INFINITELY RENORMALIZABLE QUADRATIC POLYNOMIALS

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Abstract. We prove that the Julia set of a quadratic polynomial which admits an infinite sequence of unbranched, simple renormalizations with complex bounds is locally connected. The method in this study is three-dimensional puzzles.

1. Introduction

Let \( P(z) = z^2 + c \) be a quadratic polynomial where \( z \) is a complex variable and \( c \) is a complex parameter. The filled-in Julia set \( K \) of \( P \) is, by definition, the set of points \( z \) which remain bounded under iterations of \( P \). The Julia set \( J \) of \( P \) is the boundary of \( K \). A central problem in the study of the dynamical system generated by \( P \) is to understand the topology of a Julia set \( J \), in particular, the local connectivity for a connected Julia set. A connected set \( J \) in the complex plane is said to be locally connected if for every point \( p \) in \( J \) and every neighborhood \( U \) of \( p \) there is a neighborhood \( V \) of \( p \) such that \( V \cap J \) is connected.

We first give the definition of renormalizability. A quadratic-like map \( F: U \rightarrow V \) is a holomorphic, proper, degree two branched cover map, where \( U \) and \( V \) are two domains isomorphic to a disc and \( \overline{U} \subset V \). Then \( K_F = \bigcap_{n=0}^{\infty} F^{-n}(U) \) and \( J_F = \partial K_F \) are the filled-in Julia set and the Julia set of \( F \), respectively. We only consider those quadratic-like maps whose Julia sets are connected. Let us assume the only branch point of \( F \) is 0. A quadratic-like map \( F: U \rightarrow V \) is said to be (once) renormalizable if there are an integer \( n > 1 \) and an open subdomain \( U' \) containing 0 such that \( U' \subset U \) and such that \( F_1 = F^{\text{on}}: U' \rightarrow V' \subset V \) is a quadratic-like map with connected Julia set \( J_{F_1} = J(n',U',V') \). The choice of \( (U',V') \) is called an \( n' \)-renormalization of \( (U,V) \). An annulus \( A \) is a double connected domain. The definition of the modulus \( \text{mod}(A) \) of an annulus \( A \) is defined in many books in complex analysis (see, for example, [AL]). It is \( \log r \) if \( A \) is holomorphically diffeomorphic to the annulus \( A_r = D_r \setminus \overline{D_1} \), where \( D_r \) is the open disk centered at 0 with radius \( r > 1 \). The sets \( U \setminus \overline{U'} \) and \( V \setminus \overline{V} \) are annuli. In \( \S 3 \), we prove a modulus inequality in renormalization as follows.

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Theorem 1. Suppose $F : U \to V$ is a renormalizable quadratic-like map. Consider any $n'$-renormalization $(U', V')$, $n' > 1$. Then
\[
\text{mod}(U \setminus U') \geq \frac{1}{2} \text{mod}(V \setminus U).
\]

A quadratic-like map $F : U \to V$ is said to be twice renormalizable if $F$ is once renormalizable, and there is an $m_1$-renormalization $(U', V')$ of $(U, V)$ such that $F_1 = F^{m_1} : U' \to V'$ is once renormalizable. Consequently, we have renormalizations $F_1 = F^{m_1} : U_1 \to V_1$ and $F_2 = F^{m_2} : U_2 \to V_2$ and their Julia sets $(J_{F_1}, J_{F_2})$. Similarly, we can define a $k$-times renormalizable quadratic-like map $F : U \to V$ and renormalizations $\{F_i = F^{m_i} : U_i \to V_i\}_{i=1}^k$, where $m_1 < m_2 < \cdots < m_k$. A quadratic-like map $F : U \to V$ is infinitely renormalizable if it is $k$-times renormalizable for every $k > 0$.

For an $n'$-renormalization $(U', V')$ of a renormalizable quadratic-like map $F : U \to V$, suppose that $F_1 = F^{m_1'} : U' \to V'$ has two repelling fixed points $\alpha$ and $\beta$ in the filled-in Julia set $K_{F_1}$. One fixed point $\beta$ does not disconnect $K_{F_1}$, i.e., $K_{F_1} \setminus \{\beta\}$ is still connected, and the other fixed point $\alpha$ disconnects $K_{F_1}$, i.e., $K_{F_1} \setminus \{\alpha\}$ is disconnected. McMullen [MC1] discovered that different types of renormalizations can occur in the renormalization theory of quadratic-like maps.

Let $K(i) = F^i(K_{F_i})$, $1 \leq i < n'$. An $n'$-renormalization $(U', V')$ is
- $\alpha$-type if $K(i) \cap K(j) = \{\alpha\}$ for some $i \neq j$, $0 \leq i, j < n'$;
- $\beta$-type if $K(i) \cap K(j) = \{\beta\}$ for some $i \neq j$, $0 \leq i, j < n'$;
- disjoint type if $K(i) \cap K(j) = \emptyset$ for all $i \neq j$, $0 \leq i, j < n'$.

The $\beta$-type and the disjoint type are also called simple. If $F : U \to V$ is infinitely renormalizable, i.e., it has an infinite sequence of renormalizations, then it will have an infinite sequence of simple renormalizations. We will show a construction of a most natural infinite sequence
\[
\{F_i = F^{m_i} : U_i \to V_i\}_{i=1}^\infty,
\]
where $m_1 < m_2 < \cdots < m_k < \cdots$, of simple renormalizations by using two-dimensional puzzles in §2. Henceforth, we will always use this sequence as a sequence of renormalizations for an infinitely renormalizable quadratic-like map.

Now let $F : U \to V$ be an infinitely renormalizable quadratic-like map and let $\{F_i = F^{m_i} : U_i \to V_i\}_{i=1}^\infty$ be the most natural infinite sequence of simple renormalizations in the previous paragraph. Suppose $\{J_{F_i}\}_{i=1}^\infty$ is the corresponding infinite sequence of Julia sets. (For an infinitely renormalizable quadratic-like map, its filled-in Julia set equals its Julia set.) We prove in §2 that $J_{F_i}$ is independent of the choice of $m_i$-renormalizations $(U_i, V_i)$ (Theorem 2). Thus $J_{F_i}$ can be denoted as $J_{m_i}$, and called a renormalization of $J_F$. The map $F : U \to V$ is said to have complex bounds if there are an infinite subsequence of simple renormalizations
\[
\{F_{i_s} = F^{m_{i_s}} : U_{i_s} \to V_{i_s}\}_{s=1}^\infty
\]
and a constant $\lambda > 0$ such that the modulus of the annulus $V_{i_s} \setminus U_{i_s}$ is greater than $\lambda$ for every $s > 0$ (see Definition 1). It is said to be unbranched if there are an infinite subsequence of renormalizations $\{J_{m_{i_l}}\}_{l=1}^\infty$ of the Julia set $J_F$, neighborhoods $W_l$ of $J_{m_{i_l}}$, and a constant $\mu > 0$, such that the modulus of the annulus $W_l \setminus J_{m_{i_l}}$ is greater than $\mu$ and $W_l \setminus J_{m_{i_l}}$ is disjoint with the critical orbit $\{F^{m_{i_l}}(0)\}_{n=0}^\infty$ for every $l > 0$ (see Definition 2).

