C^{1} CONNECTING LEMMAS

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Abstract. Like the closing lemma, the connecting lemma is of fundamental importance in dynamical systems. Hayashi recently proved the C^{1} connecting lemma for stable and unstable manifolds of a hyperbolic invariant set. In this paper, we prove several very general C^{1} connecting lemmas. We simplify Hayashi’s proof and extend the results to more general cases.

1. Introduction

We give a simpler proof for the C^{1} connecting lemma of Hayashi [1] in this paper. This is Theorem E below. The theorem is also more general than the original one of Hayashi. It provides certain answers to some old problems as consequences. Since these consequences are also various kinds of C^{1} connecting lemmas, and their statements are easier to formulate than Theorem E itself, we first state them as Theorems A–D. Let M be a compact manifold without boundary, and let f : M \to M be a diffeomorphism. Denote by Diff^{1}(M) the set of diffeomorphisms of M, endowed with the C^{1} topology. We state these results for diffeomorphisms, but the corresponding results are also true for flows. We further remark that these results generalize easily to symplectic and volume preserving diffeomorphisms.

**Theorem A.** Let p and q be two points of M with \( \omega(p) \cap \alpha(q) \neq \emptyset \). We also assume that \( \omega(p) \cap \alpha(q) \) contains some non-periodic point z. Then for any C^{1} neighborhood \( \mathcal{U} \) of f, there is \( g \in \mathcal{U} \) such that q is on the positive g-orbit of p. More precisely, for any C^{1} neighborhood \( \mathcal{U} \) of f, there is a positive integer L such that for any \( \delta > 0 \), there is a \( g \in \mathcal{U} \) such that \( g = f \) outside the tube \( \bigcup_{i=1}^{L} B(f^{-i}(z), \delta) \) and such that q is on the positive g-orbit of p.

This gives a partial answer to an old problem of Pugh [9]. Roughly, Theorem A says that two points p and q can get connected via C^{1} perturbations if the original forward orbit of p and the original backward orbit of q are nearly connected at some non-periodic point z. Note that the C^{0} connecting problem is trivial, just like the C^{0} closing problem. Also note that the compactness of M is essential; it guarantees the Lift Axiom formulated in [11]. If M is not compact, then Theorem A and the other results of this paper are not true even if the Whitney strong topology is used. An instructive counterexample is given by Pugh [10]. We remark that the original problem of Pugh does not assume that \( \omega(p) \cap \alpha(q) \) contains some non-periodic points. This assumption is a technical one. It is demanded by our method, but not

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by the nature of the problem. Thus a complete answer to Pugh’s original problem needs a separate treatment for the case that every point of $\omega(p) \cap \alpha(q)$ is periodic, which we do not know how to deal with at this point. Similarly, the problem of $C^1$ closing the $n^{th}$ order prolongational recurrence, also raised by Pugh in [9], is solved under the same sort of non-periodicity assumption. More precisely, $p \in M$ is $n^{th}$ order prolongationally recurrent if there are $n$ points $p = p_1, p_2, \cdots, p_n \in M$ such that $\omega(p_i) \cap \alpha(p_{i+1}) \neq \emptyset$ for each $i = 1, 2, \cdots, n$, where $p_{n+1} = p_1$. We have:

**Theorem B.** Let $p$ be $n^{th}$ order prolongationally recurrent. Assume that $\omega(p_i) \cap \alpha(p_{i+1})$ contains non-periodic points for each $i = 1, 2, \cdots, n$, where $p_{n+1} = p_1$. Then for any $C^1$ neighborhood $U$ of $f$, there is a $g \in U$ such that $p$ is a periodic point of $g$.

The statement of the second half of Theorem A, which is more precise than the first half, has the advantage that, in case $p$ is not negatively recurrent under $f$ and $q$ is not positively recurrent under $f$, then $\delta$ can be chosen small enough so that the support tube $\bigcup_{i=1}^\infty B(f^{-i}(z), \delta)$ is disjoint from $\text{Orb}^-(p, f)$ and from $\text{Orb}^+(q, f)$; hence $\text{Orb}^-(p, f) = \text{Orb}^-(p, g)$ and $\text{Orb}^+(q, f) = \text{Orb}^+(q, g)$. For instance, $p$ could go backward to a hyperbolic fixed point $p_0$ under $f$, and $q$ could go forward to a hyperbolic fixed point $q_0$ under $f$. Then this $C^1$ perturbation creates a heteroclinic connection from $p_0$ to $q_0$, respecting $g$. This gives Theorem C, which answers another old problem of Liao [3] and Mañé [5] about creating homoclinic points.

**Theorem C.** Let $\Lambda$ be an isolated hyperbolic set of $f$. Assume $W^s(\Lambda) \cap \overline{W^u(\Lambda)} - \Lambda \neq \emptyset$. Then for any $C^1$ neighborhood $U$ of $f$, there is a $g \in U$ such that $g = f$ on a neighborhood $U$ of $\Lambda$ and such that $W^s(\Lambda, g) \cap W^u(\Lambda, g) - \Lambda \neq \emptyset$.

Note that if the assumption in Theorem C is replaced by a weaker one, namely $W^s(\Lambda) \cap \overline{W^u(\Lambda)} - \Lambda \neq \emptyset$, the conclusion is still true, as long as $W^s(\Lambda) \cap W^u(\Lambda) - \Lambda$ contains non-periodic points, as is easily seen from Theorem E below. Another similar problem of this type appears in the study of the $C^1$ stability conjecture. We state an answer as Theorem D.

**Theorem D.** Let $\Lambda$ be an isolated hyperbolic set of $f$. Assume that periodic orbits outside $\Lambda$ accumulate on $\Lambda$. Then for any $C^1$ neighborhood $U$ of $f$, there is a $g \in U$ such that $g = f$ on a neighborhood of $\Lambda$ and $W^s(\Lambda, g) \cap W^u(\Lambda, g) - \Lambda \neq \emptyset$.

All these theorems are straightforward consequences of the following general version of the $C^1$ connecting lemma.

**Theorem E.** Let $z \in M$ not be a periodic point of $f$. For any $C^1$ neighborhood $U$ of $f$, there are $\rho > 1$, $L \in \mathbb{N}$ and $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$, and any two points $p$ and $q$ outside the tube $\Delta = \bigcup_{i=1}^L f^{-n}B(z, \delta)$, if the positive $f$-orbit of $p$ hits the ball $B(z, \delta/\rho)$ after $p$, and if the negative $f$-orbit of $q$ hits the same ball, then there is $g \in U$ such that $g = f$ off $\Delta$ and $q$ is on the positive $g$-orbit of $p$.

**Remark.** Here we require that the positive orbit of $p$ hits the ball after $p$ just to guarantee that the positive orbit of $p$ goes through the tube at least once before that hit. We do not require this for the negative orbit of $q$, because the tube $\Delta$ has been taken along the negative orbit of $z$. Symmetricly, we can restate Theorem E for a tube along the positive orbit of $z$, and require that the negative orbit of $q$ goes through the tube at least once before the hit. The same is true for Theorem F below.
Note that the formulation of Theorem E is more complicated than Theorems A–D. However, Theorem E assumes less. It does not assume any limit behavior for \( z \) except that it is not a periodic point, and neither \( \omega \)-limit set nor hyperbolicity is involved in the statement of the theorem. Hence it is more general than Theorems A–D, and more flexible in applications. Note that \( \delta \) can always be chosen so small that \( B(z, \delta) \) is disjoint from \( \Delta \), because \( z \) is non-periodic and \( \delta \) is independent of \( L \), and because the tube \( \Delta \) covers iterates from 1 to \( L \), but not from 0 to \( L \). Thus the special case of Theorem E for which the point \( q \) itself is in \( B(z, \delta/\rho) \) will read as the following Theorem F, which is convenient for some applications (see §7).

**Theorem F.** Let \( z \in M \) be a non-periodic point of \( f \). For any \( C^1 \) neighborhood \( U \) of \( f \), there are \( \rho > 1 \), \( L \in \mathbb{N} \) and \( \delta_0 > 0 \) such that for any \( 0 < \delta \leq \delta_0 \), and for any point \( p \) outside the tube \( \Delta = \bigcup_{n=1}^{L} f^{-n} B(z, \delta) \) and any point \( q \in B(z, \delta/\rho) \), if the positive \( f \)-orbit of \( p \) hits \( B(z, \delta/\rho) \) after \( p \), then there is \( g \in U \) such that \( g = f \) off \( \Delta \) and \( q \) is on the positive \( g \)-orbit of \( p \).

