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TREES AND VALUATION RINGS

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In Memoriam Karl Mathiak

Abstract. A subring B of a division algebra D is called a valuation ring of D if $x \in B$ or $x^{-1} \in B$ holds for all nonzero x in D. The set $\mathcal{B}$ of all valuation rings of D is a partially ordered set with respect to inclusion, having D as its maximal element. As a graph $\mathcal{B}$ is a rooted tree (called the valuation tree of D), and in contrast to the commutative case, $\mathcal{B}$ may have finitely many but more than one vertices. This paper is mainly concerned with the question of whether each finite, rooted tree can be realized as a valuation tree of a division algebra D, and one main result here is a positive answer to this question where D can be chosen as a quaternion division algebra over a commutative field.

1. Introduction

Let D be a finite-dimensional division algebra with centre F and let B be a valuation ring of D. Then each overring of B in D is a valuation ring too, and the set of all overrings of B in D is totally ordered by inclusion. This means that the set $\mathcal{B}$ of all valuation rings of D is a partially ordered set with respect to inclusion which has D as its maximal element, and as a graph $\mathcal{B}$ is a tree where D has been designated as a special vertex, i.e., $\mathcal{B}$ is a rooted tree with D as its root (cf. [O, p. 59]). $\mathcal{B}$ together with this ordering is called the valuation tree of D denoted by $\mathcal{T}$. If $D = F$ is a commutative field, then $\mathcal{B}$ has exactly one or an infinite number of elements.

The first case happens precisely when F is an algebraic extension of a finite field. If F is not of this type, then F has a subfield K which is a global field, and each valuation of K can be extended to F. In the commutative case $\mathcal{B}$ as well as subsets of $\mathcal{B}$ have been investigated in connextion with abstract Riemann surfaces (cf. [ZS]). In the noncommutative situation $\#\mathcal{B}$ can be any natural number. Indeed, class field theory shows that for each $n \in \mathbb{N}$ there exists a division algebra D finite-dimensional over a global field having precisely n valuation rings (cf. [P]), and moreover D can be chosen as a quaternion algebra. Since each nontrivial valuation of a division algebra D over a global field is real (rank 1), the valuation tree $\mathcal{T}$ of D has the very special property that each vertex of $\mathcal{T}$ is the root of $\mathcal{T}$ or a lower neighbour of the root.

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One main result of this paper is Theorem 4.2, which states that each finite, rooted tree can be realized as a valuation tree of a finite-dimensional division algebra D. Moreover, if $\#\mathcal{B} > 1$, then D can be chosen as a symbol algebra of arbitrary degree, and each valuation ring of D is invariant (we say that the valuation tree $\mathcal{T}$ of D is invariant). An immediate consequence of this result is that each finite, rooted tree can be realized as an invariant valuation tree of a quaternion algebra.

The situation is much more complicated if one is also interested in noninvariant valuation trees. For instance, if B is a noninvariant valuation ring, $aBa^{-1} \neq B$ say, then the two subtrees $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{T}$ consisting of all valuation rings lying in B resp. aBa^{-1} are isomorphic. This indicates that finite, rooted trees cannot be realized as a noninvariant valuation tree in an arbitrary manner. To clarify this situation, we introduce in section 6 the notation of the central image $\mathcal{T}_C$ of a valuation tree $\mathcal{T}$ of D consisting of the centres of all valuations rings of D, i.e., of all valuation rings of F which can be extended to D. As a graph $\mathcal{T}_C$ is also a rooted tree which is labeled naturally in the following way: The vertex V (valuation ring of F) is labeled by l if V has exactly l extensions to D, i.e., V is the centre of exactly l valuation rings of D. For example, if $\mathcal{T}$ is invariant, then $l = 1$ for each vertex, and this happens if and only if $\mathcal{T}$ and $\mathcal{T}_C$ are isomorphic. If $\mathcal{T}$ is a valuation tree of a finite-dimensional division algebra, then $\mathcal{T}_C$ is labeled in a special way (which will be described later), and we say that $\mathcal{T}_C$ is a labeled tree of valuation type. It turns out that the labeled tree $\mathcal{T}_C$ determines the valuation tree $\mathcal{T}$ completely. The main result in this connextion is Theorem 6.1, which states that each finite, labeled, rooted tree of valuation type can be realized as the central image $\mathcal{T}_C$ of the valuation tree $\mathcal{T}$ of a finite-dimensional division algebra D. Additionally, D can be chosen as a tensor product of symbol algebras.

2. Extending valuation rings

Throughout the paper all rings are associative having a unit-element 1. By a division algebra D we shall understand a ring in which all nonzero elements have a two-sided inverse, and D is said to be a finite-dimensional division algebra if D is finite-dimensional over its centre F, i.e., $\dim_F D < \infty$. In this paper D always denotes a finite-dimensional division algebra with centre F, and $\mathcal{B}$ the set of all valuation rings of D, i.e., the set of all subrings B of D with the property $x \in B$ or $x^{-1} \in B$ for all nonzero x in D. If B is a valuation ring of D, then $V := B \cap F$ is a valuation ring of F and B is called extension of V to D. In this paper we deal with division algebras D having more than one but only a finite number of valuation rings, i.e., the centre F of D has an infinite number of valuation rings but only a finite number of those are extendible to D. In this section we will provide the main ideas which lead to such constructions.

First of all, we have to distinguish between invariant and noninvariant valuation rings.

Definition. A valuation ring B of D is called invariant if $dBd^{-1} = B$ for all nonzero d in D.

Invariant valuation rings are closely related to valuations: Let Γ be an ordered multiplicatively written group and $\hat{\Gamma} = \Gamma \cup \{0\}$, where $\gamma \cdot 0 = 0 \cdot \gamma = 0$ and $0 < \gamma$ for all $\gamma \in \Gamma$. Then a surjective mapping $v : D \longrightarrow \hat{\Gamma}$ is called a valuation (of D) if the following are satisfied for all $a, b \in D$:

(V1) $v(a) = 0 \Leftrightarrow a = 0$.
(V2) $v(a \cdot b) = v(a) \cdot v(b)$.
(V3) $v(a + b) \leq \max\{v(a), v(b)\}$.

Γ is called the valuation group of v, and often we write Γ_v instead of Γ if necessary. v is said to be a real valuation whenever Γ_v is a nontrivial subgroup of the multiplicative group $\mathbb{R}^+$ of all positive real numbers, and a real valuation v is said to be discrete if Γ_v is cyclic. Each valuation v of D defines an invariant valuation ring of D, namely $B_v = \{x \in D \mid v(x) \leq 1\}$, and B_v is called the valuation ring of v. Each invariant valuation ring is the valuation ring of a suitable valuation.

A valuation v of a division algebra D is called an extension of a valuation v' of a subdivision algebra D' if $v(x) = v'(x)$ holds for all $x \in D'$. In this case $\Gamma_{v'}$ is a subgroup of Γ_v, and the index $[\Gamma_v : \Gamma_{v'}]$ is called the ramification index of v, denoted by $e(v/v')$ or just by e_v. In contrast to this numerical invariant, the residue degree can also be defined for total valuation rings which are not necessarily invariant. Let B be a total valuation ring of D. Then B has a unique maximal ideal, which is also the Jacobson radical $J(B)$, and $\overline{B} = B/J(B)$ is a finite-dimensional division algebra. $\overline{B}$ is called the residue division algebra of B, and its elements are written as $\overline{b}$, where $b \in B$ and $\overline{b} = b + J(B)$. Furthermore, the characteristic of $\overline{B}$ is said to be the residue characteristic of B or v if v is a valuation with corresponding valuation ring B. If D' is a subdivision algebra of D and $B' = B \cap D'$, then B' is a total valuation ring of D', and $\overline{B'}$ can be understood as a subdivision algebra of $\overline{B}$ since B covers B' and $J(B') = J(B) \cap B'$. The degree $[\overline{B} : \overline{B'}]$ is said to be the residue class degree of B, denoted by $f(B/B')$ or just by f_B. If B is invariant, then $B = B_v$ for some valuation v of D, and $B' = B_{v'}$, where v' is the restriction of v to D'. In this case we also write $f(v/v')$ instead of $f(B/B')$ and f_v instead of f_B.

In general, a valuation v of F need not be extendible to the entire division algebra D. Even if v has no extension to D, the valuation ring B_v can be extendible to a total (non-invariant) valuation ring of D. To deal with this situation, let B_w be a valuation ring of F which has a maximal valuation overring B_v, i.e., B_w is contained in the valuation ring of a real valuation v of F (such a maximal valuation ring need not exist, but in our construction only valuation rings of finite rank will appear—that means valuation rings having only a finite number of overrings). With respect to the extendibility of w or B_w the following results are required.

Theorem 2.1. *With the notations as above the following hold:*

i) v can be extended to a (real) valuation of D if and only if v can be extended uniquely to each (maximal) commutative subfield of D.

ii) Every valuation ring B of D extending B_v is invariant, i.e., v is extendible to D if and only if B_v can be extended to a total valuation ring.

iii) All extensions of B_w to D are conjugate, i.e., if B, B' are total valuation rings of D such that $B \cap F = B' \cap F = B_w$, then $B' = aBa^{-1}$ for some nonzero a in D.

iv) w can be extended to a valuation of D if and only if w can be extended uniquely to each (maximal) commutative subfield of D.

v) If B is an extension of B_w, then B_v has a unique extension to D which also contains B.

vi) If v' is an extension of v to D and if there exists an extension B' of B_w to D such that $B'/M_{v'}$ is invariant in $B_{v'}/M_{v'}$, then the correspondence

$$B \longmapsto (B/M_{v'}) \cap Z$$

defines a bijection between all extensions B of B_w and all extensions of B_w/M_v to Z. In particular, B is an invariant, and therefore the only extension of B_w, if and only if B_w/M_v can be extended uniquely to Z.

More about extensions of valuation rings can be found in [BG]. The next step will be to develop construction principles which provide division algebras with a finite number of *real* valuations. Furthermore, we show that under certain conditions a given finite-dimensional division algebra with real valuations $v_1, \dots, v_n$ can be enlarged to a division algebra D' with only a finite number of real valuations $w_1, \dots, w_n$, where in addition w_i and v_i have the same value group and residue division algebra. But first we need another definition.

Definition. Let F be a commutative field with real valuation v and let $\tilde{F}_v$ be the completion of F with respect to v. Then, the algebraic closure of F in $\tilde{F}_v$ is called the Henselization of v, denoted by $\hat{F}_v$. If $\tilde{v}$ is the canonical extension of v to $\tilde{F}_v$, then $\hat{v}$ denotes the restriction of $\tilde{v}$ to $\hat{F}_v$ and is called the canonical extension of v to $\hat{F}_v$.

Since $\tilde{F}_v$ is an immediate extension of F with respect to $\tilde{v}$, the same holds for the Henselization $\hat{F}_v$, i.e., v and $\hat{v}$ have the same value group and the same residue division algebra. Often we consider the Henselizations of more than one real valuation of F, and there need not be any connextion between them. But if F_a is an algebraic closure of F, then we can find copies of all of them inside F_a, and we can form their intersection, for instance. For the rest of this paper the phrase *Let $\hat{F}_{v_1}, \dots, \hat{F}_{v_n}$ be the Henselizations of $v_1, \dots, v_n$* always assumes that $\hat{F}_{v_1}, \dots, \hat{F}_{v_n}$ lie in a common field extension of F. To simplify notations we do not distinguish between the canonical extension $\hat{v}_i$ of v_i to $\hat{F}_{v_i}$ and the restriction of $\hat{v}_i$ to a subfield of $\hat{F}_{v_i}$. For instance, if K is the intersection of the Henselizations of $v_1, \dots, v_n$, then the restriction of $\hat{v}_i$ to K is also denoted by $\hat{v}_i$.

