^2\theta}\,d\theta\,,$$

if $n = 2$ and

$$\begin{aligned} I_j &= a_1^3 \cdots a_{j-1}^3 a_j a_{j+1}^3 \cdots a_n^3 \int_{-\pi}^{\pi} d\theta_{n-1}\\ &\quad \times \int_{-\pi/2}^{\pi/2} d\theta_{n-2} \cdots \int_{-\pi/2}^{\pi/2} d\theta_1 \frac{\cos^{n-2}\theta_1 \cdots \cos\theta_{n-2}\mu_j^2}{a_2^2 \cdots a_n^2\mu_1^2 + \cdots + a_1^2 \cdots a_{n-1}^2\mu_n^2}\,, \end{aligned}$$

for $j = 1, \cdots, n$ if $n \ge 3$. We shall prove that $I_1 > I_2$. When $n = 2$, we have

$$a_1^2 < a_2^2\cos^2\theta + a_1^2\sin^2\theta < a_2^2\,, \quad \text{for } \theta \in (-\pi, \pi) \setminus \{0, \pm\frac{\pi}{2}\}.$$

Now assume that $n \ge 3$. For $\theta_{n-1} \in (-\pi, 0) \cup (0, \pi)$ and $-\frac{\pi}{2} < \theta_1, \cdots, \theta_{n-2} < \frac{\pi}{2}$ we have

$$\begin{aligned} &a_2^2 \cdots a_n^2\mu_1^2 + a_1^2a_3^2 \cdots a_n^2\mu_2^2\\ &\quad = a_3^2 \cdots a_n^2 \cos^2\theta_1 \cdots \cos^2\theta_{n-2}(a_2^2\cos^2\theta_{n-1} + a_1^2\sin^2\theta_{n-1})\\ &\quad < a_2^2 \cdots a_n^2 \cos^2\theta_1 \cdots \cos^2\theta_{n-2}\,. \end{aligned}$$

Repeating this argument we obtain

$$a_2^2 \cdots a_n^2\mu_1^2 + \cdots + a_1^2 \cdots a_{n-1}^2\mu_n^2 \;<\; a_2^2 \cdots a_n^2\,,$$

for $\theta_{n-1} \in (-\pi, 0) \cup (0, \pi)$ and $-\frac{\pi}{2} < \theta_1, \cdots, \theta_{n-2} < \frac{\pi}{2}$. In the same way we have

$$a_2^2 \cdots a_n^2\mu_1^2 + \cdots + a_1^2 \cdots a_{n-1}^2\mu_n^2 \;>\; a_1^2a_3^2 \cdots a_n^2\,,$$

for $\theta_{n-1} \in [-\pi, , \pi) \setminus \{\pm\frac{\pi}{2}\}$ and $-\frac{\pi}{2} < \theta_1, \cdots, \theta_{n-2} < \frac{\pi}{2}$. Then we obtain

$$I_1 > a_1 a_2 \pi > I_2 \quad \text{if } n = 2,$$

and

$$I_1 \; > \; a_1 \cdots a_n 2^{n-2} \pi \int_0^{\pi/2} \cos^n \theta \, d\theta \cdots \int_0^{\pi/2} \cos^3 \theta \, d\theta \; > \; I_2 \, ,$$

if $n \geq 3$. The case $a_1 > a_2$ can be treated using analogous arguments. The proof of the theorem is complete.

Remark 1. Notice that (1.4) always hold for ellipsoids.

4. Solid tori in $\mathbb{R}^n$

We consider a special kind of tori in $\mathbb{R}^n$, $n \geq 3$. Let $a > R > 0$ and let $D(a, R)$ denote the disk of center $(a, 0, \cdots, 0)$ and radius R in the plane $x_2 = \cdots = x_{n-1} = 0$ of $\mathbb{R}^n$. By rotating this disk about the x_n-axis in $\mathbb{R}^n$ we obtain a torus Ω of equation

$$(\sqrt{x_1^2 + \cdots + x_{n-1}^2} - a)^2 + x_n^2 \; < \; R^2 \, . \tag{4.1}$$

Theorem 3. *Let $a > R > 0$ and let Ω be the solid torus in $\mathbb{R}^n$ defined by* (4.1)*, then Ω has the Pompeiu property.*