For a quadratic polynomial $P(z) = z^2 + c$, let $U$ be a fixed domain bounded by an equipotential curve of $P$ (see §2) and let $V = P(U)$. Then $P : U \to V$ is a
quadratic-like map whose Julia set is always $J$. We say $P$ is infinitely renormalizable if $P : U \to V$ is infinitely renormalizable. The main result in this paper is

**Theorem 3.** The Julia set of an unbranched infinitely renormalizable quadratic polynomial having complex bounds is locally connected.

Two quadratic-like maps $F : U \to V$ and $G : W \to X$ are hybrid equivalent if there is a quasiconformal homeomorphism $H : V \to X$ such that $H \circ F = G \circ H$ and $H|K_F$, the restriction of $H$ on the filled-in Julia set $K_F$ of $F$, is conformal. The reader may refer to Ahlfors’ book [AL] about definitions of quasiconformal and conformal maps. Any quadratic-like map with connected Julia set is hybrid equivalent to a unique quadratic polynomial, as shown in [DH]. Therefore, Theorem 3 applies to quadratic-like maps too. An example of an unbranched infinitely renormalizable quadratic polynomial having complex bounds is a real quadratic polynomial of bounded type (for example, the Feigenbaum polynomial) (see [SU], [MV], [MC2], [JI]). Before going into the next section, we mention some background information about renormalization and local connectivity in the study of the dynamics of quadratic polynomials. The renormalization technique was introduced into the study of dynamical systems by physicists Feigenbaum [FE1], [FE2] and Coullet and Tresser [CT] when the period doubling bifurcations following a universal law in any family of one-dimensional maps like the family of quadratic polynomials was observed about two decades ago. The technique is extensively used in the recent study of one-dimensional and complex dynamical systems (see, for example, [SU], [MV], [MC1], [MC2], [JI]). During this same period, computer-generated pictures of Julia sets of quadratic polynomials and the Mandelbrot set, which is the set of parameters $c$ such that the Julia sets $J_c$ of quadratic polynomials $P_c(z) = z^2 + c$ are connected, showed a fascinating world of fractal geometry (see [MA]). Douady and Hubbard (see [CG]) proved that the Mandelbrot set is connected. The next important problem in this direction is to show that the Mandelbrot set is locally connected. This has been a long-standing conjecture. The study of the local connectivity of the connected Julia set of a quadratic polynomial will give some important information on this conjecture. Moreover, if the Julia set of a quadratic polynomial is locally connected, then the *combinatorics* of this polynomial, that is the landing pattern of external rays (see §2 for the definition), determines completely the *topology* of the Julia set. Recently Yoccoz constructed a puzzle for the connected Julia set of a quadratic polynomial. Using these puzzles, he showed that the connected Julia set of a quadratic polynomial having no indifferent periodic points (see §2 for the definition) is locally connected if it is not infinitely renormalizable (see [HU]). Further, he translated these puzzles into a puzzle on the parameter space and showed that the Mandelbrot set is locally connected at all non-ininitely-renormalizable points (see [HU]). The remaining points to be verified are all infinitely renormalizable ones in the Mandelbrot set. We study these infinitely renormalizable quadratic polynomials. We construct a three-dimensional puzzle for the Julia set of an infinitely renormalizable quadratic polynomial. By using these three-dimensional puzzles we prove in this paper that the Julia set of a quadratic polynomial which admits an infinite sequence of unbranched, simple renormalizations with complex bounds is locally connected. The local connectivity of the Julia set of an infinitely renormalizable quadratic polynomial is not always guaranteed. Actually, Douady and Hubbard have constructed an infinitely renormalizable quadratic polynomial whose Julia set is not locally connected by a method called tuning (see [MI2]).
Figure 1. A computer picture of the Julia set of the Feigenbaum polynomial and three enlargements around 0.
this method and the Yoccoz inequality (see [HU]), Douady constructed a generic set of infinitely renormalizable points on the boundary of the Mandelbrot set such that the Mandelbrot set is locally connected at this set of points and such that the corresponding Julia sets are not locally connected (see [PM]). Douady's construction can be summarized as follows: In any copy of the Mandelbrot set, the rational limb $p/q$ with $q >> p$ are small due to the Yoccoz inequality. Considering a nested sequence of copies of the Mandelbrot set which belong to an infinite number of limbs as described above, one sees that the intersection is a singular point at which the Mandelbrot set is locally connected. By further choosing orders of tunings higher and higher in the construction, one can make such a set such that the corresponding Julia sets are not locally connected. On the other hand, by translating the three-dimensional puzzles in this paper into a three-dimensional puzzle on the parameter space, we proved in [JIM] that there is a subset of infinitely renormalizable points in the Mandelbrot set such that the subset is dense on the boundary of the Mandelbrot set, and the Mandelbrot set is locally connected at this set of points, and the corresponding Julia sets are also locally connected. The reader may also refer to McMullen's recent book [MC2] for the latest developments in this direction and for an excellent dictionary between the study of complex dynamical systems and the study of Kleinian groups and hyperbolic geometry. The series of computer pictures in Figure 1 shows us how complicated the Julia set of an infinitely renormalizable quadratic-like map can be.

This paper is organized as follows: We study some properties of renormalizable quadratic polynomials in §2, and prove Theorem 2. In §3, we prove Theorem 1 and Theorem 3.

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2. Renormalization on quadratic polynomials

2.1. Equipotential curves and external rays. Let $P(z) = z^2 + c$ be a quadratic polynomial and let $J$ be its Julia set. A point $p$ in $C$ is called a periodic point of $P$ of period $n$, where $n \geq 1$ is an integer, if $P^n(p) \neq p$ for $1 \leq i < n$ and $P^n(p) = p$. The number $E_p = (P^n)'(p)$ is called the multiplier of $P$ at a periodic point $p$ of period $n$. Then $p$ is called attractive if $|E_p| < 1$; repelling if $|E_p| > 1$; and indifferent if $|E_p| = 1$. The point 0 is the critical point of $P$ in the complex plane $C$. Let $c_i = P^{n_i}(0)$, $i \geq 1$, be the $i^{th}$ critical value of $P$. Then $PCO = \{c_i\}_{i=1}^{\infty}$ is the post-critical orbit and $CO = PCO \cup \{0\}$ is the critical orbit. The critical point 0 is said to be recurrent if for any neighborhood $W$ of 0 there is a critical value $c_i$, $i \geq 1$, in $W$. Henceforth we will only consider those quadratic polynomials whose critical points are recurrent. We will also assume that $P$ has no attractive and indifferent periodic points. These are assumed for a quadratic-like map too. Therefore all periodic points of $P$ will be repelling, and the Julia set $J$ is connected and equal to its filled-in Julia set $K$.