The following lemma provides one main step in the process of constructing valuation trees.

Lemma 2.2. *Let F be a commutative field and let D be a finite-dimensional division algebra over F such that* $\operatorname{char} F > \sqrt{\dim_F D}$. *Furthermore, let $v_1, \dots, v_n$ be real valuations of F having extensions $w_1, \dots, w_n$ to D, and let $K = \hat{F}_{v_1} \cap \dots \cap \hat{F}_{v_n}$.*

i) $Q = D \otimes_F K$ is a division algebra.

ii) For each $i = 1, \dots, n$ there exists a real valuation u_i of Q extending $\hat{v}_i$ as well as w_i, such that $e(u_i/\hat{v}_i) = e(w_i/v_i)$, $f(u_i/\hat{v}_i) = f(w_i/v_i)$, and $e(u_i/w_i) = f(u_i/w_i) = 1$.

iii) No real valuation v of K different from $\hat{v}_1, \dots, \hat{v}_n$, i.e. $B_v \neq B_{\hat{v}_i}$, is extendible to Q.

iv) If v is an arbitrary nontrivial valuation of K having an extension u to Q, then $B_u \subseteq B_{u_i}$ for some $i = 1, \dots, n$.

Proof. By Cohn's theorem [C] the statements i) and ii) hold for $\tilde{F}_{v_i}$ instead of K. Thus, i) and ii) hold with $\hat{F}_{v_i}$ instead of K, and this shows i) as well as ii). Furthermore, iv) implies iii), so it remains to show iv). Let v be a nontrivial valuation of K. Let us assume that v can be extended to a valuation u of Q such that $B_u \not\subseteq B_{u_i}$ for all i. Then $B_v \not\subseteq B_{\hat{v}_i}$ for all i, i.e., $B_v, B_{\hat{v}_1}, \dots, B_{\hat{v}_n}$ are comaximal since $\hat{v}_1, \dots, \hat{v}_n$ are real. We show that v cannot be extended uniquely

to any maximal commutative subfield L of Q, i.e., by Theorem 2.1 we obtain the contradiction that v cannot be extended to a valuation of Q at all. Let L be a maximal commutative subfield of Q and let us assume that v has a unique extension v' to L. Then $[L:K] = e_{v'} \cdot f_{v'} \cdot p^d$ by [E, 20.21 Theorem], where $p = \operatorname{char} \overline{B_v}$ and d is a nonnegative integer. F as well as K have a positive characteristic, and therefore $p = \operatorname{char} K$. By assumption, p does not divide $\dim_F D = \dim_K Q$, and that means that p does not divide $[L:K]$ since $\dim_K Q = [L:K]^2$. We conclude $[L:K] = e_{v'} \cdot f_{v'}$ and we are done as soon as we can show that $e_{v'} = f_{v'} = 1$.

First we consider $e_{v'}$ and show that each $\gamma < 1$ in $\Gamma_{v'}$ lies in Γ_v. To simplify notations we put $e = e_{v'}$ and obtain $\gamma^e \in \Gamma_v$. Let a be in K and such that $v(a) = \gamma^e < 1$. By the approximation theorem for independent valuations (cf. [E, 11.16 Theorem]) there exist $a_0, \ldots, a_{e-1} \in B_v \cap B_{\hat{v}_1} \cap \ldots \cap B_{\hat{v}_n}$ such that

$$x^e + a_{e-1}x^{e-1} + \cdots + a_1 x + a_0 \equiv x^e - a \bmod a \cdot M_v[x]$$

and

$$x^e + a_{e-1}x^{e-1} + \cdots + a_1 x + a_0 \equiv (x-1) \cdot \ldots \cdot (x-e) \bmod M_{v_i}[x]$$

for $i = 1, \ldots, n$. We want to show that $h(x) = x^e + a_{e-1}x^{e-1} + \cdots + a_1 x + a_0$ has a root α in K, and this means that we must show that h splits completely into e distinct linear factors in each $\hat{F}_{v_i}$. Obviously, h has e distinct roots in $\hat{F}_{v_i}$ if h has e distinct roots in $\tilde{F}_{v_i}$, and this can be verified by Hensel's lemma (cf. [E, 16.7 Lemma]). It remains to prove that h splits in $\overline{B_{v_i}}$. But this is clear by the definition of h, since the inequality

$$\operatorname{char} \overline{B_{v_i}} = p > \dim_F D = \dim_K Q \geq [L:K] \geq e = e_{v_i}$$

ensures that $1, \ldots, e$ are distinct in $\overline{B_{v_i}}$. We conclude that α exists in K, where $v(\alpha^e - a) < v(a)$, i.e., $v(\alpha)^e = v(\alpha^e) = v(a) = \gamma^e$, and therefore $\gamma = v(\alpha) \in \Gamma_v$.

Second, we consider $f_{v'}$ and show that $\overline{B_v} = \overline{B_{v'}}$. Since $[L:K] = f_{v'} = [\overline{B}_{v'} : \overline{B}_v]$, the field $\overline{B_{v'}}$ is a separable extension of $\overline{B_v}$ and there exists $a \in B_{v'}$ such that $\overline{B_{v'}} = \overline{B_v}\,(\overline{a})$, where $\overline{a} = a + M_{v'}$. Let

$$h(x) = x^f + a_{f-1}x^{f-1} + \ldots + a_1 x + a_0 \in B_v[x]$$

be the minimal polynomial of a over K, where $f = f_{v'}$. Then

$$\overline{h}(x) = x^f + \overline{a_{f-1}}x^{f-1} + \ldots + \overline{a_1}x + \overline{a_0} \in \overline{B_v}\,[x]$$

is the minimal polynomial of $\overline{a}$ over $\overline{B_v}$, and therefore $\overline{h(x)}$ is irreducible over $\overline{B_v}$. We are done once we show that $\overline{h(x)}$ has a root in $\overline{B_v}$. Again by the approximation theorem there exist $b_0, \ldots, b_{f-1} \in B_v \cap B_{\hat{v}_1} \cap \ldots \cap B_{\hat{v}_n}$ such that

$$g(x) \equiv x^f + a_{f-1}x^{f-1} + \cdots + a_1 x + a_0 \bmod M_v[x]$$

and

$$g(x) \equiv (x-1) \cdot \ldots \cdot (x-f) \bmod M_{v_i}[x],$$

where $g(x) = x^f + b_{f-1}x^{f-1} + \cdots + b_1 x + b_0$. The same arguments as above ensure that g has a root α in K. Furthermore, α lies in B_v and $\overline{\alpha} \in \overline{B_v}$ is a root of $\overline{h} = \overline{g}$. Since $\overline{h}$ is irreducible over $\overline{B_v}$, we obtain $f = \deg h = 1$, and this completes the proof. $\square$

3. Symbol algebras

Some basic facts about symbol algebras are collected in this section; division algebras of this type or tensor products of them will be used to realize trees as valuation trees. If one is interested just in *invariant* valuation trees only the first part of this section should be read; the second part is needed to construct rooted, labeled trees of valuation type.

Definition. Let F be a commutative field, $n \in \mathbb{N}$ a natural number not divided by $\operatorname{char} F$, and let $\omega_n \in F$ be a primitive nth root of unity. If $a, b \in F^\times$, then $(a,b)_n$ denotes the algebra generated by the two symbols i and j subject to $i^n = a$, $j^n = b$ and $ij = \omega_n ji$.

In this section and in the rest of the paper $\omega_n \in F$ always means that $\operatorname{char} F$ does not divide n and that F contains a primitive nth root of unity denoted by ω_n. Furthermore, the appearance of $(a,b)_n$ in a definition or theorem also signifies $\omega_n \in F$. The algebra $(a,b)_n$ is called symbol algebra, and $(a,b)_n$ is a central-simple algebra over F of dimension n^2 over F. If $n = 2$ then $(a,b)_n = (a,b)_2$ is a quaternion algebra as usual. Generally, it is not easy to decide whether a given symbol algebra $(a,b)_n$ is a division algebra or not, but the construction of division symbol algebras can be very easy if one uses valuation theoretical methods.

Proposition 3.1. *Let F be a commutative field, v a real valuation of F, and $n \in \mathbb{N}$, as well as $\omega_n \in F$. Furthermore, let $\alpha, \beta \in \mathbb{R}^+$ be such that $\alpha^n, \beta^n \in \Gamma_v$ and*

$$(*) \qquad \alpha^\nu \beta^\mu \in \Gamma_v \Longleftrightarrow \nu = \mu = 0$$

holds for all $0 \le \nu, \mu < n$. Then $D = (a,b)_n$ is a division algebra for each $a, b \in F^\times$ whenever $v(a) = \alpha^n$ and $v(b) = \beta^n$. In addition, v can be extended to a valuation w of D such that

$$\Gamma_w = \Gamma_v \cdot \langle \alpha \rangle \cdot \langle \beta \rangle.$$

In particular, $e(w/v) = n^2$.

Proof. An element $d \in D$ can be written as

$$d = \sum_{0 \le \nu, \mu < n} k_{\nu,\mu} i^\nu j^\mu,$$

where $k_{\nu,\mu} \in F$ are uniquely determined. We define

$$w(d) = \max_{\nu,\mu} \{ v(k_{\nu,\mu}) \alpha^\nu \beta^\mu \},$$

and $w(d)$ is a real number. Clearly, the axioms (V1) and (V3) of a valuation are satisfied, and it remains to prove (V2) and that D is a division algebra. But (V2) implies that D has no zero-divisors, and this means that D is a division algebra since D is finite-dimensional over its centre F. To varify (V2), we first observe by condition $(*)$ that

$$(**) \qquad w(k_{\nu,\mu} i^\nu j^\mu) \ne w(l_{\sigma,\tau} i^\sigma j^\tau)$$

for all $\nu, \mu, \sigma, \tau \in \{0, \dots, n-1\}$ and all $k_{\nu,\mu}, l_{\sigma,\tau} \in F^\times$ if $(\nu,\mu) \ne (\sigma,\tau)$. Thus, if $d \ne 0$ is as above, there are uniquely determined ν and μ such that $w(d) = v(k_{\nu,\mu}) \alpha^\nu \beta^\mu$. Now, let

$$d = \sum_{0 \le \nu, \mu < n} k_{\nu,\mu} i^\nu j^\mu \quad \text{and} \quad d' = \sum_{0 \le \sigma, \tau < n} l_{\sigma,\tau} i^\sigma j^\tau$$

be nonzero elements of D such that $w(d) = v(k_{\nu',\mu'})\alpha^{\nu'}\beta^{\mu'}$ as well as $w(d') = v(l_{\sigma',\tau'})\alpha^{\sigma'}\beta^{\tau'}$. Since $v(a) = \alpha^n, v(b) = \beta^n$, and $v(\omega_n) = 1$, exactly one of the n^2 summands of $d \cdot d'$ has maximal value, namely $k_{\nu',\mu'}i^{\nu'}j^{\mu'}l_{\sigma',\tau'}i^{\sigma'}j^{\tau'}$. This means

$$w(d \cdot d') = v(k_{\nu',\mu'}l_{\sigma',\tau'})\alpha^{\nu'\sigma'}\beta^{\mu'\tau'} = w(d)w(d'),$$

and this proves (V2). □

If the application of Proposition 3.1 leads to the construction of $(a,b)_n$, we will refer to this algebra as a *ramified symbol division algebra (with respect to v)*. There are also two other ways division algebras can be obtained.

