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CONFORMAL ACTIONS OF $\mathfrak{sl}_n(\mathbb{R})$ AND $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ ON LORENTZ MANIFOLDS

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Abstract. We prove that, for $n \geq 3$, a locally faithful action of $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ or of $\mathrm{SL}_n(\mathbb{R})$ by conformal transformations of a connected Lorentz manifold must be a proper action.

1. Introduction

In [Kow96], N. Kowalsky proved (among other things) that any nontrivial isometric action of $\mathrm{SL}_n(\mathbb{R})$, $n \geq 3$, on a Lorentz manifold must be proper. In [AS97], we proved the stronger result that any nontrivial isometric action of $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, $n \geq 3$, on a Lorentz manifold is proper. In this paper, we extend both of these results to conformal actions. Specifically, we prove (Theorems 8.2 and 15.1) that a locally faithful action of $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$, $n \geq 3$, by conformal transformations of a Lorentz manifold M must be a proper action.

Many of the the techniques used in this paper can be applied to other groups, and may lead to our ultimate goal of characterizing the connected Lie groups that admit a nonproper, faithful action on a Lorentz manifold by isometries or by conformal transformations.

The problem of characterizing the connected Lie groups that can act nonproperly and isometrically on a *compact* Lorentz manifold was solved in [AS95] and [Zeg95].

Since there is a two-sheeted covering map from $\mathrm{SL}_2(\mathbb{R})$ to the identity component of $\mathrm{SO}(1,2)$, there is a locally faithful action by isometric linear transformations of $\mathrm{SL}_2(\mathbb{R})$ on the Minkowski 3-space. This action fixes the origin, and is therefore nonproper. Thus the restriction $n \geq 3$ cannot be relaxed.

Note that if A is a Lie group and there is a conformal action of $Aut(A)^0 \ltimes A$ on a Lorentz manifold such that the restriction to A is nonproper, then *any* connected Lie group with a normal subgroup A' isomorphic to A admits a conformal action such that A' is nonproper. This fact will be proved in a future paper. (See Corollary 4.4 and Lemma 3.6 of [A99].) Since $\mathrm{Aut}(\mathbb{R}^n)$ is $GL(n,\mathbb{R})$, it is reasonable to ask whether $GL(n,\mathbb{R})^0 \ltimes \mathbb{R}^n$ has an action such that $\mathbb{R}^n$ is nonproper. This paper shows that the answer is no for $n \geq 3$.

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2. Global definitions

Throughout this paper, "vector space" means "real vector space", "manifold" means "real manifold", and "Lie group" means "real Lie group". All tensors will be assumed to be smooth (C^∞).

Let V be a vector space. If $v \in V$ and if v_i is a sequence in V, then v_i **converges in direction to** v if $\mathbb{R}v_i \to \mathbb{R}v$ in the topological space of linear subspaces of V.

We denote by $\mathrm{SBF}(V)$ the set of symmetric bilinear forms on V.

If V and W are vector spaces and $f : V \to W$ is a function, then f **is a polynomial map** if, for every linear map $L : \mathbb{R} \to V$ and for every linear map $T : W \to \mathbb{R}$, the composition $T \circ f \circ L : \mathbb{R} \to \mathbb{R}$ is a polynomial function.

Let S and T be tensors on a manifold M defined near a point $m \in M$. We say that S **vanishes to order** k **at** m if S vanishes at m and if, for all $l \in \{1, \dots, k\}$, for all vector fields $X_1, \dots, X_l$ on M, we have that $(L_{X_1} L_{X_2} \cdots L_{X_l})(S)$ vanishes at m. We say that S **and** T **agree to order** k **at** m if $S - T$ vanishes to order k at m.

A vector field X on $\mathbb{R}^n$ is **homogeneous of order k** if $X(x)=\sum_{i=1}^n p_i(x)(\partial/\partial x_i)$, where each p_i is a homogeneous polynomial of degree k. We say that X is **constant** if it is homogeneous of degree 0 (*i.e.*, a constant linear combination of the coordinate vector fields $\partial/\partial x_i$); **linear** if it is homogeneous of degree 1; **quadratic** if it is homogeneous of degree 2; and **cubic** if it is homogeneous of degree 3.

A vector field X on a vector space V defined near zero is **constant** (resp. **linear**, **quadratic**, **cubic**) if there is an isomorphism between V and $\mathbb{R}^{\dim V}$ under which X corresponds to a constant (resp. linear, quadratic, cubic) vector field near zero.

A **quadratic differential** on a manifold is a smoothly varying system of quadratic forms, one on each tangent space of the manifold.

A quadratic differential on a vector space V is **constant** if, for all $v \in V$, it is invariant under $w \mapsto v + w : V \to V$.

Let V be a vector space with quadratic form Q. Then $\mathrm{SO}(Q)$ is the Lie group of orientation-preserving linear transformations of V preserving Q, and $\mathfrak{so}(Q)$ is the Lie algebra of $\mathrm{SO}(Q)$. We can identify $\mathfrak{so}(Q)$ with the Lie algebra of linear vector fields X on V whose associated flow preserves Q. Also, $\mathrm{CO}(V)$ is the Lie group of orientation-preserving linear transformations of V that preserve the line $\mathbb{R}Q$ in the space of quadratic forms on V. The corresponding Lie algebra is $\mathfrak{co}(V)$. If $\overline{g}$ is the flat pseudo-Riemannian metric on V corresponding to Q, then $\mathfrak{co}(Q)$ can be identified with the collection of all linear vector fields X on V such that $L_X\overline{g} = c\overline{g}$ for some constant $c \in \mathbb{R}$.

If X is a locally compact first-countable topological space and $\{x_i\}$ is a sequence in X, then x_i **goes to infinity** if, for every compact set $K \subseteq X$, for all but a finite number of i, we have $x_i \notin K$.

A continuous action of a locally compact first-countable group G on a locally compact, first-countable topological space X is **proper** if $\{g \in G \,|\, gK \cap K \neq \emptyset\}$

is compact for every compact $K \subseteq X$. A sequence $\{g_i\}$ in G is a **nonproper sequence** if:

1. $g_i \to \infty$ in G; and
2. there exists a sequence $\{x_i\}$ in X such that $\{x_i\}$ and $\{g_i x_i\}$ are both convergent sequences in X.

Note that the G-action is nonproper if and only if there is a nonproper sequence in G. If $\{g_i\}$ is a nonproper sequence, then so is $\{g_i^{-1}\}$.

If $\{g_i\}$, $\{h_i\}$ are sequences in a locally compact, first-countable group G, then h_i **is a bounded perturbation of** g_i if there are convergent sequences $\{k_i\}$, $\{l_i\}$ in G with $h_i = k_i g_i l_i$ for all i. A bounded perturbation of a nonproper sequence is again a nonproper sequence.

Let a Lie group G act on a pseudo-Riemannian manifold (M, g). For $m \in M$ let $E_m : G \to M$, $E_m(h) = hm$ be the orbit map, and let $e_m : \mathfrak{g} \to T_m M$ be the differential of E_m at 1_G. Let $B_m := e_m^*(g_m)$, so B_m is a symmetric bilinear form on $\mathfrak{g}$. If $V \subseteq \mathfrak{g}$ is a subspace and if $m \in M$, then V is **isotropic at** m if V is B_m-isotropic, *i.e.*, $B_m|V$ is zero. A subspace $V \subseteq \mathfrak{g}$ is **somewhere isotropic** if there exists $m \in M$ such that V is isotropic at m.

For $m \in M$, let $G_m := \mathrm{Stab}_G(m)$ and let $\mathfrak{g}_m$ be the Lie algebra of G_m. Let $\Phi_m : G_m \to (0, \infty)$ be the homomorphism defined by

$$g_{hm}(hv, hw) = [\Phi_m(h)][g_m(v, w)],$$

for $v, w \in T_m M$, $h \in G_m$. Let $\phi_m : \mathfrak{g}_m \to \mathbb{R}$ be the differential of Φ_m at 1_G. Let G_m^0 be the kernel of Φ_m, and let $\mathfrak{g}_m^0$ be the corresponding Lie algebra.

Let G be a connected semisimple Lie group with finite center and let $\mathfrak{a}$ be a maximal split torus in $\mathfrak{g}$. Then $\Gamma(\mathfrak{g}, \mathfrak{a})$ is the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. For each $\alpha \in \Gamma$, let $\mathfrak{g}_\alpha$ be the root space of α. For $A \in \mathfrak{a}$, let $\Gamma_A^+ = \Gamma_A^+(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \Gamma \mid \alpha(A) > 0\}$ and $\mathfrak{n}_A^+(\mathfrak{g}, \mathfrak{a}) := \bigoplus_{\alpha \in \Gamma_A} \mathfrak{g}_\alpha$.

3. Miscellaneous algebraic results

Lemma 3.1. *Suppose that Q is a quadratic form on a vector space W. Let $X \in \mathfrak{co}(Q)$ and let $V \subseteq W$ be a subspace. Assume that $X^2(V) \subset \ker Q$. Then $X(V)$ is isotropic.*

Proof. Let $\langle \cdot, \cdot \rangle$ be the symmetric bilinear form associated to Q. Fix $v_0 \in V$. We wish to show that $\langle X(v_0), X(v_0) \rangle = 0$.

Let $I : W \to W$ be the identity transformation. Choose $\alpha \in \mathbb{R}$ and $X_0 \in \mathfrak{so}(Q)$ such that $X = X_0 + \alpha I$. Since $X_0 \in \mathfrak{so}(Q)$, we have $\langle X_0(v), X_0(v) \rangle = -\langle X_0^2(v), v \rangle$ and $\langle X_0(v), v \rangle = 0$ for all $v \in V$. Thus

$$\begin{aligned} \langle X(v), X(v) \rangle &= \langle X_0(v), X_0(v) \rangle + 2\alpha \langle X_0(v), v \rangle + \alpha^2 \langle v, v \rangle \\ &= -\langle X_0^2(v), v \rangle + \alpha^2 \langle v, v \rangle, \end{aligned}$$

and

$$\begin{aligned} 0 = \langle X^2(v), v \rangle &= \langle X_0^2(v) + 2\alpha X_0(v) + \alpha^2 v \,,\, v \rangle \\ &= \langle X_0^2(v), v \rangle + \alpha^2 \langle v, v \rangle, \end{aligned}$$

so $-\langle X_0^2(v), v\rangle = \alpha^2\langle v, v\rangle$. Thus $\langle X(v), X(v)\rangle = 2\alpha^2\langle v, v\rangle$. Replacing v by $X(v_0)$ in this last equation, we have

$$0 = \langle X^2(v_0), X^2(v_0)\rangle = 2\alpha^2\langle X(v_0), X(v_0)\rangle = 4\alpha^2\langle v_0, v_0\rangle.$$

Thus, either $\alpha = 0$ or $\langle v_0, v_0\rangle = 0$. In either case, we have $2\alpha^2\langle v_0, v_0\rangle = 0$, so $\langle X(v_0), X(v_0)\rangle = 2\alpha^2\langle v_0, v_0\rangle = 0$. □

The proof of the following lemma is a simple exercise.

Lemma 3.2. *Let $n \geq 3$ be an integer. Then $SL_n(\mathbb{R})$ has no closed subgroups of codimension-one.*

Lemma 3.3. *Let $\mathfrak{g} := \mathfrak{sl}_3(\mathbb{R})$ and let $\mathfrak{a} \subseteq \mathfrak{g}$ be the maximal split torus consisting of all diagonal matrices of trace zero. Suppose that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{a} \subseteq \mathfrak{h}$. Suppose that the codimension in $\mathfrak{g}$ of $\mathfrak{h}$ is 2. Then there exists a Lie subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{h}$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.*

Proof. Because $\mathfrak{a} \subseteq \mathfrak{h}$ it follows that $\mathfrak{h}$ is a direct sum of $\mathfrak{a}$ and a collection of root spaces in $\mathfrak{g}$ with respect to $\mathfrak{a}$. That is, there is a subset $S \subseteq \{E_{12}, E_{23}, E_{13}, E_{21}, E_{32}, E_{31}\}$ such that

$$\mathfrak{h} = \mathfrak{a} \oplus \left(\bigoplus_{s\in S} \mathbb{R}s\right),$$

where $E_{ij} \in \mathfrak{gl}_3(\mathbb{R})$ is the matrix with a 1 in the (i, j) entry, and with zeroes everywhere else.

Since the codimension in $\mathfrak{g}$ of $\mathfrak{h}$ is 2, we conclude that $|S| = 4$. Therefore, there exist $i, j \in \{1, 2, 3\}$ such that $i \neq j$ and $E_{ij}, E_{ji} \in S$. Therefore, $\mathfrak{h}$ contains the Lie subalgebra $\mathfrak{g}_0$ generated by E_{ij} and E_{ji}. Since $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, we are done. □

4. Nonproper conformal actions

The results in this section are basically due to Kowalsky.

Let a Lie group G act on a pseudo-Riemannian manifold (M, g) by conformal transformations. Let $\Phi : G \times M \to (0, \infty)$ be the conformal cocycle defined by

$$g_{hm}(hv, hw) = \Phi(h, m)[g_m(v, w)].$$

A sequence $g_i \to \infty$ in G is **nonproper with bounded conformal factors** if there exist two convergent sequences $m_i \to m_\infty$ and $m'_i \to m'_\infty$ in M such that $g_i m_i = m'_i$ and such that, for some $C > 0$, for all i, we have $\Phi(g_i, m_i) \leq C$.

A sequence $g_i \to \infty$ in G is **nonproper with conformal factors bounded away from zero** if there exist $m_i \to m_\infty$ and $m'_i \to m'_\infty$ in M such that $g_i m_i = m'_i$ and, for some $C > 0$, for all i, we have $\Phi(g_i, m_i) \geq C$. The proofs of the next two lemmas are straightforward, and are left to the reader.