Proof. We can parametrize $\partial\Omega$ by

$$\begin{aligned}
x_1 &= (a + R\cos\theta_{n-1})\cos\theta_1 \cdots \cos\theta_{n-3}\cos\theta_{n-2} \\
x_2 &= (a + R\cos\theta_{n-1})\cos\theta_1 \cdots \cos\theta_{n-3}\sin\theta_{n-2} \\
&\vdots \\
x_{n-2} &= (a + R\cos\theta_{n-1})\cos\theta_1 \sin\theta_2 \\
x_{n-1} &= (a + R\cos\theta_{n-1})\sin\theta_1 \\
x_n &= R\sin\theta_{n-1}
\end{aligned}$$

where $-\frac{\pi}{2} < \theta_1, \cdots, \theta_{n-3} < \frac{\pi}{2}$ (if $n \geq 4$) and $-\pi \leq \theta_{n-2}, \theta_{n-1} < \pi$. Then the exterior normal to $\partial\Omega$ is given by $\nu = (\nu_1, \cdots, \nu_n)$:

$$\begin{aligned}
\nu_1 &= \cos\theta_{n-1}\cos\theta_1\cos\theta_2 \cdots \cos\theta_{n-3}\cos\theta_{n-2} \\
\nu_2 &= \cos\theta_{n-1}\cos\theta_1\cos\theta_2 \cdots \cos\theta_{n-3}\sin\theta_{n-2} \\
&\vdots \\
\nu_{n-2} &= \cos\theta_{n-1}\cos\theta_1\sin\theta_2 \\
\nu_{n-1} &= \cos\theta_{n-1}\sin\theta_1 \\
\nu_n &= \sin\theta_{n-1}
\end{aligned}$$

It is enough to show that (1.3) does not hold for some $y \in \mathbb{R}^n$. Set

$$I_j = \int_{\partial\Omega} \nu_j^2(x)(x.\nu(x)) \, ds, \quad j = 1, \cdots, n.$$

We have

$$ds = R(a + R\cos\theta_2) d\theta_1 d\theta_2 \quad \text{if } n = 3,$$

$$ds = R(a + R\cos\theta_{n-1})^{n-2} \cos^{n-3}\theta_1 \cdots \cos\theta_{n-3} d\theta_1 \cdots d\theta_{n-1} \quad \text{if } n \geq 4,$$

and

$$x.\nu(x) = (R + a\cos\theta_{n-1}) \quad \text{on } \partial\Omega.$$

Define

$$L_1 = L_2 = 1\,, \quad L_m = \int_0^{\pi/2} \cos^{m-1}\theta\, d\theta \cdots \int_0^{\pi/2} \cos\theta\, d\theta \quad \text{for } m \geq 3\,.$$

Lemma 4. $L_m = \dfrac{\pi}{2(m-1)} L_{m-2}$ *for* $m \geq 3$.

Proof. $L_3 = \pi/4$. Now we have

$$\begin{aligned}\int_0^{\pi/2} \cos^{m-1}\theta\, d\theta &= \int_0^{\pi/2} (1-\sin^2\theta)\cos^{m-3}\theta\, d\theta \\ &= \int_0^{\pi/2} \cos^{m-3}\theta\, d\theta + \frac{1}{m-2}\int_0^{\pi/2} (\cos^{m-2}\theta)' \sin\theta\, d\theta \\ &= \int_0^{\pi/2} \cos^{m-3}\theta\, d\theta - \frac{1}{m-2}\int_0^{\pi/2} \cos^{m-1}\theta\, d\theta\,,\end{aligned}$$

from which we get

$$\int_0^{\pi/2} \cos^{m-1}\theta\, d\theta = \frac{m-2}{m-1}\int_0^{\pi/2} \cos^{m-3}\theta\, d\theta.$$

Then, for $m \geq 4$, we can write

$$\begin{aligned}\frac{L_m}{L_{m-2}} &= \int_0^{\pi/2} \cos^{m-1}\theta\, d\theta \int_0^{\pi/2} \cos^{m-2}\theta\, d\theta \\ &= \frac{m-2}{m-1}\int_0^{\pi/2} \cos^{m-2}\theta\, d\theta \int_0^{\pi/2} \cos^{m-3}\theta\, d\theta \\ &= \frac{m-2}{m-1}\frac{L_{m-1}}{L_{m-3}}\,,\end{aligned}$$

and the lemma follows easily.