Proposition 3.2. *Let F be a commutative field with real valuation v, and let $n \in \mathbb{N}$, as well as $\omega_n \in F$, be such that $\overline{\omega_n}$ is a primitive nth root of unity in the residue field $\overline{B_v}$. For all units a,b of B_v the symbol algebra $D = (a,b)_n$ is a division algebra if $(\overline{a},\overline{b})_n$ is a division algebra (over $\overline{B_v}$). In addition, v can be extended to a valuation w of D such that $\overline{B_w} = (\overline{a},\overline{b})_n$. In particular, $f(w/v) = n^2$.*

Proof. We define

$$B := \sum_{\nu,\mu=0}^{n-1} i^{\nu}j^{\mu}B_v,$$

and this is a subring of $(a,b)_n$ where

$$M := \sum_{\nu,\mu=0}^{n-1} i^{\nu}j^{\mu}M_v$$

is an ideal of B such that $B/M \cong (\overline{a},\overline{b})_n$. To show that $(a,b)_n$ is a division algebra we prove that $(a,b)_n$ has no zero-divisors. Thus, let d, d' be nonzero elements of $(a,b)_n$. Then there exist $x, x' \in F$ such that $xd, x'd' \in B \setminus M$ and $xd \cdot x'd'$ cannot be in M since B/M is a division algebra, i.e., $xd \cdot x'd'$ as well as dd' cannot be zero. Finally, v is extendible if there is an invariant valuation ring B' of $(a,b)_n$ satisfying $B' \cap F = B_v$. We are done if we show that B is a total valuation ring of $(a,b)_n$, and then B is invariant for v is real. Let d be not in B and let x be as above. Then x must be in M_v. If $d^{-1} \notin B$, then $yd^{-1} \in B \setminus M$ for some $y \in M_v$, and we get $xy = xd \cdot yd^{-1} \in B \setminus M$, a contradiction. □

If the application of Proposition 3.2 leads to the construction of $(a,b)_n$ we will refer to this algebra as an *inert symbol division algebra (with respect to v)*. The third type of division algebra is the *mixed symbol division algebra*, which can be obtained as follows:

Proposition 3.3. *Let F be a commutative field with real valuation v, and let $n \in \mathbb{N}$, as well as $\omega_n \in F$, be such that $\overline{\omega_n}$ is a primitive nth root of unity in the residue field $\overline{B_v}$. Furthermore, let a be a unit in B_v such that $x^n - \overline{a}$ is irreducible over $\overline{B_v}$, and let $b \in F^{\times}$ be such that $v(b)^{\frac{\nu}{n}} \notin \Gamma_v$ for all $\nu \in \{1, \dots, n-1\}$. Then, $(a,b)_n$ is a division algebra, and v can be extended to a valuation w of $(a,b)_n$ such that $\overline{B_w} = \overline{B_v}(\alpha)$, where $Irr(\alpha, \overline{B_v}) = x^n - \overline{a}$ and $\Gamma_w = \Gamma_v \cdot \langle v(b)^{\frac{1}{n}} \rangle$. In particular, $e(w/v) = f(w/v) = n$.*

The proof of this proposition is a combination of the proofs given above, and will be omitted. Often we apply Proposition 3.3 when v is discrete. Then, if we

choose b such that $v(b)$ is a generator of Γ_v, the condition $v(b)^{\frac{\nu}{n}} \notin \Gamma_v$ for all $\nu \in \{1, \dots, n-1\}$ obviously holds.

The three types of division algebras described so far are said to be of valuation type (with respect to v), and they form the main components of our constructions in this paper. Frequently, it can happen that several algebras must be "combined" to define a new "bigger" one with very similar properties. This leads to the following:

Definition. Let $(a,b)_n$ and $(c,d)_n$ be symbol division algebras over the same centre F of valuation type with respect to v. Then $(a,b)_n$ and $(c,d)_n$ are valuation-equivalent ($(a,b)_n \overset{v}{\sim} (c,d)_n$) if

$$(*) \qquad v(a-c) < \max\{v(a), v(c)\} \quad \text{and} \quad v(b-d) < \max\{v(b), v(d)\}.$$

If $(a,b)_n \overset{v}{\sim} (c,d)_n$, then v has an extension u to $(a,b)_n$ and an extension w to $(c,d)_n$, where u and w have the same residue division algebras and value groups. More precisely, we have the following situation:

i) If $(a,b)_n$ is ramified, then $(c,d)_n$ is ramified too and $\Gamma_u = \Gamma_w$.

ii) If $(a,b)_n$ is inert, then $(c,d)_n$ is inert too and $\overline{B_u} = (\overline{a}, \overline{b})_n = (\overline{c}, \overline{d})_n = \overline{B_w}$, where $\overline{a} = \overline{c}$ and $\overline{b} = \overline{d}$.

iii) If $(a,b)_n$ is mixed, then $(c,d)_n$ is mixed too and $\overline{B_u} = \overline{B_v}\,(\alpha) = \overline{B_v}\,(\beta) = \overline{B_w}$, where $\mathrm{Irr}(\alpha, \overline{B_v}) = x^n - \overline{a} = x^n - \overline{c} = \mathrm{Irr}(\beta, \overline{B_v})$ and $\Gamma_u = \Gamma_w$.

Furthermore, if $(a,b)_n$ is of valuation type and $c, d \in F^\times$ are such that $(*)$ holds (cf. the definition above), then $(c,d)_n$ is of valuation type too and $(a,b)_n \overset{v}{\sim} (c,d)_n$. Later on, this will be used often without any further explanations, especially in situations where several division algebras $(a_1,b_1)_n, \dots, (a_m,b_m)_n$ with distinct centres are given and $(a,b)_n$ with valuations $v_1, \dots, v_m$ is needed such that $\overline{B_{v_i}} \cong (a_i,b_i)_n$, $i = 1, \dots, m$.

While each finite rooted tree can be realized as an invariant valuation tree of a symbol division algebra, we will use tensor products of those algebras in connextion with noninvariant valuation trees (cf. section 6). One main problem here is to obtain symbol division algebras such that their tensor product is a division algebra too, and this difficulty can be overcome by a helpful result of P. Morandi (cf. [Mo]). We will describe this for our special situation: Let $(a_1,b_1)_{r_1}, \dots, (a_\rho,b_\rho)_{r_\rho}$ be ramified symbol division algebras with respect to v and let $u_1, \dots, u_\rho$ be the corresponding valuations which extend v (cf. Proposition 3.1). Furthermore, let $(c_1,d_1)_{s_1}, \dots, (c_\sigma,d_\sigma)_{s_\sigma}$ be inert symbol division algebras, where $v_1, \dots, v_\sigma$ are the corresponding valuations which extend v (cf. Proposition 3.2), and finally let $(e_1,f_1)_{t_1}, \dots, (e_\tau,f_\tau)_{t_\tau}$ be mixed symbol division algebras, where $w_1, \dots, w_\tau$ are the corresponding extentions of v (cf. Proposition 3.3). Then, Morandi's criterion (cf. [Mo, Theorem 1]) says that the tensor product of all symbol division algebras is a division algebra (which will be denoted by D) if the following are satisfied:

1) The tensor product of all residue division algebras is a division algebra, i.e.,

$$(\overline{c}_1, \overline{d}_1)_{s_1} \otimes \dots \otimes (\overline{c}_\sigma, \overline{d}_\sigma)_{s_\sigma} \otimes \overline{B_v}\,(\sqrt[t_1]{\overline{e}_1}) \otimes \dots \otimes \overline{B_v}\,(\sqrt[t_\tau]{\overline{e}_\tau})$$

is a division algebra.

2) $[\Gamma : \Gamma_v] = r_1^2 \cdot \ldots \cdot r_\rho^2 \cdot t_1 \cdot \ldots \cdot t_\tau$, where $\Gamma = \Gamma_{u_1} \cdot \ldots \cdot \Gamma_{u_\rho} \cdot \Gamma_{w_1} \cdot \ldots \cdot \Gamma_{w_\tau}$ is the subgroup of $\mathbb{R}^+$ which is generated by $\Gamma_{u_1}, \dots, \Gamma_{u_\rho}, \Gamma_{w_1}, \dots, \Gamma_{w_\tau}$.

3) $\mathrm{char}\,\overline{B_v}$ does not divide $r_1 \cdot \ldots \cdot r_\rho \cdot s_1 \cdot \ldots \cdot s_\sigma \cdot t_1 \cdot \ldots \cdot t_\tau$.

Furthermore, there exists a real valuation w of D that extends each u_i, v_j, and w_k. The residue division algebra of w is canonically isomorphic to the tensor product

of the residue division algebra as described in 1), and the group Γ introduced in 2) is the value group of w.

The tensor product of symbol division algebras as described above is very useful, and we will call it a *tensor product of valuation type*. For instance, if D' is another central simple algebra over F which is obtained from D by replacing some of the symbol algebras with valuation-equivalent symbol algebras with respect to v, then D' is obviously a division algebra too. Furthermore, D' has a real valuation w' which extends v such that w and w' have the "same" value group and the "same" residue division ring, and we say that D and D' are *valuation-equivalent*. This principle will be used for constructing division algebras (which are tensor products of symbol algebras) with real valuations such that their residue division algebras (which are also tensor products of symbol algebras) are of a predetermined type.

4. Invariant valuation trees

In this section we will show that each finite, rooted tree with more than one vertex can be realized as a valuation tree of a symbol division algebra D of arbitrary degree where the centre F of D has finite transcendence degree (the transcendence degree of a field F over its prime field will be abbreviated to $\mathrm{tr}F$). Furthermore, in our constructions F has a positive characteristic which can be, up to a finite number of exceptions, any prime number, i.e., F is an algebraic extension of $\mathbb{Z}_p(x_1, \dots, x_n)$.

