TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 1, Pages 391–410
S 0002-9947(00)02612-X
Article electronically published on September 13, 2000

ON MODULES OF FINITE UPPER RANK

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ABSTRACT. For a group G and a prime p, the upper p-rank of G is the supremum of the sectional p-ranks of all finite quotients of G. It is unknown whether, for a finitely generated group G, these numbers can be finite but unbounded as p ranges over all primes. The conjecture that this cannot happen if G is soluble is reduced to an analogous 'relative' conjecture about the upper p-ranks of a 'quasi-finitely-generated' module M for a soluble minimax group Γ. The main result establishes a special case of this relative conjecture, namely when the module M is finitely generated and the minimax group Γ is abelian-by-polycyclic. The proof depends on generalising results of Roseblade on group rings of polycyclic groups to group rings of soluble minimax groups. (If true in general, the above-stated conjecture would imply the truth of Lubotzky's 'Gap Conjecture' for subgroup growth, in the case of soluble groups; the Gap Conjecture is known to be false for non-soluble groups.)

INTRODUCTION

In recent years, there has been considerable progress in understanding the extent to which global structural properties of a finitely generated residually finite group are determined by properties of its finite quotients. For example, it was shown in [7] that such a group has polynomial subgroup growth if and only if it is a virtually soluble minimax group. In trying to understand the nature of groups whose subgroup growth is very slightly faster than polynomial, one is led to study an invariant called the *upper rank* of a group. The connection between subgroup growth and properties of the upper rank was indicated in [11]; the precise relationship is explained below, and established in detail in [17]. The question we are concerned with in this paper, however, relates purely to upper rank, and is of interest in itself.

A finite group F is said to have rank $r = \mathrm{rk}(F)$ if every subgroup of F can be generated by r elements, and r is minimal with this property. Now for any group G one defines the *upper rank* of G to be

$$\mathrm{ur}(G) = \sup \mathrm{rk}(\overline{G})$$

as $\overline{G}$ runs over all finite quotient groups of G; in other words, $\mathrm{ur}(G) \in \mathbb{N} \cup \{\infty\}$ is the rank of the profinite completion $\widehat{G}$ of G ([3], Chapter 1). Similarly, for a prime p we define the *p-rank* $\mathrm{r}_p(F)$ of a finite group F to be the rank of a Sylow p-subgroup of F, and set

$$\mathrm{ur}_p(G) = \sup \left\{ \mathrm{r}_p(\overline{G}) \mid \overline{G} \text{ a finite quotient of } G \right\};$$

Received by the editors March 3, 1999 and, in revised form, June 25, 1999.
2000 *Mathematics Subject Classification.* Primary 20C07, 20F16, 20E07.

this is the *upper p-rank* of G, and is equal to the rank of a Sylow pro-p-subgroup of $\widehat{G}$.

A theorem of Lucchini [9] shows that

$$\mathrm{rk}(F) \leq 1 + \max_p \mathrm{r}_p(F)$$

for every finite group F (this was originally proved for soluble F by Kovács [5]). It follows that for any group G we have

$$\mathrm{ur}(G) \leq 1 + \sup_p \mathrm{ur}_p(G);$$

thus a group has finite upper rank if and only if its upper p-ranks are *bounded* as p ranges over all primes. It is a striking fact that no example is known of a finitely generated group of infinite upper rank all of whose upper p-ranks are finite; so the question arises: can these upper p-ranks be finite for all p, and yet unbounded?

A surprising application of the Odd Order Theorem, discovered by Lubotzky and Mann [6], shows that any group G such that $\mathrm{ur}_2(G)$ is finite has a subgroup G_0 of finite index all of whose finite quotients are *soluble*. As $\mathrm{ur}_p(G_0)$ differs from $\mathrm{ur}_p(G)$ for only finitely many primes p, we see that our question is essentially one about groups all of whose finite quotients are soluble: in other words, those whose profinite completions are *prosoluble*. It is not hard to construct a 2-generator prosoluble group with finite Sylow subgroups of unbounded ranks; whether such a group can be the profinite completion of a finitely generated abstract group seems to be a delicate problem. As a first step it is reasonable to consider the case of groups which are *soluble*, and the aim of this paper is to present evidence in favour of

Conjecture A. *Let G be a finitely generated soluble group. If $\mathrm{ur}_p(G)$ is finite for every prime p, then G has finite upper rank.*

If true, this would generalise a theorem of D. J. S. Robinson [14] which says that a finitely generated soluble group with finite sectional p-rank for every prime p (an $\mathfrak{S}_0$ group) must have finite rank. It is shown in [17] that the conjecture is true for groups G which are *virtually residually nilpotent* (even without assuming solubility). It may be that in fact *every* residually finite finitely generated soluble group is virtually residually nilpotent: if this can be proved, it will thus confirm Conjecture A. However the proposition is highly speculative, and while unaware of a counterexample I see little evidence for its truth. We must therefore tackle the conjecture directly.