Lemma 4.1. *Let $g_i \in G$ be a sequence. Then g_i is a nonproper sequence with bounded conformal factors if and only if g_i^{-1} is nonproper with conformal factors bounded away from zero.*

Lemma 4.2. *Let g_i be a nonproper sequence. Then, after passing to a subsequence, either g_i is nonproper with bounded conformal factors, or else g_i^{-1} is nonproper with bounded conformal factors.*

Versions of the remaining results were proved in [AS97] for *isometric* actions. For isometric actions, $\Phi(g,m)=1$. By Lemma 4.2, any nonproper sequence g_i can be replaced by a nonproper sequence with bounded conformal factors so that $\Phi(g_i,m_i)$ is uniformly bounded. Since boundedness is the only property of $\Phi(g_i,m_i)$ that was used in the proofs of [AS97, 3.1–3.5], the same proofs work in the conformal case.

Lemma 4.3. *Let W_i and X_i be convergent sequences in $\mathfrak{g}$ and let $Y,Z\in\mathfrak{g}$. Let g_i be a sequence in G, let m_i be a convergent sequence in M and let $m\in M$. Assume that $g_im_i\to m$ in M. Assume that there is some $C>0$ such that, for all i, we have $\Phi(g_i,m_i)\le C$. Assume that $\{(\mathrm{Ad}\,g_i)W_i\}_i$ goes to infinity in $\mathfrak{g}$, but converges in direction to Y. Assume that $(\mathrm{Ad}\,g_i)X_i$ does not converge to zero in $\mathfrak{g}$, and converges in direction to Z as $i\to\infty$. Then $B_m(Y,Z)=0$.*

Corollary 4.4. *Let $\{X_i^1\}_i,\dots,\{X_i^k\}_i$ be k convergent sequences in $\mathfrak{g}$ and let $Y^1,\dots,Y^k\in\mathfrak{g}$. Let g_i be a nonproper sequence in G with bounded conformal factors. Assume, for all $j\in\{1,\dots,k\}$, that $\{(\mathrm{Ad}\,g_i)X_i^j\}_i$ is divergent in $\mathfrak{g}$, but converges in direction to Y^j. Then the span of $Y^1,\dots,Y^k$ is somewhere isotropic.*

Corollary 4.5. *Let $S\subseteq\mathfrak{g}$ be a subset. Let g_i be a nonproper sequence in G with bounded conformal factors. Assume, for all $X\in S$, that $(\mathrm{Ad}\,g_i)X\to\infty$ in $\mathfrak{g}$. Assume, for all $X\in S$, that $\{(\mathrm{Ad}\,g_i)X\}_i$ converges in direction to a vector $Y_X\in\mathfrak{g}$ as $i\to\infty$. Then the span of $\{Y_X\mid X\in S\}$ is somewhere isotropic.*

Corollary 4.6. *Let $S\subseteq\mathfrak{g}$ be a subset. Let g_i be a nonproper sequence in G with conformal factors bounded away from zero. Assume, for all $X\in S$, that $(\mathrm{Ad}\,g_i)X\to 0$ in $\mathfrak{g}$. Then the span of S is somewhere isotropic.*

Recall that $\mathfrak{n}_A^+(\mathfrak{g},\mathfrak{a})$ is defined in §2.

Lemma 4.7. *Assume that G is a connected semisimple Lie group with finite center. Let $\mathfrak{a}$ be a maximal split torus in $\mathfrak{g}$. If the action of G on M is nonproper, then there exists $A_0\in\mathfrak{a}\backslash\{0\}$ such that $\mathfrak{n}_{A_0}^+(\mathfrak{g},\mathfrak{a})$ is somewhere isotropic.*

5. Conformal flows and transformations

Let (M,g) be a pseudo-Riemannian manifold.

Lemma 5.1. *Suppose that $\mathbb{R}$ acts by conformal transformations of (M,g) fixing $m_0\in M$. Let $(t,v)\mapsto t_*v:\mathbb{R}\times T_{m_0}M\to T_{m_0}M$ be the differential action at m_0. If there is a nonisotropic vector $v_0\in T_{m_0}M$ such that $t\mapsto t_*v_0:\mathbb{R}\to T_{m_0}M$ is a polynomial map, then $g_{m_0}(t_*v,t_*w)=g_{m_0}(v,w)$ for all $v,w\in T_{m_0}M$ and $t\in\mathbb{R}$.*

Proof. Choose $\alpha\in\mathbb{R}$ such that, for all $v,w\in T_{m_0}M$, for all $t\in\mathbb{R}$, we have $g_{m_0}(t_*v,t_*w)=e^{\alpha t}g_{m_0}(v,w)$.

Since $g_{m_0}(t_*v_0,t_*v_0)=e^{\alpha t}g(v_0,v_0)$ is a polynomial function of t and $g_{m_0}(v_0,v_0)\neq 0$, it follows that α must be 0. □

The next result is the pseudo-Riemannian analogue of [Kow96, Lemma 6.4].

Lemma 5.2. *Suppose that a Lie group G acts by conformal transformations of (M,g). Suppose that H is a connected subgroup of G such that $\mathrm{Ad}_{\mathfrak{g}}(H)\subseteq GL(\mathfrak{g})$ consists of unipotent transformations of $\mathfrak{g}$ and H fixes a point $m_0\in M$. Let $(h,v)\mapsto h_*v:H\times T_{m_0}M\to T_{m_0}M$ be the isotropy representation at m_0. Assume that $B_{m_0}\neq 0$. Then, for all $v,w\in T_{m_0}M$, for all $h\in H$, we have $g_{m_0}(h_*v,h_*w)=g_{m_0}(v,w)$.*

Proof. Because $B_{m_0} \neq 0$, we can choose $X_0 \in \mathfrak{g}$ such that $B_{m_0}(X_0, X_0) \neq 0$. Let $v_0 := e_{m_0}(X_0) \in T_{m_0}M$. For any $X \in \mathfrak{h}$, the map

$$\begin{aligned} t &\mapsto g_{m_0}(\exp(tX)_* v_0, \exp(tX)_* v_0) \\ &= B_{m_0}(\mathrm{Ad}(\exp(tX))_* X_0, (\mathrm{Ad}(\exp(tX))_* X_0) \\ &= B_{m_0}\left((\exp t \operatorname{ad} X)_* X_0, (\exp t \operatorname{ad} X)_* X_0\right) \end{aligned}$$

is a polynomial since $\operatorname{ad} X$ is nilpotent. The result now follows from Lemma 5.1. □

Lemma 5.3. *Suppose M is connected of dimension greater than two and X is a vector field on M. If $L_X(g) = f \cdot g$ for some $f \in C^\infty(M)$, and X vanishes to order two at some $m_0 \in M$, then $X = 0$.*

Proof. By [Gro88, §0.3.A', Example (4)], the conformal class of g is 2-rigid. In [Gro88], the second of the two sections labeled §0.3.A contains an assertion which implies that because of this 2-rigidity, if X vanishes to order two at some $m \in M$, then X vanishes at every point in a neighborhood of m, and therefore vanishes to order two at every point of that neighborhood.

Consequently, the set of points in M where X vanishes to order two is open. It is also closed, and so, by connectedness of M, we conclude that $S = M$. Therefore $X = 0$. □

Let G be a connected Lie group acting locally faithfully by conformal transformations of a pseudo-Riemannian manifold (M, g). Fix a point $m_0 \in M$. Let $\Phi_0 : G_{m_0} \to (0, \infty)$ be the homomorphism defined by

$$g(hv, hw) = [\Phi_0(h)][g(v, w)].$$

Let $G^0_{m_0}$ be the kernel of Φ_0.

Lemma 5.4. *If $X \in \mathfrak{g}^0_{m_0}$, then*

$$[(\operatorname{ad}_{\mathfrak{g}} X)(\mathfrak{g})] \cap [(\operatorname{ad}_{\mathfrak{g}} X)^{-1}(\mathfrak{g}_{m_0})]$$

is isotropic at m_0.

Proof. Fix $Y \in [(\operatorname{ad}_{\mathfrak{g}} X)(\mathfrak{g})] \cap [(\operatorname{ad}_{\mathfrak{g}} X)^{-1}(\mathfrak{g}_{m_0})]$. We wish to show that $B_{m_0}(Y, Y) = 0$. Since $Y \in (\operatorname{ad}_{\mathfrak{g}} X)(\mathfrak{g})$, we can choose $W \in \mathfrak{g}$ such that

$$Y = (\operatorname{ad}_{\mathfrak{g}} X)W = [X, W].$$

Since $X \in \mathfrak{g}^0_{m_0}$, we have

$$B_{m_0}([X, W], Y) + B_{m_0}(W, [X, Y]) = 0.$$

Since $Y \in (\operatorname{ad}_{\mathfrak{g}} X)^{-1}(\mathfrak{g}_{m_0})$, we conclude that

$$[X, Y] = (\operatorname{ad}_{\mathfrak{g}} X)Y \in \mathfrak{g}_{m_0} \subseteq \ker(B_{m_0}).$$

Thus, as desired,

$$B_{m_0}(Y, Y) = B_{m_0}([X, W], Y) = -B_{m_0}(W, [X, Y]) = 0. \quad \square$$

6. Dynamics on Lorentz manifolds

Let (M, g) be a Lorentz manifold, and let G be a connected Lie group acting on M. Recall that $E_0 : G \to M$ is defined by $E_0(h) = hm_0$, and $e_0 : \mathfrak{g} \to T_{m_0}M$ is the differential at 1_G of E_0.

The following lemma was proved in [AS97, Lemma 4.1].

Lemma 6.1. *If $V \subseteq \mathfrak{g}$ is a subspace isotropic at $m_0 \in M$, then $\mathfrak{g}_{m_0}$ contains a codimension-one subspace of V.*

Lemma 6.2. *If $\mathfrak{n} \subseteq \mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$ that is isotropic at m_0, then $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{g}_{m_0}$.*

Proof. Let $\mathfrak{n}_0 := \mathfrak{g}_{m_0} \cap \mathfrak{n}$. By Lemma 6.1, we know that the codimension in $\mathfrak{n}$ of $\mathfrak{n}_0$ is ≤ 1. Since a proper Lie subalgebra of a nilpotent Lie algebra is never self-normalizing, $\mathfrak{n}_0$ is an ideal in $\mathfrak{n}$. Since $\mathfrak{n}/\mathfrak{n}_0$ has dimension ≤ 1, we conclude that $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{n}_0 \subseteq \mathfrak{g}_{m_0}$. □

Lemma 6.3. *Let $V \subseteq \mathfrak{g}$ be a subspace that is isotropic at m_0. If $W \subsetneq V$ is a proper subspace and $(V \backslash W) \cap \mathfrak{g}_{m_0} = \emptyset$, then $W \subseteq \mathfrak{g}_{m_0}$.*

Proof. By Lemma 6.1, the codimension in V of $V \cap \mathfrak{g}_{m_0}$ is ≤ 1. As $V \cap \mathfrak{g}_{m_0} = W \cap \mathfrak{g}_{m_0}$, we conclude that the codimension in V of $W \cap \mathfrak{g}_{m_0}$ is ≤ 1, so $W \subseteq \mathfrak{g}_{m_0}$. □

Lemma 6.4. *Suppose that a Lie group G acts by conformal transformations on M. Let $m_0 \in M$. Suppose that H is a connected Lie subgroup of G_{m_0} such that $\mathrm{Ad}_{\mathfrak{g}}(H) \subseteq GL(\mathfrak{g})$ consists of unipotent transformations of $\mathfrak{g}$. Assume that $\dim G_{m_0} \leq (\dim G) - 2$. Then, for all $v, w \in T_{m_0}M$, for all $h \in H$, we have $g_{m_0}(h_* v, h_* w) = g_{m_0}(v, w)$.*

Proof. We have that $\dim(e_0(\mathfrak{g})) = \dim(\mathfrak{g}) - \dim(\ker(e_0)) \geq 2$, so $e_0(\mathfrak{g}) \subseteq T_{m_0}M$ is not isotropic. By Lemma 5.2 we are done. □

7. Properness of $\mathrm{SL}_3(\mathbb{R})$

We show here that a locally faithful, conformal action of $G := \mathrm{SL}_3(\mathbb{R})$ on a connected Lorentz manifold (M, g) must be proper. Much of this argument is motivated by Kowalsky's proof that this is true under the stronger assumption that the action be isometric.

Let $H_{ij} := E_{ii} - E_{jj}$, and let I_3 be the 3×3 identity matrix. Let $\mathfrak{n}^+ := \mathbb{R}E_{12} \oplus \mathbb{R}E_{23} \oplus \mathbb{R}E_{13}$ be the Lie subalgebra of upper triangular nilpotent 3×3 matrices in $\mathfrak{sl}_3(\mathbb{R})$ and let $\mathfrak{n}^- := \mathbb{R}E_{21} \oplus \mathbb{R}E_{32} \oplus \mathbb{R}E_{31}$ be the subalgebra of lower triangular nilpotent 3×3 matrices.

Let $\mathfrak{a}$ be the maximal split torus in $\mathfrak{g}$ consisting of diagonal matrices of trace zero. Recall from §2 the definitions of $\Gamma_A^+ = \Gamma_A^+(\mathfrak{g}, \mathfrak{a})$ and of $\mathfrak{n}_A^+ = \mathfrak{n}_A^+(\mathfrak{g}, \mathfrak{a})$, for $A \in \mathfrak{a}$.

Define $\alpha_1, \alpha_2 \in \mathfrak{a}^*$ by

$$\alpha_1(xE_{11} + yE_{22} + zE_{33}) = x - y,$$
$$\alpha_2(xE_{11} + yE_{22} + zE_{33}) = y - z.$$

Lemma 7.1. *If G acts nonproperly on M, then there exists $m \in M$ such that $E_{13} \in \mathfrak{g}_m$.*

Proof. By Lemma 4.7, we can choose $A_0 \in \mathfrak{a}\backslash\{0\}$ and $m_0' \in M$ such that $\mathfrak{n}^+_{A_0}$ is isotropic at m_0'.

Choose $g_0 \in N_G(\mathfrak{a})$ such that $A_1 := (\operatorname{Ad} g_0)A_0$ satisfies $\alpha_1(A_1) \geq 0$ and $\alpha_2(A_1) \geq 0$. Then $\mathfrak{n}^+_{A_1}$ is isotropic at $m_1 := g_0 m_0'$.