The \( C^1 \) connecting lemma is a long-desired result. Many authors have made important contributions to this problem. For the case that \( M \) is the 2-sphere, Robinson [12] and Pixton [7] solved the problem for any \( C^r \), \( 1 \leq r \leq \infty \). For volume-preserving diffeomorphisms, Takens [15] solved the problem for \( r = 1 \), and Oliveira [6] solved the problem for any \( C^r \), \( 1 \leq r \leq \infty \), when \( M \) is the 2-torus. Mañé [5] solved the problem for \( r = 1, 2 \) with an additional measure theoretic assumption. Liao [3] solved the problem for \( r = 1 \) with an additional topological assumption. A recent surprising result came when Hayashi [1] proved a general \( C^1 \) connecting lemma which solved Theorems C and D. The present paper proves another general version of the \( C^1 \) connecting lemma (Theorem E), which enables us to also obtain Theorems A and B. What encourages us is that the proof of Theorem E presented in this paper turns out not to be very long. We first formulate a basic \( C^1 \) perturbation theorem, which can be extracted from the work of Liao, Pugh, and Robinson (see [2], [8] and [11]) on the \( C^1 \) closing lemma. This is Theorem 3.1 below, which serves as a fundamental preliminary for our proof of the \( C^1 \) connecting lemma. A key ingredient then added in is Hayashi’s brilliant idea of “cutting” [1]. Finally, with an intensive use of these beautiful ideas in a series of combinatorial selections (Xia [16]), we are able to cut short some disjoint orbital arcs with \( C^1 \) perturbations, and eventually get the two points \( p \) and \( q \) connected.

This work is an outgrowth of an earlier paper [16] by the second author, which contains a self-contained proof of the \( C^1 \) connecting lemma for an important case, and contains all the crucial ideas of the present paper. We wish to thank J. Palis, Charles Pugh, and C. Robinson for many good critical comments and suggestions.

In §2 we introduce Theorem 2.2, which is a linear version of Theorem E. In §3 we formulate a basic \( C^1 \) perturbation theorem. In §4 we give an arrangement of boxes. These two sections serve as preparations for proving Theorem 2.2. The proof of Theorem 2.2 itself will be given in §5. In §6 we describe a linearization process needed to realize Theorem 2.2 on manifolds to obtain Theorem E. §7 is devoted to applications of Theorem E.

2. A linear version of the \( C^1 \) connecting lemma

We introduce a linear version of Theorem E in this section. This is Theorem 2.2 below. Its formulation uses a geometrical notion called \( \varepsilon \)-kernel avoiding transition, which is due to Mai [4] and is the basic pattern for the \( C^1 \) perturbation constructed
below. This way of constructing perturbations in proving the $C^1$ closing lemma actually appeared very early ([8]). It is just the unified notion of $\varepsilon$-kernel avoiding transition that appeared relatively late ([2], [13]). First we define $\varepsilon$-kernel lifts, which serve as the basic elements of our $C^1$ perturbations.

Let $B \subset \mathbb{R}^m$ be a closed ball with radius $r$ and let $\varepsilon > 0$. We denote by $\varepsilon B$ the ball of the same center and of radius $\varepsilon r$. We call $\varepsilon B$ the $\varepsilon$-kernel of $B$. Thus the number $\varepsilon$ here gives a relative ratio but not an absolute size. For any $x$ and $y$ in the interior of $B$, there is a (in fact many) $C^\infty$ diffeomorphism $h: \mathbb{R}^m \to \mathbb{R}^m$ that is the identity outside $B$, which takes $x$ to $y$. If $x$ and $y$ are in $\varepsilon B$, we call such an $h$ an $\varepsilon$-kernel lift that lifts $x$ to $y$, supported on $B$. The following simple but fundamental lemma tells how $\varepsilon$ controls the first derivatives of $h - id$ for certain $\varepsilon$-kernel lifts $h$. The formal formulation of this fact with the proof on manifolds can be found in [11, Theorem 6.1].

**Lemma 2.1.** For any $\beta > 0$, there is an $\varepsilon > 0$ such that for any closed ball $B$ in $\mathbb{R}^m$, and any $x$ and $y$ in $\varepsilon B$, there is an $\varepsilon$-kernel lift $h$ that lifts $x$ to $y$, supported on $B$, such that all partial derivatives of $h - id$ have absolute values less than $\beta$.

**Proof.** The proof is easy. We only need to consider the case that $x$ is the center of the ball, because the composition of two lifts of this type gives what we want. Fix a $C^\infty$ bump function $\alpha: \mathbb{R}^m \to \mathbb{R}$ such that $0 \leq \alpha \leq 1$ on all $\mathbb{R}^m$, $\alpha = 1$ on $B(0, \frac{1}{2})$, $\alpha = 0$ off $B(0, \frac{3}{4})$, and such that all partial derivatives of $\alpha$ have absolute values less than or equal to $6$. Let $r$ be the radius of $B$, and let $y \in \varepsilon B$. Define $h: \mathbb{R}^m \to \mathbb{R}^m$ by

$$h(u) = u + \alpha\left(\frac{u - x}{r}\right)(y - x).$$

Then $h$ is a diffeomorphism if $\varepsilon$ is small enough, and for any $i$,

$$|\frac{\partial}{\partial u_i}(h - id)| \leq \frac{1}{r} \cdot 6 \cdot \varepsilon r = 6\varepsilon.$$

This proves the lemma.

Roughly, the number $\varepsilon$ controls the size of the first derivatives of $h - id$. Note that the radius $r$ of $B$ is not mentioned in the statement of Lemma 2.1, which clearly controls the $C^0$ size of $h - id$. Therefore the $\varepsilon$-kernel lift $h$ can be defined to be $C^1$ close to the identity if both $\varepsilon$ and $r$ are small, and the composition $h \circ f$ hence gives a $C^1$ perturbation of $f$. The $C^1$ perturbations used in this paper will be a composition of $f$ with finitely many of this kind of $\varepsilon$-kernel lifts with disjoint supports. By virtue of Lemma 2.1, we will not mention the $\varepsilon$-kernel lift $h$ explicitly, but only mention the ball $B$ and the two points $x, y \in \varepsilon B$. Whenever such $B$, $x$, and $y$ are specified, we can perform a suitable $\varepsilon$-kernel lift $h$ at any time. In this way we define $\varepsilon$-kernel avoiding transitions, which are the basic patterns of $C^1$ perturbations used below. Let $V_0, V_1, \ldots, V_n, \ldots,$ be a sequence of $m$-dimensional inner product spaces, and let $T_n: V_n \to V_{n-1}, n = 1, 2, \ldots,$ be a sequence of linear isomorphisms. Let $\varepsilon > 0$, $u, v \in V_0$, $L \in \mathbb{N}$, $Q \subset V_0$, and $G \subset V_0$ be given. By an $\varepsilon$-kernel transition of $\{T_n\}$ from $u$ to $v$ of length $L$ contained in $Q$ and avoiding $G$, we mean $L + 1$ points $c_n, 0 \leq n \leq L$, together with $L$ balls $B_n \subset V_n, 0 \leq n \leq L - 1$, such that

1. $c_0 = v$ and $c_L = F^{-1}_L(u)$, where $F_n = T_1 \circ T_2 \circ \cdots \circ T_n$.
2. $c_n \in \varepsilon B_n$ and $T_{n+1}(c_{n+1}) \in \varepsilon B_n, 0 \leq n \leq L - 1$. 

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3. $B_n \subset F^{-1}_n(Q)$, $0 \leq n \leq L - 1$.
4. $B_n \cap F^{-1}_n(G) = \emptyset$, $0 \leq n \leq L - 1$.

Two $\varepsilon$-kernel transitions $c_0, c_1, \ldots, c_L$; $B_0, B_1, \ldots, B_{L-1}$ and $c'_0, c'_1, \ldots, c'_L$; $B'_0, B'_1, \ldots, B'_{L-1}$, contained respectively in $Q_1$ and $Q_2$, are disjoint if $B_n \cap B'_n = \emptyset$ for all $0 \leq n \leq L - 1$.

Roughly, a transition of length $L$ consists of $L + 1$ points that form a pseudo-orbit with $L$ jumps. The associated $L$ balls indicate that these $L$ jumps are $\varepsilon$-kernel lifts. The containing set $Q$ and the avoidance set $G$ are constraints put on the transition. Note that the terminologies defined here are abbreviated ones. Such an $\varepsilon$-kernel transition actually is from $F^{-1}_L(u)$ to $v$, and is contained in the tube $\bigcup_{n=1}^{L-1} F^{-1}_n(Q)$, and avoid a set of orbital arcs $\bigcup_{n=1}^{L-1}(G)$. We emphasize that, in the definition of disjointness of two transitions, we do not require that $Q_1$ and $Q_2$ are disjoint. We merely require that $B_n$ and $B'_n$ in $V_n$ are disjoint (and hence the $2L$ balls are mutually disjoint, since the $V_n$’s are distinct vector spaces. Later on the $V_n$’s will correspond to disjoint neighborhoods on the manifold). This is sufficient for our purpose because it is these balls $B_n$ that support our $C^1$ perturbation.