As both hypothesis and conclusion depend only on the finite quotients of G, we may as well assume that G in the conjecture is a residually finite group. In that case, G having finite upper rank is equivalent to G having finite rank, in which case it is a *minimax* group [10, Theorem A]. Let us recall that an *abelian* group A is minimax if A has a finitely generated subgroup A_0 such that A/A_0 satisfies the minimal condition (it is always possible to choose A_0 so that A/A_0 is a direct product of finitely many quasi-cyclic groups (groups of type C_{p^∞})). A soluble group G is minimax if there is a finite chain

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that G_i/G_{i-1} is an abelian minimax group for $i = 1, \ldots, n$.

Suppose now that G is a finitely generated, residually finite soluble group of derived length $\ell > 1$, and that $\mathrm{ur}_p(G)$ is finite for every prime p. Let A be a

maximal abelian normal subgroup of G containing the last non-trivial term of the derived series of G. Then G/A is again residually finite, by an elementary lemma, and the derived length of G/A is $\ell - 1$. Assuming inductively that the conjecture is true for groups of derived length less than ℓ, we infer that G/A is a minimax group. The conjugation action of G/A on A makes A into a module for the group ring $\mathbb{Z}(G/A) = R$, say. We define

$$\mathrm{ur}_p(A_R) = \sup\{\dim_{\mathbb{F}_p}(\overline{A})\},$$

where $\overline{A}$ runs over all finite R-module images of A/Ap, and

$$\mathrm{ur}(A_R) = \sup_p \mathrm{ur}_p(A_R).$$

It is easy to see that $\mathrm{ur}(G) \leq \mathrm{ur}(G/A) + \mathrm{ur}(A_R)$; on the other hand, if B is an R-submodule of finite index in A then G/B is residually finite, because the class of residually finite minimax groups is closed under extensions [13, §9.3], and this implies that $\mathrm{ur}_p(A_R) \leq \mathrm{ur}_p(G)$ for each prime p. To establish the conjecture, then, it would suffice to show that if $\mathrm{ur}_p(A_R)$ is finite for every prime p, then $\mathrm{ur}_p(A_R)$ is uniformly bounded over all p. Thus the conjecture is reduced to a problem about modules for the group ring of a finitely generated minimax group.

Let Γ be a minimax group, and M a $\mathbb{Z}\Gamma$-module. I shall write $\mathrm{ur}_p(M)$ for $\mathrm{ur}_p(M_{\mathbb{Z}\Gamma})$. In general, it is perfectly possible for the upper p-ranks $\mathrm{ur}_p(M)$ to be finite but unbounded as p ranges over all primes: for example M could be the direct sum, over all p, of vector spaces $(\mathbb{F}_p)^p$, with trivial Γ-action. However, this example cannot occur as A_R, above; for A_R satisfies an additional finiteness condition. Let us say that the $\mathbb{Z}\Gamma$-module M is *quasi-finitely generated* if there exists a finitely generated group which is an extension of M by Γ. Conjecture A is thus equivalent to

Conjecture B. *Let Γ be a finitely generated minimax group and M a quasi-finitely generated $\mathbb{Z}\Gamma$-module. If $\mathrm{ur}_p(M)$ is finite for every prime p, then M has finite upper rank.*

In order to establish a result like Conjecture B, one needs a technique for producing finite images of a module like M: supposing that M has infinite upper rank, the technique should enable us to find finite images of large rank with a given prime exponent. Now just such a technique was invented by P. Hall in the 1950s, and greatly developed by J. E. Roseblade in the 1970s, in their study of modules over polycyclic group rings. The ingredients are a group Γ, an abelian normal subgroup G of Γ, and a $\mathbb{Z}\Gamma$-module M, and the idea is to find out as much as possible about the structure of M as a $\mathbb{Z}G$-module, using the structure of $\mathbb{Z}G$ as a commutative ring with Γ as a group of operators.

In the 'classical' case, where Γ is a polycyclic group, G is a finitely generated abelian group and so $\mathbb{Z}G$ is a finitely generated commutative ring; thus the whole arsenal of 'Noetherian' commutative algebra can be brought to bear. This is no longer available when G is merely an abelian minimax group. To replace it, I have developed in [16] the beginnings of a structure theory for modules over the group ring of an abelian minimax group. Although more restricted than the 'classical' theory, it goes far enough to be useful in the following investigation.

The main argument of Roseblade's paper [15] hinged on two crucial theorems, 'Theorem C' and 'Theorem E'. The first of these, generalising Hall's work, describes the kG-module structure of a finitely generated $k\Gamma$-module, when Γ is a polycyclic

group containing an abelian normal subgroup G, and k is a commutative Noetherian ring. This was generalised by K. A. Brown in [2], who obtained essentially the same result assuming merely that Γ/G is polycyclic, G being now an arbitrary abelian normal subgroup of Γ; a self-contained proof of Brown's theorem is given in Section 1 below.