Let $D_1$ be the open unit disk in the complex plane $C$, and let $P_0(z) = z^2$. There is a Riemann mapping $h$ from $\overline{C} \setminus D_1$ onto $\overline{C} \setminus J$ such that $h(z)/z \to 1$ as $z$ tends to infinity and

$$h \circ P_0 = P \circ h$$
on \( C \setminus \overline{D_1} \) (see [MC1], [MI1]). The image \( S_t \) of a circle \( s_t = \{te^{2\pi i \theta} \mid 0 \leq \theta < 1 \} \) for \( 1 < t < \infty \) under \( h \) is called an equipotential curve of \( P \). If we consider the Green’s function \( G(z) = \log |h^{-1}(z)| = \lim_{n \to \infty} (\log^+ |P^n(z)|)^{1/n} \) defined on \( C \setminus J \), then \( G(P(z)) = 2G(z) \), and it takes the constant value \( \log t \) when it is restricted on \( S_t \), where \( \log^+ x = \max\{\log x, 0\} \). This implies that
\[
P(S_t) = S_t^2.
\]
Let \( U_t \) be the open domain bounded by \( S_t \). Then \( P : U_t \to U_t \) is a quadratic-like map. The image \( R_\theta \) of a ray \( r_\theta = \{te^{2\pi i \theta} \mid 1 < t < \infty \} \) under \( h \) is called an external ray of angle \( \theta \). Thus,
\[
P(R_\theta) = R_{2\theta \pmod{1}}.
\]
An external ray \( R_\theta \) is called periodic if \( P^k(R_\theta) = R_\theta \) for some \( k \), and the smallest such \( k \) is called the period of \( R_\theta \). Douady and Yoccoz proved that every repelling periodic point of \( P \) is a landing point of finitely many periodic external rays of the same period, and furthermore, these landing rays are in the same orbit because \( P \) is a quadratic polynomial (see [HU], [MI2]).

### 2.2. Construction of Yoccoz puzzles

Take a quadratic polynomial \( P(z) = z^2 + c \) as in §2.1. Let us fix an equipotential curve \( S_t \) and the domain \( U = U_t \) bounded by \( S_t \). Then we have a quadratic-like map \( F = P : U \to V = P(U) \) whose Julia set is \( J \). Then \( F \) has two fixed points. One of them, say \( \beta \), is non-separating, and the other, say \( \alpha \), is separating, i.e., \( J \setminus \{\beta\} \) is still connected and \( J \setminus \{\alpha\} \) is not. There are at least two, but a finite number, external rays of \( P \) landing at \( \alpha \). Let \( \Gamma_0^\alpha \) be the union of a cycle of external rays landing at \( \alpha \). Then \( \Gamma_0^\alpha \) cuts \( U_0^\alpha = U \) into finitely many domains. Let \( \eta_0 \) be the collection of the closure of these domains. Let \( \Gamma_n^\alpha = F^{-n}(\Gamma_0^\alpha) \) for any \( n > 0 \). Then \( \Gamma_n^\alpha \) cuts \( U_n^\alpha = F^{-n}(U_0^\alpha) \) into finitely many domains. Let \( \eta_n \) be the collection of the closures of these domains. The sequence \( \xi_0 = \{\eta_n\}_{n=0}^{\infty} \) is called the Yoccoz puzzle for \( J \) (refer to [BI], [HU], [MI2], [JI]). The domain \( C_n \) in \( \eta_n \) containing \( 0 \) is called the critical piece in \( \eta_n \). It is clear that \( P \) restricted to all domains but \( C_n \) is bijective to domains in \( \eta_{n-1} \), and \( P|C_n \) is a degree two branched cover map onto a domain in \( \eta_{n-1} \). Let
\[
J_1 = \bigcap_{n=0}^{\infty} C_n.
\]

The following result follows directly from the result of Yoccoz about local connectivity of nonrenormalizable quadratic polynomials (refer to [HU], [MI2], [JI]), and gives an equivalent definition of renormalizability:

**Theorem A** (Yoccoz). Suppose \( P(z) = z^2 + c \) has recurrent critical orbit. Then \( P \) is renormalizable if and only if \( J_1 \) consists of more than one point.

We will use \( N(X, \epsilon) = \{x \in \mathbb{C} \mid d(x, X) < \epsilon\} \) to denote the \( \epsilon \)-neighborhood of \( X \) in the complex plane in this paper. Suppose \( P \) is renormalizable. We have two integers \( n_1 \geq 0, m_1 > 1 \) such that \( F_1 = F^{m_1} : C_{m_1+n_1} \to C_{n_1} \) is a degree two branched cover map and such that \( C_{m_1+n_1} \subseteq N(J_1, 1) \) (refer to [HU], [MI2], [JI]). We can further take domains \( C_{n_1+m_1} \subseteq U_1 \subseteq U \) and \( C_{n_1} \subseteq V_1 \subseteq V \) such that
\[
F_1 = F^{m_1} : U_1 \to V_1
\]
is a quadratic-like map. Then its Julia set is \( J_1 \). (Note that \( F_1 = F^{m_1} : U_1 \to V_1 \) is a simple renormalization of \( F : U \to V \).)
Theorem 2. Suppose $G = P_m : U' \to V'$ is any $m_1$-renormalization of $P$. Then its filled-in Julia set is always $J_1$.

Proof. The point 0 is in the intersection $U' \cap U_1$. Suppose $U''$ is the connected component of $U' \cap U_1$ containing 0. Then $H = P_{m_1} : U'' \to V'' \cap V_1$ is also a renormalization of $P$, and its Julia set $J_H$ is connected. It is easy to check that $J_H \subset J_G \cap J_{F_1}$. Let $\beta_{F_1}$ and $\alpha_{F_1}$ be the non-separating and separating fixed points of $F_1$, and let $\beta_G$ and $\alpha_G$ be the non-separating and separating fixed points of $G$. All $\beta_{F_1}, \alpha_{F_1}, \beta_G,$ and $\alpha_G$ are in $J_H$, since each of $G, F$ and $H$ has exactly two fixed points in its domain, and every fixed point of $H$ has to be a common fixed point of $G$ and $F$. Since $F_1 = P_{m_1} : U_1 \to V_1$ and $G = P_{m_1} : U' \to V'$ are both degree two branched cover maps and $\beta_{F_1}$ is in both $U'$ and $U_1$, all preimages of $\beta_{F_1}$ under iterates of $G$ and $F_1$ are in both $U''$ and $U_1$. But each of $J_H, J_1,$ and $J_G$ is the closure of the set of all preimages of $\beta_{F_1}$ under iterates of $H, G,$ and $F_1$. Therefore, $J_H = J_G = J_1$.