We shall assume that $\alpha_1(A_1) > 0$ (the case $\alpha_2(A_1) > 0$ is similar). Now

$$\alpha_1, \alpha_1 + \alpha_2 \in \Gamma^+_{A_1},$$

so $\mathfrak{g}_{\alpha_1} \subseteq \mathfrak{n}^+_{A_1}$ and $\mathfrak{g}_{\alpha_1+\alpha_2} \subseteq \mathfrak{n}^+_{A_1}$.

By Lemma 6.1, $sE_{12} + tE_{13} \in \mathfrak{g}_{m_1}$ for some $(s,t) \in \mathbb{R}^2\backslash\{(0,0)\}$. If $s = 0$, then $E_{13} \in \mathfrak{g}_{m_1}$, and we are done. If $s \neq 0$, let $g_1 := E_{11} - E_{23} + E_{32} + (t/s)E_{33}$, and let $m := g_1 m_1$. Then

$$sE_{13} = (Ad_{\mathfrak{g}} g_1)(sE_{12} + tE_{13}) \in \mathfrak{g}_m,$$

so $E_{13} \in \mathfrak{g}_m$. □

Lemma 7.2. *There are no fixed points for the action of G on M, i.e., $G_m \neq G$ for all $m \in M$.*

Proof. Suppose that G fixes m. We wish to obtain a contradiction.

Because G has split rank two, while $\mathrm{SO}(g_m)$ has split rank one, the isotropy representation $F : G \to \mathrm{CO}(g_m)$ cannot be injective. As G is a simple center-free Lie group, we conclude that F is trivial.

Then, by Thurston's Stability Theorem [Thur74, Theorem 3, p. 348], there is a nontrivial homomorphism $G \to \mathbb{R}$. However, $[G,G] = G$, so this is impossible. □

Recall from §2 that G^0_m is the kernel of the homomorphism $\Phi_m : G_m \to (0,\infty)$, and $\phi_m : \mathfrak{g}_m \to \mathbb{R}$ is the differential at the identity of Φ_m.

Lemma 7.3. *If $X \in \mathfrak{g}_{m_0}$ is nilpotent (as a matrix), then $X \in \mathfrak{g}^0_{m_0}$.*

Proof. Lemma 3.2 and Lemma 7.2 imply that $\dim G_{m_0} \leq (\dim G) - 2$. Let $H := \{\exp(tX) \,|\, t \in \mathbb{R}\}$. Since X is nilpotent, $\mathrm{Ad}_{\mathfrak{g}}(H) \subseteq \mathrm{GL}(\mathfrak{g})$ consists of unipotent transformations. Therefore, by Lemma 6.4, we are done. □

Lemma 7.4. *Assume that G acts nonproperly on M. Then there exists $m_0 \in M$ such that $H_{13} \in \mathfrak{g}_{m_0}$.*

Proof. By Lemma 7.1, choose $m \in M$ such that $E_{13} \in \mathfrak{g}_m$. It suffices to show, for some $t > 0$ and some $g \in G$, that $(\mathrm{Ad}_{\mathfrak{g}}\, g)(tH_{13}) \in \mathfrak{g}_m$.

For all $t > 0$, let S_t be the set of elements $X \in \mathfrak{g}$ such that the eigenvalues of the matrix $X \in \mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ are t, 0 and $-t$. Let $S := \bigcup_{t>0} S_t$. For all $t > 0$ and all $X \in S_t$, there exists $g \in G$ such that $(\mathrm{Ad}_{\mathfrak{g}}\, g)X = tH_{13}$. So it suffices to show that $S \cap \mathfrak{g}_m \neq \emptyset$.

We therefore assume that $S \cap \mathfrak{g}_m = \emptyset$, and aim for a contradiction.

Let $V_1 := \mathbb{R}H_{13} \oplus \mathbb{R}E_{12} \oplus \mathbb{R}E_{23}$ and $W_1 := \mathbb{R}E_{12} \oplus \mathbb{R}E_{23}$. Matrix computation shows that

$$V_1 \subseteq [(\mathrm{ad}_{\mathfrak{g}}\, E_{13})(\mathfrak{g})] \cap [(\mathrm{ad}_{\mathfrak{g}}\, E_{13})^{-1}(\mathbb{R}E_{13})].$$

Since $E_{13} \in \mathfrak{g}_m$, it follows from Lemma 7.3 that $E_{13} \in \mathfrak{g}^0_m$. So, by Lemma 5.4, we see that V_1 is isotropic at m. Since $V_1\backslash W_1 \subseteq S$, we conclude, by Lemma 6.3, that $W_1 \subseteq \mathfrak{g}_m$. So $E_{12}, E_{23} \in \mathfrak{g}_m$.

Let $V_2 := \mathbb{R}H_{12} \oplus \mathbb{R}E_{32}$ and $W_2 := \mathbb{R}E_{32}$. Matrix computation shows that

$$V_2 \subseteq [(\mathrm{ad}_{\mathfrak{g}} E_{12})(\mathfrak{g})] \cap [(\mathrm{ad}_{\mathfrak{g}} E_{12})^{-1}(\mathbb{R}E_{12})].$$

Since $E_{12} \in \mathfrak{g}_m$, it follows from Lemma 7.3 that $E_{12} \in \mathfrak{g}_m^0$. So, by Lemma 5.4, we see that V_2 is isotropic at m. Since $V_2 \backslash W_2 \subseteq S$, we conclude, by Lemma 6.3, that $W_2 \subseteq \mathfrak{g}_m$. So $E_{32} \in \mathfrak{g}_m$.

Let $V_3 := \mathbb{R}H_{23} \oplus \mathbb{R}E_{21}$ and $W_3 := \mathbb{R}E_{21}$. An argument similar to the one in the preceding paragraph shows that $E_{21} \in \mathfrak{g}_m$.

Since $E_{12}, E_{23}, E_{32}, E_{21} \in \mathfrak{g}_m$ generate the Lie algebra $\mathfrak{g}$, we conclude that $\mathfrak{g} = \mathfrak{g}_m$, so $G = G_{m_0}$. This contradicts Lemma 7.2. □

Lemma 7.5. *Let $m_0 \in M$. Assume that $H \in \mathfrak{a} \cap \mathfrak{g}_{m_0}^0$ and $\alpha_1(H) > 0$ and $\alpha_2(H) > 0$.*

1. *If $\phi_{m_0}(H) \leq 0$, then $\mathfrak{n}^+$ is isotropic at m_0.*
2. *If $\phi_{m_0}(H) < 0$, then $\mathfrak{a} \oplus \mathfrak{n}^+$ is isotropic at m_0.*
3. *If $\phi_{m_0}(H) \geq 0$, then $\mathfrak{n}^-$ is isotropic at m_0.*
4. *If $\phi_{m_0}(H) > 0$, then $\mathfrak{a} \oplus \mathfrak{n}^-$ is isotropic at m_0.*

Proof. We only prove 1; the proofs of 2, 3 and 4 are similar.

Fix $E, E' \in \{E_{12}, E_{23}, E_{13}\}$. Let $T := \mathrm{ad}_{\mathfrak{g}}(H) : \mathfrak{g} \to \mathfrak{g}$. Choose $a, a' \in \mathbb{R}$ such that $T(E) = aE$ and $T(E') = a'E'$. Then $a > 0$ and $a' > 0$. Let $b := \phi_{m_0}(H) \leq 0$.

By differentiating the definition of Φ_{m_0}, we get

$$B_{m_0}(T(E), E') + B_{m_0}(E, T(E')) = [\phi_{m_0}(H)][B_{m_0}(E, E')].$$

Thus $aB_{m_0}(E, E') + a'B_{m_0}(E, E') = bB_{m_0}(E, E')$. Since $a, a' > 0$ and $b \leq 0$, we conclude that $B_{m_0}(E, E') = 0$, as desired. □

Corollary 7.6. *Let $m_0 \in M$. If $H \in \mathfrak{a} \cap \mathfrak{g}_{m_0}^0$, $\alpha_1(H) > 0$ and $\alpha_2(H) > 0$, then $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are isotropic at m_0.*

Lemma 7.7. *Let $m_0 \in M$. If $H_{13} \in \mathfrak{g}_{m_0}^0$, then $\mathfrak{a} \subseteq \mathfrak{g}_{m_0}$.*

Proof. Since $\mathfrak{a}$ is two-dimensional and since $\mathbb{R}H_{13} \subseteq \mathfrak{g}_{m_0}^0 \subseteq \mathfrak{g}_{m_0}$, it suffices to show that $(\mathfrak{a} \backslash \mathbb{R}H_{13}) \cap \mathfrak{g}_{m_0} \neq \emptyset$.

Since $H_{13} \in \mathfrak{g}_{m_0}^0$, by Corollary 7.6, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are isotropic at m_0.

By Lemma 6.2, since $\mathfrak{n}^+$ is isotropic at m_0, we have $[\mathfrak{n}^+, \mathfrak{n}^+] \subseteq \mathfrak{g}_{m_0}$, so $E_{13} \in \mathfrak{g}_{m_0}$. Moreover, by Lemma 6.1, we can choose $p, q \in \mathbb{R}$ such that $pE_{12} + qE_{23} \in \mathfrak{g}_{m_0}$ and $(p, q) \neq (0, 0)$.

Similarly, $\mathfrak{n}^-$ is isotropic at m_0, so $E_{31} \in \mathfrak{g}_{m_0}$ and we can choose $r, s \in \mathbb{R}$ such that $rE_{21} + sE_{32} \in \mathfrak{g}_{m_0}$ and $(r, s) \neq (0, 0)$.

Then $pE_{32} - qE_{21} = [E_{31}, pE_{12} + qE_{23}] \in \mathfrak{g}_{m_0}$, so

$$pq(-E_{11} + 2E_{22} - E_{33}) = [pE_{12} + qE_{23}, pE_{32} - qE_{21}] \in \mathfrak{g}_{m_0}.$$

If $pq \neq 0$, then $pq(-E_{11} + 2E_{22} - E_{33}) \in \mathfrak{a} \backslash \mathbb{R}H_{13}$, and we are done.

Let $Q := g_{m_0}$ and $J := H_{13}$. Then $J \in \mathfrak{g}_{m_0}^0$ by assumption. We consider two cases:

Case 1. Suppose $p = 0$ and $q \neq 0$. Then $E_{23} \in \mathfrak{g}_{m_0}$, so

$$sH_{23} = [E_{23}, rE_{21} + sE_{32}] \in \mathfrak{g}_{m_0}.$$

If $s \neq 0$, then $sH_{23} \in \mathfrak{a} \backslash \mathbb{R}H_{13}$, and we are done. Therefore, we can assume that $s = 0$, which implies that $E_{21} \in \mathfrak{g}_{m_0}$. Set $Z := -E_{21}$, $J' := H_{12}$, $Y := E_{12}$ and $W := E_{12}$.

Case 2. Suppose $p \neq 0$ and $q = 0$. Then $E_{12} \in \mathfrak{g}_{m_0}$, so

$$rH_{12} = [E_{12}, rE_{21} + sE_{32}] \in \mathfrak{g}_{m_0}.$$

If $r \neq 0$, then $rH_{12} \in \mathfrak{a} \backslash \mathbb{R}H_{13}$, and we are done. Therefore, we can assume that $r = 0$, which implies that $E_{32} \in \mathfrak{g}_{m_0}$. Set $Z := -E_{32}$, $J' := H_{23}$, $Y := E_{23}$ and $W := E_{23}$.

In both Case 1 and Case 2, $Z \in \mathfrak{g}_{m_0}$, $Z \in \mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ is nilpotent, $J' \in \mathfrak{a} \backslash \mathbb{R}H_{13}$, $W \in \mathfrak{n}^+$ and

$$[J, W] = W, \qquad [J', Z] = -2Z, \qquad [Z, Y] = J' = [Z, W].$$

We conclude from Lemma 7.3 that $Z \in \mathfrak{g}^0_{m_0}$. As $J' \in \mathfrak{a} \backslash \mathbb{R}H_{13}$, it suffices to show that $J' \in \mathfrak{g}_{m_0}$.

Thus, following [Kow96, pp. 631–632]

$$\begin{aligned}
Q(W, W) &= 0 \qquad \text{because } \mathfrak{n}^+ \text{ is isotropic,} \\
Q(J', W) &= Q(J', [J, W]) = -Q([J, J'], W) \\
&= -Q(0, W) = 0 \qquad \text{because } J \in \mathfrak{g}^0_{m_0}, \\
Q(J', J') &= Q([Z, Y], J') = -Q(Y, [Z, J']) \\
&= Q(Y, 2Z) = 0 \qquad \text{because } Z \in \mathfrak{g}^0_{m_0} \subseteq \ker(Q).
\end{aligned}$$

This proves that $\mathbb{R}J' \oplus \mathbb{R}W$ is isotropic at m_0. By Lemma 6.1, choose $a, b \in \mathbb{R}$, such that $aJ' + bW \in \mathfrak{g}_{m_0}$ and $(a, b) \neq (0, 0)$.

If $b = 0$, then $J' \in \mathfrak{g}_{m_0}$ and we are done. On the other hand, if $b \neq 0$, then $bW = [J, aJ' + bW] \in \mathfrak{g}_{m_0}$, so $W \in \mathfrak{g}_{m_0}$, so $J' = [Z, W] \in \mathfrak{g}_{m_0}$. □

Lemma 7.8. *If $H_{13} \in \mathfrak{g}_{m_0}$, then $\mathfrak{a} \subseteq \mathfrak{g}_{m_0}$.*

Proof. If $H_{13} \in \mathfrak{g}^0_{m_0}$, the preceding lemma implies that $\mathfrak{a} \subset \mathfrak{g}^0_{m_0}$. So we may assume that $H_{13} \notin \mathfrak{g}^0_{m_0}$, *i.e.*, $\Phi_{m_0}(H_{13}) \neq 0$. We shall assume that $\Phi_{m_0}(H_{13}) < 0$. The proof when $\Phi_{m_0}(H_{13})\, 0$ is similar.