Roseblade's 'Theorem E' concerns the behaviour of maximal ideals in kG under the action of Γ, when G is a finitely generated free abelian group, k is a finite field and Γ is a polycyclic group acting on G. As the Jacobson radical of kG is zero, it is clear that for each non-zero element $\lambda \in kG$ there exists a maximal ideal of kG not containing λ; what Theorem E shows is that (under certain conditions) there exists a maximal ideal of kG not containing λ^γ for any $\gamma \in \Gamma$. The key technical result of this paper is the following generalisation, proved in Section 2:

Theorem 2.1. *Let G be a reduced abelian minimax group, Γ a soluble group acting on G, k a finitely generated commutative ring, and P a Γ-invariant faithful regular prime ideal of kG. Let $\lambda \in kG \setminus P$. Then there exists a maximal ideal L of finite index in kG, with $P \subseteq L$, such that $\lambda^\gamma \notin L$ for all $\gamma \in \Gamma$.*

(Here, G is *reduced* if its torsion subgroup is finite, and to say that P is *regular* means that G has no section of type C_{p^∞} for $p = \mathrm{char}(kG/P)$.) While modelled on Roseblade's original argument, the proof depends heavily on the results of [16] and on work of C. J. B. Brookes [1].

The tools developed in the first two sections and in [16] are put to work in Section 3, which is devoted to proving a special case of Conjecture B (slightly generalised by allowing an arbitrary finitely generated commutative ring k to take the place of $\mathbb{Z}$):

Theorem 3.1. *Let Γ be a minimax group which is abelian-by-polycyclic, and let M be a finitely generated $k\Gamma$-module. If $\mathrm{ur}_p(M_{k\Gamma})$ is finite for every prime p, then $\mathrm{ur}(M_{k\Gamma})$is finite.*

To prove Conjecture B in full will require further work: (1) the hypothesis on Γ must be relaxed, and (2) the condition that M be finitely generated must be weakened to M being quasi-finitely generated. Nevertheless, it is my feeling that the techniques introduced here to establish this special case will prove essential to further progress on this problem. They may also have a significant role to play in other investigations concerning the structure of finitely generated soluble groups.

As regards the application to Conjecture A, we can at least deduce from Theorem 3.1 the following corollary. While rather limited in application, it does suggest that a counterexample to the conjecture may be quite hard to come by.

Corollary. *The statement of Conjecture A holds for every finitely generated group G which is the semi-direct product of an abelian group by an abelian-by-polycyclic minimax group.*

To conclude this introduction, let me briefly explain the connection with problems of *subgroup growth*. Let G be a finitely generated residually finite group. For each n, the number $\mathrm{s}_n(G)$ of subgroups of index at most n in G is finite. G is said to have *polynomial subgroup growth* (PSG) if $\mathrm{s}_n(G)$ is bounded above by some fixed power of n as $n \to \infty$; it was proved in [7] that this holds if and only if G is virtually a soluble minimax group, which is equivalent to G having finite upper rank ([10], Theorem A). In general, one says that G has *subgroup growth of type*

$\leq f$, for a function f, if

$$\log \mathrm{s}_n(G) = O(\log f(n)).$$

If in fact $\log \mathrm{s}_n(G) = o(\log f(n))$, we say that G has subgroup growth of type *strictly less than* f; and if the type is $\leq f$ but not strictly less than f, then the type is f. (Thus PSG means subgroup growth of type $\leq n$, for example.) Finitely generated groups with subgroup growth of type

$$n^{\frac{\log n}{(\log\log n)^2}}$$

were constructed in [8]. But until very recently, no finitely generated group was known whose subgroup growth is of a type strictly intermediate between n and $n^{\log n/(\log\log n)^2}$. Thus the question arose of whether there is a *gap* in the subgroup growth types of finitely generated groups: indeed, Lubotzky, Pyber and Shalev conjectured in [8] that $n^{\log n/(\log\log n)^2}$ is the minimal type of subgroup growth for finitely generated groups not having PSG. Some evidence for this is provided by the following, proved in [17]:

Proposition A. *Let G be a finitely generated group which is virtually residually nilpotent. If G has subgroup growth of type strictly less than $n^{\log n/\log\log n}$, then G has PSG.*

Just before the end of the millennium, however, I found a way to construct finitely generated groups of arbitrarily slow non-polynomial subgroup growth [18]: these groups are very far from soluble, having in fact no abelian upper chief factors. The 'gap problem' for soluble groups (and, more generally, for groups all of whose finite quotients are soluble) therefore remains a challenge. Let us consider the case of soluble (not necessarily residually nilpotent) groups. The theorem of [10] mentioned above also has a converse, which says the following: *if G has PSG, then there is a finite upper bound for the ranks of all finite soluble quotients of G.* What the proof actually shows is that for a soluble group, 'slow' subgroup growth entails a bound on the upper p-rank of the group, for each prime p. If 'slow' means 'polynomial', the bound obtained is uniform in p. Under a weaker hypothesis, this no longer holds, and what the argument gives is

Lemma B ([17])**.** *Let G be a group such that*

$$(*) \qquad \mathrm{s}_n(G) \leq n^{\frac{c\log n}{(\log\log n)^2}}$$

for all $n \geq n_0$, where $c > 0$. Then $\mathrm{r}_p(\overline{G}) \leq \max\{703, \log n_0\}$ for every finite soluble quotient $\overline{G}$ of G and every prime $p \leq 2^{1/20c}$.