**Remark.** We insert an informal illustration here on what these $V_n$ and $T_n$ have to do with $M$ and $f$. Applied to the manifold via some standard linearization along a finite orbit of length $L$, these $V_n$, $n = 0, 1, \ldots, L - 1$, simply correspond to disjoint neighborhoods of the iterates along a backward orbit of $f$, and these $T_n$ simply correspond to $f$ itself. To see the dynamics we mix them up just for illustration. Thus the transition transits a point from one orbit of $f$ to another orbit of $f$ via $L$ lifts which form a pseudo-orbit. If $Q$ is small (which bounds the $C^0$ size of the perturbation), and if $\varepsilon$ is small too, then the transition gives a $C^1$ small perturbation. An important case is that $u$ is on the positive orbit of $v$ before perturbation, say $u = f^N(v)$. Then $N$ must be much larger than $L$, since it needs more than $L$ iterates for $v \in Q$ to get back, for the first time, to $Q$ at some point $v_1 \in Q$. Then it again needs more than $L$ iterates for $v_1 \in Q$ to get back to $Q$ at some $v_2 \in Q$, etc. Since $u$ is some return of $v$, say, $u = v_k$, $k \geq 1$, the $L$th pull-back $f^{-L}(u)$ of $u$, which corresponds to $F^{-1}_L(u)$, is still on the positive orbit of $v$ under $f$. Now in case the avoidance set $G$ contains the set of intermediate returns $\{v_1, v_2, \ldots, v_{k-1}\}$, then the old orbit from $v$ to $f^{-L}(u)$ remains unchanged. Hence this perturbation creates a periodic orbit through $v$, which goes from $v$ to $f^{-L}(u)$ via the old orbits, and from $f^{-L}(u)$ to $v$ via the transition. This is the way the perturbations are constructed in proving the $C^1$ closing lemma. The novelty of the perturbation constructed in proving the $C^1$ connecting lemma is that, while the perturbation for the closing case consists of a single transition in a tube, the perturbation for the connecting case consists of finitely many disjoint transitions in the same tube. Besides, for all but one transition in the connecting case the situation is somewhat the opposite: $v$ is on the positive orbit of $u$, and the avoidance set is the set of returns which are non-intermediate to all (not just one) of the transitions. See below for more comments about this. Thus what the transition from $F^{-1}_L(u)$ to $v$ does is not a closing, but a shortcut. This is formulated in the following linear version of the $C^1$ connecting lemma, whose proof will be given in §5.

**Theorem 2.2** (The linear version of the $C^1$ connecting lemma). Given any sequence of isomorphisms $\{T_n\}$, and any $\varepsilon > 0$, there are $\sigma > 1$ and $L \in \mathbb{N}$ such that, for any two sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^L$ in $V_0$, with an order $< defined on
there exist two points
\[ x \in \{ x_i \}_{i=1}^s \cap B(x_s, \sigma|x_s - y_t|) \quad \text{and} \quad y \in \{ y_i \}_{i=1}^t \cap B(x_s, \sigma|x_s - y_t|), \]

\[ \text{together with some ordered pairs } \{ p_i, q_i \} \subset X \cap B(x_s, \sigma|x_s - y_t|), \text{ say, } k \text{ of them}, \]

\[ x_1 \leq p_1 \leq q_1 < p_2 \leq q_2 < \cdots < p_{k'} \leq q_{k'} < x \]

\[ < y < p_{k'+1} \leq q_{k'+1} < \cdots < p_k \leq q_k \leq y_1 \]

such that the following four conditions are satisfied.
1. There is an \( \varepsilon \)-kernel transition from \( x \) to \( y \) of length \( L \) that is contained in \( B(x_s, \sigma|x_s - y_t|) \).
2. For each \( i = 1, 2, \cdots, k \), there is an \( \varepsilon \)-kernel transition from \( p_i \) to \( q_i \) of length \( L \), contained in \( B(x_s, \sigma|x_s - y_t|) \).
3. These \( k + 1 \) transitions each avoid \( X - [x, y] - [p_1, q_1] - \cdots - [p_k, q_k] \), where the closed interval notation \( [x, y] \) denotes \( \{ z \in X \mid x \leq z \leq y \} \), etc.
4. These \( k + 1 \) transitions are mutually disjoint.

The formulation of Theorem 2.2 is complicated, because it describes in detail how the connection is made. Note that the difference between the pair \( \{ x, y \} \) and the pairs \( \{ p_i, q_i \} \) is that \( x \) and \( y \) belong to different sequences, while \( p_i \) and \( q_i \) belong to the same sequence. While \( x \) and \( y \) are different points, \( p_i \) and \( q_i \) may be the same point. In this case the transition from \( p_i \) to \( q_i \) is understood as the trivial one, i.e. no lifts at all. For convenience we will call the pair \( \{ x, y \} \) the connecting pair, and \( \{ p_i, q_i \} \) the cutting pairs.

Remark. Let us give an informal illustration for Theorem 2.2. We visualize the sequence \( \{ x_i \}_{i=1}^s \) as returns of the positive orbit of \( p \) to a neighborhood of \( z \) and \( \{ y_i \}_{i=1}^t \) as returns of the negative orbit of \( q \) to the same neighborhood, where we think of \( p, q \) and \( z \) as the three points in Theorem A. Note that while \( x_s \) and \( y_t \) can be arbitrarily close to \( z \), the numbers \( \sigma \) and \( L \) are independent of \( |x_s - y_t| \). Hence all the \( k + 1 \) \( \varepsilon \)-kernel transitions can be put in an arbitrarily thin tube of length \( L \). Roughly, the existence of such a \( \sigma \) guarantees control of the \( C^0 \) size of the perturbation. After perturbations, the positive orbit of \( p \) will go through the following points successively:

\[ p, \cdots, x_1, \cdots, F_L^{-1}(p_1), \cdots, q_1, \cdots, F_L^{-1}(p_2), \cdots, q_2, \cdots, \]

\[ F_L^{-1}(p_{k'}), \cdots, q_{k'}, \cdots, F_L^{-1}(x), \cdots, y, \cdots, \]

\[ F_L^{-1}(p_{k'+1}), \cdots, q_{k'+1}, \cdots, F_L^{-1}(p_k), \cdots, q_k, \cdots, y_1, \cdots, q. \]

That is, it takes an old orbit from \( p \) to \( F_L^{-1}(p_1) \), then takes a transition (a shortcut) from \( F_L^{-1}(p_1) \) to \( q_1 \), then takes an old orbit from \( q_1 \) to \( F_L^{-1}(p_2) \), then takes a transition from \( F_L^{-1}(p_2) \) to \( q_2 \), etc. All the transitions here are shortcuts except for one: the transiton associated with \( \{ x, y \} \) goes from the orbit of \( p \) to a different
orbit of $q$. This is how $p$ and $q$ get connected. We emphasize that, according to condition 3 in the theorem, the transition from $p_i$ to $q_i$ does not need to avoid the points of $X$ that are intermediate to the other pairs $(p_j, q_j)$. Only those points of $X$ that are non-intermediate to all of the pairs are to be avoided. In this case, as long as these transitions are disjoint, they make the connections.

3. A basic $C^1$ perturbation theorem

In this section we formulate a basic $C^1$ perturbation theorem, which can be extracted from the work of Liao, Pugh, and Robinson on the $C^1$ closing lemma. This is Theorem 3.1 below, which serves as a fundamental preliminary for the proof of Theorem 2.2.

Let $V$ be an $m$-dimensional inner product space and $e = (e_1, e_2, \cdots, e_m)$ an orthonormal basis of $V$. An $e$-box $Q$ of center $x \in V$ and of size $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ is defined as

$$Q = \{ y \in V | |y_i - x_i| \leq \lambda_i, \ 1 \leq i \leq m \},$$

where $x_i$ and $y_i$ are coordinates of $x$ and $y$, with respect to the basis $e$. For $\alpha > 0$, define

$$\alpha Q = \{ y \in V | |y_i - x_i| \leq \alpha \lambda_i, \ 1 \leq i \leq m \}.$$

If $\alpha < 1$, we say that $\alpha Q$ is the $\alpha$-kernel of $Q$. We say that a box $Q'$ is of type $Q$, if

$$Q' = z + \alpha Q$$

for some $z \in V$ and some $\alpha > 0$.

**Theorem 3.1.** For any sequence of isomorphisms $T_n: V_n \to V_{n-1}$, $n = 1, 2, \cdots$, there is an orthonormal basis $e = (e_1, e_2, \cdots, e_m)$ in $V_0$ such that for any $\varepsilon > 0$, and any $0 < \alpha < 1$, there exist an $e$-box $A$ and an integer $L \in \mathbb{N}$ such that for any $e$-box $Q$ of type $A$ and any two points $x, y \in \alpha Q$, there is an $\varepsilon$-kernel transition $e_0; e_1, \cdots, e_L; B_0, B_1, \cdots, B_{L-1}$ of $\{T_n\}$ from $x$ to $y$ of length $L$, contained in $Q$. Moreover, the radius of $B_n$ is less than or equal to half of the distance between $\partial(F^{-1}_{n}(Q))$ and $\partial(F^{-1}_{n}(\alpha Q))$.