Then, (2) of Lemma 7.5 implies that $\mathfrak{a} \oplus \mathfrak{n}^+$ is isotropic at m_0. Therefore, by Lemma 6.1, $\mathfrak{g}_{m_0} \cap (\mathfrak{a} \oplus \mathfrak{n}^+)$ has codimension at most 1 in $\mathfrak{a} \oplus \mathfrak{n}^+$.

Since $\mathrm{ad}_{\mathfrak{g}}(H_{13}) : \mathfrak{g} \to \mathfrak{g}$ is real diagonalizable and since this map preserves $\mathfrak{g}_{m_0} \cap (\mathfrak{a} \oplus \mathfrak{n}^+)$, it follows that $\mathfrak{g}_{m_0}(\mathfrak{a} \cap \mathfrak{n}^+) = (\mathfrak{g}_{m_0} \cap \mathfrak{a}) \oplus (\mathfrak{g}_{m_0} \oplus \mathfrak{n}^+)$. If $\mathfrak{g}_{m_0} \cap \mathfrak{a} = \mathfrak{a}$, then we are done. So assume that $\mathfrak{g}_{m_0} \cap \mathfrak{a} \neq \mathfrak{a}$. Then $\mathfrak{g}_{m_0} \cap \mathfrak{n}^+ = \mathfrak{n}^+$, so $\mathfrak{n}^+ \subset \mathfrak{g}_{m_0}$.

It follows from Lemma 7.3 that $\mathfrak{n}^+ \subset \mathfrak{g}^0_{m_0}$. We calculate that

$$\mathbb{R}H_{12} \oplus \mathbb{R}E_{32} \subseteq [(\mathrm{ad}_{\mathfrak{g}}\, E_{12})(\mathfrak{g})] \cap [(\mathrm{ad}_{\mathfrak{g}}\, E_{12})^{-1}(\mathbb{R}E_{12})].$$

Lemma 5.4 implies that $\mathbb{R}H_{12} \oplus \mathbb{R}E_{32}$ is isotropic at m_0. By Lemma 6.1, choose $a, b \in \mathbb{R}$, such that $aH_{12} + bE_{32} \in \mathfrak{g}_{m_0}$ and $(a, b) \neq (0, 0)$.

If $b = 0$, we are done. If not, then since $E_{23} \in \mathfrak{g}_{m_0}$,

$$-bH_{23} = aE_{23} + [aH_{12} + bE_{32}, E_{23}] \in \mathfrak{g}_{m_0}. \quad \square$$

Lemma 7.9. *If $m_0 \in M$ and if $\mathfrak{a} \subset \mathfrak{g}_{m_0}$, then there is a codimension-two subspace W of $\mathfrak{n}^+ \oplus \mathfrak{n}^-$ such that $W \subseteq \mathfrak{g}_{m_0}$.*

Proof. Let $X := H_{13} + 2H_{23}$. If $X \in \mathfrak{g}^0_{m_0}$, then by Corollary 7.6, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are isotropic at m_0. By Lemma 6.1, $(\mathfrak{g}_{m_0} \cap \mathfrak{n}^+) \oplus (\mathfrak{g}_{m_0} \cap \mathfrak{n}^-)$ has codimension at most two in $\mathfrak{n}^+ \oplus \mathfrak{n}^-$, and the lemma follows.

We therefore assume that $X \notin \mathfrak{g}^0_{m_0}$, *i.e.*, that $\Phi_{m_0}(X) \neq 0$.

Let $S := \{\pm 1, \pm 4, \pm 5\}$ and let $\mathcal{E} := \{E_{12}, E_{23}, E_{13}, E_{21}, E_{32}, E_{31}\}$. For all $a \in S$, there exists a unique $E \in \mathcal{E}$ such that $(\mathrm{ad}_{\mathfrak{g}} X)E = aE$. Moreover, the linear span of $\mathcal{E}$ is $\mathfrak{n}^+ \oplus \mathfrak{n}^-$.

By Lemma 6.1, it suffices to show that there is a codimension-one subspace $V \subset \mathfrak{n}^+ \oplus \mathfrak{n}^-$ that V is isotropic at m_0.

Let $T := \{(a, a') \in S \times S \mid a \leq a'\}$. Let

$$U := \{(a, a') \in S \times S \mid a \leq a' \text{ and } a + a' \neq 0\}.$$

Define $A : T \to \mathbb{Z}$ by $A(a, a') = a + a'$. Then $A|U$ is injective, so we can choose $(a_0, a_0') \in U$ such that $\Phi_{m_0}(X) \notin A(U \backslash \{(a_0, a_0')\})$.

Let $\tilde{T} := T \backslash \{(a_0, a_0')\}$. Let $\tilde{U} := U \backslash \{(a_0, a_0')\}$. Then $\Phi_{m_0}(X) \notin A(\tilde{U})$. Moreover, $\Phi_{m_0}(X) \neq 0$, so $\Phi_{m_0}(X) \notin A(\tilde{U}) \cup \{0\} = A(\tilde{T})$.

Let E_0 be the unique element of $\mathcal{E}$ such that $(\mathrm{ad}_{\mathfrak{g}} X)E_0 = a_0 E_0$. It suffices to prove that the linear span of $\tilde{\mathcal{E}} := \mathcal{E} \backslash \{E_0\}$ is isotropic at m_0.

Fix $E, E' \in \tilde{\mathcal{E}}$. Choose $a, a' \in S$ such that $(\mathrm{ad}_{\mathfrak{g}} X)E = aE$ and $(\mathrm{ad}_{\mathfrak{g}} X)E' = a'E'$. By interchanging E and E', if necessary, we may assume that $a \leq a'$. Then $(a, a') \in T$.

Now $E \in \tilde{\mathcal{E}}$, so $E \neq E_0$, so $a \neq a_0$, so $(a, a') \in T \backslash \{(a_0, a_0')\} = \tilde{T}$. Then $a + a' = A(a, a') \in A(\tilde{T})$, while $\Phi_{m_0}(X) \notin A(\tilde{T})$. Thus

$$a + a' \neq \Phi_{m_0}(X).$$

By differentiating the definition of Φ_{m_0}, we get

$$B_{m_0}((\mathrm{ad}_{\mathfrak{g}} X)E, E') + B_{m_0}(E, (\mathrm{ad}_{\mathfrak{g}} X)E') = [\Phi_{m_0}(X)][B_{m_0}(E, E')].$$

Since $(\mathrm{ad}_{\mathfrak{g}} X)E = aE$ and $(\mathrm{ad}_{\mathfrak{g}} X)E' = a'E'$, we have

$$a[B_{m_0}(E, E')] + a'[B_{m_0}(E, E')] = [\Phi_{m_0}(X)][B_{m_0}(E, E')],$$

so $[a + a' - \Phi_{m_0}(X)][B_{m_0}(E, E')] = 0$. Since $a + a' - \Phi_{m_0}(X) \neq 0$, we conclude that $B_{m_0}(E, E') = 0$, as desired. □

Lemma 7.10. *If $m_0 \in M$ and if $\mathfrak{a} \subseteq \mathfrak{g}_{m_0}$, then the codimension in $\mathfrak{g}$ of $\mathfrak{g}_{m_0}$ is at most 2.*

Proof. Choose m_0 as in Lemma 7.8, so that $\mathfrak{a} \subseteq \mathfrak{g}_{m_0}$. By Lemma 7.9, there is a codimension-two subspace W of $\mathfrak{n}^+ \oplus \mathfrak{n}^-$ such that $W \subseteq \mathfrak{g}_{m_0}$.

Now $\dim \mathfrak{n}^+ = \dim \mathfrak{n}^- = 3$, so $\dim W = 4$ and $\dim(\mathfrak{a} \oplus W) = 6$. Since $\dim \mathfrak{g} = 8$, the codimension in $\mathfrak{g}$ of $\mathfrak{a} \oplus W$ is 2. We conclude that the codimension in $\mathfrak{g}$ of $\mathfrak{g}_{m_0}$ is ≤ 2 since $\mathfrak{a} \oplus W \subseteq \mathfrak{g}_{m_0}$. □

Theorem 7.11. *The action of $G = SL_3(\mathbb{R})$ on M is proper.*

Proof. Assume for a contradiction that G acts nonproperly. Choose m_0 as in Lemma 7.4. Then by Lemmas 7.8 and 7.10, $\mathfrak{a} \subseteq \mathfrak{g}_{m_0}$ and $\dim \mathfrak{g}_{m_0} \leq (\dim \mathfrak{g}) - 2$. By Lemmas 3.2 and 7.2, we see that the codimension in $\mathfrak{g}$ of $\mathfrak{g}_{m_0}$ must be 2.

Define $E_0 : G \to M$ by $E_0(g) = gm_0$ and let $e_0 : \mathfrak{g} \to T_{m_0}M$ be the differential at 1_G of E_0. Let $V := e_0(\mathfrak{g})$. Then e_0 descends to a G_{m_0}-equivariant isomorphism of vector spaces $\overline{e}_0 : \mathfrak{g}/\mathfrak{g}_{m_0} \to V$ since $\mathfrak{g}_{m_0} = \ker(e_0)$. Because $\dim V = \dim(\mathfrak{g}/\mathfrak{g}_{m_0}) = 2$, the restriction to V of the Minkowski form g_{m_0} is nonzero.

Because $\mathfrak{a} \subseteq \mathfrak{g}_{m_0}$ and because $\dim \mathfrak{g}_{m_0} = (\dim \mathfrak{g}) - 2$, it follows from Lemma 3.3 that there is a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}_{m_0}$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

The isotropy representation of $\mathfrak{h}$ on $T_{m_0}M$ preserves V and a nonzero symmetric bilinear form on V. The representation of $\mathfrak{h} \cong \mathfrak{sl}_2(\mathbb{R})$ on V is completely reducible.

Since $\mathfrak{sl}_2(\mathbb{R})$ has no nontrivial one-dimensional representations, this is either trivial or irreducible. The canonical representation of $\mathfrak{sl}_2(\mathbb{R})$ on $\mathbb{R}^2$ is, up to isomorphism, the only irreducible two-dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$, and it does not preserve a nonzero symmetric bilinear form; therefore, the representation of $\mathfrak{h}$ on V is trivial.

Because $\overline{e}_0$ provides an isomorphism between the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{g}_{m_0}$ and the representation of $\mathfrak{h}$ on V, we conclude that the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{g}_{m_0}$ is trivial.

However, because $\mathfrak{g}$ is simple, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}/\mathfrak{g}_{m_0}$ is faithful. This gives the desired contradiction. □

8. Properness of $\mathrm{SL}_n(\mathbb{R})$

We show here that a locally faithful, conformal action of $G := \mathrm{SL}_n(\mathbb{R})$ on a connected Lorentz manifold M must be proper if $n \geq 3$.

Let $\mathfrak{a}$ be the maximal split torus in $\mathfrak{g}$ consisting of diagonal matrices of trace zero. Recall from §2 the definitions of $\Gamma = \Gamma(\mathfrak{g}, \mathfrak{a})$, $\Gamma_A^+ = \Gamma_A^+(\mathfrak{g}, \mathfrak{a})$ and of $\mathfrak{n}_A^+ = \mathfrak{n}_A^+(\mathfrak{g}, \mathfrak{a})$ for $A \in \mathfrak{a}$. For each $\alpha \in \Gamma$ let $\mathfrak{g}_\alpha$ be the root space of α.

Lemma 8.1. *Let $A_0 \in \mathfrak{a}\backslash\{0\}$. Then there exists a closed subgroup H of G such that H is isomorphic to $SL_3(\mathbb{R})$ and* $\dim(\mathfrak{h} \cap \mathfrak{n}_{A_0}^+) \geq 2$.

Proof. Let $a_1, \ldots, a_n$ be, respectively, the $(1,1), \ldots, (n,n)$ entries of A_0, *i.e.*, the entries down the diagonal. We may assume, after permutation of vectors in the standard basis (if necessary), that $a_1 > a_2 \geq a_3$ or that $a_1 \geq a_2 > a_3$. In the former case, $E_{12}, E_{13} \in \mathfrak{n}_{A_0}^+$. In the latter case, $E_{23}, E_{13} \in \mathfrak{n}_{A_0}^+$. In either case, we may let H be the copy of $\mathrm{SL}_3(\mathbb{R})$ embedded in the upper left corner of G. □

Theorem 8.2. *The action of $G = SL_n(\mathbb{R})$ on M is proper.*

Proof. Assume for a contradiction that the action is nonproper. By Lemma 4.7, we can choose $A_0 \in \mathfrak{a}$ and $m_0 \in M$ such that $\mathfrak{n}_{A_0}^+$ is isotropic at m_0.

Choose H as in Lemma 8.1. Then H is isomorphic to $\mathrm{SL}_3(\mathbb{R})$ and $\mathfrak{h} \cap \mathfrak{n}_{A_0}^+$ is isotropic at m_0 and has dimension at least 2. Thus by Lemma 6.1, $\mathfrak{h}_{m_0} \cap \mathfrak{n}_{A_0}^+ = \mathfrak{h} \cap \mathfrak{n}_{A_0}^+ \cap \mathfrak{g}_{m_0}$ has dimension at least 1.

Since every element of $\mathfrak{n}_{A_0}^+$ is $\mathrm{Ad}_{\mathfrak{g}}$-nilpotent, every element of the corresponding connected subgroup of G is $\mathrm{Ad}_{\mathfrak{g}}$-unipotent. Thus the subgroup of H generated by $\mathfrak{h}_{m_0} \cap \mathfrak{n}_{A_0}^+$ is not precompact. Thus H acts nonproperly on M, contradicting Theorem 7.11. □

9. Actions of $\mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$

Fix an integer $n \geq 3$. Let M be a connected Lorentz manifold and suppose the Lie group $G := \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ acts locally faithfully on M by conformal transformations. Eventually, we will show that the action of G is proper. In this section we show that if the action of G is nonproper, then the action of the subgroup $\mathbb{R}^n$ on M is nonproper.

Let $K := \mathrm{SO}(n)$. Let A be the maximal split torus in $\mathrm{SL}_n(\mathbb{R})$ consisting of $n \times n$ diagonal matrices with positive diagonal entries and determinant one.