(Here log means logarithm to base 2). Now suppose that G has subgroup growth of type strictly less than $n^{\log n/(\log\log n)^2}$. Given any prime p, we can then find n_0 such that $(*)$ holds with $c = (20\log p)^{-1}$ for all $n \geq n_0$, and so infer

Proposition C. *Let G be a finitely generated soluble group, having subgroup growth of type strictly less than $n^{\log n/(\log\log n)^2}$. Then $\mathrm{ur}_p(G)$ is finite for every prime p. Hence if Conjecture A is true, then G has finite upper rank, and so has PSG.*

Thus Conjecture A implies the 'gap conjecture' of [8] for the special case of soluble groups. In fact a more elaborate argument, given in [17], shows that if Conjecture A is true then the slowest non-polynomial subgroup growth for any finitely generated soluble group is of type at least $n^{\log n}$.

Acknowledgement. I must thank the painstaking anonymous referee of an earlier version of this paper for the detection and correction of several mathematical errors.

1. Brown's Nullstellensatz

Throughout this section, Γ will denote a group, S a ring and $R = S * \Gamma$ a crossed product of S with Γ; that is,

$$R = \bigoplus_{\gamma \in \Gamma} S\overline{\gamma},$$

where each $\overline{\gamma}$ is a unit of R normalising S, $\overline{1} = 1_R$ and $S\overline{\alpha}\overline{\beta} = S\overline{\alpha\beta}$ for α, $\beta \in \Gamma$. For $s \in S$ and $\gamma \in \Gamma$, I shall write $s^\gamma = \overline{\gamma}^{-1} s \overline{\gamma}$.

We consider an R-module M generated by an S-submodule A; writing $A\gamma = A\overline{\gamma}$, we have $M = \sum_{\gamma \in \Gamma} A\gamma$. For any subset X of Γ, we put $AX = \sum_{\gamma \in X} A\gamma$.

We fix a well-ordering $<$ on (the underlying set of) Γ, and for $\beta \in \Gamma$ put $\Gamma_\beta = \{\alpha \in \Gamma \mid \alpha < \beta\}$, $\quad M_\beta = A\Gamma_\beta$. Writing β^+ for the successor of β, we thus have $\Gamma_{\beta^+} = \Gamma_\beta \cup \{\beta\}$ and $M_{\beta^+} = M_\beta + A\beta$.

Two S-modules U and V are said to be *conjugate* if there exist an additive isomorphism $\theta : U \to V$ and an element ξ in the subgroup of R^* generated by $\overline{\Gamma}$ such that $(us)\theta = u\theta s^\xi$ $(u \in U, s \in S)$.

Definition. Let $(X, \preceq)$ be a partially ordered set. If Y is a subset of X, a *base* for Y is a subset $Y^\flat$ of Y such that for each $y \in Y$ there exists $y^\flat \in Y^\flat$ with $y^\flat \preceq y$. We say that $(X, \preceq)$ has the *finite base property* if *every* subset of X has a *finite* base.

1.1. Lemma. *Suppose that Γ has a partial ordering $\preceq$ such that*

$$\alpha \preceq \beta \in \Gamma \Rightarrow \Gamma_\alpha \alpha^{-1} \subseteq \Gamma_\beta \beta^{-1}. \tag{1.1}$$

Let $Y \subseteq \Gamma$ and let $Y^\flat$ be a base for Y. Then for every $\beta \in Y$ there exists $\alpha \in Y^\flat$ such that the S-module M_{β^+}/M_β is conjugate to a quotient of M_{α^+}/M_α.

(Here and throughout, the notations Γ_α etc. refer to the *original* ordering $<$ on Γ.)

Proof. By definition, there exists $\alpha \in Y^\flat$ with $\alpha \preceq \beta$. Then $\Gamma_\alpha \alpha^{-1} \subseteq \Gamma_\beta \beta^{-1}$, and then $M_\alpha \leq M_\beta \beta^{-1}\alpha$. Consequently, $A\alpha/(A\alpha \cap M_\beta \beta^{-1}\alpha)$ is a quotient of $A\alpha/(A\alpha \cap M_\alpha) \cong M_{\alpha^+}/M_\alpha$. On the other hand,

$$\begin{aligned} A\alpha/(A\alpha \cap M_\beta \beta^{-1}\alpha) &\cong (M_\beta \beta^{-1}\alpha + A\alpha)/M_\beta \beta^{-1}\alpha \\ &= M_{\beta^+}\beta^{-1}\alpha / M_\beta \beta^{-1}\alpha, \end{aligned}$$

which is conjugate as an S-module to M_{β^+}/M_β. □

1.2. Corollary. *Assume that Γ has a partial ordering $\preceq$ with the finite base property and satisfying (1.1). Suppose that S is a commutative domain and that A is cyclic as an S-module. Then there exists a non-zero element λ of S such that for each $\beta \in \Gamma$, either M_{β^+}/M_β is $\langle \lambda^\Gamma \rangle$-torsion as S-module, or M_{β^+}/M_β is free of rank 1 as an S-module.*