Note that Theorem 3.1 does not consider the avoidance of the transition. To actually prove the $C^1$ closing lemma using Theorem 3.1, one needs to carefully select a sequence of points in such a way that certain avoidance is realized. This requires some combinatorial considerations. For the connecting case some additional considerations are needed below on the disjointness of different transitions. In the statement of Theorem 3.1 the requirement that the support balls be uniformly small in ratio deserves a special attention. More precisely, in addition to the requirement that the ball $B_n$ should be contained in $F^{-1}_{n}(Q)$, the last sentence of this theorem requires that the ball $B_n$ should also be small enough relative to the parallelepiped $F^{-1}_{n}(Q)$ that, via a parallel translation, it can be inserted into the gap between the two parallelepipeds $F^{-1}_{n}(Q)$ and $F^{-1}_{n}(\alpha Q)$. This is crucial to the proof of the $C^1$ connecting lemma given below.

There is a beautiful proof for the $C^1$ closing lemma by Mai [4] with a different approach, which was later generalized to some non-invertible maps by Wen [14]. The proof is fairly simple. It does not yield Theorem 3.1, however, because the radii of the balls used there do not have to satisfy the last requirement of Theorem 3.1.
4. An arrangement of boxes

We describe in this section a simple arrangement of boxes in $V_0$. The construction will be used in the next section to realize certain avoidance in the proof of Theorem 2.2.

Let $e$ be an orthonormal basis of $V_0$, and let $A$ be an $e$-box. All boxes considered in this section will be $e$-boxes of the same type $A$.

Given a box $H_0$ of type $A$, there are $4^m - 2^m$ boxes $H_i$, $1 \leq i \leq 4^m - 2^m$, of the same type $A$, with size reduced by $1/2$, which fill out $2H_0 - H_0$ as the figure shows.

![Diagram of boxes](image)

**Figure 1.** $e$-boxes

In other words, we can enclose $H_0$ with boxes of the same type $A$ but of half-size to build up $2H_0$. Then we can enclose $2H_0$ with boxes ($10^m - 8^m$ of them) of the same type but of $1/4$-size to build up $(2 + 1/2)H_0$. This process continues, and gives a sequence

$$H_0, H_1, H_2, \cdots,$$

where $H_1$ through $H_{4^m - 2^m}$ (we may call them the boxes of the second generation) are those boxes of half-size, $H_{4^m - 2^m + 1}$ through $H_{10^m - 8^m}$ (we may call them the boxes of the third generation) are those boxes of $1/4$-size, etc. The precise formulas for the subscripts like $4^m - 2^m$ are not essential to us, and will not be calculated explicitly below. It is clear that the union of all $H_i$ is int$(3H_0)$. This open $e$-box will be somewhat important to us, because all $e$-kernel transitions below will be contained in int$(3H_0)$ for a suitably chosen $H_0$.

Now let

$$D_i = 2H_i, \quad i \geq 1.$$  

The following three properties are clear.

D1) $\bigcup_{i \geq 0} D_i = \bigcup_{i \geq 0} H_i$.

D2) Each $D_i$ is contained in int$(3H_0)$.

D3) There is a universal constant $N^*$, independent of $e$, $A$, and $H_0$, such that each $D_i$ intersects at most $N^*$ of the other $D_j$.

Properties D1) and D2) are obvious. To see property D3), we first note that if $H_i$ and $H_j$ are three generations apart, then $D_i \cap D_j = \emptyset$. Then each $D_i$ can intersect the other boxes $D_j$ in at most five generations. For each of the five generations, it is easy to see that there is a constant $N$ such that $D_i$ intersects at most $N$ of the other boxes $D_j$ in this generation. Moreover, $N$ can be chosen independent of $D_i$, and independent of $e$, $A$, and $H_0$ as well.
5. The proof of Theorem 2.2

Proof. Let \( \{T_n\} \) and \( \varepsilon > 0 \) be given. Let \( e \) be the orthonormal basis given by Theorem 3.1. Let \( N^* \) be the universal constant in property D3) in \( \S 4 \). Let \( 0 < \alpha < 1 \) be a number that satisfies the inequality

\[
\left( \frac{1}{\alpha} - 1 \right)(\frac{3}{\varepsilon})^{2N^*+3} \leq 1.
\]

For \( \varepsilon, \alpha \) as chosen, take the \( e \)-box \( A = (\lambda_1, \cdots, \lambda_m) \) and the integer \( L \in \mathbb{N} \) as Theorem 3.1 claims. Then let

\[
\sigma = 6 \max\{\lambda_i/\lambda_j \mid 1 \leq i, j \leq m\},
\]

that is, 6 times the \textit{ballicity} of \( A \). \( \Box \)

Now we verify that \( \sigma \) and \( L \) so chosen satisfy the conditions of Theorem 2.2. Given any two sequences \( \{x_i\}_{i=1}^n \) and \( \{y_i\}_{i=1}^n \) in \( V_0 \), we need to select from \( X \) a connecting pair \( \{x, y\} \) and some cutting pairs \( \{p_i, q_i\} \) in \( B(x_i, \sigma|x_i - y_i|) \) that satisfy the four conditions in Theorem 2.2. This will be done in the following three steps.

Step 1. A preliminary selection.

This step selects a pair \( \{\xi, \zeta\} \), which is a candidate for \( \{x, y\} \), and some pairs \( \{u_i, v_i\} \), which are candidates for \( \{p_i, q_i\} \). The final selection for \( \{x, y\} \) and \( \{p_i, q_i\} \) will be done in Step 3.

The selection of \( \{\xi, \zeta\} \) and \( \{u_i, v_i\} \) proceeds through a series of trial selections as follows.

Let \( H_0 \) be an \( e \)-box of type \( A \) such that \( x_s, y_t \in H_0 \subset 3H_0 \subset B(x_i, \sigma|x_i - y_i|) \). Such an \( H_0 \) exists because of the choice of \( \sigma \). This can be illustrated as follows. The ball \( B(x_i, \sigma|x_i - y_i|) \) contains an \( e \)-cube \( C \) of size \( \sigma|x_i - y_i| \) (in fact an \( e \)-cube of size \( \sqrt{m}\sigma|x_i - y_i| \)) centered at \( x_i \). Shrinking with a suitable ratio on each side, one obtains in \( C \) an \( e \)-box of type \( A \) of center \( x_s \) with longest side \( \sigma|x_s - y_t| \) and with shortest side \( 6|x_s - y_t| \). It is easy to check that this box could be our \( 3H_0 \). Let \( H_0, H_1, \cdots \) be the sequence of boxes determined by \( H_0 \), as arranged in \( \S 4 \). Everything we do below is in those \( D_n \), and hence in \( 3H_0 \) by D2), which in turn is contained in \( B(x_s, \sigma|x_s - y_t|) \).

First we look at \( H_0 \) and choose \( \xi \) and \( \zeta \) to be the smallest and the largest points of \( X \cap H_0 \). Here the terms smallest and largest are respecting the order \( < \) of \( X \). Thus \( \xi \leq x_s < y_t \leq \zeta \) because \( x_s, y_t \in H_0 \). We emphasize that this selection is a trial one. It is subject to change.

Then we look at \( H_1 \) and let \( a \) and \( b \) be the smallest and the largest point of \( X \) within \( H_1 \), subtracting the closed interval \( [\xi, \zeta] \). In other words, \( a \) and \( b \) are the smallest and the largest points in \( (X - [\xi, \zeta]) \cap H_1 \). Note that \( a \) and \( b \) could be the same, and then \( (X - [\xi, \zeta]) \cap H_1 \) would reduce to a single point. This corresponds to the case that some cutting pairs \( \{p_i, q_i\} \) are a single point and the corresponding transition is the trivial one. Also, \( (X - [\xi, \zeta]) \cap H_1 \) could be empty; in this case we simply go on to \( H_2 \). There are two cases to consider.

Case 1. \( a \) and \( b \) belong to the same sequence. That is, \( a \leq b < \xi < \zeta \), or \( \xi < \zeta < a \leq b \).
In this case we make a trial selection of \( \{a, b\} \) as one of the cutting pairs, say \( \{u, v\} \). Note that \( b \neq \xi \) and \( \zeta \neq a \), because the closed interval \([\xi, \zeta]\) has been subtracted out from \( X \).

Case 2. \( a \) and \( b \) belong to different sequences. That is, \( a < \xi < \zeta < b \).

In this case we select \( \{a, b\} \) as a better candidate for the connecting pair, and drop the open interval \((a, b)\), including the old candidates \( \xi \) and \( \zeta \), from our considerations, forever. Thus we rename \( a \) as \( \xi \), and \( b \) as \( \zeta \). Note that this is still subject to change.

This finishes our observation in \( H_1 \). We obtain a candidate connecting pair \( \{\xi, \zeta\} \), and a (or no) candidate cutting pair \( \{u, v\} \). We emphasize that the closed intervals determined by these pairs are mutually disjoint.