Remark 1. From Theorem 2, for any $m_1$-renormalization $(U', V')$ of $F : U \to V$, there is $C_{m_1+n} \subset U'$ such that $F_1 = F^{m_1} : C_{m_1+n} \to C_n \subset V'$ is a degree two branched cover map.

Remark 2. From the Douady and Hubbard theorem [DH] that every quadratic-like map $F : U \to V$ whose Julia set is connected is hybrid equivalent to a unique quadratic polynomial, all arguments in this section can be applied to quadratic-like maps (by considering induced external rays and equipotential curves from the hybrid equivalent quadratic polynomial).

2.3. Construction of three-dimensional puzzles. Now we assume $P(z) = z^2 + c$ is an infinitely renormalizable quadratic polynomial. Let $k_1 = m_1, \alpha_1 = \alpha, \beta_1 = \beta, \eta_0^n = \eta_n, C_0^n = C_n,$ and let $n_1, J_1, \Gamma_{n_1}^0, U_{n_1}^0, \xi_0^1, F : U \to V$ be as in the previous subsection. Suppose $\beta_2$ and $\alpha_2$ are the non-separating and separating fixed points of $F_1$, i.e., $J_1 \setminus \{\beta_2\}$ is still connected and $J_1 \setminus \{\alpha_2\}$ is not. The points $\beta_2$ and $\alpha_2$ are also repelling periodic points of $P$. There are at least two, but finitely many, external rays of $P$ landing at $\alpha_2$. Let $\Gamma_0^1$ be the union of a cycle of external rays landing at $\alpha_2$. Then $\Gamma_0^1$ cuts $U_0^1 = C_{n_1+m_1}$ into finitely many domains. Let $\eta_0^1$ be the collection of the closures of these domains. Let $\Gamma_1^1 = F^{-n}(\Gamma_0^1)$ for any $n > 0$. Then $\Gamma_1^1$ cuts $U_n^1 = F^{-n}(U_0^1)$ into finitely many domains. Let $\eta_1^n$ be the collection of the closures of these domains. The sequence $\xi_1^1 = \{\eta_1^n\}_{n=0}^\infty$ is the two-dimensional puzzle for $J_1$. We call it the first puzzle. (We also call $\xi_0^1$ the 0th puzzle.)

The domain $C_1^n$ in $\eta_1^n$ containing 0 is called the critical piece in $\eta_1^n$. It is clear that $F_1$ restricted to all domains but $C_1^n$ is bijective to domains in $\eta_1^{n-1}$, and $P|C_1^n$ is a degree two branched cover map onto a domain in $\eta_1^{n-1}$. Let

$$J_2 = \bigcap_{n=0}^{\infty} C_1^n.$$ 

There are two integers $n_2 \geq 0, k_2 > 1$ such that

$$F_2 = F_1^{n_2+k_2} : C_1^{n_2+k_2} \to C_1^{n_2}$$

is a degree two branched cover map and such that $C_1^{n_2+k_2} \subset N(J_2, 1/2)$. We take
domains \( C_{n_2+k_2} \subseteq U_2 \subset U_1 \) and \( C_{n_2} \subseteq V_2 \subset V_1 \) such that
\[
F_2 = F^{0k_2} : U_2 \to V_2
\]
is a quadratic-like map. Then its Julia set is \( J_2 \).
Inductively, for every \( i \geq 2 \), suppose we have constructed
\[
F_i = F^{0k_i} : C_{n_i+k_i} \to C_{n_i} \quad \text{and} \quad F_i = F^{0k_i} : U_i \to V_i,
\]
whose Julia set is \( J_i \). Let \( \beta_{i+1} \) and \( \alpha_{i+1} \) be the non-separating and separating fixed points of \( F_i \); i.e., \( J_i \setminus \{ \beta_{i+1} \} \) is still connected and \( J_i \setminus \{ \alpha_{i+1} \} \) is not. The points \( \beta_{i+1} \) and \( \alpha_{i+1} \) are also repelling periodic points of \( F_i \). There are at least two, but a finite number of, external rays of \( F_i \) landing at \( \alpha_{i+1} \). Let \( \Gamma_0^i \) be the union of a cycle of external rays landing at \( \alpha_{i+1} \). Then \( \Gamma_0^i \) cuts \( U_0^i = C_{n_i+k_i} \) into finitely many domains. Let \( \eta_n^i \) be the collection of the closures of these domains. Let \( \eta_n^i = F_i^{-n}(\Gamma_0^i) \) for any \( n > 0 \). Then \( \eta_n^i \) cuts \( U_n^i = F_i^{-n}(U_0^i) \) into finitely many domains. Let \( \eta_n^i \) be the collection of the closures of these domains. The domain \( C_n^i \) in \( \eta_n^i \), containing 0 is called the critical piece in \( \eta_n^i \). It is clear that \( F_i \) restricted to all domains but \( C_n^i \) is bijective to domains in \( \eta_{n-1}^i \), and \( P|C_n^i \) is a degree two branched cover map onto a domain in \( \eta_{n-1}^i \). Let
\[
J_{i+1} = \bigcap_{n=0}^{\infty} C_{n_i}.
\]
There are two integers \( n_{i+1} \geq 0 \), \( k_{i+1} > 1 \) such that
\[
F_{i+1} = F^{0k_{i+1}} : C_{n_{i+1}+k_{i+1}} \to C_{n_{i+1}}
\]
is a degree two branched cover map and such that \( C_{n_{i+1}+k_{i+1}} \subset N(J_{i+1}, 1/(i+1)) \).
We take domains \( C_{n_{i+1}+k_{i+1}} \subseteq U_{i+1} \subset U_i \) and \( C_{n_{i+1}} \subseteq V_{i+1} \subset V_i \) such that
\[
F_{i+1} = F^{0k_{i+1}} : U_{i+1} \to V_{i+1}
\]
is a quadratic-like map. Then its Julia set is \( J_{i+1} \). Let \( \xi_i^j = \{ \eta_n^j \}_{n=0}^{\infty} \). It is the two-dimensionnal puzzle for \( J_i \). We call it the \( i \)th partition.

Remark 3. For any \( k_{i+1}-\text{renormalization} \) \( (U', V') \) of \( F_i : U_i \to V_i \), we have an integer \( n > 0 \) such that \( C_{n_i+k_{i+1}} \cap N(J_{i+1}, 1/(i+1)) \) and \( C_{n_i} \subseteq V' \), and such that \( F_{i+1} = F^{0k_{i+1}} : C_{n_{i+1}+k_{i+1}} \to C_n \) is a degree two branched cover map. We will still use \( \xi_i^j \) to mean \( \xi_i^j \cap C_{n_{i+1}+k_{i+1}} \). Therefore, \( (U_{i+1}, V_{i+1}) \) can be an arbitrary \( k_{i+1}-\text{renormalization} \) of \( F_i : U_i \to V_i \).