(Note that when S is *commutative*, the formula $s^\gamma = \overline{\gamma}^{-1} s \overline{\gamma}$ defines an action of Γ on S. A module is $\langle \lambda^\Gamma \rangle$-*torsion* if every element is annihilated by some product $\lambda^{\gamma_1} \cdots \lambda^{\gamma_n}$ with $\gamma_1, \ldots, \gamma_n \in \Gamma$.)

Proof. For each $\gamma \in \Gamma$, the S-module M_{γ^+}/M_γ is cyclic, as it is isomorphic to $A\gamma/(A\gamma \cap M_\gamma)$. So if it is not a torsion module for S, then M_{γ^+}/M_γ is free of rank 1. Now let Y be the set of all $\gamma \in \Gamma$ such that M_{γ^+}/M_γ is a torsion module for S, and let $Y^\flat$ be a finite base for Y. As S is a domain and $Y^\flat$ is finite, there exists an element λ such that

$$0 \neq \lambda \in \bigcap_{\alpha \in Y^\flat} \operatorname{ann}_S(M_{\alpha^+}/M_\alpha).$$

Let $\beta \in Y$. Then 1.1 shows that M_{β^+}/M_β is conjugate to a quotient of M_{α^+}/M_α for some $\alpha \in Y^\flat$; it follows that M_{β^+}/M_β is $\langle \lambda^\Gamma \rangle$-torsion. □

1.3. Proposition. *Assume that Γ has a partial ordering $\preceq$ with the finite base property and satisfying (1.1). Suppose that S is a commutative domain, and that M is a finitely generated R-module. Then M contains a free S-submodule F such that M/F is $\langle \lambda^\Gamma \rangle$-torsion for some non-zero $\lambda \in S$.*

Proof. Let A be the S-submodule generated by a finite generating set for the R-module M. Suppose first that A is cyclic, and keep the notation of the preceding proof. Put

$$F = \sum_{\alpha \in \Gamma \setminus Y} A\alpha.$$

Then F is a free S-module, since for each $\alpha \in \Gamma \setminus Y$ the S-module $A\alpha/(A\alpha \cap M_\alpha)$ is both cyclic and torsion-free. Now suppose M/F is not $\langle \lambda^\Gamma \rangle$-torsion. Then there exists a minimal $\beta \in \Gamma$ (relative to $<$) such that $(M_{\beta^+} + F)/F$ is not $\langle \lambda^\Gamma \rangle$-torsion. If $\beta \notin Y$ then $M_{\beta^+} + F = M_\beta + A\beta + F = M_\beta + F$; while if $\beta \in Y$ then M_{β^+}/M_β is $\langle \lambda^\Gamma \rangle$-torsion, by the choice of λ. In either case it follows that $(M_{\beta^+} + F)/(M_\beta + F)$ is $\langle \lambda^\Gamma \rangle$-torsion; but so is $(M_\beta + F)/F$, by the minimal choice of β (note that if β is not a successor, then M_β is the union of the M_{γ^+} for $\gamma < \beta$). This implies that $(M_{\beta^+} + F)/F$ is $\langle \lambda^\Gamma \rangle$-torsion, a contradiction. Thus M/F is $\langle \lambda^\Gamma \rangle$-torsion as required.

In the general case, we have $A = B + C$, say, where C is a cyclic S-module and B needs fewer generators than A. Arguing by induction, we may suppose that the R-submodule BR of M contains a free S-submodule E_1 such that BR/E_1 is $\langle \lambda_1^\Gamma \rangle$-torsion for some non-zero element λ_1 of S. As M/BR is generated by its cyclic S-submodule $(C + BR)/BR$, the previous paragraph shows that M/BR contains a free S-submodule E_2/BR such that M/E_2 is $\langle \lambda_2^\Gamma \rangle$-torsion for some non-zero element λ_2. Let E_3 be an S-module complement to BR in E_2. We put $F = E_1 + E_3$ and $\lambda = \lambda_1\lambda_2$ to complete the proof. □

Of course, these observations only acquire some interest if we can produce examples of orderings $\preceq$ having the required properties. Let us say that a group Γ can be *doubly ordered* if Γ has a well-ordering $<$ and a partial ordering $\preceq$ such that $\preceq$ has the finite base property and such that condition (1.1) holds.