Lemma 9.1. *The action of $A \ltimes \mathbb{R}^n$ on M is nonproper.*

Proof. Since $\mathrm{SL}_n(\mathbb{R}) = KAK$, it follows that $G = K(A \ltimes \mathbb{R}^n)K$. So every sequence in G has a bounded perturbation in $A \ltimes \mathbb{R}^n$. □

The next lemma was proved in [AS97, Lemma 6.2].

Lemma 9.2. *Let $X \in \mathfrak{sl}_n(\mathbb{R})\backslash\{0\}$ and let $v \in \mathbb{R}^n$. Let $m_0 \in M$. Assume that every row of X vanishes except the first. Assume that every entry of v vanishes except possibly the first. Then $X + v \notin \mathfrak{g}_{m_0}$.*

Lemma 9.3. *The action of $\mathbb{R}^n$ on M is nonproper.*

Proof. By Lemma 9.1, there exists a nonproper sequence g_i in $A \ltimes \mathbb{R}^n$. By Lemma 4.2, we may assume that g_i has bounded conformal factors. For all i, choose a_i in A and v_i in $\mathbb{R}^n$ such that $g_i = a_i v_i$.

For $j, k \in \{1, \dots, n\}$, let E_{jk} be the matrix with a one in the (j, k) entry and zeroes everywhere else. Let u_j be the vector in $\mathbb{R}^n$ with a one in the jth entry and zeroes everywhere else.

If $\{a_i\}$ has a convergent subsequence, then after passing to this subsequence and making a bounded perturbation, we conclude that v_i is a nonproper sequence; this would imply that $\mathbb{R}^n$ is nonproper on M, and we would be done. We may therefore assume that $a_i \to \infty$ in A.

For each i and each $j \in \{1, \dots, n\}$, let a_i^j be the (j, j) entry of a_i and let v_i^j be the jth entry of v_i; note that $a_i^j > 0$. Reordering coordinates (if necessary), we may assume that $a_i^1 \to \infty$ and $a_i^2 \to 0$.

If $a_i^3 \to \infty$, then both $(\mathrm{Ad}\, g_i)u_1 = a_i^1 u_1$ and $(\mathrm{Ad}\, g_i)u_3 = a_i^3 u_3$ go to infinity and converge in direction to u_1 and u_3, respectively. By Corollary 4.5, we conclude that the span $\mathbb{R}u_1 + \mathbb{R}u_3$ is somewhere isotropic. By Lemma 6.1, a codimension-one subspace in $\mathbb{R}u_1 + \mathbb{R}u_3$ is contained in the stabilizer of some point, so $\mathbb{R}^n$ acts nonproperly and we are done.

We assume then that a_i^3 does not go to infinity. Passing to a subsequence, we may assume that a_i^3 is bounded.

For all i, we have $(\mathrm{Ad}\, g_i)E_{12} = (a_i^1/a_i^2)E_{12} - a_i^1 v_i^2 u_1$, which, after passing to a subsequence, converges in direction. Choose $X \in \mathbb{R}E_{12}$ and $u \in \mathbb{R}u_1$ such that $(\mathrm{Ad}\, g_i)E_{12}$ converges in direction to $X + u$.

Similarly, $(\mathrm{Ad}\, g_i)E_{13} = (a_i^1/a_i^3)E_{13} - a_i^1 v_i^3 u_1$, which goes to infinity, and, after passing to a subsequence, converges in direction to a vector $Y + v$ for some $Y \in \mathbb{R}E_{13}$ and $v \in \mathbb{R}u_1$.

We consider first the case $X \neq 0$. Because E_{12} and E_{13} are linearly independent, $X + u$ and $Y + v$ are as well. It follows from Corollary 4.5 and Lemma 6.1 that we can choose $m_0 \in M$ and $s, t \in \mathbb{R}$ such that $s(X + u) + t(Y + v) \in \mathfrak{g}_{m_0}\backslash\{0\}$. Let $Z := sX + tY$ and $w := su + tv$.

Then $Z + w \in \mathfrak{g}_{m_0}\backslash\{0\}$ and $Z \in \mathbb{R}E_{12} + \mathbb{R}E_{13}$ and $w \in \mathbb{R}u_1$. So, all but the first row of Z vanishes and all but possibly the first entry of w vanishes. In this case, by Lemma 9.2, we must have $Z = 0$. Thus we obtain a nontrivial (hence noncompact) stabilizer for the $\mathbb{R}^n$-action, so $\mathbb{R}^n$ acts nonproperly, as desired, provided that $X \neq 0$.

A similar argument works when $Y \neq 0$. We may therefore assume that $X = Y = 0$. Since $(a_i^1/a_i^3)E_{13} - a_i^1 v_i^3 u_1$ converges in direction to $Y + v = v \in \mathbb{R}u_1$, it follows that $(a_i^1/a_i^3)/(a_i^1 v_i^3) \to 0$.

As $X = 0$, it follows that $X + u = u \in \mathbb{R}u_1$. So $(\operatorname{Ad} g_i)E_{12}$ converges in direction to u_1. Now, for all i, we have $(\operatorname{Ad} g_i)E_{23} = (a_i^2/a_i^3)E_{23} - a_i^2 v_i^3 u_2$, and, since

$$(a_i^2/a_i^3)/(a_i^2 v_i^3) = (a_i^1/a_i^3)/(a_i^1 v_i^3) \to 0,$$

we see that $(\operatorname{Ad} g_i)(E_{23})$ converges in direction to u_2.

We therefore conclude, from Corollary 4.5 (with $S := \{E_{12}, E_{23}\}$), that the span of u_1 and u_2 is somewhere isotropic. By Lemma 6.1, this implies that a nontrivial (hence noncompact) subgroup of $\mathbb{R}^n$ stabilizes some point of M. Thus $\mathbb{R}^n$ acts nonproperly on M. □

10. Codimension-one stabilizers in $\mathbb{R}^n$

Fix an integer $n \geq 2$. Let M be a Lorentz manifold and let the Lie group $G := \mathrm{SL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ act locally faithfully on M by conformal transformations. Suppose that $\mathbb{R}^n$ acts nonproperly on M. (Eventually, we will show that this is impossible.) In this section, we prove:

Lemma 10.1. *The subalgebra $\mathbb{R}^n$ of $\mathfrak{g}$ is somewhere isotropic.*

Proof. By assumption, there is a nonproper sequence v_i in $\mathbb{R}^n$. By Lemma 4.2, we may assume that v_i has bounded conformal factors.

Fix a norm on $\mathbb{R}^n$. Choose sequences $\{w_i\}$ in the unit sphere of $\mathbb{R}^n$ and $\{t_i\}$ in $(0, \infty)$ such that $t_i \to \infty$ and $v_i = t_i w_i$. Passing to a subsequence, we may assume that there is a unit vector w_∞ such that $w_i \to w_\infty$.

For $X \in \mathfrak{sl}_n(\mathbb{R})$, let $\tilde{X} : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding endomorphism; we then have

$$(\operatorname{Ad} v_i)X = X + [v_i, X] = X - \tilde{X} v_i = X - t_i(\tilde{X} w_i).$$

For $X \in P := \{X \in \mathfrak{sl}_n(\mathbb{R}) \mid \tilde{X} w_\infty \neq 0\}$, we conclude that the sequence $\{(\operatorname{Ad} v_i)X\}_i$ goes to infinity, but converges in direction to $\tilde{X} w_\infty$.

Note that $\mathbb{R}^n$ is the span of $\{\tilde{X} w_\infty \mid X \in \mathfrak{sl}_n(\mathbb{R})\}$, and therefore is also the span of $S := \{\tilde{X} w_\infty \mid X \in P\}$. Thus, by Corollary 4.5, we conclude that $\mathbb{R}^n$ is somewhere isotropic, as desired. □

11. Some representation-theoretic results

Roughly speaking, the lemmas in this section show that the Lie algebra $\mathfrak{co}(1, d-1) \ltimes \mathbb{R}^d$ does not contain $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ as a subalgebra. More precisely, the following lemma, together with Lemma 11.2, shows that there is no nonzero linear map from $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ to $\mathfrak{co}(1, d-1) \ltimes \mathbb{R}^d$ that preserves the Lie bracket of elements of $\mathfrak{sl}_2(\mathbb{R})$ with elements of $\mathbb{R}^2$. This fact will be used in §14.

Lemma 11.1 was proved in [AS97, Lemma 7.1]. Lemma 11.2 was proved in [AS97, Lemma 7.2] for $\mathfrak{so}(1, d-1) \ltimes \mathbb{R}^d$ instead of $\mathfrak{co}(1, d-1) \ltimes \mathbb{R}^d$, but the same proof works for $\mathfrak{co}(1, d-1) \ltimes \mathbb{R}^d$.

Lemma 11.1. *Let $\mathfrak{b}$ and $\mathfrak{h}$ be Lie algebras and let $W \subseteq \mathfrak{b}$ be a subspace that is not contained in any proper Lie subalgebra of $\mathfrak{b}$. Let $\phi : W \to \mathfrak{h}$ be a linear map. Let $\mathfrak{h}_0$ be the smallest Lie subalgebra of $\mathfrak{h}$ that contains $\phi(W)$. Let V be a vector space, and let $\rho : \mathfrak{b} \to \mathfrak{gl}(V)$ be a representation. For all $X \in \mathfrak{b}$, let $\tilde{X} := \rho(X) : V \to V$ be the endomorphism corresponding to X. Let $\tilde{\mathfrak{b}} := \rho(\mathfrak{b}) \subseteq \mathfrak{gl}(V)$. Let $\psi : V \to \mathfrak{h}$ be a linear map. Assume, for all $X \in W$, for all $U \in V$, that*

$$[\phi(X), \psi(U)] = \psi(\tilde{X} U). \tag{11.1}$$

Then

1. *if $\rho : \mathfrak{b} \to \mathfrak{gl}(V)$ is irreducible and if $\psi \neq 0$, then $\psi : V \to \mathfrak{h}$ is injective;*
2. *if ψ is injective, then $\tilde{\mathfrak{b}}$ is a Lie quotient of $\mathfrak{h}_0$, i.e., there exists a surjective Lie algebra homomorphism $\sigma : \mathfrak{h}_0 \to \tilde{\mathfrak{b}}$; and*
3. *if ψ is injective and if $\tilde{\mathfrak{b}}$ is semisimple, then there exists a Lie algebra homomorphism $\phi' : \mathfrak{b} \to \mathfrak{h}_0$ such that and $X \in \mathfrak{b}$, for all $U \in V$, we have*

$$[\phi'(X), \psi(U)] = \psi(\tilde{X}U). \tag{11.2}$$

Lemma 11.2. *Let $d \geq 1$ be an integer and let $\mathfrak{h} = \mathfrak{co}(1, d-1) \ltimes \mathbb{R}^d$. For all $X \in \mathfrak{sl}_2(\mathbb{R})$, let $\tilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map corresponding to X. Assume that $\phi : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{h}$ is a Lie algebra homomorphism and that $\psi : \mathbb{R}^2 \to \mathfrak{h}$ is a nonzero linear map. Then there exist $X \in \mathfrak{sl}_2(\mathbb{R})$ and $U \in \mathbb{R}^2$ such that $[\phi(X), \psi(U)] \neq \psi(\tilde{X}U)$.*

12. Jets of vector fields of conformal actions

Let g be the metric on a pseudo-Riemannian manifold M and let $m_0 \in M$. Let N be a neighborhood of zero in $T_{m_0}M$ such that $\exp_{m_0}|N$ is a diffeomorphism onto a neighborhood M_0 of m_0. Let $\hat{g} := \exp_{m_0}^*(g)$. Let $\overline{g}$ be the flat pseudo-Riemannian metric on $T_{m_0}M$ corresponding to the inner product g_{m_0} on $T_{m_0}M$.

The following two results are straightforward facts about the calculus of vector fields, and are proved in [AS97]. The statements are reproduced here for the convenience of the reader.

Lemma 12.1. *$\hat{g}$ and $\overline{g}$ agree to order one at zero.*

Lemma 12.2. *Let X be a vector field defined near zero in a vector space V. Then for any $k > 0$, there exist vector fields $X_0, \ldots, X_k$ and R on V such that X_i is homogeneous of degree i, R vanishes to order k at zero, and $X = X_0 + \cdots + X_k + R$ on a neighborhood of zero.*

Let G be a connected Lie group acting locally faithfully on a connected pseudo-Riemannian manifold (M, g). Fix $m_0 \in M$. Let N be a starlike neighborhood of 0 in $T_{m_0}M$ such that $\exp_{m_0}$ is defined on N and $\exp_{m_0}$ is a diffeomorphism of N onto a neighborhood M_0 of m_0 in M.

Let $\Phi_0 : G_{m_0} \to \mathbb{R}$ be the homomorphism defined by $g(h_*(v), h_*(w)) = [\Phi_0(h)][g(v, w)]$, for all $v, w \in T_{m_0}M$, for all $h \in G_{m_0}$. Let $G_{m_0}^0$ be the kernel of Φ_0.

For all $X \in \mathfrak{g}$, let X_M be the vector field on M corresponding to X, and let $\hat{X} := (\exp_{m_0}^{-1})_*(X_M)$. For $X \in \mathfrak{g}$, using Lemma 12.2, we can write $\hat{X} = \hat{X}_C + \hat{X}_L + \hat{X}_Q + \hat{X}_K + \hat{X}_R$ where $\hat{X}_C$, $\hat{X}_L$, $\hat{X}_Q$, $\hat{X}_K$ are homogeneous of orders 0, 1, 2, and 3, respectively, and $\hat{X}_R$ vanishes to order three at zero.