Lemma 4.1. *Let $F_1, \dots, F_n$ be commutative fields of finite transcendence degree with the same characteristic $p > n$. Then there exists a commutative field F with* $\mathrm{char}\, F = p$ *and finite transcendence degree which has discrete valuations $v_1, \dots, v_n$ such that $\overline{B_{v_i}} \cong F_i$, $i = 1, \dots, n$. Furthermore, if $\gamma_1, \dots, \gamma_n \in \mathbb{R}^+$ are given with $\gamma_1, \dots, \gamma_n < 1$, then $\Gamma_{v_i} =< \gamma_i >$ can also be realized for $i = 1, \dots, n$.*

Proof. Let $N = \max_{i=1,\dots,n}\{n_i\}$ where $n_i = \mathrm{tr}F_i$, and let $L = \mathbb{Z}_p(x_1, \dots, x_N)(x)$ be the rational function field over $\mathbb{Z}_p$ in $N + 1$ indeterminates. First of all, we define for each $i \in \{1, \dots, n\}$ a discrete valuation u_i of L such that $\mathrm{tr}\overline{B_{u_i}} = n_i$. Clearly, if u_i denotes the $(x - i)$-adic valuation of $L_i = \mathbb{Z}_p(x_1, \dots, x_{n_i})(x)$ with $u_i(x - i) = \gamma_i$, then $\Gamma_{u_i} = \langle \gamma_i \rangle$ as well as $\overline{B_{u_i}} = \mathbb{Z}_p(x_1, \dots, x_{n_i})$, and we consider $\overline{B_{u_i}}$ as a subfield of F_i. Let $\tilde{L}_i$ be the completion of L_i with respect to u_i. Since L_i is countable where $\tilde{L}_i$ is not, $L = L_i(x_{n_i+1}, \dots, x_N)$ can be understood as a subfield of $\tilde{L}_i$ and u_i has an extension to L (which is also denoted by u_i) such that L is immediate over L_i, i.e., u_i is a discrete valuation of L with $\overline{B_{u_i}} = \mathbb{Z}_p(x_1, \dots, x_{n_i})$. By assumption $p > n$, and therefore $u_1, \dots, u_n$ are distinct in the sense that they have pairwise distinct valuation rings. Now, let A be an algebraic closure of L; we show that there exists a subfield F of A containing L such that each u_i can be extended to a discrete valuation v_i of F satisfying $\overline{B_{v_i}} \cong F_i$ and $\Gamma_{v_i} = \langle \gamma_i \rangle$. Symmetry and transfinite methods allow us to reduce the proof of the lemma to the verification of the following statement:

If K is a subfield of A containing L such that each u_i can be extended to a valuation w_i of K, where $\Gamma_{w_i} = \langle \gamma_i \rangle$ and $\overline{B_{w_i}}$ is a subfield of F_i, and if $\overline{B_{w_1}}(\alpha) \subseteq F_1$ is a finite extension of $\overline{B_{w_1}}$ inside F_1, then there exists $\beta \in A$ such that the following hold:

a) w_1 can be extended to a valuation v_1 of $K(\beta)$ such that $\overline{B_{v_1}} = \overline{B_{w_1}}(\alpha)$ and $e(v_1/w_1) = 1$.

b) w_i can be extended to a valuation v_i of $K(\beta)$ such that $e(v_i/w_i) = f(v_i/w_i) = 1$ for $i \neq 1$.

To prove this statement, let $h(x) = x^m + h_{m-1}x^{m-1} + \ldots + h_1 x + h_0 \in B_{w_1}[x]$ be such that $\overline{h}(x) = x^m + \overline{h_{m-1}}x^{m-1} + \ldots + \overline{h_1}x + \overline{h_0}$ is the minimal polynomial of α over $\overline{B_{w_1}}$. By the approximation theorem there exist $g_0, \ldots, g_{m-1} \in B_{w_1} \cap \ldots \cap B_{w_n}$ such that

a′) $g(x) = x^m + g_{m-1}x^{m-1} + \ldots + g_1 x + g_0 \equiv x^m + h_{m-1}x^{m-1} + \ldots + h_1 x + h_0 \bmod M_{w_1}[x]$, and

b′) $g(x) \equiv x(x-1)^{m-1} \bmod M_{w_i}[x]$ for $i > 1$.

Clearly, $g(x)$ is irreducible over K since $\overline{g}(x)$ is irreducible mod M_{w_1}, and if $\beta \in A$ is a root of $g(x)$ then a) holds by a′), and b) holds since $g(x)$ has a root in the completion of K with respect to w_i $(i > 1)$ by b′). □

With the notations as in the lemma above we can assume $\omega_l \in F$ whenever $\omega_l \in F_i$ for all $i = 1, \ldots, n$, since in this case we can modify the proof by defining $L = \mathbb{Z}_p(\omega_l)(x_1, \ldots, x_N)(x)$.

Theorem 4.2. *Each finite, rooted tree $\mathcal{T}$ is the valuation tree of a finite-dimensional division algebra D over a commutative field F. Moreover, if $\mathcal{T}$ has more than one vertex then the following additional conditions can be realized:*

i) F has finite transcendence degree.

ii) $\sqrt{\dim_F D} = n$ can be any natural number greater than one.

iii) D is a symbol division algebra.

iv) Each total valuation ring of D is invariant.

v) With at most a finite number of exceptions, $\operatorname{char} F$ *can be any prime p.*

Proof. Let N be the number of vertices of $\mathcal{T}$. If $N = 1$, a finite-dimensional division algebra is required which does not have any proper valuation ring. This special example will be given right after the proof of the theorem in Remark 1. Thus, let $N > 1$; we prove the theorem by induction on N, where the case $N = 2$ is included in the following process (we will refer to this later on). To simplify the notation we do not distinguish between the vertices of $\mathcal{T}$ and the corresponding valuation rings, althought misinterpretations are foreseeable. So, let D be the root of $\mathcal{T}$ and let $B_1, \ldots, B_r$ be the lower neighbours of D, and $\mathcal{T}_1, \ldots, \mathcal{T}_r$ the subtrees of $\mathcal{T}$ which are the lower sections of $\mathcal{T}$ defined by each B_i. Later on $\mathcal{T}_i$ will be the set of all total valuation rings of D contained in B_i. Now, it can happen that $\mathcal{T}_i$ only contains B_i, and these cases must be treated separately. To do this let $\mathcal{T}_i$ have more than one element exactly for $i = 1, \ldots, s$, where $s \leq r$. The next step is the construction of a division algebra D with real valuations $w_1, \ldots, w_r$ such that the valuation tree of the residue division algebra of w_i is precisely $\mathcal{T}_i$.

By the induction hypothesis, for each $i \leq s$ there exist a commutative field F_i of finite transcendence degree and a symbol division algebras $D_i = (a_i, b_i)_n$ over F_i with $\mathcal{T}_i$ as its valuation tree. For $i > s$ put $F_i = \mathbb{Z}_p(\omega_n)$ and apply Lemma 4.1 to obtain a commutative field F of finite transcendence degree with discrete valuations $v_1, \ldots, v_r$ such that $\overline{B_{v_i}} \cong F_i$. According to the remark after Lemma 4.1 we can also assume $\omega_n \in F$. Now, we can write $\overline{B_{v_i}}$ instead of F_i for all i, and for $i \leq s$ we have in addition $D_i = (\overline{a_i}, \overline{b_i})_n$, where $a_i, b_i \in B_{v_i}$, as well as $\overline{a_i} = a_i + M_{v_i}$, $\overline{b_i} = b_i + M_{v_i}$. To complete our construction we define $a_i, b_i \in B_{v_i}$ for the remaining $i > s$ as follows. Let K_i be a commutative field extension of $\mathbb{Z}_p(\omega_n)$ of degree n. Then there exists $\alpha_i \in K_i$ such that $K_i = \mathbb{Z}_p(\omega_n, \alpha_i)$ and

$\alpha_i^n = a_i \in \mathbb{Z}_p(\omega_n)$, i.e., $x^n - a_i$ is the minimal polynomial of α_i over F, and also over $F_i = \overline{B_{v_i}}$. Finally, let $b_i \in F$ be such that $v_i(b_i)$ is a generator of the cyclic group Γ_{v_i}. Now, we are ready to define a symbol division algebra D which is almost the desired one. By the approximation theorem there exist $a, b \in F^\times$ such that

$$v_i(a - a_i) < v_i(a_i), \qquad v_i(b - b_i) < v_i(b_i)$$

for all $i = 1, \dots, r$. For $i \leq s$ we obtain the symbol algebra $(a_i, b_i)_n$ over F, which turns out to be an inert symbol division algebra with respect to v_i by Proposition 3.2, and then $(a, b)_n$ is an inert symbol division algebra over F with respect to v_i such that $(a_i, b_i)_n \overset{v_i}{\sim} (a, b)_n$. For $i > s$ we obtain the symbol algebra $(a_i, b_i)_n$ over F, which is a mixed symbol division algebra with respect to v_i by Proposition 3.3, and $(a_i, b_i)_n \overset{v_i}{\sim} (a, b)_n$. Each v_i has an extension w_i according to Propositions 3.2, 3.3.

We summarize our construction obtained so far. $D = (a, b)_n$ is a symbol division algebra over F (where F has finite transcendence degree) with real valuations $w_1, \dots, w_r$ such that the following hold:

a) $\overline{B_{w_i}}$ has the valuation tree $\mathcal{T}_i$ for $i \leq s$.

b) $\overline{B_{w_i}}$ is a finite field, i.e., $\overline{B_{w_i}}$ has also the valuation tree $\mathcal{T}_i$ for $i > s$.

c) $\overline{B_{v_i}}$ is the centre of $\overline{B_{w_i}}$ for $i \leq s$.

Now, we apply Lemma 2.2, and hence, in addition to a), b), and c), the following condition can also be assumed:

d) If B is a proper total valuation ring of D, then $B \subseteq B_{w_i}$ for some i.

Actually, Lemma 2.2 only provides d) where B is an invariant valuation ring, but each proper total valuation ring of D is contained in an invariant one by [BG, Lemma 4] and therefore d) indeed holds.

It remains to check that D has all properties as claimed. Clearly, in the valuation tree of D the lower neighbours of D are precisely $B_{w_1}, \dots, B_{w_r}$, and if B is any proper total valuation ring of D, then $B \subseteq B_{w_i}$ for some i and B/M_{w_i} fits into the valuation tree $\mathcal{T}_i$ of $\overline{B_{w_i}}$. It remains to check that each total valuation ring of D is invariant. But this is an immediate consequence of Theorem 2.1 vi) and of c).

The arguments above also provide a *direct* proof of a special version of the theorem where $\mathcal{T}$ only consists of a root and lower neighbours of the root, since this is the special case $s = 0$ where the induction hypothesis is not used. As promised before, this covers the omitted part $N = 2$. □

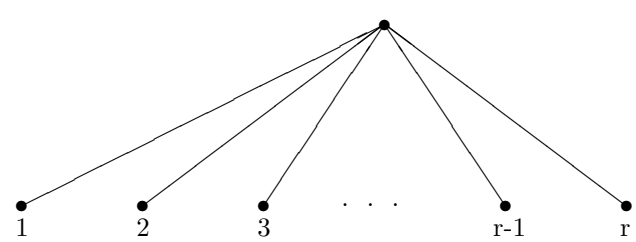
Remark. 1) A finite-dimensional division algebra which does not have any proper valuation ring can be obtained as follows: We are done if a finite-dimensional division algebra over a global field is given which does not have any real valuation, since each nontrivial total valuation ring of such an algebra is invariant and belongs to a real valuation. An example of this type can be obtained as follows: Let $(-1, -1)_2$ be Hamilton's quaternion algebra over $\mathbb{Q}$. It is well-known that only the $2-$adic valuation v_2 of $\mathbb{Q}$ is extendible to $(-1, -1)_2$. We define $D := (-1, -1)_2 \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})$, which is obviously a division algebra and it remains to show that v_2 cannot be extended to D, i.e. , $(-1, -1)_2 \otimes_{\mathbb{Q}} \mathbb{Q}_2(\sqrt{3})$ is not a division algebra, where $\mathbb{Q}_2$ denotes the completion of $\mathbb{Q}$ with respect to v_2. But this is clear if $x^2 + y^2 + 3z^2 = 0$ has a nontrivial solution in $\mathbb{Q}_2$ (cf. [P, 1.6. Proposition]). The last statement follows by [BS, Kap. I, §6, Satz 4] and the fact that $3^2 + 2^2 + 3 \cdot 1^2 \equiv 0 \bmod 16$.