1.4. Lemma. *Let Γ be a group and Δ a normal subgroup of finite index. If Δ can be doubly ordered, then so can Γ.*

Proof. Suppose Δ has the well-ordering $<$ and a partial ordering $\preceq$. Let $\{t_1, \ldots, t_m\}$ be a transversal to the cosets of Δ in Γ. We extend $<$ to Γ by setting

$$\alpha t_i < b t_j \Leftrightarrow i < j \text{ or } (i = j) \& (\alpha < \beta),$$

and we extend $\preceq$ to Γ by setting

$$\alpha t_i \preceq b t_j \Leftrightarrow i = j \text{ and } \alpha \leq \beta.$$

It is easy to see that the finite base property is preserved: if $Y = \bigcup Y_i t_i$ with $Y_i \subseteq \Delta$ for each i, and $Y_i^\flat$ is a finite base for Y_i, then $Y^\flat = \bigcup Y_i^\flat t_i$ is a finite base for Y. As for property (1.1), note that for $\alpha \in \Delta$,

$$\Gamma_{\alpha t_i} = \bigcup_{j<i} \Delta t_j \cup \Delta_\alpha t_i;$$

as $\Delta \lhd \Gamma$ this implies that

$$\Gamma_{\alpha t_i}(\alpha t_i)^{-1} = \bigcup_{j<i} \Delta t_j t_i^{-1} \cup \Delta_\alpha \alpha^{-1}.$$

If $\beta \in \Delta$ and $\beta \preceq \alpha$ then $\Delta_\beta \beta^{-1} \subseteq \Delta_\alpha \alpha^{-1}$; it follows that $\Gamma_{\beta t_i}(\beta t_i)^{-1} \subseteq \Gamma_{\alpha t_i}(\alpha t_i)^{-1}$. □

1.5. Proposition. *Every polycyclic-by-finite group can be doubly ordered.*

Proof. In view of the preceding lemma, it will suffice to deal with groups which are poly-(infinite cyclic). Such a group Γ can be written in the form

$$\Gamma = \langle x_1 \rangle \langle x_2 \rangle \ldots \langle x_h \rangle,$$

where $\Gamma_i = \langle x_1 \rangle \langle x_2 \rangle \ldots \langle x_i \rangle$ is normalised by x_{i+1}, which has infinite order modulo Γ_i, for $i = 1, \ldots, h-1$. We also put $\Gamma_0 = 1$. To the element $\gamma(\mathbf{e}) = \prod_{i=1}^h x_i^{e_i}$ of Γ we associate the $2h$-tuple of non-negative integers

$$\widetilde{\mathbf{e}} = (e_1^+, e_1^-, \ldots, e_h^+, e_h^-),$$

where for $e \in \mathbb{Z}$ we write

$$\begin{aligned} e^+ &= e, \quad e^- = 0 \qquad \text{if } e \geq 0, \\ e^+ &= 0, \quad e^- = |e| \qquad \text{if } e < 0. \end{aligned}$$

These $2h$-tuples are ordered lexicographically from the right, and we define a well-ordering on Γ by setting

$$\gamma(\mathbf{e}) < \gamma(\mathbf{f}) \Leftrightarrow \widetilde{\mathbf{e}} < \widetilde{\mathbf{f}}$$

(thus, for example, $x_1^{m_1} < x_1^{m_2} x_2^3 < x_1^{m_3} x_2^7 < x_1^{m_4} x_2^{-1} < x_1^{m_5} x_2^{-2}$).

The partial ordering $\preceq$ is defined by setting $\gamma(\mathbf{e}) \preceq \gamma(\mathbf{f}) \Leftrightarrow \widetilde{\mathbf{e}} \preceq \widetilde{\mathbf{f}}$, where

$$(a_1, \ldots, a_{2h}) \preceq (b_1, \ldots, b_{2h}) \Leftrightarrow a_i \leq b_i \text{ for } i = 1, \ldots, 2h. \tag{1.2}$$

It is clear that $\preceq$ satisfies the descending chain condition; so if $Y \subseteq \Gamma$ then the set of all minimal elements of $(Y, \preceq)$ is a base for Y. The finite base property is therefore a direct consequence of the fact that $(\Gamma, \preceq)$ contains no infinite set of pairwise incomparable elements: this follows from Lemma 1.6 below.

To verify that (1.1) holds, observe that if $\alpha = \gamma(\mathbf{e})$, then

$$\Gamma_\alpha = \bigcup_{i=1}^h \Gamma_{i-1} x_i^{S(e_i)} x_{i+1}^{e_{i+1}} \ldots x_h^{e_h},$$

where

$$S(e) = \begin{cases} [0, e-1] \cap \mathbb{Z} & \text{if } e > 0, \\ \emptyset & \text{if } e = 0, \\ [e+1, \infty) \cap \mathbb{Z} & \text{if } e < 0. \end{cases}$$