Then we look at \( H_2 \). Let \( a \) and \( b \) be the smallest and the largest points in \((X - [\xi, \zeta] - [u, v]) \cap H_2\), where \( \{u, v\} \) is the candidate cutting pair obtained in Case 1 above. For Case 2, we simply do not have this term. There are still two cases to consider.

Case 1. \( a \) and \( b \) belong to the same sequence.

In this case we select \( \{a, b\} \) as a candidate cutting pair. Note that \( a \) and \( b \) do not belong to any of the closed intervals of the previously chosen pairs, since they have been subtracted out from \( X \). Thus either \([a, b]\) is disjoint from all these intervals, or its interior \((a, b)\) covers any of these intervals that intersect \([a, b]\). In the latter case, we drop \( \{a, b\} \) from our considerations. Thus the closed intervals of all pairs so obtained are mutually disjoint.

Case 2. \( a \) and \( b \) belong to different sequences.

In this case we select \( \{a, b\} \) as a better candidate for \( \{\xi, \zeta\} \), and drop \( \{a, b\} \) from our considerations.

Remark. This might be a good place to indicate Hayashi’s brilliant idea of “cutting” [1]. As mentioned earlier, \( \{x_i\} \) will be returns of the positive orbit of \( p \) to a neighborhood of \( z \), and \( \{y_i\} \) will be returns of the negative orbit of \( q \) to the same neighborhood of \( z \), where \( p, q, \) and \( z \) are the three points in Theorem A. These returns are ordered in a way that fits our aim of connecting. Now \( a \) and \( b \) are two of the returns, and \( a \) is smaller than or equal to \( b \). Whenever we can transit from \( a \) to \( b \), or more precisely, from \( F_L^{-1}(a) \) to \( b \), directly via a transition, the old iterates after \( F_L^{-1}(a) \) and before \( b \) (which could form a single orbital segment if \( \{a, b\} \) is a cutting pair, or two orbital segments if \( \{a, b\} \) is, a connecting pair), including in particular those returns between \( a \) and \( b \) in \( X \), would be irrelevant to our aim of connecting. The farther \( a \) and \( b \) are apart in \( X \), the better the transition would be, whatever \( \{a, b\} \) is a connecting or cutting pair. This is why when a pair covers some other pairs in the selection process above, we simply drop those pairs from our considerations. This beautiful idea of Hayashi cutting will be used throughout the proof of Theorem 2.2.

This finishes our observation in \( H_2 \), and we go on to \( H_3 \), etc. After each stage, we obtain a unique candidate connecting pair, together with some candidate cutting pairs such that the closed intervals determined by these pairs are mutually disjoint. We may visualize the ordered set \( X \) as a line, and draw a bridge across each of these closed intervals, whether connecting type or cutting type, as Figure 2 shows.

Then the rule can be formulated as follows. Whenever a new bridge (the solid line in the figure) appears, its two end points must not be on any of the old closed bridges because they are subtracted before we choose a new pair, and we drop the
whole open interval down the new bridge, all the old bridges down the new bridge in particular, from our considerations. Note that the new bridge may have both end points chosen from a box $H_i$, but covers some points of $X$ that are in some other boxes $H_j$, because the order in $X$ does not reflect the location in $V_0$. We must drop these points as well. This is important to keep bridges mutually disjoint.

This process terminates, because $X$ is finite. This finishes Step 1 and gives us a connecting pair $\{\xi, \zeta\}$, which is a candidate for $\{x, y\}$, and some cutting pairs $\{u_i, v_i\}$, say, $l$ of them, which are candidates for $\{p_i, q_i\}$, ordered as $x_1 \leq u_1 \leq v_1 < \cdots < u_l \leq v_l < \xi < \zeta < u_{l+1} \leq v_{l+1} < \cdots < u_l \leq v_l \leq y_1$.

Note that the index $i$ of $\{u_i, v_i\}$ is determined by the order of $X$, and not the order in which $\{u_i, v_i\}$ was chosen in the above selection process. Moreover, some boxes $H_i$ may produce no new pairs. Thus the box from which $\{u_i, v_i\}$ was chosen may not be $H_i$ at all. Let us denote it as $H_i^*$, and denote $D_i = 2H_i^*$.

To keep the notations uniform, we use also $\{u_0, v_0\}$ to denote $\{\xi, \zeta\}$. The following four properties are clear.

$D^{*1}$) $u_i, v_i \in H_i^* \subset D_i^*$, $0 \leq i \leq l$.

$D^{*2})$ $D_i^* \subset B(x_s, \sigma|x_s - y_l|)$, $0 \leq i \leq l$.

$D^{*3}$) Each $D_i^*$ intersects at most $N^*$ of the other $D_j^*$.

$D^{*4}$) $(X - [\xi, \zeta] - [u_1, v_1] - \cdots - [u_l, v_l]) \cap \text{int}(3H_0) = \emptyset$.

Properties $D^{*1}$, $D^{*2}$, and $D^{*3}$ are obvious by construction. Property $D^{*4}$ holds because int$(3H_0)$ is the union of all $H_i$, $i = 0, 1, \cdots$, and hence if the intersection were not empty, the selection process would still continue, and produce more new pairs.

**Step 2.** Basic balls and jumbo balls in $V_0$.

We define the so-called jumbo balls in $V_n$ for every $n$, which will be used to form the $\varepsilon$-kernel transitions required by Theorem 2.2. First we look at $V_0$. The other $V_n$ will be treated in the same way in Step 4 at the end of the proof of Theorem 2.2. Let

$$Q_i^* = \frac{1}{\alpha}H_i^*, \quad 0 \leq i \leq l,$$

where $\alpha$ is the number determined in the beginning of the proof of Theorem 2.2, which is less than but very close to one. Then

$$u_i, v_i \in \alpha Q_i^*, \quad 0 \leq i \leq l.$$
The box $Q_i^*$ is between $D_i^*$ and $H_i^*$. It is only a little bit larger than $H_i^*$. More precisely, $H_i^* = \alpha Q_i^*$. Since the type box $A$ has been chosen according to $\varepsilon$ and this $\alpha$, Theorem 3.1 applies by treating $Q_i^*$ as $Q$ and $H_i^*$ as $\alpha Q$. That is, for each $i = 0, 1, \cdots, l$, there is an $\varepsilon$-kernel transition

$$c_{i0}, c_{i1}, \cdots, c_{iL}; \quad B_{i0}, B_{i1}, \cdots, B_{iL-1},$$

from $u_i$ to $v_i$, contained in $Q_i^*$, where $L$ is the number determined at the beginning of the proof of Theorem 2.2. Since the gap between $Q_i^*$ and $H_i^*$ is very narrow, the balls $B_{in}$ are very small relative to $F_n^{-1}(D_i^*)$, and the precise ratio is given by the inequality that defines the number $\alpha$, whose geometrical meaning will become clear shortly. This will be crucial to what follows. Recall that $u_0 = \xi$, $v_0 = \zeta$.

Note that the first two conditions in Theorem 2.2 are satisfied if we treat $u_i$, $v_i$ as $p_i$, $q_i$. The third condition is also satisfied because of $D^4$). Another condition in Theorem 2.2 about the locations of $x$, $y$, $p_i$ and $q_i$ is guaranteed by $D^2$). The only problem now left is that the fourth condition in Theorem 2.2 is not satisfied. Clearly, these transitions are not necessarily disjoint. In fact they are not the transitions we want. The transitions we want will use the so-called jumbo balls defined below. The key to this will be the fact that the universal number $N^*$ bounds all the multiplicities of overlaps of $D_i^*$.

We first define jumbo balls in $V_0$. We will define jumbo balls in other $V_n$ later. There are $l + 1$ balls

$$B_{00}, B_{10}, \cdots, B_{l0}$$

in $3H_0 \subset V_0$. Let us call them basic balls in $V_0$. Each basic ball $B_{i0}$ contains two interesting points $c_{i0}$ and $T_1(c_{i1})$ in its $\varepsilon$-kernel. Let us call these $l + 1$ pair of points $c_{i0}, T_i(c_{i1}), 0 \leq i \leq l$, basic points in $V_0$. Recall that $c_{i0} = v_i$.