Now we have

$$I_1 = 2\pi R \int_0^{\pi} (a + R\cos\theta)(R + a\cos\theta)\cos^2\theta\, d\theta$$

if $n = 3$, and

$$\begin{aligned}I_1 &= \int_{-\pi}^{\pi} d\theta_{n-1} \int_{-\pi}^{\pi} d\theta_{n-2} \int_{-\pi/2}^{\pi/2} d\theta_{n-3} \cdots \int_{-\pi/2}^{\pi/2} d\theta_1 [R(a + R\cos\theta_{n-1})^{n-2} \\ &\quad \times (R + a\cos\theta_{n-1})\cos^2\theta_{n-1}\cos^{n-1}\theta_1 \cdots \cos^2\theta_{n-2}] \\ &= 2^{n-1} R \int_0^{\pi} (a + R\cos\theta)^{n-2}(R + a\cos\theta)\cos^2\theta\, d\theta \int_0^{\pi/2} \cos^{n-1}\theta\, d\theta \\ &\quad \cdots \int_0^{\pi/2} \cos^3\theta\, d\theta \int_0^{\pi} \cos^2\theta\, d\theta\,,\end{aligned}$$

if $n \geq 4$. Therefore we can write

$$I_1 = 2^n R\Big(\int_0^{\pi} (a + R\cos\theta)^{n-2}(R + a\cos\theta)\cos^2\theta\, d\theta\Big) L_n\,.$$

In the same way we obtain

$$I_n = 2^{n-1}\pi R(\int_0^\pi (a+R\cos\theta)^{n-2}(R+a\cos\theta)\sin^2\theta\, d\theta)L_{n-2}\,.$$

Using Lemma 4 we can write

$$\begin{aligned} I_n - I_1 =& 2^n R L_n \int_0^\pi (a+R\cos\theta)^{n-2}(R+a\cos\theta)((n-1)\sin^2\theta - \cos^2\theta)\, d\theta \\ =& \sum_{j=0}^{n-1} c_j \int_0^\pi ((n-1)\sin^2\theta - \cos^2\theta)\cos^j\theta\, d\theta\,, \end{aligned}$$

with $c_j > 0$ for $j = 0, \cdots, n-1$. We have

$$\int_0^\pi \cos^j\theta\, d\theta = 0\,, \quad \text{for } j \text{ odd}\,,$$

and

$$\int_0^\pi \cos^j\theta\sin^2\theta\, d\theta = -\frac{1}{j+1}\int_0^\pi (\cos^{j+1}\theta)'\sin\theta\, d\theta = \frac{1}{j+1}\int_0^\pi \cos^{j+2}\theta\, d\theta\,.$$

We deduce that

$$I_n - I_1 = \sum_{0\le 2j\le n-1} c_{2j}(\frac{n-1}{2j+1} - 1)\int_0^\pi \cos^{2j+2}\theta\, d\theta\,. \tag{4.1}$$

If $n = 2p$ ($p \ge 2$) is even (4.1) implies that $I_n > I_1$. If $n = 2p+1$ is odd we have

$$I_n - I_1 = \frac{3aR^2\pi^2}{4} > 0$$

when $p = 1$, and

$$\begin{aligned} I_n - I_1 =& \sum_{0\le 2j\le n-5} c_{2j}(\frac{n-1}{2j+1} - 1)\int_0^\pi \cos^{2j+2}\theta\, d\theta \\ &+ \frac{1}{n-2}c_{n-3}\int_0^\pi \cos^{n-1}\theta d\theta - \frac{1}{n}c_{n-1}\int_0^\pi \cos^{n+1}\theta d\theta\,, \end{aligned}$$

for $p \ge 2$. Since

$$\int_0^\pi \cos^{n+1}\theta\, d\theta \;=\; \frac{n}{n+1}\int_0^\pi \cos^{n-1}\theta\, d\theta\,,$$

and

$$c_{n-3} = a2^n L_n R^{n-3}((n-2)R^2 + \frac{(n-2)(n-3)}{2}a^2)\,, \quad c_{n-1} = a2^n L_n R^{n-1}\,,$$

we still have $I_n > I_1$. The proof of the theorem is complete.