2) The proof of the theorem above signifies that statement v) can be sharpened a little bit in the following way: *If t is the number of vertices of $\mathcal{T}$, then* $\operatorname{char} F$ *can be any prime* $p > \max\{n, t\}$.

3) The proof also shows that F can be chosen such that $\omega_l \in F$ for all $l \in \mathbb{N}$, $l \leq n$.

We close this section with some examples. Since the application of Lemma 4.1 in the proof of Theorem 4.2 indicates that the centres of the desired division algebras can be very complicated (as they are intersections of Henselizations), we restrict ourselves to a class of rooted trees with simple structure which can be realized by specific quaternion division algebras.

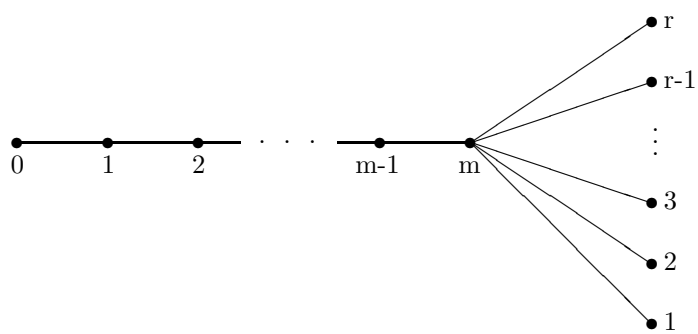
We start with a quaternion division algebra over the global field $\mathbb{Z}_p(x)$. Let $F = \mathbb{Z}_p(x)$ be the rational function field over $\mathbb{Z}_p$ in one variable, where $p > 2$ is such that -1 is a square in $\mathbb{Z}_p$, i.e. $p \equiv 1 \bmod 4$. Furthermore, let $a \in \mathbb{Z}_p^\times$ be not a square and let $b = (x-1) \cdot \ldots \cdot (x-r)$, where r is even and $r < p$. We show that $(a,b)_2$ is a division algebra with r proper valuation rings, or equivalently that exactly r valuations of $\mathbb{Z}_p(x)$ can be extended to $(a,b)_2$. Then, the valuation tree of $(a,b)_2$ has apart from the root exactly r vertices, which are in addition lower neighbours of the root. The set of all valuations of F consists of the f-adic valuations v_f, where f is an irreducible, monic polynomial in $\mathbb{Z}_p[x]$, and v_∞. The valuation $v = v_f$ or $v = v_\infty$ can be extended to $(a,b)_2$ whenever $(a,b)_2 \otimes_F \tilde{F}$ is a division algebra; here $\tilde{F}$ is the completion of F with respect to v. We consider $v = v_f$, where $f(x) \notin \{x-1, \ldots, x-r\}$. If b is a square mod M_v, then b is a square in $\tilde{F}$ and $(a,b)_2 \otimes_F \tilde{F}$ is not a division algebra. The same holds if a is a square mod M_v. Thus, let us assume that a, b are nonsquares mod M_v, which means that $a \cdot b$ is a square mod M_v (since $\overline{B_v}$ is finite and p odd) and therefore a square in $\tilde{F}$. We apply [S, 11.13 Corollary] and obtain $(a,b)_2 \cong (a(ab),b)_2 \cong (b,b)_2 \cong (b,-1)_2$ as algebras over $\tilde{F}$. Since -1 is a square in $\mathbb{Z}_p$, we conclude that $(a,b)_2 \otimes_F \tilde{F}$ is not a division algebra. The same is true for $v = v_\infty$, since r is even and therefore $b = x^r(1-\frac{1}{x})(1-\frac{2}{x}) \cdot \ldots \cdot (1-\frac{r}{x})$ is a square in $\tilde{F}$. Thus, the only candidates for extendible valuations are $v_{x-1}, \ldots, v_{x-r}$. But for each of these the value of $b = (x-1) \cdot \ldots \cdot (x-r)$ is a generator of the corresponding value group. We can apply Proposition 3.3 to conclude that with the notations as above $(a,b)_2$ is indeed a quaternion division algebra with the following valuation tree.



Let $Q = (a,b)_2$ be as above and let $D = Q((x))$ be the division algebra of all formal Laurent series in a central indeterminate x. Then D is a quaternion division algebra with centre $F((x))$, and

$$B_w = \{q_0 + q_1 x + q_2 x^2 + \ldots \mid q_0, q_1, q_2 \ldots \in Q\}$$

is an invariant valuation ring of D which belongs to a real valuation w such that $\overline{B_w} = (a,b)_2$. If v is the restriction of w to $F((x))$, then $F((x))$ is complete with respect to v. Thus, v is a Henselian valuation, and Lemma 2.2 shows that each proper valuation ring B of D is contained in B_w. The valuation tree $\mathcal{T}$ of D can be described as follows: $\mathcal{T}$ has $r+2$ vertices; D is the root of $\mathcal{T}$, and B_w is the only lower neighbour of B_w, where B_w has exactly r lower neighbours. We can repeat this process and obtain the quaternion division algebra $D = (a,b)_2((x_1))((x_2))\dots((x_m))$. The valuation tree of D has $r+m+1$ vertices and has the following representation.



5. The p-part of Henselizations and their intersections

This section contains some technical results which are needed later on. We introduce the notation of the p-part of Henselizations and their intersections. This construction is useful if one wants to obtain division algebras with only a finite number of real valuations $v_1, \dots, v_n$ such that the valuation rings B_{v_i} may contain noninvariant valuation rings. We will apply these results in the next section, where we deal with noninvariant valuation trees.

Let F be a commutative field with real valuations $v_1, \dots, v_n$, and let $\hat{F}_{v_1}, \dots, \hat{F}_{v_n}$ be the Henselizations of $v_1, \dots, v_n$, which are subfields of a common algebraic closure of F. Furthermore, let $\hat{v}_i$ be the canonical extension of v to $\hat{F}_{v_i}$, $i = 1, \dots, n$. We define $\hat{F}(v_1, \dots, v_n)$ to be the intersection $\hat{F}_{v_1} \cap \dots \cap \hat{F}_{v_n}$ of all Henselizations, and for a fixed prime p we define the field $\hat{F}^p(v_1, \dots, v_n)$ as a maximal p-extension of F in $\hat{F}(v_1, \dots, v_n)$, i.e., if K is a finite field extension of F in $\hat{F}^p(v_1, \dots, v_n)$, then $[K : F]$ is a p-power, and with this property $\hat{F}^p(v_1, \dots, v_n)$ is maximal. The restriction of $\hat{v}_i$ to $\hat{F}^p(v_1, \dots, v_n)$ or any subfield of $\hat{F}^p(v_1, \dots, v_n)$ will also be denoted by $\hat{v}_i$, $i = 1, \dots, n$. We call $\hat{F}^p(v_1, \dots, v_n)$ a p-part of the intersection of the Henselizations of $v_1, \dots, v_n$. In the rest of this section these notations will be fixed.

Lemma 5.1. *If* $\operatorname{char} F > p$, *then for each real valuation* v *of* $\hat{F}^p(v_1, \dots, v_n)$ *different to* $\hat{v}_1, \dots, \hat{v}_n$, *the following hold:*

i) There does not exist any finite field extension of $\overline{B_v}$ *of degree* p.

ii) For each $\gamma \in \Gamma_v$ *there exists* $\delta \in \Gamma_v$ *such that* $\delta^p = \gamma$.

Proof. To prove i) we assume that there exists

$$f(x) = x^p + a_{p-1}x^{p-1} + \dots + a_1 x + a_0 \in B_v[x]$$

such that

$$\overline{f}(x) = x^p + \overline{a_{p-1}}x^{p-1} + \ldots + \overline{a_1}x + \overline{a_0} \in \overline{B_v}\ [x]$$

is irreducible over $\overline{B_v}$. By the approximation theorem there are $b_0, \ldots, b_{p-1} \in \hat{F}^p(v_1, \ldots, v_n)$ such that

$(*)$ $g(x) := x^p + b_{p-1}x^{p-1} + \ldots + b_1x + b_0 \equiv (x-1) \cdot \ldots \cdot (x-p) \bmod M_{\hat{v}_i}[x]$

and

$$g(x) \equiv f(x) \bmod M_v[x].$$

By $(*)$, since $\operatorname{char} \overline{B_{\hat{v}_i}} = \operatorname{char} F > p$, we see that g splits completely into p distinct linear factors mod $M_{\hat{v}_i}$. By Hensel's lemma (cf. [E, 16.7 Lemma]) g splits completely into p distinct linear factors over each $\hat{F}_{v_i}$, and therefore g has a root α in $\hat{F}(v_1, \ldots, v_n)$. But $g(x) \equiv f(x) \bmod M_v$ shows that g is irreducible over $\hat{F}^p(v_1, \ldots, v_n)$ and $\hat{F}^p(v_1, \ldots, v_n)(\alpha)$ is a field extension of $\hat{F}^p(v_1, \ldots, v_n)$ of degree p inside $\hat{F}(v_1, \ldots, v_n)$, which contradicts the maximality of $\hat{F}^p(v_1, \ldots, v_n)$.

The proof of ii) is very similar. Let us assume that $\gamma < 1$ and there exists $a \in \hat{F}^p(v_1, \ldots, v_n)$ with $v(a) = \gamma < 1$. We define $f(x) = x^p - a \in \hat{F}^p(v_1, \ldots, v_n)[x]$, and again the approximation theorem yields a monic polynomial $g \in \hat{F}^p(v_1, \ldots, v_n)[x]$ of degree p such that

$$g(x) \equiv (x-1) \cdot \ldots \cdot (x-p) \bmod M_{\hat{v}_i}[x]$$

and

$$g(x) \equiv f(x) \bmod a \cdot M_v[x].$$

As above, g has a root α in $\hat{F}(v_1, \ldots, v_n)$. If w denotes an extension of v to $\hat{F}^p(v_1, \ldots, v_n)(\alpha)$, then $w(\alpha) \leq 1$ and $w(\alpha^p - a) < w(a)$, i.e., $w(\alpha)^p = w(\alpha^p) = w(a) = \gamma$. It remains to show that $w(\alpha)$ lies in the value group of v. If this is not the case then $e(w/v) \geq p$, i.e., $e(w/v) = p$ and g must be irreducible over $\hat{F}^p(v_1, \ldots, v_n)$, which is a contradiction as in the proof of i). □

The main application of the p-part of Henselizations and their intersections is the following:

Theorem 5.2. *Let D be a finite-dimensional division algebra with centre F and let $v_1, \ldots, v_n$ be real valuations of F with extensions $w_1, \ldots, w_n$ to D. Then the following hold.*

i) $D^p = D \otimes_F \hat{F}^p(v_1, \ldots, v_n)$ is a division algebra.

ii) For each $i = 1, \ldots, n$ there exists a real valuation w_i^p of D^p which extends w_i as well as the valuation $\hat{v}_i$ of $\hat{F}^p(v_1, \ldots, v_n)$ such that

$$e(w_i^p/\hat{v}_i) = e(w_i/v_i),$$
$$f(w_i^p/\hat{v}_i) = f(w_i/v_i),$$
$$e(w_i^p/w_i) = f(w_i^p/w_i) = 1.$$

iii) If p divides $\sqrt{\dim_F D}$ but p^2 does not, and if $\operatorname{char} F > p$, then $w_1^p, \ldots, w_n^p$ are the only real valuations of D^p.