Let $b_i$ and $r_i$ be the center and the radius of $B_{i0}$, respectively, $0 \leq i \leq l$. Consider the $2N^* + 3$ balls

$$B(b_i, (3/\varepsilon)^nr_i), \quad 1 \leq n \leq 2N^* + 3,$$

of the same center $b_i$. The largest of them is still in $D_i^*$, by Theorem 3.1 and the choice of $\alpha$. Indeed, the inequality that defines $\alpha$ just means geometrically that the gap between $H_i^*$ and $1/\alpha H_i^*$ should be so narrow that by enlarging $2N^* + 3$ times a ball $B$, which is contained in $1/\alpha H_i^*$ and is small enough that it can be inserted via a parallel translation into the gap, each time by a factor $3/\varepsilon$, we can never get out of $2H_i^*$. Then there is one of these $2N^* + 3$ balls, call it $\beta_i$, such that $\beta_i - (\frac{2}{3})\beta_i$ does not contain any basic points. This is because each $D_i^*$ contains at most $2N^* + 2$ basic points. In fact, each $D_i^*$ intersects at most $N^*$ of the other $D_j^*$ by property $D^3$), and $c_{j0}, T_j(c_{j1}) \in \varepsilon B_{j0} \subset Q_j^* \subset D_j^*$. Let

$$J_i = \beta_i / 3.$$

We will call $J_i$ the prejumbo ball in $V_0$ associated with $B_{i0}$. Thus each basic ball $B_{i0}$ gives rise to a prejumbo ball $J_i, i = 0, 1, \cdots, l$. Of course $J_i$ is contained in $D_i^*$, and hence in $\text{int}(3H_0)$.

Claim 1. Two prejumbo balls either are disjoint, or the $\varepsilon$-kernel of one contains all the basic points contained in the other.

In fact, Let $J_i$ and $J_j$ be two prejumbo balls. Without loss of generality we assume that the radius of $J_i$ is less than or equal to the radius of $J_j$. If $J_i \cap J_j \neq \emptyset$,
then \( J_i \subset \beta_j \). So all basic points in \( J_i \) are in \( \beta_j \), hence are in \( \varepsilon J_j = \frac{1}{3} \beta_j \), since \( \beta_j - \frac{1}{3} \beta_j \) contains no basic points. This proves the claim.

Let us call a collection \( \mathcal{C} \) of prejumbo balls regular, if for every \( i = 0, 1, \cdots, l \) there is a prejumbo ball \( J(i) \) in \( \mathcal{C} \) such that the pair of basic points \( c_{i0} \) and \( T_1(c_{i1}) \) is contained in \( \varepsilon J(i) \). For instance, the whole set of prejumbo balls \( J_0, J_1, \cdots, J_l \) is regular, because every such pair of basic points are contained in the \( \varepsilon \)-kernel of some basic ball, which in turn is contained in the \( \varepsilon \)-kernel of some prejumbo ball.

**Claim 2.** There is a regular collection of prejumbo balls

\[
J_{i_1}, J_{i_2}, \cdots, J_{i_d}
\]

which are mutually disjoint.

In fact, if all prejumbo balls are mutually disjoint, we are done. Otherwise there is a prejumbo ball, say \( J_i \), such that all basic points contained in \( J_i \) are contained in the \( \varepsilon \)-kernel of some other prejumbo ball, by Claim 1. Then we drop \( J_i \) from the collection. (Note that at this stage we do not drop any other prejumbo balls even if their basic points may be contained in \( J_i \). For instance, \( J_i \) and \( J_{i-1} \) might contain each other’s basic points, and do not intersect any other prejumbo balls. In this case we certainly do not want drop both of them.) Then the collection of the remaining prejumbo balls \( J_0, J_1, \cdots, J_{i-1} \) is still regular, and we go on to the next stage. In this way Claim 2 is proved by induction.

From now on we fix such a disjoint regular collection of prejumbo balls stated in Claim 2, and call them jumbo balls in \( V_0 \). Clearly, each pair of basic points \( c_{i0} \) and \( T_1(c_{i1}) \) in \( V_0 \) is contained in the \( \varepsilon \)-kernel of a unique jumbo ball in \( V_0 \). Note that all jumbo balls defined in \( V_0 \) are contained in \( \text{int}(3H_0) \).

**Step 3. A combining process.**

We now combine the transitions whose basic points in \( V_0 \) are contained in the same jumbo ball in \( V_0 \) into a single \( \varepsilon \)-kernel transition, and further adjust the candidates for the connecting and cutting pairs. Since this process will appear later for other \( V_n \) as well, we first describe it in a more general way. Let \([a_1, b_1], [a_2, b_2], \cdots, [a_k, b_k]\) be \( k \) disjoint closed intervals in \( X \), ordered as

\[x_1 \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k \leq y_1,\]

where \( X \) is the set in Theorem 2.2. Assume that for each \( i = 1, \cdots, k \) we have an \( \varepsilon \)-kernel transition

\[P_{i0}, P_{i1}, \cdots, P_{iL} ; \quad P_{i0}, P_{i1}, \cdots, P_{iL-1}\]

from \( a_i \) to \( b_i \) of length \( L \). Here \( P_{in} \) could be any ball in \( V_n \), not necessarily basic, nor jumbo (as mentioned before, we will define basic and jumbo balls in \( V_n \) for every \( n \) later). We allow such a generality here because later we will deal with transitions that use both basic balls and jumbo balls in a mixed way. After all, the balls used for transitions in Theorem 2.2 are not specified to be basic, nor jumbo.

Now let \( n \) be an integer with \( 0 \leq n \leq L - 1 \), and let \( J \) be a ball in \( V_n \), whose \( \varepsilon \)-kernel \( \varepsilon J \) contains the two points \( T_{n+1}(p_{1,n+1}) \) and \( p_{k,n} \). We can combine these transitions into a single transition as follows. We first use the old \( \varepsilon \)-kernel lifts of index \( i = 1 \) from \( L \) up to \( n + 1 \). When we get to the point \( T_{n+1}(p_{1,n+1}) \) in \( V_n \), instead of jumping onto the point \( p_{1n} \) within \( P_{1n} \), we jump onto the point \( p_{kn} \)
within $J$. Then we continue the rest of the old $\varepsilon$-kernel lift of index $i = k$. That is, we make a new $\varepsilon$-kernel transition

$$p_{k0}, \ldots, p_{k,n-1}, p_{k,n}, p_{k,n+1}, \ldots, p_{1,L}; \quad P_{k0}, \ldots, P_{k,n-1}, P_{k,n}, P_{k,n+1}, \ldots, P_{1,L-1}.$$  

This new transition is from $a_1$ to $b_k$, or more precisely, from $F^{-1}_L(a_1)$ to $b_k$. Then Hayashi's cutting idea applies. That is, we make a new pair $\{a_1, b_k\}$, and drop everything in the open interval $(a_1, b_k)$. This includes in particular all the old pairs in $(a_1, b_k)$, together with all their transitions. We call this process combining transitions and adjusting pairs via $\{a_1, b_k\}$.

Now we do this process in $V_0$. We start with the first jumbo ball $J_{i_1}$. Its $\varepsilon$-kernel $\varepsilon J_{i_1}$ contains some basic points, but $J_{i_1} - \varepsilon J_{i_1}$ contains no basic points. Let $s_1$ and $l_1$ be the smallest and the largest index of basic points contained in $\varepsilon J_{i_1}$. By the regularity, basic points in $\varepsilon J_{i_1}$ appear in pairs. In particular the two points $T_1(c_{s_1,1})$ and $c_{l_1,0}$ are in $\varepsilon J_{i_1}$. This gives a new $\varepsilon$-kernel lift that pushes $T_1(c_{s_1,1})$ onto $a_{l_1,0}$ within the jumbo ball $J_{i_1}$ in $V_0$. As described above, we make a new $\varepsilon$-kernel transition

$$c_{l_1,0}, c_{s_1,1}, c_{s_1,2}, \ldots, c_{s_1,L}; \quad J_{i_1}, B_{s_1,1}, B_{s_1,2}, \ldots, B_{s_1,L-1}.$$  

This new transition is from $u_{s_1}$ to $v_{l_1}$ (more precisely, from $F^{-1}_L(u_{s_1})$ to $v_{l_1}$). It uses the original $\varepsilon$-kernel lifts in $V_0$, $n \geq 1$, but a new $\varepsilon$-kernel lift with a jumbo ball in $V_0$.

As described above, we simply take $\{u_{s_1}, v_{l_1}\}$ as a new candidate pair, and drop all pairs of index $i$ with $s_1 \leq i \leq l_1$, together with all their transitions, from our further considerations. This is just the combining and adjusting process described above caused by a jumbo ball in $V_0$. That is, we combine all the transitions associated with the pairs which are covered by the bridge of the new pair $\{u_{s_1}, v_{l_1}\}$, including the two transitions associated with the two pairs $\{u_{s_1}, v_{s_1}\}$ and $\{u_{l_1}, v_{l_1}\}$ themselves, into this single new transition. Note that in Step 1 we did not have transitions yet and what we dropped were pairs in $V_0$, while now we drop the transitions. But this is not really a difference because, when we dropped a pair $(a, b)$ in Step 1, we had ignored forever all possible transitions from $a$ to $b$. Thus what are left now after this combination are still some transitions, one of which uses a jumbo ball at $V_0$. Also note that, when we drop all pairs of index $i$ with $s_1 \leq i \leq l_1$ together with their transitions, we may have dropped at the same time some transitions associated with (i.e. whose basic points in $V_0$ are contained in) some other jumbo balls rather than just with $J_{i_1}$, because the inequality $s_1 \leq i \leq l_1$ reflects the order in $X$ but not the location in $V_0$. This is similar to the situation we had in Step 1, and is important to keep bridges mutually disjoint. Finally, note that this new pair $\{u_{s_1}, v_{l_1}\}$ could be of the connecting type, and becomes a better candidate for $\{x, y\}$. Or, it could be of the cutting type, and becomes a better candidate for a $\{p_i, q_i\}$. In either case, we still have a unique candidate connecting pair and some candidate cutting pairs in $V_0$ with disjoint bridges, together with their transitions, one of which is the combined one. We remark that this is still subject to change.