Lemma 12.3. *Assume that G acts on M by conformal transformations. Let $T, U, X \in \mathfrak{g}$:*

1. *if $U \in \mathfrak{g}_{m_0}$, then $\hat{U}_C = 0$;*
2. *if $U \in \mathfrak{g}_{m_0}^0$, then $\hat{U}_L \in \mathfrak{so}(g_{m_0})$;*
3. *$\hat{X}_L \in \mathfrak{co}(g_{m_0})$;*
4. *if $\hat{U}_C = 0$, then there exists a linear function f on N such that $L_{\hat{U}_Q}\overline{g} = f \cdot \overline{g}$;*

5. *if* $\hat{U}_C = 0$ *and if* $T = [X, U]$, *then*
$$\hat{T}_C = [\hat{X}_C, \hat{U}_L] \qquad \textit{and} \qquad \hat{T}_L = [\hat{X}_L, \hat{U}_L] + [\hat{X}_C, \hat{U}_Q];$$
6. *if* $\dim M \geq 3$ *and* $\hat{U}_C = U_L = 0$, *then* $\hat{U}_K = 0$;
7. *if* $\dim M \geq 3$ *and* $\hat{U}_C = \hat{U}_L = 0$ *and* $T = [X, U]$, *then*
$$\hat{T}_C = 0, \qquad \hat{T}_L = [\hat{X}_C, \hat{U}_Q] \qquad \textit{and} \qquad \hat{T}_Q = [\hat{X}_L, \hat{U}_Q];$$
8. *if* $\dim M \geq 3$ *and* $\hat{U}_C = \hat{U}_L = \hat{U}_Q = 0$, *then* $U = 0$.

Proof. For any $Z \in \mathfrak{g}$, let f^Z be the C^∞ function on N satisfying $L_{\hat{Z}}(\hat{g}) = f^Z \cdot \hat{g}$. Let f^Z_C, f^Z_L and f^Z_Q be the constant, linear and quadratic parts, respectively, of f^Z, so $f^Z - f^Z_C - f^Z_L - f^Z_Q$ vanishes to order two at zero.

Let $\hat{g}_C := \overline{g}$. By Lemma 12.1, $\hat{g}_L = 0$.

Proof of 1: Since $U \in \mathfrak{g}_{m_0}$, U_M vanishes at m_0, so $\hat{U}_C = 0$.

Proof of 2: Since $U \in \mathfrak{g}^0_{m_0} \subseteq \mathfrak{g}_{m_0}$, we conclude that $\exp tU \in G_{m_0}$, for all $t \in \mathbb{R}$, and $(\exp tU)_* : T_{m_0}M \to T_{m_0}M$ preserves g_{m_0}. Thus $(\exp tU)_* \in \mathrm{SO}(g_{m_0})$.

Let W be the vector field of the flow
$$(t, v) \to (\exp tU)_* v : \mathbb{R} \times T_{m_0}M.$$
Then $W \in \mathfrak{so}(g_{m_0})$. Moreover, W agrees to order one with $\hat{U}$ at zero, so $\hat{U}_L = W \in \mathfrak{so}(g_{m_0})$.

Proof of 3: Taking the constant parts of both sides of the equation $L_{\hat{X}}(\hat{g}) = f^X \cdot \hat{g}$, we have
$$L_{\hat{X}_L}(\hat{g}_C) = f^X_C \cdot \hat{g}_C.$$
Since $\hat{g}_C = \overline{g}$, we conclude that $L_{\hat{X}_L}\overline{g} = f^X_C \cdot \overline{g}$, so $\hat{X}_L \in \mathfrak{co}(g_{m_0})$.

Proof of 4: We have $L_{\hat{U}}(\hat{g}) = f^U \cdot \hat{g}$, so
$$f^U_L \cdot \hat{g}_C + f^U_C \cdot \hat{g}_L = L_{\hat{U}_Q}(\hat{g}_C) + L_{\hat{U}_L}(\hat{g}_L) + L_{\hat{U}_C}(\hat{g}_Q).$$
Since $\hat{g}_L = 0$ and $\hat{U}_C = 0$, we have $f^U_L \cdot \hat{g}_C = L_{\hat{U}_Q}(\hat{g}_C)$. Since $\overline{g} = \hat{g}_C$, we are done.

Proof of 5: We compute $\hat{T}_C = [\hat{X}_L, \hat{U}_C] + [\hat{X}_C, \hat{U}_L]$ and
$$\hat{T}_L = [\hat{X}_Q, \hat{U}_C] + [\hat{X}_L, \hat{U}_L] + [\hat{X}_C, \hat{U}_Q].$$
Since $\hat{U}_C = 0$, we are done.

Proof of 6: Taking the constant terms of both sides of the equation $L_{\hat{U}}(\hat{g}) = f^U \cdot \hat{g}$ we get
$$f^U_C \cdot \hat{g}_C = L_{\hat{U}_L}(\hat{g}_C) + L_{\hat{U}_C}(\hat{g}_L).$$
Since $\hat{U}_C = 0$ and $\hat{U}_L = 0$, we conclude that $f^U_C \cdot \hat{g}_C = 0$, so $f^U_C = 0$.

Taking the quadratic terms yields
$$\begin{aligned} f^U_Q \cdot \hat{g}_C + f^U_L \cdot \hat{g}_L + f^U_C \cdot \hat{g}Q \\ = L_{\hat{U}_K}(\hat{g}_C) + L_{\hat{U}_Q}(\hat{g}_L) + L_{\hat{U}_L}(\hat{g}_Q) + L_{\hat{U}_C}(\hat{g}_K). \end{aligned}$$
Since $\hat{g}_L = 0$ and $f^U_C = 0$ and $\hat{U}_L = 0$ and $\hat{U}_C = 0$, we see that $f^U_Q \cdot \hat{g}_C = L_{\hat{U}_K}(\hat{g}_C)$. Replacing X by $\hat{U}_K$, g by $\hat{g}_C$, f by f^U_Q and M by N in Lemma 5.3, we find that $\hat{U}_K = 0$.

Proof of 7: We compute $\hat{T}_C = [\hat{X}_L, \hat{U}_C] + [\hat{X}_C, \hat{U}_L]$ and
$$\hat{T}_L = [\hat{X}_Q, \hat{U}_C] + [\hat{X}_L, \hat{U}_L] + [\hat{X}_C, \hat{U}_Q]$$

and

$$\hat{T}_Q = [\hat{X}_K, \hat{U}_C] + [\hat{X}_Q, \hat{U}_L] + [\hat{X}_L, \hat{U}_Q] + [\hat{X}_C, \hat{U}_K].$$

We have $\hat{U}_C = \hat{U}_L = 0$ by assumption. By 6 of Lemma 12.3, we have $\hat{U}_K = 0$, so we are done.

Proof of 8: Replacing X by $\hat{U}$, g by $\hat{g}$, f by f^U and M by N in Lemma 5.3, we find that $\hat{U} = 0$. Local faithfulness implies that $U = 0$. □

13. Conformal transformations of flat Lorentz space

Let $\langle \cdot, \cdot \rangle$ be a nondegenerate symmetric bilinear form on a vector space V. For $v \in V$, let C^v be the constant vector field on V corresponding to the vector v, *i.e.*, $C^v_w = [d/dt]_{t=0}[w + tv]$. Let $\overline{g}$ be the constant Lorentz metric on V corresponding to g, *i.e.*, $\overline{g}(C^v, C^w)$ is the constant function on V with value $g(v, w)$. Note that for constant vector fields $C = C^v$, $C' = C^w$, we have $L_C\overline{g} = 0$ and $[C, C'] = L_C C' = 0$.

Let R be the radial vector field on V defined by $R_w = [d/dt]_{t=0}[w + tw]$. Let $q : V \to \mathbb{R}$ be the quadratic form defined by $q(v) = g(v, v)$. For $v \in V$, let f^v be the linear function on V defined by $f^v(w) = g(v, w)$. Let $Q^v := f^v R - (q/2)C^v$. Note that $C^w f^v$ is the constant function with value $g(v, w)$. Moreover, $L_{C^w} R = C^w$ and $C^w q = 2f^w$.

Lemma 13.1. *For all $v, w \in V$, $[C^w, Q^v] = g(v, w)R + f^v C^w - f^w C^v$.*

Proof. We compute:

$$\begin{aligned}[C^w, Q^v] &= [C^w, f^v R - (q/2)C^v] \\ &= (C^w f^v)R + f^v[C^w, R] - (1/2)(C^w q)C^v - (q/2)[C^w, C^v] \\ &= g(v, w)R + f^v C^w - f^w C^v. \quad \square\end{aligned}$$

Lemma 13.2. *For all constant vector fields C and C' on V and all vector fields X on V, we have $(L_X\overline{g})(C, C') = C(g(X, C')) + C'(g(X, C))$.*

Proof. We compute:

$$\begin{aligned}(L_X\overline{g})(C, C') &= X(\overline{g}(C, C')) - \overline{g}([X, C], C) - \overline{g}(C, [X, C']) \\ &= \overline{g}([C, X], C') + \overline{g}(C, [C', X]) = C(\overline{g}(X, C')) + C'(\overline{g}(C, X)),\end{aligned}$$

since $\overline{g}(C, C') = 0$, $L_C\overline{g} = L'_C\overline{g} = 0$ and $L_C C' = 0$. □

Lemma 13.3. *For all $v \in V$, we have $L_{Q^v}\overline{g} = 2f^v \cdot \overline{g}$.*

Proof. Fix $v, w \in V$.

$$\begin{aligned}(L_{Q^v}\overline{g})(C^w, C^w) &= Q_v(\overline{g}(C^w, C^w)) - 2\overline{g}(L_{Q^v}C^w, C^w) \\ &= 2\overline{g}(g(v, w)R + f^v C^w - f^w C^v, C^w) \\ &= 2g(v, w)\overline{g}(R, C^w) + 2\overline{g}(f^v C^w, C^w) - 2\overline{g}(f^w C^v, C^w) \\ &= 2\overline{g}(C^v, C^w)f^w + 2f^v\overline{g}(C^w, C^w) - 2\overline{g}(C^v, C^w)f^w \\ &= 2f^v\overline{g}(C^w, C^w). \quad \square\end{aligned}$$

Lemma 13.4. *For every linear function f on V, there is a unique quadratic vector field Q on V such that $L_Q\overline{g} = f \cdot \overline{g}$.*

Proof. For any linear function f on V, there exists $v \in V$ such that $f = 2f^v$, so existence follows from Lemma 13.3.

It remains to prove uniqueness. Assume that Q is a quadratic vector field on V and $L_Q\overline{g} = 0$. The flow of Q fixes the origin (zero) in V and fixes every tangent vector at zero. It is also a flow by isometries, so it preserves the Levi-Civita connection of $\overline{g}$. Therefore, it fixes all the rays out of zero, and therefore fixes all points in V, since V is starlike about zero. Thus $Q = 0$. □

Corollary 13.5. *Suppose that Q is a quadratic vector field on V. Then the following are equivalent:*

1. *There exists $f \in C^\infty(V)$ such that $L_Q\overline{g} = f \cdot \overline{g}$.*
2. *There exists $v \in V$ such that $Q = Q^v$.*

Proof. (1) *implies* (2): Because Q is quadratic and $\overline{g}$ is constant, $L_Q\overline{g}$ is linear. But $L_Q\overline{g} = f \cdot \overline{g}$, so f is linear, and so there exists $v \in V$ such that $f = 2f^v$. By Lemma 13.3, we have $L_{Q^v}\overline{g} = 2f^v \cdot \overline{g} = f \cdot \overline{g} = L_Q\overline{g}$. By Lemma 13.4, we conclude that $Q = Q^v$.

(2) *implies* (1): This is immediate from Lemma 13.3. □

Corollary 13.6. *For any $v, w \in V$, $[Q^v, Q^w] = 0$.*

Proof. The result follows from a straightforward calculation (or from Lemma 5.3 for $\dim V > 2$). □

14. Restrictions on stabilizers

Suppose $G := \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ acts locally faithfully by conformal transformations of a connected Lorentz manifold (M, g). Throughout this section we assume that $G_m \cap \mathrm{SL}_2(\mathbb{R})$ is compact for all $m \in M$.

For all $X \in \mathfrak{sl}_2(\mathbb{R})$, let $\tilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$ be the corresponding linear map.

Fix $m_0 \in M$. Define $E_0 : G \to M$ by $E_0(g) = gm_0$ and let $e_0 : \mathfrak{g} \to T_{m_0}M$ be the differential at 1_G of E_0.

Fix a starlike neighborhood N of $0 \in T_{m_0}M$ such that $\exp_{m_0}$ is defined on N and $\exp_{m_0}$ is a diffeomorphism of N onto a neighborhood M_0 of m_0 in M. For $X \in \mathfrak{g}$, let X_M be the conformal Killing vector field on M corresponding to X, and let $\hat{X} := (\exp_{m_0}^{-1})_*(X_M)$.

Since $N \subseteq T_{m_0}M$ is a neighborhood of 0, there is a natural identification of $T_0N \subset T(T_{m_0}M)$ with $T_{m_0}M$. In particular, if Y is a vector field on N, then $Y_0 \in T_0N$ can also be considered as a vector in $T_{m_0}M$. We will freely make such identifications without further comment.

For all $v \in T_{m_0}M$, let C^v be the constant vector field on $T_{m_0}M$ corresponding to the vector v, *i.e.*, $C^v_w = [d/dt]_{t=0}[w + tv]$. Let R be the radial vector field on $T_{m_0}M$ defined by $R_w = [d/dt]_{t=0}[w + tw]$. Let q be the quadratic function on $T_{m_0}M$ defined by $q(v) = g_{m_0}(v, v)$. For $v \in T_{m_0}M$, let f^v be the linear function on $T_{m_0}M$ defined by $f^v(w) = g_{m_0}(v, w)$. Let $Q^v := f^vR - (q/2)C^v$.

The natural identification of $T_{m_0}M$ with $T(T_{m_0}M)$ yields a natural identification between linear maps from $T_{m_0}M$ to $T_{m_0}M$ and linear vector fields on $T_{m_0}M$, as follows: a linear map $X : T_{m_0}M \to T_{m_0}M$ defines a linear vector field X by the rule $X_v = -Xv$; conversely, a linear vector field Y on $T_{m_0}M$ defines a linear map $Y : T_{m_0}M \to T_{m_0}M$ by the rule $Yv = -Y_v$. This identification is an isomorphism of Lie algebras. To avoid a proliferation of notation, we will use this identification

without further comment. Note that the radial vector field R is the vector field associated to the linear map $-I$.