Proof. By Cohn's theorem [C] the statements i) and ii) hold for $\tilde{F}_{v_i}$ instead of $\hat{F}^p(v_1, \ldots, v_n)$. Thus, i) and ii) hold with $\hat{F}_{v_i}$ instead of $\hat{F}^p(v_1, \ldots, v_n)$, and this verifies i) as well as ii). To prove iii) we show that no real valuation v of $\hat{F}^p(v_1, \ldots, v_n)$ different to $\hat{v}_1, \ldots, \hat{v}_n$ can be extended to D^p. Since p divides $\sqrt{\dim_F D}$ but p^2 does not, we conclude that $D^p = D_1 \otimes D_2$, where $\deg D_1 = p$,

by the primary decomposition theorem for central simple algebras (cf. [P, §14.4]). We are done if we show that v cannot be extended uniquely to any maximal commutative subfield L of D_1, since then v cannot be extended to D_1 and therefore not to $D_1 \otimes D_2$ either by Theorem 2.1. Now, if v has a unique extension w to L, then $p = [L : \hat{F}^p(v_1, \dots, v_n)] = e(w/v) \cdot f(w/v)$ by [E, 20.21 Theorem] and $\operatorname{char} \overline{B_v} = \operatorname{char} F > p$. We conclude that $e(w/v) = p$ or $f(w/v) = p$, which contradicts Lemma 5.1. □

The results of this section are very similar to those stated in Lemma 2.2. In the construction of invariant valuation trees Lemma 2.2 provides the basic tools to obtain division algebras with only a finite number of real valuations, and the use of Henselizations seems to be more natural than the p-parts of Henselizations and their intersections. But the noninvariant case is more delicate, as we will see in the next section.

6. The central image of a valuation tree

In section 4 we have seen that each finite, rooted tree can be realized as an invariant valuation tree of a division algebra which is finite-dimensional over its centre. But in general the valuation rings of those division algebras need not be invariant, and therefore some of the "vertices" of the valuation tree are conjugate. Of course, conjugate valuation rings have the same rank, and therefore the conjugacy defines an equivalence relation on each set of valuation rings with the same rank. Since the rank of a valuation ring equals the number of proper overrings, we define the rank of a vertex to be the distance between this vertex and the root, and with this notation the conjugacy defines an equivalent relation on each set of vertices of the valuation tree which have the same rank. In the following we are mainly concerned with the question of which finite, rooted trees "with equivalent relations" can be realized as valuation trees where the equivalent relations are defined by the conjugacy.

The first observation is that two valuation rings B, B' of a division algebra D finite-dimensional over its centre F are conjugate if and only if B and B' have the same centre $V = B \cap F = B' \cap F$ (cf. [BG, Theorem 2]). Thus, let $\mathcal{T}_C$ be the set of all $B \cap F$ where $B \in \mathcal{T}$ is a valuation ring of D. Then each element of $\mathcal{T}_C$ is a valuation ring of F, and $\mathcal{T}_C$ is a partially ordered set with respect to the inclusion, and is also a rooted tree. We call $\mathcal{T}_C$ the central image and

$$\mathcal{C} : \mathcal{T} \longrightarrow \mathcal{T}_C, \quad B \longmapsto F \cap B$$

the central map of $\mathcal{T}$. It turns out that $\mathcal{C}$ is an order-preserving map in the sense that $B \subseteq B'$ implies $\mathcal{C}(B) \subseteq \mathcal{C}(B')$. Furthermore, if $B \in \mathcal{T}$ is a lower neighbour of $B' \in \mathcal{T}$ in $\mathcal{T}$, then $\mathcal{C}(B)$ is a lower neighbour of $\mathcal{C}(B')$ in $\mathcal{T}_C$, and the other implication is true whenever $B \subseteq B'$. Finally, if $V = \mathcal{C}(B) \in \mathcal{T}_C$ is a lower neighbour of $V' \in \mathcal{T}_C$, then there exists $B' \in \mathcal{T}$ such that $V' = \mathcal{C}(B')$ and $B \subset B'$. These properties show that $\mathcal{C}$ is rank-preserving in the sense that $\operatorname{rank}(B) = \operatorname{rank}(\mathcal{C}(B))$ for all B in $\mathcal{T}$.

First of all, we will investigate the connextion between $\mathcal{T}$ and $\mathcal{T}_C$ in the case when $\mathcal{T}$ is finite, and we will show how $\mathcal{T}$ is determined by $\mathcal{T}_C$. There are two extreme situations:

First, $\mathcal{C}$ is an order-isomorphism. This means that two distinct valuation rings B and B' of D have distinct centres. This occurs precisely when each valuation ring B of D is invariant, since two valuation rings with the same centre are conjugate

(cf. [BG, Theorem 2]). Thus, Theorem 4.2 shows that each finite, rooted tree can be realizied as a valuation tree of a finite-dimensional division algebra such that the central map is an order-isomorphism.

Second, $\mathcal{C}$ maps two valuation rings onto the same image if both have the same rank. Here $\mathcal{T}_C$ is a chain, and we will see later that this case can occur also even if $\mathcal{T}$ is not a chain.

These examples show that $\mathcal{T}$ is not determined by $\mathcal{T}_C$, and we may ask what additional information is needed to get $\mathcal{T}$ from $\mathcal{T}_C$. Here, the approach is to label $\mathcal{T}_C$ in such a way that the label $l(V)$, $V \in \mathcal{T}_C$, counts the number of $B \in \mathcal{T}$ such that $B \cap F = V$, i.e., $l(V)$ shows how often V occurs as a centre of a valuation ring of D or how many extensions V has. For instance, if $V = F$ is the trivial valuation ring, then $l(V) = 1$; and also if V has rank 1 we obtain $l(V) = 1$ by [C] or [G]. Now, let $V \in \mathcal{T}_C$ be a nontrivial valuation ring of F and let $W \in \mathcal{T}_C$ be a lower neighbour of V in $\mathcal{T}_C$. We claim that $m := l(V)$ is a divisor of $n := l(W)$. To see this, let $R_1, \dots, R_m$ be all distinct extensions of V to D, with each extension B of W contained in exactly one R_i. Since $R_1, \dots, R_m$ are conjugate, we conclude that all R_j contain the same number of extensions of W to D, i.e., m divides n.

Definition. A finite, labeled, rooted tree is of valuation type if the following conditions are satisfied:

i) Each vertex a is labeled by a natural number $l(a) \in \mathbb{N}$.

ii) $l(a) = 1$ if a is the root or a lower neighbour of the root.

iii) If b is a lower neighbour of a, then $l(a)$ is a divisor of $l(b)$.

If $\mathcal{T}$ is a finite valuation tree of a finite-dimensional division algebra, then the above observation shows that the central image $\mathcal{T}_C$ is a finite, labeled, rooted tree of valuation type. Furthermore, $\mathcal{T}_C$ determines $\mathcal{T}$ completely by the following construction: If V is a vertex of $\mathcal{T}_C$ of rank k, i.e., V is a valuation ring of F of rank k, then V defines exactly $m = l(V)$ valuation rings $B_1, \dots, B_m$ of D, which are exactly the extensions of V to D. Now, let $V_1, \dots, V_r$ be the lower neighbours of V in $\mathcal{T}_C$ and let $n_i = \frac{l(V_i)}{l(V)}$ for $i = 1, \dots, r$. By assumption, n_i is a natural number and each B_j has exactly $n_1 + \ldots + n_r$ lower neighbours, where n_i of those are extensions of V_i to D, $i = 1, \dots, r$. Since each finite, rooted tree is completely determined if for all vertices the number of the lower neighbours are known, we have seen how $\mathcal{T}$ can be obtained from the labeled tree $\mathcal{T}_C$.

The main result of this section is

Theorem 6.1. *Let $\mathcal{T}_C$ be a finite, labeled, rooted tree of valuation type. Then there exist a commutative field F and a finite-dimensional division algebra D over F such that $\mathcal{T}_C$ is the central image of the valuation tree $\mathcal{T}$ of D.*

We will prove a sharpened version of this theorem by claiming some additional statements. First of all, let each vertex a of $\mathcal{T}_C$ be labeled by $l(a) \in \mathbb{N}$, and define

$$g(\mathcal{T}_C\ ) := \prod_a l(a),$$

where the product runs over all terminal vertices of $\mathcal{T}_C$. For each prime p with $p > g(\mathcal{T}_C)$ and p greater than the number of vertices of $\mathcal{T}_C$, also the following can be realized:

i) $\operatorname{char} F$ can be any prime number greater than p.

ii) F has finite transcendence degree.

iii) F contains a primitive qth root of unity $\omega_q \in F$ for each prime q with $q \leq p$.

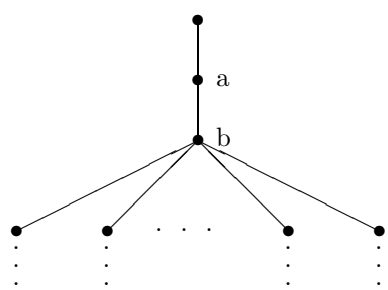
iv) $D = F$ is a finite field, or D is a tensor product of symbol algebras of prime degrees with $[D : F] = (d \cdot p)^2$, where $d \leq g(\mathcal{T}_C)$.

We prove all statements by induction on the number of vertices of $\mathcal{T}_C$. If $\mathcal{T}_C$ has exactly one vertex, then $g(\mathcal{T}_C) = 1$, and we choose F as a finite field with char $F > p$ which contains all primitive qth roots of unity for all primes $q \leq p$, and $D = F$ satisfies all conditions.

Thus, let $\mathcal{T}_C$ have more than one vertex. We will divide the proof in three steps, where each step needs special techniques to be developed.

Step 1. Here, we will realize $\mathcal{T}_C$ as the central image of a valuation tree under the special assumption that the root of $\mathcal{T}_C$ has precisely one lower neighbour a. If a has no lower neighbour, i.e., $\mathcal{T}_C$ has two vertices, then $g(\mathcal{T}_C) = 1$, and for each prime $p > g(\mathcal{T}_C) = 1$ and $p > 2$ there exist a commutative field F of finite transcendence degree and a symbol division algebra D over F with $[D : F] = p^2$ and with exactly two valuation rings (the trivial one and a real one) by Theorem 4.2. Furthermore, we can also assume that i) and iii) are satisfied by remarks 2 and 3 after Theorem 4.2.

Thus, let us assume that a has at least one lower neighbour. In this step we will restrict to the case that a has exactly one lower neighbour b, where b is labeled by $l(b) = q$.