Then we deal with the transitions that are left after the combination, and look at the second jumbo ball $J_{i_2}$ in $V_0$. In the same way, we combine the transitions whose basic points in $V_0$ are contained in $J_{i_2}$ into a new transition. This gives a new pair, or what is the same, a new bridge in $X$, which either is disjoint from the old (here the word “old” means the ones that are just left after the last combination) bridges, or wholly covers some old bridges. Then we adjust the pairs in $X$ as before.
We go on dealing with the transitions that are left and look at the third jumbo ball \( J_{3} \), etc. In this way we will end up with a collection of disjoint bridges in \( X \), together with their \( \varepsilon \)-kernel transitions of length \( L \), each using one jumbo ball at \( V_{0} \). The number of transitions that are left could be equal to \( d \), which is the number of jumbo balls in \( V_{0} \), or less than \( d \), because when we deal with \( J_{i} \), say, we may have dropped some transitions associated with some other jumbo balls as well, as noticed above. This finishes our work in \( V_{0} \).

**Step 4.** The other \( V_{n} \) with \( n \geq 1 \). The final selection of pairs.

Now we treat \( V_{1} \). By basic ball and basic points in \( V_{1} \) we mean the balls \( B_{3i} \) and the points \( c_{3i}, T_{2}(c_{2}) \), where the indices \( i \) are rearranged ones which run from one to however many transitions are left. These basic balls are contained in the parallelepiped \( F_{i}^{-1}(3H_{0}) \). In the same way, we define prejumbo balls and jumbo balls in \( V_{1} \), then combine all the transitions (each has exactly one jumbo ball at \( V_{0} \)) that are associated with (i.e. whose basic points in \( V_{1} \) are contained in) a jumbo ball in \( V_{1} \) into a single new transition, and adjust the pairs in \( V_{0} \) accordingly, as we did in \( V_{0} \). The only difference is that the boxes \( D_{i}^{*}, Q_{i}^{*} \) and \( H_{i}^{*} \) in \( V_{0} \) become parallelepipeds \( F_{i}^{-1}(D_{i}^{*}), F_{i}^{-1}(Q_{i}^{*}) \), and \( F_{i}^{-1}(H_{i}^{*}) \) in \( V_{1} \), respectively. But the parallelepipeds \( F_{i}^{-1}(D_{i}^{*}) \) still overlap with multiplicity no more than \( N^{*} \) as the boxes \( D_{i}^{*} \) in \( V_{0} \) do. Hence each \( F_{i}^{-1}(D_{i}^{*}) \) contains at most \( 2N^{*} + 2 \) basic points, since \( c_{3i}, T_{2}(c_{2}) \in F_{i}^{-1}(Q_{j}^{*}) \subset F_{i}^{-1}(D_{j}^{*}) \). Moreover, when we consider the \( 2N^{*} + 3 \) concentric balls surrounding a basic ball \( B_{3i} \), which is contained in \( F_{i}^{-1}(Q_{j}^{*}) \) and is small enough so that it can be inserted into the gap between \( F_{i}^{-1}(Q_{j}^{*}) \) and \( F_{i}^{-1}(H_{i}^{*}) \) by Theorem 3.1, the largest one is still in \( F_{i}^{-1}(D_{j}^{*}) \) by the choice of \( \alpha \). Thus all the arguments still go through, and we get a unique connecting pair and some cutting pairs in \( V_{0} \), together with their \( \varepsilon \)-kernel transitions of length \( L \), each of which uses two jumbo balls at \( V_{0} \) and \( V_{1} \).

Inductively, we treat \( V_{2}, V_{3}, \ldots, V_{L} \) the same way. This eventually gives us a unique connecting pair \( x, y \) and some cutting pairs \( p_{i}, q_{i} \) in \( V_{0} \) such that the balls used for \( \varepsilon \)-kernel transitions are all jumbo balls. All these transitions are contained in \( 3H_{0} \), and hence in \( B(x_{s}, \sigma|x_{s} - y|) \). The four conditions of Theorem 2.2 are easily checked. In fact, the first two conditions are obvious. The third condition (the avoidance condition) is satisfied because, by the way those combinations are done in Step 3, the avoidance set \( X - [x, y] - [p_{1}, q_{1}] - \cdots - [p_{k}, q_{k}] \) is a subset of the avoidance set \( X - [\xi, \zeta] - [u_{1}, v_{1}] - \cdots - [u_{n}, v_{n}] \), which is outside \( \text{int}(3H_{0}) \), and because all jumbo balls are contained in some pull-back of \( \text{int}(3H_{0}) \). The fourth condition is satisfied because, being jumbo balls, those balls are mutually disjoint. Thus all conditions of Theorem 2.2 are satisfied, and Theorem 2.2 is proved.

## 6. An illustration for the linearization process

Via a linearization process, Theorem E reduces to Theorem 2.2. This linearization process is rather standard, and the details can be found, for instance, in [14]. For convenience we give in this section a brief illustration of this process. Thus we are in a position of having the assumptions of Theorem E, and trying to prove Theorem E by using Theorem 2.2.

Let \( \mathcal{U} \) be any \( C^{1} \) neighborhood of \( f \) in \( \text{Diff}^{1}(M) \). First we take a number \( \eta > 0 \) such that the \( \eta \)-ball of \( f \) in \( \text{Diff}^{1}(M) \) is contained in \( \mathcal{U} \). Then we take two numbers
r > 0 and ε > 0 such that hg is within the η/2-ball of g for any g that is within the
1-ball of f in Diff^1(M), where h is any ε-kernel lift supported on a ball of radius r
as defined in Lemma 2.1.

Since z is not periodic of f, all terms in the negative orbit of z are hence distinct.
Treating the tangent spaces \( T_{f^{-n}(z)} M \) as \( V_n \), and the tangent maps \( T_{f^{-n}(z)} f \) as \( T_n \),
we get a sequence of isomorphisms \( \{ T_n \} \). Applying Theorem 2.2 to the sequence
\( \{ T_n \} \) and the number ε determined above, we get two numbers \( \sigma > 1 \) and \( L \in \mathbb{N} \).
Since \( L \) is specified now, we can take a small number \( \delta_0 > 0 \) such that the ball
\( B(z, \delta_0) \) and its negative iterates \( f^{-n} B(z, \delta_0) \), \( 0 \leq n \leq L \), are mutually disjoint,
and are all of radius less than \( r \). Actually \( \delta_0 \) may have to be smaller to meet another
requirement of some linearization process below. For \( 0 < a \leq \delta_0 \), let us denote by
\( \Delta(a) \) the tube \( \bigcup_{n=1}^{L} f^{-n} B(z,a) \).

Let \( 0 < \delta \leq \delta_0 \) be given. As a preliminary \( C^1 \) perturbation we take a linearization
of \( f \), which is a diffeomorphism that agrees with \( f \) off the tube \( \Delta(\delta) \), and agrees
with the tangent maps \( T_{f^{-n}(z)} f \) on the thinner tube \( \Delta(\delta/2) \). Here we identify a
neighborhood of the iterates \( f^{-n}(z) \) in the manifold with a neighborhood of the
origin in the tangent space \( T_{f^{-n}(z)} M \), via the exponential map. Also, as noticed
above, here we may assume that \( \delta_0 \) has been chosen so small that the linearization
along the \( \delta \)-tube \( \Delta(\delta) \) of length \( L \) is within the \( \eta/2 \)-ball of \( f \). To keep the behavior
of some orbits unchanged we need to cancel out the change brought in by this
linearization. See [14] for details. For simplicity we still denote this linearization
by \( f \), and let \( \rho = 2\sigma \).

Now let \( 0 \leq \delta \leq \delta_0 \) be given, and let \( p \) and \( q \) be two points outside the tube \( \Delta(\delta) \)
such that the positive orbit of \( p \) hits the ball \( B(z, \delta/\rho) \) after \( p \), and the negative
orbit of \( q \) hits the ball \( B(z, \delta/\rho) \) too, at two points \( f^{\alpha}(p) \) and \( f^{\beta}(q) \), respectively.
Collect the iterates of \( p \) from \( f(p) \) up to \( f^{\alpha}(p) \) that are in the ball \( B(z, \delta/2) \) as
\( x_1, x_2, \ldots, x_s \), and the iterates of \( q \) from \( q \) up to \( f^{\beta}(q) \) that are in the same ball as
\( y_1, y_2, \ldots, y_t \). Thus \( x_s \) and \( y_t \) are both in \( B(z, (\delta/2)/\sigma) \). Clearly, the point \( f^{s-L}(x_1) \)
is still on the positive orbit of \( p \). (This is why we require that the positive orbit
hits the ball \( B(z, \delta/\rho) \) after \( p \), and why we have collected the iterates of \( p \) from
\( f(p) \) to \( f^{\alpha}(p) \) but not from \( p \) to \( f^{\alpha}(p) \) that are in the ball \( B(z, (\delta/2)/\sigma) \).) Then a direct
application of Theorem 2.2 gives Theorem E.