Let \( m_i = \prod_{j=1}^{i} k_j \), \( 1 \leq i < \infty \). We have thus constructed a most natural infinite sequence of simple renormalizations of \( F : U \to V \),
\[
\{ F_i = F^{0m_i} : U_i \to V_i \}_{i=1}^{\infty},
\]
and the nested-nested sequence \( \{ \xi_i^j \}_{i=0}^{\infty} \) of partitions for \( \{ J_i \}_{i=0}^{\infty} \) (where \( J_0 = J \)),
which we call a three-dimensional puzzle. Henceforth, we will fix all notations in this subsection.

Definition 1. We say an infinitely renormalizable quadratic polynomial \( P(z) = z^2 + c \) has complex bounds if there are a constant \( \lambda > 0 \) and an infinite sequence of simple renormalizations \( \{ F_{i,s} = F^{0m_{is}} : U_{i,s} \to V_{i,s} \}_{s=1}^{\infty} \) such that the modulus \( \text{mod}(V_{i,s} \setminus U_{i,s}) \) is greater than \( \lambda \) for every \( s \geq 1 \).
Definition 2. We say an infinitely renormalizable quadratic polynomial $P(z) = z^2 + c$ is unbranched if there are an infinite subsequence of renormalizations $\{J_i\}_{i=1}^\infty$ of the Julia set $J$ of $P$, neighborhoods $W_l$ of $J_i$ for every $l > 0$, and a constant $\mu > 0$ such that the modulus $\text{mod}(W_l \setminus J_i)$ is greater than $\mu$ and $W_l \setminus J_i$ contains no point in the critical orbit $CO$ of $P$.

Remark 4. For the same reason as that in Remark 2, all arguments in this section can be applied to an infinitely renormalizable quadratic-like map.

3. Three-dimensional puzzles and local connectivity

Before we use the three-dimensional puzzle for the Julia set $J$ of an infinitely renormalizable quadratic polynomial $P(z) = z^2 + c$ to study the local connectivity of $J$, we first prove the following result, which we call a modulus inequality in renormalization (see Figure 2). Remember that an $n'$-renormalization $(U', V')$ of a quadratic-like map $F : U \to V$ means a pair of domains $U' \subset U$ and $V' \subset V$ such that $F_1 = F^{\circ n'} : U' \to V'$ is a quadratic-like map with connected filled-in Julia set.

Theorem 1. Suppose $F : U \to V$ is a renormalizable quadratic-like map. Consider any $n'$-renormalization $(U', V')$, $n' > 1$ (which may or may not be simple). Then

$$\text{mod}(U \setminus \overline{U'}) \geq \frac{1}{2} \text{mod}(V \setminus \overline{V'}).$$

Proof. Since $F_1 = F^{\circ n'} : U' \to V'$ is quadratic-like, its only critical point is 0. So the first critical value $c_1 = F(0)$ of $F$ is not in $V'$. (Otherwise, there will be a point $x \neq 0$ in $U'$ such that $F^{\circ (n'-1)}(x) = 0$, since $c_1$ has only one preimage 0 under $F$ and 0 is not a periodic point of $F$ as we assumed in §2.1. Then $x$ will be a critical point of $F_1$.) Since $V'$ is simply connected, $F$ has two analytic inverse branches,

$$g_0 : V' \to g_0(V') \subset U \quad \text{and} \quad g_1 : V' \to g_1(V') \subset U.$$

One of them is $F^{\circ (n'-1)}(U')$. Therefore $F(U')$ is a domain inside $U$ and containing $c_1$. Consider the annuli $U \setminus \overline{U'}$ and $V \setminus \overline{F(U')}$. Then

$$F : U \setminus \overline{U'} \to V \setminus \overline{F(U')}$$
is a degree two cover map. This implies that
\[ \text{mod}(U \setminus \overline{U'}) = \frac{1}{2} \text{mod}(V \setminus \overline{F(U')}). \]

Since \( F(U') \) is a subset of \( U \), the annulus \( V \setminus \overline{U} \) is a sub-annulus of the annulus \( V \setminus \overline{F(U')} \), i.e., \( V \setminus \overline{U} \subseteq V \setminus \overline{F(U')} \). Thus we have that
\[ \text{mod}(U \setminus \overline{U'}) \geq \frac{1}{2} \text{mod}(V \setminus \overline{U}). \]

Now we prove the main result about local connectivity by using three-dimensional puzzles.

**Theorem 3.** The Julia set of an unbranched infinitely renormalizable quadratic polynomial having complex bounds is locally connected.

We break the proof of the theorem into four lemmas. We use the same notations as in the previous section.

Suppose \( P(z) = z^2 + c \) is an infinitely renormalizable quadratic polynomial. Let \( J \) be its Julia set and \( \{ \xi^i \}_{i=\infty}^{0} \) be the three-dimensional puzzle. We first classify points in \( J \) into two categories. For a point \( x \) in \( J \), let \( O(x) = \{ P^n(x) \}_{i=0}^{\infty} \) be its orbit and \( \overline{O(x)} \) the closure of its orbit. Then \( x \) is non-recurrent to \( 0 \) in the three-dimensional puzzle if \( \overline{O(x)} \cap J_i = \emptyset \) for some \( i \geq 1 \); otherwise it is recurrent to \( 0 \) in the three-dimensional puzzle. For examples, all non-separating and separating fixed points \( \alpha_{j+1} \) and \( \beta_{j+1} \) of \( F_j \), \( 1 \leq j < \infty \), and their preimages under iterations of \( P \) are non-recurrent, and the critical orbit and its preimages under iterations of \( P \) are recurrent. There are many other non-recurrent and recurrent points.

**Lemma 1.** For any domain \( D \) in \( \eta_n^1 \), \( i, n \geq 0 \), \( D \cap J \) is connected.

**Proof.** Since the domain \( D \) is bounded by finitely many external rays \( \Pi = \{ r_{\theta_j} \}_{j=1}^{m} \) and by some equipotential curve, then \( \partial D \cap J \) consists of a finite number of points \( \{ p_j \}_{j=1}^{m} \). Every \( p_j \) is a landing point of two external rays in \( \Pi \). Suppose \( D \cap J \) is not connected for some \( D \) in \( \eta_n^1 \). Then there are two disjoint open sets \( X \) and \( Y \) such that \( D \cap J = (D \cap J \cap X) \cup (D \cap J \cap Y) \). Suppose that \( p_1, \ldots, p_m \) are in \( X \) and that \( p_{m+1}, \ldots, p_{m'} \) are in \( Y \). The two external rays in \( \Pi \) landing at \( p_j \) cut \( \mathbb{C} \) into two domains. One of them, denoted by \( Z_j \), is disjoint with \( D \), i.e., \( D \cap Z_j = \emptyset \). Then \( U' = \bigcup_{j=1}^{m'} Z_j \) and \( V' = V \cup \bigcup_{m'+1}^{m''} Z_j \) are two disjoint open sets, and \( J = (U' \cap J) \cup (V' \cap J) \). This contradicts the fact that \( J \) is connected. \( \square \)