We consider the subtree $\mathcal{T}'_C$ which can be obtained from $\mathcal{T}_C$ by removing the root of $\mathcal{T}_C$. Clearly, $\mathcal{T}'_C$ is also a rooted tree with root a, and each vertex c of $\mathcal{T}'_C$ different from a will be labeled by $l'(c) = l(c) \cdot q^{-1}$. By assumption $l'(c)$ is a natural number, and $\mathcal{T}'_C$ becomes a finite, labeled, rooted tree of valuation type which can be realized as the central image of the valuation tree of a finite-dimensional division algebra D' over a commutative field F' such that i),...,iv) hold. In particular, $D' = (a_1, b_1)_{p_1} \otimes \ldots \otimes (a_r, b_r)_{p_r}$ is a tensor product of symbol division algebras of prime degrees. One should recall that p is a prime number such that $p > g(\mathcal{T}_C) \geq g(\mathcal{T}'_C)$, and that $[D' : F'] = (d' \cdot p)^2$, $d' \leq g(\mathcal{T}'_C)$. Let w be the only real valuation of D', where w' is the restriction of w to F'. We will show that there exists a Kummer extension L' of F' of degree q such that w' has exactly q extensions to L' and each of those can be extended to $D' \otimes L'$, which is a division algebra. Unfortunately, this need not be true in general, but we show that D' and F' can be chosen this way. To simplify notations we assume that q is a prime number; it is not hard to see how the general case can be treated. Now, we show that the following can be assumed:

($*$) There exists a cyclic field extension $L' = F'(\alpha)$ of F' of degree q, where $\alpha^q \in F'$, such that w' has exactly q extensions to $L' = F'(\alpha)$ and each of those can be extended to $D' \otimes L'$, which is a division algebra.

To prove $(*)$, let $F'(x)$ be the rational function field over F' in the indeterminate x. Since F' is countable while the completion $\tilde{F}'_{w'}$ of F' with respect to w' is not, $F'(x)$ can be understood as a subfield of $\tilde{F}'_{w'}$, where $D'(x) = D' \otimes F'(x)$ can be understood as a subdivision algebra of $D' \otimes \tilde{F}'_{w'}$ and w has an extension w_x to $D'(x)$ such that $D'(x)$ is immediate over D'. By assumption, $\operatorname{char} F' > p > g(\mathcal{T}_C\ ) \geq q$, and furthermore F' contains a primitive qth root of unity by iii). Thus, $F'(x)$ is a cyclic field extension of $F'(x^q)$ of degree q, and the restriction w'_{x^q} of w_x to $F'(x^q)$ has exactly q extensions to $F'(x)$ by [E, 20.21 Theorem]. Now, we consider $D'(x^q)$, which is a finite-dimensional division algebra over $F'(x^q)$ such that $D'(x^q) = D' \otimes F'(x^q)$, and we are done once we show that the (unique) extension of w'_{x^q} (which is the restriction of w_x to $D'(x^q)$) is the only real valuation of $D'(x^q)$. Of course, this need not be true, but before we show how to modify our construction we summarize the results we have obtained so far: We have proved $(*)$ with the exception that $\mathcal{T}'_C$is not the central image of the valuation tree of D', but if we consider the set of all valuation rings of F' consisting of F' and all valuation rings contained in $B_{w'}$ as a labeled tree (where again the label counts the number of extensions to D'), then this tree coincides with $\mathcal{T}'_C$. It remains to "eliminate" all real valuations different from w, but this is easily done by Theorem 5.2: Let $\hat{F'^p}(w')$ be a p-part of the Henselization of F' with respect to w'; then $D' \otimes \hat{F'^p}(w')$ is a division algebra with $\mathcal{T}'_C$as the central image of the corresponding valuation tree. Finally, $\hat{F'^p}(w') \otimes_{F'} F'(\alpha)$ is a commutative field since $[F'(\alpha) : F'] = q$ is a prime number different to p, and $D' \otimes_{F'} \hat{F'^p}(w') \otimes_{F'} F'(\alpha)$ is a division algebra since $\hat{F'^p}(w') \otimes_{F'} F'(\alpha)$ is a subfield of the Henselization of F' with respect to w'.

In order to realize $\mathcal{T}_C$ as the central image of a valuation tree, the above division algebra D' must be the residue division algebra of a real valuation of a suitable division algebra. Thus, we define $F = F'(x)$, where x is an indeterminate over F' and also a central indeterminate over D', and we consider the x-adic valuation v of $D = D'(x)$. To apply Proposition 3.3 we need a cyclic field extension $F(\beta)$ of $F = F'(x)$ of degree q, where $\beta^q \in F$ is such that $F(\beta)$ is totally ramified over F with respect to v', where v' is the restriction of v to $F = F'(x)$, i.e., v' is the x-adic valuation of $F'(x)$. This is easily done by $F(\sqrt[q]{x})$, but if q is not a prime such a simple construction will not be enough to get a division algebra which is a tensor product of symbol algebras of prime degrees. In the general situation we deal with $D(y)$, where y is a central indeterminate and $F(y)$ is the centre of $D(y)$. There exists $\gamma \in \mathbb{R}^+$such that $\gamma^n \in \Gamma_v, n \in \mathbb{N}$, if and only if $n = 0$, and v has an extension to $D(y)$ which maps $d_n y^n + \ldots + d_1 y + d_0$ onto $\max\{v(d_n)\gamma^n, \ldots, v(d_1)\gamma, v(d_0)\}$, where the corresponding residue division algebra is equal to $\overline{B_v}$. This extension is also denoted by v, and in the following we will refer to such an extension as a *Gauss γ-extension of v, where γ is free over* Γ_v. Then, $F(y)(\sqrt[q]{y})$ is nearly the desired cyclic extension of $F(y)$. But again, v is not the only real valuation of $D(y)$, and we consider $\hat{F^p}(v')$ instead of $F(y)$, $D \otimes \hat{F^p}(v')$ instead of D, and apply Theorem 5.2 to overcome this problem.

Now we are almost done, and again we summarize our results: We have a commutative field F which satisfies conditions i), ii), and iii) of the additional statements of Theorem 6.1, and a finite-dimensional algebra D over F such that the following hold:

a) D has a unique real valuation v and a unique valuation u of rank 2.

Let v', resp. u', be the restriction of v, resp. u, to F.

b) $D = (a_1, b_1)_{p_1} \otimes \ldots \otimes (a_r, b_r)_{p_r}$ is a tensor product of symbol division algebras of prime degrees, and each factor is inert with respect to v'.

c) There exists $b \in F$ such that $x^q - b$ is irreducible over F and $\gamma^{\frac{1}{q}}$ is not in Γ_v, where $\gamma = v(b)$.

d) $F' := \overline{B_{v'}}$ is the centre of $D' := \overline{B_v}$.

e) There exists an $a \in B_{v'}$ such that $x^q - \overline{a}$ is irreducible over F', where $\overline{a} = a + M_{v'}$, and $D' \otimes F'(\alpha)$ is a division algebra, where α is a root of $x^q - \overline{a}$.

f) If w' denotes the real valuation of $F' = \overline{B_{v'}}$ with valuation ring $B_{u'}/M_{v'}$, then w' has q extensions to $F'(\alpha)$, and each of them can be extended to $D' \otimes F'(\alpha)$.

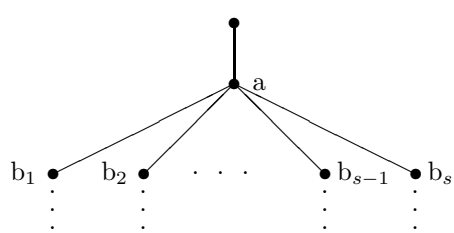
g) $\mathcal{T}'_C$ labeled as above is the central image of the valuation tree of D'.

h) $[D : F] = (d' \cdot p)^2$, where $d' \leq g(\mathcal{T}'_C)$.

We apply Proposition 3.3, and see that $(a, b)_q$ is a symbol division algebra over F and v has an extension $v_{a,b}$ to $(a, b)_q$ such that $F'(\alpha)$ is the corresponding residue division algebra and $\Gamma_v \cdot \langle \gamma^{\frac{1}{q}} \rangle$ the corresponding value group. Furthermore, $D \otimes (a, b)_q = (a_1, b_1)_{p_1} \otimes \ldots \otimes (a_r, b_r)_{p_r} \otimes (a, b)_q$ is a division algebra by e) and the comments at the end of section 3, where each factor is of valuation type with respect to v'. Additionally, v can be extended to $D \otimes (a, b)_q$ with residue division algebra $D' \otimes F'(\alpha)$, and this extension is the only real valuation of $D \otimes (a, b)_q$. The rank 2 valuation rings of $D \otimes (a, b)_q$ are uniquely determined by the real valuation rings of $D' \otimes F'(\alpha)$ (cf. Theorem 2.1vi). By f) and g) there are exactly q real valuation rings in $D' \otimes F'(\alpha)$, they are all extensions of $B_{w'}$, and they all have the same residue division algebra as u. This means that the central image of the valuation tree of $D \otimes (a, b)_q$ coincides with $\mathcal{T}_C$. Finally, $[D \otimes (a, b)_q : F] = (d' \cdot q \cdot p)^2$, where $d' \cdot q \leq g(\mathcal{T}'_C) \cdot q \leq g(\mathcal{T}_C)$. To see the last inequality, one should recall that each valuation ring of F with rank ≥ 2 which has l extensions to D has $l \cdot q$ extensions to $D \otimes (a, b)_q$.

A few words should be added how to deal with the general situation when $q = q_1 \cdot \ldots \cdot q_t$ is the decomposition of q into primes, with $t > 1$. With the notations as above, in $(*)$ the cyclic extension L' of F' must be replaced by a Kummer extension $L' = F'(\sqrt[q_1]{a_1}, \ldots, \sqrt[q_t]{a_t})$ of degree q, and in the corresponding proof $F'(x_1, \ldots, x_t)$ must be considered instead of $F'(x)$. Furthermore, in the second part of step 1 the division algebra $D(y)$ must be replaced by $D(y_1, \ldots, y_t)$ and γ by $\gamma_1, \ldots, \gamma_t$, where $\gamma_1^{n_1} \cdot \ldots \cdot \gamma_t^{n_t} \in \Gamma_v$, $n_1, \ldots, n_t \in \mathbb{N}$, if and only if $n_1, \ldots, n_t = 0$. Since F is countable, $\gamma_1, \ldots, \gamma_t$ always exist. The rest of the proof goes off in accordance with the above arguments.

Step 2. Here, we will realize $\mathcal{T}_C$ as the central image of a valuation tree under the special assumption that a has more than one lower neighbour, $b_1, \ldots, b_s$, where the notations are as in step 1.



To simplify notations we reduce to the situation where $s = 2$, $l(b_1) = q$ is a prime and $l(b_2) = 1$. As soon as this special $\mathcal{T}_C$ is realized it is not hard to see how the general construction can be obtained. We consider the subtree $\mathcal{T}'_C$ which can be obtained from $\mathcal{T}_C$ by removing the root of $\mathcal{T}_C$, and again $\mathcal{T}'_C$ is a rooted tree with root a. Each vertex c of $\mathcal{T}'_C$ which lies below b_1 will be labeled by $l'(c) = l(c) \cdot q^{-1}$. As above, $\mathcal{T}'_C$ can be realized as the central image of the valuation tree of a finite-dimensional division algebra $D' = (a_1, b_1)_{p_1} \otimes \ldots \otimes (a_r, b_r)_{p_r}$ over a commutative field F' such that i),...,iv) hold. Now, D' has two real valuations w_1 and w_2, and similary the following can be assumed, where w'_i is the restriction of w_i to F':

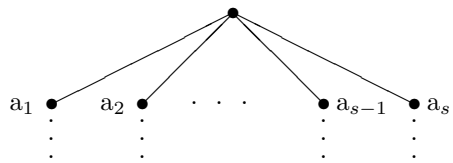
$(**)$ There exists a cyclic field extension $L' = F'(\alpha)$ of F' of degree q, where $\alpha^q \in F'$, such that w'_1 has exactly q extensions to $L' = F'(\alpha)$ and each of those can be extended to $D' \otimes L'$, which is a division algebra, where w'_2 has a unique extension to L' which can also be extended to $D' \otimes L'$.