7. Some applications of the \( C^1 \) connecting lemma

In this section we are concerned with applications of the \( C^1 \) connecting lemma.
First we note that it directly implies the celebrated \( C^1 \) closing lemma. In fact, the
non-wandering point \( z \) in the assumption of the \( C^1 \) closing lemma can be assumed
to be non-periodic, for otherwise it is already closed. Since \( z \) is non-wandering,
there are a point \( p \in B(z, \delta/\sigma) \) and an integer \( n \geq 1 \) such that \( f^n(p) \in B(z, \delta/\sigma) \).
Applying Theorem F by treating \( p \) itself as \( q \) yields a periodic orbit of \( g \). Another
perturbation takes this periodic orbit through \( z \). This is because the proofs for
the \( C^1 \) connecting lemma and the \( C^1 \) closing lemma both are in an essential way
based on the basic \( C^1 \) perturbation theorem, Theorem 3.1, and the main difference
is just that the former contains more complicated combinatorial considerations of
avoidance discussed above.

Now we prove Theorems A–D. This is fairly straightforward. We only take
Theorems B and D as examples, as they are more general than Theorems A and
C, respectively.
Proof of Theorem B. Take \( z_i \in \omega(p_i) \cap \alpha(p_{i+1}) \) such that \( z_i \) is not periodic. We treat the case that the \( 2n \) orbits \( \text{Orb}(p_i) \) and \( \text{Orb}(z_i) \) are distinct and apply Theorem E. The other cases can be treated similarly. (For instance, if \( z_i \) and \( z_j \), or \( p_i \) and \( p_j \), share the same orbit, then \( p \) is actually a reduced order prolongationally recurrent. Also, if \( z_i \) and \( p_j \) share the same orbit, then the special connecting lemma, Theorem F, applies too.)

Given any \( C^1 \) neighborhood \( \mathcal{U} \) of \( f \), by Theorem E, for each \( z_i \), there are three numbers \( p_i, L_i, \delta_0, \delta_i \) that satisfy the conditions of Theorem E. Denote \( \rho = \max\{p_i\}, L = \max\{L_i\} \), and \( \delta_0 = \min\{\delta_0, \delta_i\} \). We may assume that \( \delta_0 \) has been taken small enough so that the \( n \) tubes \( \Delta_i(\delta_0) = \bigcup_{k=1}^{L_i} B(f^{-k}(z_i), \delta_0) \) are mutually disjoint, and that \( p_i \) and \( p_{i+1} \) are outside \( \Delta_i(\delta_0) \) for all \( i \). By the prolongational recurrence, for any \( 0 < \delta \leq \delta_0 \), the positive orbit of \( p_i \) hits \( B(z_i, \delta/\rho) \) after \( p_i \), and the negative orbit of \( p_{i+1} \) hits \( B(z_i, \delta/\rho) \) too.

Now for each \( i = 1, \cdots, n \) we choose suitable \( \delta_i \) to get the desired tube \( \Delta_i(\delta_i) \). For \( i = 1 \), we simply take \( \delta_1 \) to be \( \delta_0 \) itself. Then take two integers \( a_1 \geq 1 \) and \( b_1 \geq 0 \) so that \( f^{a_1}(p_1) \) and \( f^{-b_1}(p_2) \) are both in \( B(z_1, \delta_1/\rho) \). Take \( 0 < \delta_2 \leq \delta_0 \) small enough so that the tube \( \Delta_2(\delta_2) \) is disjoint from the two finite orbits \( \{p_1, f(p_1), \cdots, f^{a_1}(p_1)\} \) and \( \{p_2, f^{-1}(p_2), \cdots, f^{-b_1}(p_2)\} \). Then take two integers \( a_2 \geq 1 \) and \( b_2 \geq 0 \) so that \( f^{a_2}(p_2) \) and \( f^{-b_2}(p_3) \) are both in \( B(z_2, \delta_2/\rho) \). Then take \( 0 < \delta_3 \leq \delta_0 \) small enough so that the tube \( \Delta_3(\delta_3) \) is disjoint from the previously chosen four finite orbits, and so on. After these \( n \) tubes \( \Delta_i(\delta_i) \) and the \( 2n \) finite orbits have been chosen, we start the connecting process as follows. First we apply Theorem E to the tube \( \Delta_n(\delta_n) \) by treating \( z_n \) as \( z \), \( p_n \) as \( p \), and \( p_1 \) as \( q \) to make \( p_1 \) on the positive orbit of \( p_n \) under a new diffeomorphism \( g_1 \). By construction this does not affect the other \( 2n-2 \) finite orbits, one of which takes \( p_n \) backwards to hit the ball \( B(z_{n-1}, \delta_{n-1}/\rho) \). Hence the negative \( g_1 \)-orbit of \( p_1 \) hits \( B(z_{n-1}, \delta_{n-1}/\rho) \). Then we apply Theorem E to the tube \( \Delta_{n-1}(\delta_{n-1}) \) by treating \( z_{n-1} \) as \( z \), \( p_{n-1} \) as \( p \), and, still, \( p_1 \) as \( q \), to make \( p_1 \) on the positive orbit of \( p_{n-1} \) for some \( g_2 \). This does not affect the rest of the \( 2n-4 \) finite orbits, and so on. This eventually makes \( p \) periodic.

Proof of Theorem D. We apply Theorem F, the special \( C^1 \) connecting lemma, twice as follows.

By assumption, there is a sequence of periodic orbits outside \( \Lambda \) that accumulate on \( \Lambda \). Since \( \Lambda \) is isolated, there is a neighborhood \( W \) of \( \Lambda \) such that any periodic orbit which is not in \( \Lambda \) cannot be entirely in \( W \). Then there must be \( z^* \in W^u(\Lambda) - \Lambda \) and \( z^* \in W^s(\Lambda) - \Lambda \) such that for any neighborhood \( U \) of \( z^* \) and any neighborhood \( V \) of \( z^* \), there are a point \( p \in U \) and an integer \( n \in \mathbb{N} \) such that \( f^n(p) \in V \). Note that \( z^* \) and \( z^* \) are not periodic points of \( f \). Also, we may assume \( z^* \) is not on the positive orbit of \( z^* \) under \( f \), as otherwise there is nothing to prove.

Let \( \mathcal{U} \) be any \( C^1 \) neighborhood of \( f \). By Theorem F, there are three numbers \( \rho^u, L^u, \) and \( \delta^u \) for \( z^* \) that satisfy the conditions of Theorem F. Also, there are three numbers \( \rho^s, L^s, \) and \( \delta^s \) for \( z^* \) that satisfy the conditions of Theorem F. Denote \( \rho = \max(\rho^u, \rho^s), L = \max(L^u, L^s), \) and \( \delta_0 = \min(\delta^u, \delta^s) \). We may assume that \( \delta_0 \) has been chosen small enough so that the tube \( \Delta^u = \bigcup_{i=1}^{L^u} B(f^{-i}(z^*), \delta_0) \) is disjoint from the tube \( \Delta^u = \bigcup_{i=1}^{L^u} B(f^{-i}(z^*), \delta_0) \), and \( \Delta^u \) is disjoint from \( \text{Orb}^+(z^*, f) \) and \( \Delta^u \) is disjoint from \( \text{Orb}^-(z^*, f) \). Now there is a point \( p \in B(z^*, \delta_0/\rho) \) and \( n \geq 1 \) such that \( f^n(p) \in B(z^*, \delta_0/\rho) \). Applying Theorem F to \( \Delta^* \) by treating \( z^* \) as \( q = z \) makes \( z^* \) on the positive orbit of \( p \) for some \( g_1 \). Let \( f^k(p) \) be the first \( f \)-iterate of \( p \) that is in the tube \( \Delta^* \). We emphasize that while this perturbation does many cuttings
(and \( \varepsilon \)-kernel transitions too) to the finite orbit \( \{ f^k(p), f^{k+1}(p), \ldots, f^n(p) \} \), it does not hurt the finite orbit \( \{ p, f(p), \ldots, f^k(p) \} \). Thus \( p \) is on the negative orbit of \( z^s \) for \( g_1 \). Then we apply Theorem F symmetrically, as mentioned in the remark after the statement of Theorem E in §1, to the tube \( \Delta^u \) by treating \( z^u \) as \( z = q \), \( z^s \) as \( p \). This makes \( z^u \) on the negative orbit of \( z^s \) for some \( g \), and proves Theorem D. \( \square \)

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