Remark 2. Notice that (1.4) is satisfied for these tori. Moreover $I_1 = \cdots = I_{n-1}$.

5. Planar domains

In this section we consider the case of planar domains. We give an example of nonconvex planar domain having the Pompeiu property. Then we examine the necessary conditions (1.3), (1.4) in the particular case of convex planar domains.

In the 2-dimensional case the necessary conditions (1.3), (1.4) are equivalent to

$$\int_{\partial\Omega} (\nu_1(x) + i\nu_2(x))^2 (x-y).\nu(x)\, ds = 0 \quad \forall y \in \mathbb{R}^n\,. \tag{5.1}$$

5.1. Nonconvex planar domains. Consider the curve whose equation in polar coordinates is given by

$$\rho(\theta) = a\cos\theta + b\,, \quad b > a > 0, \quad \theta \in [0, 2\pi).$$

This curve is real analytic. When $2a \leq b$ the curve bounds a convex domain Ω, but when $a < b < 2a$ Ω is not convex. We have

$$\nu_1(\theta) = \frac{\rho'\sin\theta + \rho\cos\theta}{(\rho^2+\rho'^2)^{1/2}}, \qquad \nu_2(\theta) = \frac{\rho\sin\theta - \rho'\cos\theta}{(\rho^2+\rho'^2)^{1/2}}$$

and

$$ds = (\rho^2+\rho'^2)^{1/2}d\theta.$$

Using the residue formula we obtain

$$\int_{\partial\Omega}(\nu_1(x)+i\nu_2(x))^2(x.\nu(x))\,ds = \int_0^{2\pi}\rho^2\frac{\rho-i\rho'}{\rho+i\rho'}e^{2i\theta}\,d\theta \;=\; -\frac{\pi a^2}{2b^5}(a^2-b^2)^3\,.$$

Therefore (5.1) is not satisfied and we conclude that Ω has the Pompeiu property.

We refer the reader to [8] for more comments on this example.

5.2. Convex planar domains. In this subsection we examine the necessary conditions (1.3), (1.4) in the case of planar convex domains.

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex open set with the origin as an interior point. We assume that $\partial\Omega$ is a C^2 curve with positive curvature. Let $x = x(s) = (x_1(s), x_2(s))$ be a parametrization of $\partial\Omega$ by arc length. For each angle θ, $0 \leq \theta < 2\pi$, let $h(\theta)$ denote the distance from the origin to the supporting line of Ω with outward normal $\nu = (\cos\theta, \sin\theta)$. We have

$$h(\theta) = x.\nu\,,$$

and h has period 2π. From the Serret-Frenet formulas we can derive the following second order ordinary differential equation involving the support function h and the radius of curvature ρ:

$$h(\theta) + h''(\theta) = \rho(\theta)\,.$$

When $0 \notin \Omega$, the support function is defined in the following way. By translation there exists $a = (a_1, a_2) \in \mathbb{R}^2$ such that $0 \in \tilde{\Omega} = a + \Omega$. If $\tilde{h}$ denotes the support function of $\tilde{\Omega}$ we have

$$h(\theta) = -a_1\cos\theta - a_2\sin\theta + \tilde{h}(\theta).$$

We refer the reader to Flanders [7] and the references therein for a detailed discussion.

Theorem 4. *Let Ω be a bounded convex open set. We assume that $\partial\Omega$ is a C^2 curve with positive curvature. Let h denote the support function of Ω. If*

$$\int_0^{2\pi} h(h+h'')e^{2i\theta}\,d\theta \neq 0,$$

then Ω has the Pompeiu property.