So

$$\begin{aligned} \Gamma_\alpha \alpha^{-1} &= \bigcup_{i=1}^{h} \Gamma_{i-1} x_i^{S(e_i)-e_i} (x_1^{e_1} \dots x_{i-1}^{e_{i-1}})^{-1} \\ &= \bigcup_{i=1}^{h} \Gamma_{i-1} x_i^{S(e_i)-e_i} \end{aligned}$$

since $\Gamma_{i-1} \lhd \Gamma_{i-1} \langle x_i \rangle$. One checks at once that if $\widetilde{\mathbf{e}} \preceq \widetilde{\mathbf{f}}$ then $S(e_i) - e_i \subseteq S(f_i) - f_i$ for each i; it follows that if $\alpha \preceq \beta \in \Gamma$ then $\Gamma_\alpha \alpha^{-1} \subseteq \Gamma_\beta \beta^{-1}$, as required. □

1.6. Lemma. *In the partially ordered set* $(\mathbb{N}_0^{2h}, \preceq)$*, defined by (1.2), every infinite subset contains an infinite chain.*

Proof. Every infinite sequence in $\mathbb{N}_0$ contains an infinite non-decreasing subsequence. So if X is an infinite subset of $\mathbb{N}_0^{2h}$, then X contains an infinite subset $\{x_1, x_2, \dots\}$ such that the sequence $\pi_1(x_n)$ is non-decreasing, where $\pi_j : \mathbb{N}_0^{2h} \to \mathbb{N}$ denotes the jth projection map. The sequence (x_n) contains a subsequence (x_{i_n}) such that the sequence $\pi_2(x_{i_n})$ is non-decreasing. Repeating this step, we arrive at an infinite subset $\{y_1, y_2, \dots\}$, say, of X such that $\pi_j(y_n) \leq \pi_j(y_{n+1})$ for all n and for $j = 1, \dots, 2h$. But this simply means that (y_n) is an ascending chain with respect to the ordering $\preceq$. □

Putting together Propositions 1.5 and 1.3, we obtain

1.7. Theorem (K. A. Brown)**.** *Let G be an abelian normal subgroup of a group Γ such that Γ/G is polycyclic-by-finite, let k be a commutative ring, and let P be a Γ-invariant prime ideal of kG. Let M be a finitely generated $k\Gamma$-module with $MP = 0$. Then M contains a free (kG/P)-submodule F such that M/F is $\langle \lambda^\Gamma \rangle$-torsion for some element $\lambda \in kG \setminus P$.*

Remark. This result was obtained by Ken Brown [2, Corollary 2.1] in 1982. I have given the present proof because it seems (to me) clearer, and in the hope that it may have other applications.

2. An E-type theorem

Throughout this section, G denotes an abelian minimax group, Γ a soluble-by-finite group acting on G, and k a finitely generated commutative ring. The set of primes p such that G has a section of type C_{p^∞} is denoted $\operatorname{spec}(G)$, and G is said to be *reduced* if its torsion subgroup $\tau(G)$ is finite (this is equivalent to G being residually finite). A prime ideal P of kG is said to be *regular* if $\operatorname{char}(kG/P) \notin \operatorname{spec}(G)$; and we write

$$P^\dagger = (1 + P) \cap G.$$

For any commutative ring S, let us denote by

$$\max_f(S)$$

the set of all maximal ideals of finite index in S. It is quite easy to see that if $S = kG$ and kG is an *integral domain*, then $\bigcap \max_f(S) = 0$. Indeed, if $F = k/L$ is a finite field image of k and $m \in \mathbb{N}$ is coprime to $\mathrm{char}(F)$, then $F(G/G^m)$ is a direct product of finite fields and $LkG + (G^m - 1)kG = I(L, m)$, say, is the kernel of $kG \to F(G/G^m)$, so $I(L, m)$ is an intersection of maximal ideals of finite index in kG. On the other hand, $\bigcap_m I(L, m) = LkG$ because $\bigcap_m G^m = 1$, and $\bigcap_{L \in \max_f(k)} LkG = 0$ because $\bigcap \max_f(k) = \mathrm{Jac}(k) = 0$. In [16], this observation was generalised as follows:

Theorem ([16, Theorem 4.2])**.** *Let P be a regular prime ideal of kG such that $G/P^\dagger$ is reduced. Then*

$$\bigcap \max_f(kG/P) = 0.$$

Here we establish a further generalisation, that takes into account the action of the operator group Γ.

Definition. Let S be a commutative ring acted on by Γ. Then

$$\mathrm{J}_\Gamma(S) = \bigcap_{L \in \max_f(S)} \left(\bigcup_{\gamma \in \Gamma} L^\gamma \right).$$

2.1. Theorem. *Let P be a Γ-invariant regular prime ideal of kG such that $G/P^\dagger$ is reduced. Then $\mathrm{J}_\Gamma(kG/P) = 0$.*

Remarks. (i) This result can be formulated in the following way: *if $\lambda \in kG \setminus P$ then there exists $L \in \max_f(kG)$ such that $L \geq P$ but $\lambda^\gamma \notin L$ for all $\gamma \in \Gamma$.* Replacing λ by $\lambda\mu$, where μ is an arbitrary element of $kG \setminus P$, we see that in fact the intersection of all such ideals L is exactly P.