Again, to prove $(**)$ we consider $F'(x)$, and w_1 will be handled as w, while w_2 has a Gauss γ-extension where γ is free over Γ_{w_2} (such a γ exists since F' is countable). The rest of the proof is almost the same as in the proof of $(*)$, with $\hat{F'^p}(w')$ replaced by $\hat{F'^p}(w'_1, w'_2)$.

After $(**)$ is shown we copy the rest of the proof in step 1 with some slight and obvious modifications. For instance, in a) we have to replace the unique valuation u of rank 2 by exactly two valuations u_1, u_2 of rank 2, and in f) the valuation w' must be replaced by w'_1, w'_2, where w'_1 has q extensions to $F'(\alpha)$ and w'_2 has exactly one, and all of those can be extended to $D' \otimes F'(\alpha)$.

After all, we have proved Theorem 6.1 with all additional statements if the root of $\mathcal{T}_C$ has exactly one lower neighbour. Furthermore, the division algebra D is a tensor product of valuation type with respect to the only real valuation, and $\deg D = d \cdot p$, where $d \leq g(\mathcal{T}_C)$. But in the next step more flexibility in the degree is required, and we will need that $\mathcal{T}_C$ can be realized such that $\deg D = d \cdot p \cdot q$, where q can be any natural number with prime factors less than p. To show this we can assume that q is a prime number less than p, and we will repeat this procedure as long as necessary. So, let D have $\mathcal{T}_C$ as the central image of its valuation tree (where in $\mathcal{T}_C$ the root has exactly one lower neighbour) and let D be a tensor product of valuation type with respect to the only valuation v. Furthermore, let F be the centre of D such that i), ii), and iii) hold, and let v' be the restriction of v to F. Since F is countable, there exist positive real numbers γ and δ such that v has a Gauss γ-extension to $D(x)$, where γ is free over Γ_v (which will also be denoted by v), and a Gauss δ-extension to $D(x, y)$, where δ is free over $\Gamma_v \cdot \langle\gamma\rangle$, which will also be denoted by v, with x, y central indeterminates over D. Let v' also be the restriction of v to $F(x, y)$. Clearly, $D \otimes \widehat{F(x,y)}^p(v')$ is a finite-dimensional division algebra with the same degree as D, where $\widehat{F(x,y)}^p(v')$ is a p-part of the Henselization of $F(x, y)$ with respect to v' and $D \otimes \widehat{F(x,y)}^p(v')$ has also $\mathcal{T}_C$ as the central image of its valuation tree by Theorem 5.2. Thus, $(x, y)_q$ is a ramified symbol division algebra over $\widehat{F(x,y)}^p(v')$ and $(D \otimes \widehat{F(x,y)}^p(v')) \otimes (x, y)_q$ is the division algebra we are looking for.

Step 3. Let $a_1, \ldots, a_s$ be the lower neighbours of the root. For each $i = 1, \ldots, s$ the subtree of $\mathcal{T}_C$ consisting of the root of $\mathcal{T}_C$ as well as a_i and all vertices of $\mathcal{T}_C$ which lie below a_i is denoted by $\mathcal{T}^i_C$.



For each i there exists a commutative field F_i with $\operatorname{char} F_i > p$ such that F_i has finite transcendence degree and F_i contains a primitive qth root of unity ω_q for each prime number $q \leq p$. Furthermore, there exists a finite-dimensional division algebra D_i over F_i with $[D_i : F_i] = (d_i \cdot p)^2, d_i \leq g(\mathcal{T}_C^i)$ such that $\mathcal{T}_C^i$ is the central image of the valuation tree of D_i. Let w_i be the unique real valuation of D_i and D_i a tensor product of valuation type where each factor has prime degree. Since $g(\mathcal{T}_C) = g(\mathcal{T}_C^1) \cdot \ldots \cdot g(\mathcal{T}_C^s)$ and $d_i \leq g(\mathcal{T}_C^i) \leq g(\mathcal{T}_C) < p$, we put $d = d_1 \cdot \ldots \cdot d_s$, and we can assume that $[D_i : F_i] = (d \cdot p)^2, d \leq g(\mathcal{T}_C)$ by the observation at the end of step 2. Each D_i is a tensor product (with factors of prime degrees)

$$D_i = (a_{i1}, b_{i1})_{p_1} \otimes \ldots \otimes (a_{ir}, b_{ir})_{p_r}$$

of valuation type with respect to w_i', the restriction of w_i to F_i. We apply Lemma 4.1 as well as the remark after it, and obtain a commutative field F which satisfies i), ii), and iii) of the additional conditions of Theorem 6.1. Furthermore, there exist real valuations $v_1', \ldots, v_s'$ of F such that $\overline{B_{v_i'}} \cong \overline{B_{w_i'}}$ and $\Gamma_{v_i'}$ is a subgroup of $\Gamma_{w_i'}$. We show that F can be enlarged to a commutative field L where L has finite transcendence degree over F such that each v_i' can be extended to a real valuation v_i where $\overline{B_{v_i}} \cong \overline{B_{w_i'}}$ and $\Gamma_{v_i} = \Gamma_{w_i'}$. Symmetry and transfinite methods allow us to reduce to the proof of the following statement: For each $\gamma \in \Gamma_{w_i'}$ there exists an extension $F(\alpha)$ of F which has immediate extensions of $v_2', \ldots, v_s'$ and an extension v_1 of v_1' with $\overline{B_{v_1}} = \overline{B_{v_1'}}$ and $\Gamma_{v_1} = \Gamma_{v_1'} \cdot \langle\gamma\rangle$. If there exists a nonzero $e \in \mathbb{N}$ such that $\gamma^e \in \Gamma_{v_1'}$, then we choose e minimal. Let $\gamma < 1$, and let a in $B_{v_1'}$ be such that $v_1'(a) = \gamma^e$. By the approximation theorem (cf. [E, 11.16 Theorem]) there exist $a_0, \ldots, a_{e-1} \in B_{v_1'} \cap \ldots \cap B_{v_s'}$ such that

$$x^e + a_{e-1}x^{e-1} + \cdots + a_1 x + a_0 \equiv x^e - a \bmod a \cdot M_{v_1'}[x]$$

and for $i = 2, \ldots, s$

$$x^e + a_{e-1}x^{e-1} + \cdots + a_1 x + a_0 \equiv x(x-1)^{e-1} \bmod M_{v_i'}[x].$$

Let α be a root of $f(x) = x^e + a_{e-1}x^{e-1} + \cdots + a_1 x + a_0 \in F[x]$ in an algebraic closure of F. If v_1 denotes an extension of v_1' to $F(\alpha)$, then $v_1(\alpha) < 1$ and $v_1(\alpha^e - a) < v_1(a)$, i.e., $v_1(\alpha)^e = v_1(\alpha^e) = v_1(a) = \gamma^e$ and $v_1(\alpha) = \gamma$. We conclude that $e(v_1/v_1') \geq e$ and also $f(v_1/v_1') = 1$, since $[F(\alpha) : F] \leq e$. This means $\overline{B_{v_1}} = \overline{B_{v_1'}}$ and $\Gamma_{v_1} = \Gamma_{v_1'} \cdot \langle\gamma\rangle$.

For $i > 1$ each v_i' has an immediate extension to $F(\alpha)$, since f has a root in $\tilde{F}_{v_i}$ by Hensel's Lemma (cf. [E, 16.7 Lemma]).

If there exists no nonzero $e \in \mathbb{N}$ such that $\gamma^e \in \Gamma_{v_1'}$, then v_1' has a Gauss γ-extension to $F(x)$, where γ is free over $\Gamma_{v_1'}$ and x is an indeterminate over F. Since F_i has finite transcendence degree, we have to deal with this case only a finite number of times. For $i > 1$ there exists an immediate extension of v_i' to $F(x)$, since $F(x)$ can be embedded into the completion $\tilde{F}_{v_i'}$ of v_i'.

Altogether we have proved that there exists a commutative field F satisfying i), ii), and iii) with real valuations $v_1, \dots, v_s$ such that $\overline{B_{v_i}} \cong \overline{B_{w_i'}}$ and $\Gamma_{v_i} = \Gamma_{w_i'}$, $i = 1, \dots, s$. Furthermore, for each i there exist $c_{i1}, \dots, c_{ir}, d_{i1}, \dots, d_{ir} \in F$ such that

$$(*) \qquad c_{ij} \equiv a_{ij} \bmod a_{ij} \cdot M_{v_i} \quad \text{and} \quad d_{ij} \equiv b_{ij} \bmod b_{ij} \cdot M_{v_i}.$$

Since F and F_i are different as sets, the above congruences must be explained: If $w_i'(a_{ij}) \neq 1$, then $c_{ij} \equiv a_{ij} \bmod a_{ij} \cdot M_{v_i}$ just means $v_i(c_{ij}) = w_i'(a_{ij})$, and if $w_i'(a_{ij}) = 1$, then $c_{ij} \equiv a_{ij} \bmod a_{ij} \cdot M_{v_i}$ means $c_{ij} + M_{v_i} = \phi_i(a_{ij} + M_{w_i'})$, where $\phi_i : \overline{B_{w_i'}} \longrightarrow \overline{B_{v_i}}$ is an isomorphism. The same holds for d_{ij} and b_{ij}.

We obtain s algebras

$$Q_i = (c_{i1}, d_{i1})_{p_1} \otimes \dots \otimes (c_{ir}, d_{ir})_{p_r}.$$

Since

$$D_i = (a_{i1}, b_{i1})_{p_1} \otimes \dots \otimes (a_{ir}, b_{ir})_{p_r}$$

is of valuation type with respect to w_i', condition $(*)$ together with the arguments given at the end of section 3 shows that Q_i is also a division algebra which is a tensor product of valuation type with respect to v_i. Furthermore, v_i has an extension to Q_i with the same value group and the "same" residue division algebra as w_i.

Now we are almost done. It remains to show how $Q_1, \dots, Q_s$ fit together. Again, we use the approximation theorem to obtain $a_1, \dots, a_r, b_1, \dots, b_r \in F$ such that the following hold for all $i = 1, \dots, s$:

$$v_i(a_1 - c_{i1}) < v_i(c_{i1}), \dots, v_i(a_r - c_{ir}) < v_i(c_{ir}),$$

$$v_i(b_1 - d_{i1}) < v_i(d_{i1}), \dots, v_i(b_r - d_{ir}) < v_i(d_{ir}).$$

Then, $D = (a_1, b_1)_{p_1} \otimes \dots \otimes (a_r, b_r)_{p_r}$ and Q_i are valuation-equivalent with respect to v_i. The unique extensions of v_i to D and Q_i, have the "same" residue division algebra, which is equal to the residue division algebra $\overline{B_{w_i}}$ of w_i. Now, if we replace F by $\hat{F}^p(v_1, \dots, v_s)$ and D by $D \otimes \hat{F}^p(v_1, \dots, v_s)$, we get the desired finite-dimensional division algebra which has $\mathcal{T}_C$ as the central image of its valuation tree.

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