Proof. Let $a = (a_1, a_2) \in \mathbb{R}^2$ be such that $0 \in \tilde{\Omega} = a + \Omega$ and denote by $\tilde{h}$ the support function of $\tilde{\Omega}$. Since

$$\begin{aligned}\int_{\partial\tilde{\Omega}} (\nu_1(x) + i\nu_2(x))^2 (x-a).\nu(x)\, ds &= \int_0^{2\pi} (\tilde{h} - a_1\cos\theta - a_2\sin\theta)(\tilde{h} + \tilde{h}'')e^{2i\theta}\, d\theta \\ &= \int_0^{2\pi} h(h+h'')e^{2i\theta}\, d\theta \neq 0,\end{aligned}$$

(5.1) is not satisfied for $\tilde{\Omega}$, hence $\tilde{\Omega}$ has the Pompeiu property. Therefore Ω has the Pompeiu property.

Example 1. Let $m \geq 2$ be an integer. Define

$$h(\theta) = \alpha\cos m\theta + \gamma\,, \quad 0 \leq \theta < 2\pi\,,$$

with $\gamma > (m^2-1)|\alpha|$ and $\alpha \neq 0$. h is of class C^2, and has period 2π and

$$h(\theta) + h''(\theta) = \gamma - \alpha(m^2-1)\cos m\theta > 0\,, \quad 0 \leq \theta < 2\pi\,.$$

Then h must be the support function of a convex set Ω which is not a disc (notice that $0 \in \Omega$). We easily verify that

$$\int_0^{2\pi} h(h+h'')e^{2i\theta}\, d\theta = \begin{cases} 0 & \text{if } m \geq 3, \\ -2\pi\alpha\gamma & \text{if } m = 2. \end{cases}$$

Therefore Theorem 4 implies that Ω has the Pompeiu property for $m = 2$. Now define

$$\tilde{h}(\theta) = \alpha\cos m\theta + \beta\cos\theta + \gamma\,, \quad 0 \leq \theta < 2\pi\,,$$

with $\beta \neq 0$. $\tilde{h}$ is the support function of $\tilde{\Omega} = (\beta, 0) + \Omega$. We have

$$\int_0^{2\pi} \tilde{h}(\tilde{h}+\tilde{h}'')e^{2i\theta}\, d\theta = \begin{cases} 0 & \text{if } m \geq 4, \\ -4\pi\alpha\beta & \text{if } m = 3, \\ -2\pi\alpha\gamma & \text{if } m = 2\,, \end{cases}$$

Theorem 4 implies that $\tilde{\Omega}$ has the Pompeiu property for $m = 2$ and 3. Therefore Ω has also the Pompeiu property for $m = 3$. When $m \geq 4$ we cannot conclude.

Notice that

$$h(\theta) + h(\theta+\pi) = 2\gamma + \alpha(1 + (-1)^m)\cos m\theta\,, \quad 0 \leq \theta < 2\pi\,.$$

Then, if $m = 2p+1$, Ω is of constant width and Ω is not a disc.

We complete this subsection with an example which is not contained in the work of Garofalo and Segala [9].

Example 2. Consider the curve whose equation in polar coordinates is given by

$$\rho(\theta) = \exp(\lambda\cos 2\theta)\,, \quad |\lambda| < \frac{1}{4}, \quad \theta \in [0, 2\pi).$$

This curve is real analytic. When $|\lambda| < \frac{1}{4}$ the curve bounds a convex domain Ω_λ. We have

$$\begin{aligned}\int_{\partial\Omega} (\nu_1(x) + i\nu_2(x))^2 (x.\nu(x))\, ds &= \int_0^{2\pi} e^{2\lambda\cos 2\theta} \frac{1 + 2i\lambda\sin 2\theta}{1 - 2i\lambda\sin 2\theta} e^{2i\theta}\, d\theta \\ &\sim -2\pi\lambda \quad \text{when} \quad \lambda \to 0\,.\end{aligned}$$

Therefore there exists $\lambda_0 > 0$ such that, for $\lambda \in (-\lambda_0, \lambda_0) \setminus 0$, (5.1) is not satisfied and we conclude that Ω_λ has the Pompeiu property (notice that Ω_0 is the unit disk). Clearly Ω_λ does not satisfy the assumptions of Theorem 1.3 in [9] for $\lambda \in (-\lambda_0, \lambda_0) \setminus 0$.