For any subspace S of $T_{m_0}M$, let $\mathfrak{gl}(S)$ be the Lie algebra of linear maps $S \to S$. Note that, using the identification of linear maps with linear vectors, for any $X \in \mathfrak{gl}(S)$ and $v \in S$, $[X, C^v] = C^{Xv}$.

A subspace S of $T_{m_0}M$ is **degenerate** if $g_{m_0}|S$ is degenerate; otherwise, S is **nondegenerate**. The subspace $\{0\}$ is considered to be nondegenerate.

For any nondegenerate subspace $S \subseteq T_{m_0}M$, let $S^\perp$ be the orthogonal complement to S. Let $\iota_S : S \to T_{m_0}M$ be the inclusion map and let $\pi_S : T_{m_0}M = S \oplus S^\perp \to S$ be the orthogonal projection map. For any linear map $L : S \to S^\perp$, let $L^* : S^\perp \to S$ be the adjoint map.

Let $\mathfrak{so}(S) \subseteq \mathfrak{gl}(S)$ be the collection of linear maps $S \to S$ that infinitesimally preserve $g_{m_0}|S$. Let I_S be the identity transformation on S. Let $\mathfrak{co}(S) := \mathfrak{so}(S) + \mathbb{R}I_S$.

For any nondegenerate subspace $S \subseteq T_{m_0}M$, let $\mathfrak{co}_Q(S)$ be the collection of quadratic vector fields Q on S such that $L_Q(g_{m_0}|S) = f \cdot (g_{m_0}|S)$ for some $f \in C^\infty(N \cap S)$. Then $\mathfrak{co}_Q(S)$ is normalized by $\mathfrak{co}(S)$ and, by Corollaries 13.6 and 13.5, is Abelian. Let $\mathfrak{co}_*(S) := \mathfrak{co}(S) \ltimes \mathfrak{co}_Q(S)$.

If $S \subseteq T_{m_0}M$ is a nondegenerate subspace and if $A : T_{m_0}M \to T_{m_0}M$ is a linear map, then define

$$\begin{aligned}
\mathrm{UL}_S(A) &:= \pi_S \circ A \circ \iota_S : S \to S,\\
\mathrm{UR}_S(A) &:= \pi_S \circ A \circ \iota_{S^\perp} : S^\perp \to S,\\
\mathrm{LL}_S(A) &:= \pi_{S^\perp} \circ A \circ \iota_S : S \to S^\perp,\\
\mathrm{LR}_S(A) &:= \pi_{S^\perp} \circ A \circ \iota_{S^\perp} : S^\perp \to S^\perp.
\end{aligned}$$

In these definitions, "UL" stands for "upper left", "UR" for "upper right", "LL" for "lower left" and "LR" for "lower right".

Lemma 14.1. *Let $S_0 \subseteq T_{m_0}M$ be a nondegenerate subspace. Let $d_0 := \dim S_0$. Then $\mathfrak{co}_*(S_0)$ is Lie algebra isomorphic either to $\mathfrak{co}(d_0) \ltimes \mathbb{R}^{d_0}$ or to $\mathfrak{co}(1, d_0-1) \ltimes \mathbb{R}^{d_0}$.*

Proof. Depending on the signature of the nondegenerate form $g_{m_0}|S_0$, $\mathfrak{co}(S_0) \ltimes S_0$ is isomorphic to either $\mathfrak{co}(d_0) \ltimes \mathbb{R}^{d_0}$ or $\mathfrak{co}(1, d_0 - 1) \ltimes \mathbb{R}^{d_0}$, so it suffices to show that $\mathfrak{co}_*(S_0)$ is isomorphic to $\mathfrak{co}(S_0) \ltimes S_0$.

Define $F : \mathfrak{co}(S_0) \ltimes S_0 \to \mathfrak{co}_*(S_0)$ by $F(X, v) = X + Q^v$. Then F is a Lie algebra isomorphism. □

Lemma 14.2. *Let $S \subseteq T_{m_0}M$ be a nondegenerate subspace. Let $A \in \mathfrak{so}(T_{m_0}M)$. Then*

1. *$UL_S(A) \in \mathfrak{so}(S)$;*
2. *$LR_S(A) \in \mathfrak{so}(S^\perp)$; and*
3. *$UR_S(A) + LL_S(A)^* = 0$.*

Proof. Since $A \in \mathfrak{so}(T_{m_0}M)$, it follows that $A + A^* = 0$, so $\mathrm{UL}_S(A) + \mathrm{UL}_S(A^*) = 0$. Because $\mathrm{UL}_S(A^*) = \mathrm{UL}_S(A)^*$, we get $\mathrm{UL}_S(A) + \mathrm{UL}_S(A)^* = 0$, so $\mathrm{UL}_S(A) \in \mathfrak{so}(S)$, proving 1.

The proof of 2 is similar to the proof of 1.

It remains to prove 3. Since $A + A^* = 0$, we have

$$\mathrm{UR}_S(A) + \mathrm{UR}_S(A^*) = 0.$$

Since $\mathrm{UR}_S(A^*) = \mathrm{LL}_S(A)^*$, the result follows. □

Lemma 14.3. *Let $S \subseteq T_{m_0}M$ be a nondegenerate subspace. Let $A \in \mathfrak{co}(T_{m_0}M)$. Then*

1. *$UL_S(A) \in \mathfrak{co}(S)$;*
2. *$LR_S(A) \in \mathfrak{co}(S^\perp)$; and*
3. *$UR_S(A) + LL_S(A)^* = 0$.*

Proof. Since $A = A' + \alpha I$ for some $\alpha \in \mathbb{R}$ and $A' \in \mathfrak{so}(T_{m_0}M)$, the result follows by a simple computation from Lemma 14.2. □

Lemma 14.4. *Let $S \subseteq T_{m_0}M$ be a nondegenerate subspace. If $A, B : T_{m_0}M \to T_{m_0}M$ are linear maps and $UR_S(B) = 0$ and $LL_S(B) = 0$, then*

$$LR_S([A, B]) = [LR_S(A), LR_S(B)].$$

Proof. By matrix multiplication, we have

$$\begin{aligned}
\mathrm{LR}_S(AB) &= \mathrm{LL}_S(A) \cdot \mathrm{UR}_S(B) + \mathrm{LR}_S(A) \cdot \mathrm{LR}_S(B) \\
&= \mathrm{LR}_S(A) \cdot \mathrm{LR}_S(B), \\
\mathrm{LR}_S(BA) &= \mathrm{LL}_S(B) \cdot \mathrm{UR}_S(A) + \mathrm{LR}_S(B) \cdot \mathrm{LR}_S(A) \\
&= \mathrm{LR}_S(B) \cdot \mathrm{LR}_S(A),
\end{aligned}$$

so $\mathrm{LR}_S(AB - BA) = \mathrm{LR}_S(A) \cdot \mathrm{LR}_S(B) - \mathrm{LR}_S(B) \cdot \mathrm{LR}_S(A)$, as desired. □

Lemma 14.5. *Let $S \subseteq T_{m_0}M$ be a nondegenerate subspace. Let $v \in S^\perp$ and $w \in S$. Then $LR_S(f^v C^w) = 0$ and $LR_S(f^w C^v) = 0$.*

Proof. We have $(f^v C^w)(S) \subseteq \mathbb{R}w \subseteq S$, so $\pi_{S^\perp} \circ f^v C^w = 0$. Thus $\mathrm{LR}_S(f^v C^w) = 0$. Similarly, $S^\perp \subseteq (\mathbb{R}w)^\perp \subseteq \ker f^w C^v$, so $f^w C^v \circ \iota_{S^\perp} = 0$. Thus $\mathrm{LR}_S(f^w C^v) = 0$. □

Lemma 14.6. *If $U \in \mathbb{R}^2 \subseteq \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ and if $U \in \mathfrak{g}_{m_0}$, then $U \in \mathfrak{g}^0_{m_0}$.*

Proof. By hypothesis, recall that $G_{m_0} \cap \mathrm{SL}_2(\mathbb{R})$ is compact. Since $U \in \mathbb{R}^2$, $\mathrm{ad}_{\mathfrak{g}}(U) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent, so $\{\mathrm{Ad}_{\mathfrak{g}}(\exp(tU)) \mid t \in \mathbb{R}\} \subseteq \mathrm{GL}(\mathfrak{g})$ consists of unipotent transformations. By Lemma 6.4, we are done. □

Lemma 14.7. *There is a nondegenerate subspace $S \subseteq e_0(\mathfrak{sl}_2(\mathbb{R}))$ such that no proper Lie subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ contains $e_*^{-1}(S)$.*

Proof. If $e_0(\mathfrak{sl}_2(\mathbb{R}))$ is nondegenerate, then we can set $S := e_0(\mathfrak{sl}_2(\mathbb{R}))$. We therefore assume that $e_0(\mathfrak{sl}_2(\mathbb{R}))$ is degenerate.

Since $e_0(\mathfrak{sl}_2(\mathbb{R}))$ is a degenerate subspace of the Minkowski space $T_{m_0}M$, it must be positive semidefinite with one-dimensional kernel.

Let $\mathcal{S}$ be the topological space of all codimension-one subspaces of $e_0(\mathfrak{sl}_2(\mathbb{R}))$. Let $\mathcal{S}_1$ be the collection of nondegenerate elements of $S \in \mathcal{S}$. Any subspace of $e_0(\mathfrak{sl}_2(\mathbb{R}))$ that does not contain the one-dimensional kernel of $g_{m_0}|e_0(\mathfrak{sl}_2(\mathbb{R}))$ is nondegenerate, so $\mathcal{S}_1$ is open and dense in $\mathcal{S}$.

Let $K := G_{m_0} \cap \mathrm{SL}_2(\mathbb{R})$. Recall that throughout this section we are assuming that K is compact. Let $\hat{\mathcal{S}}$ be the topological space of codimension-one subspaces $\hat{S} \subseteq \mathfrak{sl}_2(\mathbb{R})$ such that $\mathfrak{k} \subseteq \hat{S}$. Define $e_* : \hat{\mathcal{S}} \to \mathcal{S}$ by $e_*(\hat{S}) = e_0(\hat{S})$. Then $e_* : \hat{\mathcal{S}} \to \mathcal{S}$ is a homeomorphism, so $e_*^{-1}(\mathcal{S}_1)$ is open and dense in $\hat{\mathcal{S}}$.

Let $\hat{\mathcal{S}}_2$ be the set of codimension-one subspaces $\hat{S} \subseteq \mathfrak{sl}_2(\mathbb{R})$ such that $\mathfrak{k} \subseteq \hat{S}$ and no proper Lie subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ contains $\hat{S}$. It suffices to show that $\hat{\mathcal{S}}_2 \cap e_0^{-1}(\mathcal{S}_1)$ is nonempty.

If $\mathfrak{k} \neq \{0\}$, then K is a maximal compact subgroup, and therefore a maximal proper subgroup of $\mathrm{SL}_2(\mathbb{R})$. In this case, if $\hat{S} \in \hat{\mathcal{S}}$, then $\mathfrak{k} \subsetneq \hat{S} \subsetneq \mathfrak{sl}_2(\mathbb{R})$, so maximality of $\mathfrak{k}$ among proper Lie subalgebras of $\mathfrak{sl}_2(\mathbb{R})$ implies that $\hat{S} \in \hat{\mathcal{S}}_2$. Thus $\hat{\mathcal{S}}_2 = \hat{\mathcal{S}}$.

If $\mathfrak{k} = \{0\}$, then $\hat{\mathcal{S}}$ is the set of all codimension-one subspaces of $\mathfrak{sl}_2(\mathbb{R})$. The collection $\hat{\mathcal{T}}$ of codimension-one Lie subalgebras of $\mathfrak{sl}_2(\mathbb{R})$ is a proper algebraic subset of $\hat{\mathcal{S}}$, and so we conclude that $\hat{\mathcal{T}}$ is closed and nowhere dense in $\mathcal{S}$. Since $\hat{\mathcal{S}}_2 = \hat{\mathcal{S}} \backslash \hat{\mathcal{T}}$, we are done. □

Lemma 14.8. *If $X, U \in \mathfrak{g}$, then $\hat{U}_L(\hat{X}_0) = [\hat{U}_L, \hat{X}_C]_0$, where $\hat{U}_L$ is considered as a linear map on the left-hand side of the equation, and as a linear vector field on the right-hand side.*

Proof. Let $v := \hat{X}_0$. Since $C^v = \hat{X}_C$, we get $\hat{U}_L(v) = (C^{\hat{U}_L(v)})_0 = [\hat{U}_L, C^v]_0 = [\hat{U}_L, \hat{X}_C]_0$. □

Lemma 14.9. *Assume that $\mathbb{R}^2 \subseteq \mathfrak{g}_{m_0}$. Then for all $X \in \mathfrak{sl}_2(\mathbb{R})$, for all $U \in \mathbb{R}^2$, we have $[\hat{U}_L, \hat{X}_C] = 0$.*

Proof. Let $T := [U, X]$. Then $\hat{T} = [\hat{U}, \hat{X}]$ and $\hat{T}_C = [\hat{U}_L, \hat{X}_C] + [\hat{U}_C, \hat{X}_L]$. Because $T, U \in \mathbb{R}^2 \subseteq \mathfrak{g}_{m_0}$, we see that $\hat{T}$ and $\hat{U}$ both vanish at zero, so $\hat{T}_C = \hat{U}_C = 0$. Thus $[\hat{U}_L, \hat{X}_C] = 0$. □

Lemma 14.10. *Assume that $\mathbb{R}^2 \subseteq G_{m_0}$. Then, for all $X \in \mathfrak{sl}_2(\mathbb{R})$, for all $U \in \mathbb{R}^2$, we have $\hat{U}_L(\hat{X}_0) = 0$.*

Proof. By Lemma 14.9, $[\hat{U}_L, \hat{X}_C] = 0$, while, by Lemma 14.8, $\hat{U}_L(\hat{X}_0) = [\hat{U}_L, \hat{X}_C]_0$. □

Lemma 14.11. *If $v, x \in T_{m_0}M$, then $f^v C^x - f^x C^v \in \mathfrak{so}(T_{m_0}M)$.*

Proof. We compute that $(f^v C^x)^* = f^x C^v$ and $(f^x C^v)^* = f^v C^x$. This implies that

$$(f^v C^x - f^x C^v)^* = -(f^v C^x - f^x C^v),$$

so $f^v C^x - f^x C^v \in \mathfrak{so}(T_{m_0}M)$. □

Lemma 14.12. *Assume that $\mathbb{R}^2 \subseteq G_{m_0}$. Then for all $U \in \mathbb{R}^2$, for all $v \in T_{m_0}M$ and all $x \in e_0(\mathfrak{sl}_2(\mathbb{R}))$, if $\hat{U}_Q = Q^v$, then $g_{m_0}(v, x) = 0$.*

Proof. Choose $X \in \mathfrak{sl}_2(\mathbb{R})$ such that $x = \hat{X}_0 = e_0(X)$. Then $C^x = \hat{X}_C$. Let $T := [X, U] = \tilde{X}U \in \mathbb{R}^2$.