**Lemma 2.** The Julia set \( J \) is locally connected at any non-recurrent point.

**Proof.** Suppose \( x \) in \( J \) is non-current. Take \( n_0 = 0 \), \( k_0 = 1 \), and \( C_n^{-1} \) for all \( j \geq 1 \). Then there is the smallest integer \( i \geq 0 \) such that \( O(x) \cap C_i^{-1} \neq \emptyset \) and \( \overline{O(x)} \cap J_{i+1} = \emptyset \). Consider the \( i \)th puzzle \( \xi^i = \{ \eta_n \}_{n=0}^{\infty} \) inside \( C_{n_j}^{-1} \). Since \( J_{i+1} = \bigcap_{n=1}^{\infty} C_n^{-1} \), there is an integer \( N \geq 0 \) such that \( \overline{O(x)} \cap C_N^{-1} = \emptyset \). First let us assume that \( x \) is in \( C_{n_j}^{-1} \). Consider
\[ \eta_N = \{ C_N, B_{N, 1}, B_{N, 2}, \ldots, B_{N, q} \} \]
in the \( i \)th puzzle \( \xi^i \). Assume that \( B_{N, 1} \) contains \( P(0) \). (Then \( P(0) \) is an interior point of \( B_{N, 1} \).) The orbit \( P(O(x)) \) is disjoint with \( B_{N, 1} \), since \( O(x) \) is disjoint
with $C^i_{N+1} \subseteq C^i_N$ and $P(C^i_{N+1}) = B^i_{N,1}$. Suppose $x \in D$ is a domain in $\eta^i_{N+1}$ in the $i^{th}$ puzzle $\xi^i$. Suppose $D \subseteq B^i_{N,j}$ and $P(D) = B^i_{N,j}$ for $2 \leq i, j \leq q$. Then $P : D \to B^i_{N,j}$ has the inverse $g_{ij} : B^i_{N,j} \to D$. Therefore for each pair $(B^i_{N,i}, B^i_{N,j})$, $2 \leq i, j \leq q$, satisfying $g_{ij}(B^i_{N,i}) \subseteq B^i_{N,j}$, we can thicken $B^i_{N,i}$ and $B^i_{N,j}$ to simply connected domains $\tilde{B}^i_{N,i,ij}$ and $\tilde{B}^i_{N,j,ij}$ such that $B^i_{N,i} \subseteq \tilde{B}^i_{N,i,ij}$ and $B^i_{N,j} \subseteq \tilde{B}^i_{N,j,ij}$ and such that $g_{ij}$ can be extended to a schlicht function from $\tilde{B}^i_{N,i,ij}$ into $\tilde{B}^i_{N,j,ij}$. We still use $g_{ij}$ to denote this extended function. Now consider $\tilde{B}^i_{N,i,ij}$ and $\tilde{B}^i_{N,j,ij}$ as hyperbolic Riemann surfaces with hyperbolic distances $d_{H,i,ij}$ and $d_{H,j,ij}$. Then $g_{ij}$ on $B^i_{N,i}$ strictly contracts these hyperbolic distances; more precisely, there is a constant $0 < \lambda_{ij} < 1$ such that $d_{H,i,ij}(g_{ij}(x), g_{ij}(y)) < \lambda_{ij}d_{H,j,ij}(x,y)$ for all $x, y \in B^i_{N,i}$. Since there are only a finite number of such pairs, we have a constant $0 < \lambda < 1$ such that $d_{H,i,ij}(g_{ij}(x), g_{ij}(y)) < \lambda d_{H,j,ij}(x,y)$ for all such pairs.

Let

$$x \in \cdots \subseteq D^i_n(x) \subseteq D^i_{n-1}(x) \subseteq \cdots \subseteq D^i_1(x) \subseteq D^i_0(x)$$

be any $x$-end in the $i^{th}$ puzzle $\xi^i$, where $D^i_n(x) \in \eta^i_n$. Then $P^n_m(D^i_n(x))$ for $n - m > N$ is in $B^i_{N,j}$ for some $2 \leq j \leq q$. Therefore, there is a constant $C > 0$ such that for any $D^i_n(x)$ and for any $n > N$,

$$d(D^i_n(x)) = \max_{y,z \in D^i_n(x)} |y - z| \leq C\lambda^{n-N}.$$ 

Thus $d(D^i_n(x))$ tends to zero as $n$ goes to infinity. From Lemma 1, $J \cap D^i_n(x)$ is connected. If $x$ is an interior point of $D^i_n(x)$ for all $n$, then $\{D^i_n(x)\}_{n=1}^\infty$ is a basis of connected neighborhoods of $J$ at $x$. Thus $J$ is locally connected at $x$.

If $x$ is on the boundary of $D^i_n(x)$ but in the interior of $C^{i-1}_{n+1-k_i}$, then we consider all $x$-ends

$$x \in \cdots \subseteq D^i_{s,n}(x) \subseteq D^i_{s,n-1}(x) \subseteq \cdots \subseteq D^i_{s,1}(x) \subseteq D^i_{s,0}(x)$$

in the $i^{th}$ puzzle $\xi^i$. The number of these ends is finite. Then $x$ is an interior point of the domain $\bigcup D^i_{s,n}(x)$ for all $n \geq 1$. Also, since $\bigcup D^i_{s,n}(x)$ is bounded by external rays and equipotential curves of $P$, $J \cap (\bigcup D^i_{s,n}(x))$ is connected, following a similar proof for Lemma 1. Thus $\{\bigcup_i D^i_{s,n}(x)\}_{n=1}^\infty$ is a basis of connected neighborhoods of $J$ at $x$, and $J$ is locally connected at $x$.

If $x$ is on the boundary of $C^{i-1}_{n+1-k_i}$, then $i \geq 1$, and $x$ is a preimage of the separating fixed point $\alpha_i$ of $F_{i-1} : U_{i-1} \to V_{i-1}$ under some iterates of $F_{i-1}$. (Denote $F_0 = F$, $U_0 = U$, $V_0 = V$.) Suppose $P^q(x) = \alpha_i$. Then $P^q$ restricted on a small neighborhood of $x$ is homeomorphic. Thus we have sequences of nested domains

$$\alpha_i \in \cdots \subseteq P^q(D^i_{s,n}(x)) \subseteq P^q(D^i_{s,n-1}(x)) \subseteq \cdots \subseteq P^q(D^i_{s,0}(x))$$

for some big $N_0 > 0$ such that the domains $E_n = \bigcup P^q(D^i_{s,n}(x))$ are bounded by external rays and equipotential curves of $P$ and the diameter $diam(E_n)$ tends to 0 as $n$ goes to $\infty$. Since $\alpha_i$ is the repelling periodic point of $P$ of period $m_i$, we have a finite number of domains $E_{m_i} = P^{m_i}(E_n)$ cyclically around $\alpha_i$ such that the domain $\bigcup E_{m_i}$ has $x$ as an interior point and is bounded by external rays and equipotential curves. The diameter $diam(\bigcup E_{m_i})$ tends to 0 as $n$ goes to $\infty$. Thus, similarly to Lemma 1, $J \cap (\bigcup E_{m_i})$ is connected and $\{\bigcup E_{m_i}\}_{n=1}^{\infty}$, for some large $N_1 \geq N_0$, is a basis of connected neighborhoods of $J$ at $\alpha_i$. This basis can be pulled back by the local homeomorphism $P^q$ on a small neighborhood of $x$ to get a basis of connected neighborhoods of $J$ at $x$. Therefore, $J$ is locally connected at $x$. 