6. Hypersurfaces of revolution in $\mathbb{R}^n$

Let D be a domain in the plane $x_2 = \cdots = x_{n-1} = 0$ of $\mathbb{R}^n$ $(n \geq 3)$ bounded by a regular closed curve

$$[\alpha, \beta) \ni t \to (f(t), g(t)).$$

We assume that $D \subset \{x \in \mathbb{R}^n;\, x_2 = \cdots = x_{n-1} = 0 \text{ and } x_1 > 0\}$. By rotating D about the x_n-axis in $\mathbb{R}^n$ we obtain a domain Ω bounded by a hypersurface of revolution. We can parametrize $\partial\Omega$ by

$$\begin{aligned}
x_1 &= f(t)\cos\theta_1 \cdots \cos\theta_{n-3}\cos\theta_{n-2}\\
x_2 &= f(t)\cos\theta_1 \cdots \cos\theta_{n-3}\sin\theta_{n-2}\\
&\vdots\\
x_{n-2} &= f(t)\cos\theta_1 \sin\theta_2\\
x_{n-1} &= f(t)\sin\theta_1\\
x_n &= g(t)
\end{aligned}$$

where $-\frac{\pi}{2} < \theta_1, \cdots, \theta_{n-3} < \frac{\pi}{2}$ (if $n \geq 4$), $-\pi \leq \theta_{n-2} < \pi$ and $\alpha \leq t < \beta$. Then the exterior normal to $\partial\Omega$ is given by $\nu = (\nu_1, \cdots, \nu_n)$:

$$\begin{aligned}
\nu_1 &= \frac{g'(t)}{(f'^2(t) + g'^2(t))^{1/2}} \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-3}\cos\theta_{n-2}\\
\nu_2 &= \frac{g'(t)}{(f'^2(t) + g'^2(t))^{1/2}} \cos\theta_1 \cos\theta_2 \cdots \cos\theta_{n-3}\sin\theta_{n-2}\\
&\vdots\\
\nu_{n-2} &= \frac{g'(t)}{(f'^2(t) + g'^2(t))^{1/2}} \cos\theta_1 \sin\theta_2\\
\nu_{n-1} &= \frac{g'(t)}{(f'^2(t) + g'^2(t))^{1/2}} \sin\theta_1\\
\nu_n &= \frac{-f'(t)}{(f'^2(t) + g'^2(t))^{1/2}}.
\end{aligned}$$

We have

$$ds = f(t)(f'^2(t) + g'^2(t))^{1/2} d\theta_1 dt \quad \text{if } n = 3,$$

$$ds = f^{n-2}(t)(f'^2(t) + g'^2(t))^{1/2} \cos^{n-3}\theta_1 \cdots \cos\theta_{n-3} d\theta_1 \cdots d\theta_{n-2} dt \quad \text{if } n \geq 4\,,$$

and

$$x.\nu(x) = \frac{f(t)g'(t) - g(t)f'(t)}{(f'^2(t) + g'^2(t))^{1/2}} \quad \text{on } \partial\Omega.$$

Theorem 5. *Let Ω be as above. Assume that one of the following holds:*

(i) $\displaystyle\int_\alpha^\beta \frac{f^{n-2}(fg' - gf')((n-1)f'^2 - g'^2)}{f'^2 + g'^2}\, dt \neq 0\,,$ *or*

(ii) $\displaystyle\int_\alpha^\beta \frac{f^{n-2}g'^2 f'}{f'^2+g'^2}\,dt \neq 0\,,$ *or*

(iii) $\displaystyle\int_\alpha^\beta \frac{f^{n-2}f'^3}{f'^2+g'^2}\,dt \neq \frac{1}{n-1}\int_\alpha^\beta \frac{f^{n-2}g'^2 f'}{f'^2+g'^2}\,dt\,.$

Then Ω has the Pompeiu property.

Remark 3. Notice that (1.4) is satisfied when $y=0$ and that

$$\int_{\partial\Omega} \nu_1^2(x)(x.\nu(x))\,ds = \cdots = \int_{\partial\Omega} \nu_{n-1}^2(x)(x.\nu(x))\,ds\,.$$

Proof. We keep the notations of Section 4.