Lemma 14.6 implies that $T \in \mathfrak{g}^0_{m_0}$. Thus $\hat{T}_L \in \mathfrak{so}(T_{m_0}M)$ by 2 of Lemma 12.3. Moreover, $[\hat{X}_L, \hat{U}_L] \in [\mathfrak{co}(T_{m_0}M), \mathfrak{co}(T_{m_0}M)] = \mathfrak{so}(T_{m_0}M)$.

By 5 of Lemma 12.3, we have $\hat{T}_L = [\hat{X}_L, \hat{U}_L] + [\hat{X}_C, \hat{U}_Q]$. Since $\hat{T}_L \in \mathfrak{so}(T_{m_0}M)$ and $[\hat{X}_L, \hat{U}_L] \in \mathfrak{so}(T_{m_0}M)$, we conclude that $[\hat{X}_C, \hat{U}_Q] \in \mathfrak{so}(T_{m_0}M)$.

By Lemma 13.1, we have

$$[\hat{X}_C, \hat{U}_Q] = [C^x, Q^v] = g_{m_0}(v, x)R + f^v C^x - f^x C^v.$$

By Lemma 14.11, $f^v C^x - f^x C^v \in \mathfrak{so}(T_{m_0}M)$.

We conclude that $g_{m_0}(v,x)R \in \mathfrak{so}(T_{m_0}M)$; but $\lambda R \in \mathfrak{so}(T_{m_0}M)$ if and only if $\lambda = 0$, so $g_{m_0}(v,x) = 0$. □

Lemma 14.13. *Assume that* $\dim M \geq 3$. *If* $\mathbb{R}^2 \subseteq G_{m_0}$, *then there exist a nondegenerate subspace* $S_0 \subseteq T_{m_0}M$, *a subspace* $W \subseteq \mathfrak{sl}_2(\mathbb{R})$ *and linear maps* $\phi : W \to \mathfrak{co}_*(S_0)$ *and* $\psi : \mathbb{R}^2 \to \mathfrak{co}_*(S_0)$ *such that* $\psi \neq 0$, *no proper Lie subalgebra of* $\mathfrak{sl}_2(\mathbb{R})$ *contains* W *and, for all* $X \in W$ *and all* $U \in \mathbb{R}^2$,

$$[\phi(X), \psi(U)] = \psi(\tilde{X}U).$$

Proof. Choose $S \subseteq e_0(\mathfrak{sl}_2(\mathbb{R}))$ as in Lemma 14.7. For $U \in \mathbb{R}^2$, it follows from Lemma 14.10 that $\hat{U}_L \,|\, e_0(\mathfrak{sl}_2(\mathbb{R})) = 0$, so $\hat{U}_L|S = 0$. Then $\mathrm{UL}_S(\hat{U}_L) = \mathrm{LL}_S(\hat{U}_L) = 0$. Since $U \in \mathfrak{g}_{m_0}$, it follows from 1 and 3 of Lemma 12.3 that $\hat{U}_C = 0$ and $\hat{U}_L \in \mathfrak{co}(T_{m_0}M)$. Since $\mathrm{LL}_S(\hat{U}_L) = 0$, it follows from 3 of Lemma 14.3 that

$$\mathrm{UR}_S(\hat{U}_L) = -\mathrm{LL}_S(\hat{U}_L)^* = 0.$$

By 4 of Lemma 12.3, $\hat{U}_Q \in \mathfrak{co}_Q(T_{m_0}M)$ for all $U \in \mathbb{R}^2$. By 3 of Lemma 12.3, $\hat{X}_L \in \mathfrak{co}(T_{m_0}M)$ for all $X \in \mathfrak{sl}_2(\mathbb{R})$.

We now consider the **special case** $\mathrm{LR}_S(\hat{U}_L) = 0$ for all $U \in \mathbb{R}^2$. In this case, let $S_0 := T_{m_0}M$ and $W := \mathfrak{sl}_2(\mathbb{R})$. We define $\phi : W \to \mathfrak{co}_*(S_0)$ and $\psi : \mathbb{R}^2 \to \mathfrak{co}_*(S_0)$ by $\phi(X) = \hat{X}_L$ and $\psi(U) = \hat{U}_Q$.

For $U \in \mathbb{R}^2$, we know that UL_S, UR_S, LL_S and LR_S all vanish on $\hat{U}_L$, so $\hat{U}_L = 0$.

By 8 of Lemma 12.3, ψ is injective. By the third result in 7 of Lemma 12.3,

$$[\phi(X), \psi(U)] = [\hat{X}_L, \hat{U}_Q] = \psi([X,U]),$$

for all $X \in \mathfrak{sl}_2(\mathbb{R})$ and for all $U \in \mathbb{R}^2$, as desired, concluding the **special case**.

We therefore assume that $\mathrm{LR}_S(\hat{U}_L^0) \neq 0$ for some $U^0 \in \mathbb{R}^2$. Let $S_0 := S^\perp$ and $W := e_0^{-1}(S)$. By Lemma 14.7, no proper Lie subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ contains W. Since $\mathfrak{co}(S_0) \subseteq \mathfrak{co}_*(S_0)$, it suffices to find maps $\phi : W \to \mathfrak{co}(S_0)$ and $\psi : \mathbb{R}^2 \to \mathfrak{co}(S_0)$ such that $\psi \neq 0$ and $[\phi(X), \psi(U)] = \psi(\tilde{X}U)$ for all $X \in W$, for all $U \in \mathbb{R}^2$.

For all $X \in \mathfrak{sl}_2(\mathbb{R})$, we have $\hat{X}_L \in \mathfrak{co}(T_{m_0}M)$; applying Lemma 14.3, we conclude that $\mathrm{LR}_S(\hat{X}_L) \in \mathfrak{co}(S_0)$. Similarly, $\hat{U}_L \in \mathfrak{co}(T_{m_0}M)$ for all $U \in \mathbb{R}^2$; applying Lemma 14.3 once more, we conclude that $\mathrm{LR}_S(\hat{U}_L) \in \mathfrak{co}(S_0)$.

Define $\phi : W \to \mathfrak{co}(S_0)$ and $\psi : \mathbb{R}^2 \to \mathfrak{co}(S_0)$ by $\phi(X) = \mathrm{LR}_S(\hat{X}_L)$ and $\psi(U) = \mathrm{LR}_S(\hat{U}_L)$. Then $\psi(U^0) \neq 0$, so $\psi \neq 0$.

Fix $X \in W$ and $U \in \mathbb{R}^2$. Let $T := [X,U] = \tilde{X}U$. We wish to show that $[\phi(X), \psi(U)] = \psi(T)$. That is, we wish to show that $[\mathrm{LR}_S(\hat{X}_L), \mathrm{LR}_S(\hat{U}_L)] = \mathrm{LR}_S(\hat{T}_L)$.

By 5 of Lemma 12.3, $\hat{T}_L = [\hat{X}_L, \hat{U}_L] + [\hat{X}_C, \hat{U}_Q]$.

We have $T = \tilde{X}U \in \mathbb{R}^2 \subseteq \mathfrak{g}_{m_0}$. By Lemma 14.6, we get $T \in \mathfrak{g}_{m_0}^0$, and it follows, from 2 of Lemma 12.3, that $\hat{T}_L \in \mathfrak{so}(T_{m_0}M)$. Moreover, $[\hat{X}_L, \hat{U}_L] \in [\mathfrak{co}(T_{m_0}M), \mathfrak{co}(T_{m_0}M)] = \mathfrak{so}(T_{m_0}M)$. Therefore,

$$[\hat{X}_C, \hat{U}_Q] = \hat{T}_L - [\hat{X}_L, \hat{U}_L] \in \mathfrak{so}(T_{m_0}M).$$

Since $\hat{U}_Q \in \mathfrak{co}_Q(T_{m_0}M)$, by Corollary 13.5, we may choose $v \in T_{m_0}M$ such that $Q^v = \hat{U}_Q$. Let $w := \hat{X}_0$, so that $\hat{X}_C = C^w$.

Since $X \in W = e_0^{-1}(S)$, we see that $w \in S$. Since $S \subseteq e_0(\mathfrak{sl}_2(\mathbb{R}))$, it follows from Lemma 14.12 that $g(v,x) = 0$, for all $x \in S$. Thus $v \in S^\perp = S_0$. Since $w \in S$, we conclude that $g(v,w) = 0$.

By Lemma 13.1, we have

$$[\hat{X}_C, \hat{U}_Q] = [C^w, Q^v] = g(v,w)R + f^v C^w - f^w C^v = f^v C^w - f^w C^v.$$

Thus $\hat{T}_L = [\hat{X}_L, \hat{U}_L] + [\hat{X}_C, \hat{U}_Q] = [\hat{X}_L, \hat{U}_L] + f^v C^w - f^w C^v$.

Applying LR_S to both sides of this, and using Lemma 14.5, we get $\mathrm{LR}_S(\hat{T}_L) = \mathrm{LR}_S([\hat{X}_L, \hat{U}_L])$. Finally, replacing A by $\hat{X}_L$ and B by $\hat{U}_L$ in Lemma 14.4, we get $\mathrm{LR}_S([\hat{X}_L, \hat{U}_L]) = [\mathrm{LR}_S(\hat{X}_L), \mathrm{LR}_S(\hat{U}_L)]$.

Thus $\mathrm{LR}_S(\hat{T}_L) = [\mathrm{LR}_S(\hat{X}_L), \mathrm{LR}_S(\hat{U}_L)]$, as desired. □

Note that in the proof of the following lemma, ϕ is only linear, and cannot be used in Lemma 11.2.

Lemma 14.14. *If* $\dim M \geq 3$*, then* $\mathbb{R}^2$ *cannot be contained in* G_{m_0}*.*

Proof. Assume for a contradiction that $\mathbb{R}^2 \subseteq G_{m_0}$. Let ρ be the standard representation of $\mathfrak{b} := \mathfrak{sl}_2(\mathbb{R})$ on $V := \mathbb{R}^2$. Then $\tilde{\mathfrak{b}} := \rho(\mathfrak{b})$ is isomorphic to $\mathfrak{b}$. Let ϕ, ψ, S_0 and W be as in Lemma 14.13.

Let $d_0 := \dim S_0$. By Lemma 14.1, $\mathfrak{h} := \mathfrak{co}_*(S_0)$ is isomorphic either to $\mathfrak{so}(d_0) \ltimes \mathbb{R}^{d_0}$ or to $\mathfrak{so}(1, d_0 - 1) \ltimes \mathbb{R}^{d_0}$. By 1 of Lemma 11.1, we see that ψ is injective. By 2 of Lemma 11.1, we see that $\tilde{\mathfrak{b}}$ is a subquotient of $\mathfrak{g}$. Since $\tilde{\mathfrak{b}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, we conclude that $\mathfrak{g}$ cannot be isomorphic to $\mathfrak{so}(d_0) \ltimes \mathbb{R}^{d_0}$. Thus $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(1, d_0 - 1) \ltimes \mathbb{R}^{d_0}$.

Choose ϕ' as in 3 of Lemma 11.1. Replacing ϕ by ϕ' and d by d_0 in the statement of Lemma 11.2, we arrive at a contradiction. □

15. Conclusion

Theorem 15.1. *Let* $n \geq 3$ *be an integer. Assume that* $G := SL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ *acts locally faithfully by conformal transformations of a Lorentz manifold* M*. The action of* G *on* M *is proper.*

Proof. Suppose that the G-action on M is nonproper. We wish to obtain a contradiction.

By Lemma 10.1, there exists $m_0 \in M$ such that $\mathbb{R}^n$ is isotropic at m_0. By Lemma 6.1, the stabilizer G_{m_0} of m_0 contains a codimension-one subgroup of $\mathbb{R}^n$; since $n \geq 3$, we see that G_{m_0} contains a two-dimensional subspace S of $\mathbb{R}^n$.

By Theorem 8.2, the action of $\mathrm{SL}_n(\mathbb{R})$ is proper, and therefore has compact stabilizers. Since $n \geq 3$, every compact subgroup of $\mathrm{SL}_n(\mathbb{R})$ has codimension ≥ 3, so $\dim(M) \geq 3$.

Let S' be any vector space complement to S in $\mathbb{R}^n$. Then $\mathrm{SL}(S)$ injects into $\mathrm{SL}(S) \times \mathrm{SL}(S')$, which, in turn, injects into $\mathrm{SL}_n(\mathbb{R})$. Thus $\mathrm{SL}(S) \ltimes S$ injects into G.

Since S is two-dimensional, $\mathrm{SL}(S) \ltimes S$ is isomorphic to $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$. We therefore obtain an injective homomorphism $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 \to G$ such that $\mathbb{R}^2$ maps onto S. Thus $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ acts on M, the restriction to $\mathrm{SL}_2(\mathbb{R})$ has compact stabilizers and there exists $m_0 \in M$ such that $\mathbb{R}^2 \subseteq G_{m_0}$. This contradicts Lemma 14.14. □

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