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If \( x \) is not in \( C_{n+k}^{i-1} \), let \( r \geq 1 \) be the smallest integer such that \( y = P^{or}(x) \) is in \( C_{n+k}^{i-1} \). The above argument says \( J \) is locally connected at \( y \). Since \( x \) is not a critical point of \( P^{or} \), \( P^{or} \) restricted on a small neighborhood of \( x \) is homeomorphic. Thus \( J \) is also locally connected at \( x \). 

**Lemma 3.** If \( P \) has complex bounds, then the Julia set \( J \) is locally connected at 0.

**Proof.** To make notations simple, we assume \( \{ F_i = F^{or} : U_i \to V_i \}_{i=1}^{\infty} \) is the infinite sequence of simple renormalizations in Definition 1. Let \( \lambda > 0 \) be the constant in Definition 1. Then \( \{ U_i \}_{i=1}^{\infty} \) is a sequence of nested domains containing 0. From the modulus inequality, Theorem 1, we have

\[
\text{mod}(U_i \setminus U_{i+1}) \geq \frac{1}{2} \text{mod}(V_i \setminus U_i) > \frac{\lambda}{2}.
\]

Let \( A_i = U_i \setminus U_{i+1} \) for \( i \geq 1 \) and \( X = \bigcap_{i=1}^{\infty} U_i \). Since \( U_i \setminus X = \bigcup_{i=1}^{\infty} A_i \),

\[
\text{mod}(U_1 \setminus X) \geq \sum_{i=1}^{\infty} \text{mod}(A_i) = \infty.
\]

Thus, \( X = \{ 0 \} \). This implies that the diameter \( d(U_i) \) tends to 0 as \( i \) goes to infinity.

Consider the \( i \)th puzzle \( \xi^i = \{ \eta_n^i \}_{n=0}^{\infty} \) for every \( i \geq 0 \). Remember that \( C_n^i \) is the member in \( \eta_n^i \) containing 0. Consider the corresponding critical end

\[
0 \in \cdots \subseteq C_n^i \subseteq C_{n-1}^i \subseteq \cdots \subseteq C_1^i \subseteq C_0^i
\]

in the \( i \)th puzzle \( \xi^i \). Since \( J_i+1 = \bigcup_{j=0}^{\infty} C_j^i \) is the Julia set of \( F_{i+1} : U_{i+1} \to V_{i+1} \), there is a \( C_{n(i)}^i \) contained in \( U_{i+1} \). The diameter \( d(C_{n(i)}^i) \) of \( C_{n(i)}^i \) tends to zero as \( i \) goes to infinity. But \( C_{n(i)}^i \cap J_i \) is locally connected at 0, and \( J_i \) is locally connected at 0.

**Corollary 1.** Suppose \( \{ J_i \}_{i=1}^{\infty} \) is the infinite sequence of renormalizations of the Julia set \( J \). If \( P \) has complex bounds, then \( \bigcap_{i=1}^{\infty} J_i = \{ 0 \} \).

**Proof.** It follows that \( J_i \subseteq C_{n(i)}^i \) and \( \bigcap_{i=1}^{\infty} C_{n(i)}^i = \{ 0 \} \).

**Lemma 4.** If \( \bigcap_{i=1}^{\infty} J_i = \{ 0 \} \) and if \( P \) is unbranched, then \( J \) is locally connected at all recurrent points.

**Proof.** Consider the \( i \)th puzzle \( \xi^i = \{ \eta_n^i \}_{n=0}^{\infty} \) for every \( i \geq 0 \), and the corresponding critical end

\[
0 \in \cdots \subseteq C_n^i \subseteq C_{n-1}^i \subseteq \cdots \subseteq C_1^i \subseteq C_0^i.
\]

Consider \( F_{i+1} = F_i^{k_{i+1}} : C_{n_{i+1}+k_{i+1}}^i \to C_{n_{i+1}}^i \). Let \( k(i) = n_{i+1} + k_{i+1} \). Since \( \bigcap_{i=1}^{\infty} J_i = \{ 0 \} \) and \( C_{l(i)}^i \subseteq N(J_{i+1}, 1/(i+1)) \), \( \{ C_{l(i)}^i \}_{i=0}^{\infty} \) is a basis of connected neighborhoods of \( J \) at 0.

Let \( \mu > 0 \) be a constant and let \( \{ W_i \}_{i=1}^{\infty} \) be domains satisfying Definition 2. Without loss of generality, we assume in Definition 2 that \( l = i \) and \( i_{l+1} = i+1 \). Since \( \text{mod}(W_i \setminus J_{i+1}) \geq \mu \) and \( J_{i+1} = \bigcap_{n=0}^{\infty} C_n^i \), by choosing \( k(i) \) large enough, we can assume that

\[
\text{mod}(W_i \setminus C_{l(i)}^i) \geq \frac{\mu}{2}.
\]

for all \( i \geq 1 \). Also, by modifying \( W_i \), we can assume the diameter \( \text{diam}(W_i) \) tends to zero as \( i \) goes to \( \infty \).
We first construct a sequence of partitions for the Julia set \( J \) from the three-dimensional puzzle \( \{ \xi^j \}_{j=0}^\infty \). Denote by \( \tau_1 \) the first partition, which will be constructed as follows: Consider the 0th puzzle \( \xi^0 = \{ \eta^0_n \}_{n=0}^\infty \). Take \( C^0_{k(0)} \in \eta^0_0 \in \xi^0 \). Put all domains in \( \eta_{k(0)+1} \) which are the preimages of \( C^0_{k(0)} \) under \( F \) in \( \tau_1 \), and let \( \eta^0_{k(0)+1} \) be the rest of the domains. Consider \( \eta_{k(0)+2} \cap \eta^0_{k(0)+1} \), consisting of all domains in \( \eta_{k(0)+2} \) which are subdomains of the domains in \( \eta^0_{k(0)+1} \). Put all domains in \( \eta_{k(0)+2} \cap \eta^0_{k(0)+1} \) which are the preimages of \( C^0_{k(0)} \) under \( F^2 \) in \( \tau_1 \), and let \( \eta^0_{k(0)+2} \) be the rest of the domains. Suppose we already have \( \eta^0_{k(0)+s} \) for \( s \geq 2 \). Consider \( \eta_{k(0)+s+1} \cap \eta^0_{k(0)+s} \), consisting of all domains in \( \eta_{k(0)+s+1} \) which are subdomains of the domains in \( \eta^0_{k(0)+s} \). Put all domains in \( \eta_{k(0)+s+1} \cap \eta^0_{k(0)+s} \) which are the preimages of \( C^0_{k(0)} \) under \( F^{s+1} \) in \( \tau_1 \), and let \( \eta^0_{k(0)+s+1} \) be the rest of the domains. Thus we can construct the partition \( \tau_1 \) inductively. This partition covers the Julia set \( J \) minus all points not entering the interior of \( C^0_{k(0)} \) under all iterations of \( F \).