(i) Arguing as in the proof of Theorem 3 we arrive at

$$I_n - I_1 = 2^{n-1}L_n \int_\alpha^\beta \frac{f^{n-2}(fg'-gf')((n-1)f'^2-g'^2)}{f'^2+g'^2}\,dt\,.$$

Then (1.3) does not hold for $y=0$.

(ii) We have

$$\int_{\partial\Omega} \nu_j^2\nu_n\,ds = -\frac{2^{n-2}\pi}{n-1}L_{n-2}\int_\alpha^\beta \frac{f^{n-2}g'^2 f'}{f'^2+g'^2}\,dt\,, \quad j\in\{1,\cdots,n-1\}\,. \tag{6.1}$$

Let $y=(1,0,\cdots,0)$. Then (6.1) and Remark 3 imply that (1.4) does not hold.

(iii) We have

$$\int_{\partial\Omega} \nu_n^3\,ds = -2^{n-2}\pi L_{n-2}\int_\alpha^\beta \frac{f^{n-2}f'^3}{f'^2+g'^2}\,dt\,. \tag{6.2}$$

Let $y=(0,\cdots,0,1)$. Then (6.1), (6.2) and Remark 3 imply that (1.3) does not hold.

Remark 4. Assume that f is even and that g is odd. Then (1.4) holds and

$$\int_{\partial\Omega} \nu_1^2(x)(x-y).\nu(x)\,ds = \cdots = \int_{\partial\Omega} \nu_{n-1}^2(x)(x-y).\nu(x)\,ds$$

for any $y\in\mathbb{R}^n$.

We give below a result which extends Theorem 3.

Theorem 6. *Assume that*

$$f(t)=a+R\cos t\,, \quad g(t)=r\sin t \quad -\pi\le t<\pi,$$

with $r,R>0$ and $a>R$. Then there exists $\rho>R$ such that Ω has the Pompeiu property for $r\neq\rho$.

Proof. By Theorem 5 and Remark 4 it is enough to show that there exists $\rho>R$ such that

$$I(r)=\int_0^\pi \frac{(a+R\cos\theta)^{n-2}(R+a\cos\theta)((n-1)R^2\sin^2\theta-r^2\cos^2\theta)}{R^2\sin^2\theta+r^2\cos^2\theta}\,d\theta \neq 0$$

when $r\neq\rho$. We write

$$I(r)=\sum_{j=0}^{n-1} d_j\int_0^\pi \frac{((n-1)R^2\sin^2\theta-r^2\cos^2\theta)\cos^j\theta}{R^2\sin^2\theta+r^2\cos^2\theta}\,d\theta\,,$$

with $d_j > 0$ independent of r for $j = 0, \cdots, n-1$. We have

$$\int_0^\pi \frac{\cos^j \theta}{R^2 \sin^2 \theta + r^2 \cos^2 \theta}\, d\theta = \int_0^\pi \frac{\cos^j \theta \sin^2 \theta}{R^2 \sin^2 \theta + r^2 \cos^2 \theta}\, d\theta = 0$$

for j odd. Then

$$I(r) = \sum_{0 \le 2k \le n-1} d_{2k} \int_0^\pi \frac{((n-1)R^2 \sin^2 \theta - r^2 \cos^2 \theta) \cos^{2k} \theta}{R^2 \sin^2 \theta + r^2 \cos^2 \theta}\, d\theta\,.$$

We easily verify that $I'(r) < 0$ for $r > 0$. The proof of Theorem 3 shows that $I(R) = (I_n - I_1)/2^n R L_n > 0$. Since

$$\lim_{r \to \infty} I(r) = -\sum_{0 \le 2k \le n-1} d_{2k} \int_0^\pi \cos^{2k} \theta\, d\theta \; < 0\,,$$

the theorem follows.

Example 3. Assume that $n = 3$. Using the residue formula we can show that

$$I_1 = I_2 = (4R + 3r)\frac{ar^2 R \pi^2}{(r+R)^2} \quad \text{and} \quad I_3 = (3R + 2r)\frac{2arR^2\pi^2}{(r+R)^2}\,,$$

hence $\rho = \sqrt{2}R$.

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