Consider the first puzzle \( \xi^1 = \{ \eta^1_n \}_{n=0}^\infty \). Take \( C^1_{k(1)} \in \eta^1_1 \in \xi^1 \). We can use arguments similar to those in the previous paragraph by considering \( F_1 : C^0_{k(0)} \to C^0_{k(0)-m_1} \) to get a partition \( \tau_{1,1} \) in \( C^0_{k(0)} \). Then we use all iterations of \( F \) to pull back this partition following \( \tau_1 \) to get a partition \( \tau_2 \). It is a sub-partition of \( \tau_1 \), and covers the Julia set \( J \) minus all points not entering the interior of \( C^1_{k(1)} \) under iterations of \( F \).

Suppose we have already constructed the \( (j-1) \)th partition \( \tau_{j-1} \) for \( j > 2 \). Consider the puzzle \( \xi^j = \{ \eta^j_n \}_{n=0}^\infty \). Take \( C^j_{k(j)} \in \eta^j_1 \in \xi^j \). Similarly, by considering \( F_j : C^{j-1}_{k(j-1)} \to C^{j-1}_{k(j-1)-m_j} \), we get a partition \( \tau_{j,1} \) in \( C^{j-1}_{k(j-1)} \). Then we use all iterations of \( F_{j-1} \) to pull back this partition following \( \tau_{j-1} \) to get a partition \( \tau_{j,2} \) in \( C^{j-2}_{k(j-2)} \), and all iterations of \( F_{j-2} \) to pull back this partition following \( \tau_{j-1} \) to get a partition \( \tau_{j,3} \) in \( C^{j-3}_{k(j-3)} \), and so on, to obtain a partition \( \tau_j = \tau_{j,j} \) in \( U \). It is a sub-partition of \( \tau_{j-1} \), and covers the Julia set minus all points not entering the interior of \( C^j_{k(j)} \) under iterations of \( F \). By induction, we have a sequence of nested partitions \( \{ \tau_j \}_{j=1}^\infty \), which covers the Julia set \( J \) minus all non-recurrent points. We call \( \{ \tau_j \}_{j=1}^\infty \) the (extended) three-dimensional puzzle for the Julia set \( J \).

Suppose \( x \neq 0 \) in \( J \) is recurrent. Then the orbit \( O(x) = \{ P^n(x) \}_{n=0}^\infty \) enters every \( C^i_{k(i)} \) infinitely many times. Consider the (extended) three-dimensional puzzle \( \{ \tau_j \}_{j=1}^\infty \) and the \( x \)-end in this puzzle,

\[
x \in \cdots \subseteq D_j(x) \subseteq D_{j-1}(x) \subseteq \cdots \subseteq D_1(x),
\]

where \( D_j(x) \in \tau_j \). Let \( q_j(x) \geq 0 \) be the unique integer such that

\[
F^{\infty q_j(x)} : D_{j+1}(x) \to C^j_{k(j)}
\]

is a proper holomorphic diffeomorphism (see Figure 3). Let

\[
g_{j,x} : C^j_{k(j)} \to D_{j+1}(x)
\]

be its inverse. At this point we use the unbranched condition. Since there are no critical values \( \{ c_r = P^n(0) \}_{r=1}^\infty \) in \( W_j \setminus C^j_{k(j)} \), \( g_{j,x} \) can be extended to a proper holomorphic diffeomorphism on \( W_j \) which we still denote by \( g_{j,x} \).
Consider the \( x_i \)-end in the (extended) three-dimensional puzzle,

\[
x_i \in \cdots \subseteq D_t(x_i) \subseteq D_{t-1}(x_i) \subseteq \cdots \subseteq D_1(x_i),
\]

where \( D_t(x_i) \in \mathcal{T}_t \). Then \( P^{\alpha_{ij} + 1 - \alpha_{ij}(x_i)} : D_{j+1 - \alpha_{ij}(x_i)}(x_i) \to C^j_{k(j)} \) is a proper holomorphic diffeomorphism (see Figure 3). Let \( g_{ij} \) be its inverse. Then \( g_{ij} \) can be extended to \( W_j \) because of the unbranched condition. We still use \( g_{ij} \) to denote this extension. Since \( C^i_{k(i)} \) is bounded by external rays landing at some pre-images of \( \alpha_i \) under iterations of \( P \) and by equipotential curves of \( P \), it follows that \( W_{ij} = g_{ij}(W_j) \) is contained in \( C^i_{k(i)} \). Thus

\[
\text{mod}(W_i \setminus W_{ij}) \geq \frac{\mu}{2}.
\]

Consider \( X_i = g_{i,x}(W_i) \) and \( X_j = g_{j,x}(W_j) = g_{i,x}(W_{ij}) \). Then

\[
\text{mod}(X_i \setminus X_j) \geq \frac{\mu}{2},
\]

since \( g_{i,x} \) is conformal.

Therefore, inductively, we find an infinite sequence of nested domains \( \{X_i\}_{i=1}^\infty \) such that

\[
\text{mod}(X_i \setminus X_{i+1}) \geq \frac{\mu}{2}.
\]

**Figure 3.** The (extended) three-dimensional puzzle
for \( t \geq 1 \). Thus the diameter of \( X_{t_0} \) tends to zero as \( t \) goes to infinity. Since \( D_{i,t+1}(x) = g_i(x(C^{i'}_{k_i(t_0)})) \), we have that
\[
D_{i,t+1}(x) \subseteq X_{t_0}.
\]
So the diameter \( d(D_{i,t+1}(x)) \) tends to zero as \( t \) goes to infinity. Since each \( D_{i,t+1}(x) \) is bounded by external rays and equipotential curves of \( P \), similarly to Lemma 1, \( D_{i,t+1}(x) \cap J \) is connected. Therefore, \( \{D_{i,t+1}(x)\}_{t=1}^{\infty} \) forms a basis of connected neighborhoods of \( J \) at \( x \), and \( J \) is locally connected at \( x \).

**References**


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