(ii) One obtains the same result without assuming that $G/P^\dagger$ is reduced, provided the definition of $\mathrm{J}_\Gamma(S)$ is modified to include *all* maximal ideals of S, rather than just those of finite index. The proof is along similar lines; unlike that of Theorem 2.1, it does not depend on [16, Theorem 4.2].

(iii) The special case of Theorem 2.1 where G is a free abelian 'plinth' for Γ and $P = 0$ is precisely Theorem E of Roseblade's paper [15].

The rest of this section is devoted to the proof of Theorem 2.1. We begin with a reduction lemma:

2.2. Lemma. *Let S be a commutative ring acted on by Γ. Let (I_α) be a family of Γ-invariant prime ideals of S and (Δ_a) a family of subgroups of finite index in Γ. If $\bigcap_\alpha I_\alpha = 0$ and $\mathrm{J}_{\Delta_\alpha}(S/I_\alpha) = 0$ for each α, then $\mathrm{J}_\Gamma(S) = 0$.*

Proof. Let λ be a non-zero element of S. Then $\lambda \notin I_\alpha$ for some α. Let T be a (finite) transversal to the right cosets of Δ_α in Γ, and put

$$\mu = \prod_{t \in T} \lambda^{t^{-1}}.$$

Then $\overline{\mu} = \mu + I_\alpha$ is a non-zero element of S/I_α, so there exists $\overline{L} = L/I_\alpha \in \max_f(S/I_\alpha)$ such that $\overline{\mu} \notin \bigcup_{\delta \in \Delta_\alpha} \overline{L}^\delta$. It is clear that then $\lambda \notin \bigcup_{\gamma \in \Gamma} L^\gamma$, and the result follows. □

Remark. Taking $\{I_\alpha\} = \{0\}$, we see in particular that if $\mathrm{J}_\Delta(S) = 0$ for some subgroup Δ of finite index in Γ, then $\mathrm{J}_\Gamma(S) = 0$.

2.3. Lemma. *If G is reduced, then $G = T \times H$, where $T = \tau(G)$ is finite and H is torsion-free, and $\mathrm{C}_\Gamma(T) \cap \mathrm{N}_\Gamma(H)$ has finite index in Γ.*

Proof. The first claim is standard. Since $\mathrm{Hom}(G/T, T)$ is finite, T has only finitely many complements in G; as these are permuted by Γ it follows that $|\Gamma : \mathrm{N}_\Gamma(H)|$ is finite, and the second claim follows since $|\Gamma : \mathrm{C}_\Gamma(T)|$ is also finite. □

The proof of Theorem 2.1 starts with a series of reductions. I shall use the notation

$$S = U \uparrow_K^G$$

to indicate that the kG-module S is induced from its kK-submodule U, where K is a subgroup of G; that is, the natural map

$$U \otimes_{kK} kG \to S$$

is an isomorphism.

First reduction: to the special case where $P = 0, G$ is torsion-free and k is a finite field.

Put $S = kG/P$. We may clearly assume that $P^\dagger = 1$. By Lemma 2.3, we then have $G = T \times H$, where T is finite and H is torsion-free, and replacing Γ by $\mathrm{C}_\Gamma(T) \cap \mathrm{N}_\Gamma(H)$, as we may in view of the remark after 2.2, we shall suppose that Γ centralises T and fixes H. Putting $P_1 = P \cap kT$, we have

$$S = \frac{k(T \times H)}{P} = \frac{(kT/P_1)H}{P/(P_1H)} = (k_1H)/\overline{P},$$

where $k_1 = kT/P_1$ and $\overline{P}$ denotes the image of P; and Γ is now acting by automorphisms on H and trivially on k_1. Replacing k by k_1 and G by H, we may therefore suppose that k is an integral domain and that G is torsion-free.

Let K be the subgroup of Γ-orbital elements in G; that is,

$$K = \{g \in G \mid |\Gamma : \mathrm{C}_\Gamma(g)| \text{ is finite}\}.$$

Now K has a finitely generated subgroup K_0 such that K/K_0 is periodic; it is easy to see that $|\Gamma : \mathrm{C}_\Gamma(K_0)|$ is finite and $\mathrm{C}_\Gamma(K_0) = \mathrm{C}_\Gamma(K)$, so replacing Γ by $\mathrm{C}_\Gamma(K)$ we may assume that in fact Γ centralises K. It is clear also that G/K is torsion-free. Put $P_1 = P \cap kK$. It follows from Theorem B of [1] that $P = P_1kG$ (Brookes states his theorem for the case where k is a field; an easy localisation argument extends this result to the case where k is an arbitrary integral domain. As a matter of fact, Chris Brookes has pointed out to me that his proof is only valid when the prime ideal P is *regular*, fortunately the case which concerns us here). Hence if $S_1 = kK/P_1$, then $S = S_1 \uparrow_K^G$.

Suppose $L \in \max_f(S_1)$, and put $k_1 = S_1/L$. Since $K/L^\dagger$ is finite, we have $G/L^\dagger = K/L^\dagger \times H_L$ for some torsion-free subgroup H_L of $G/L^\dagger$, and then

$$S/LS = (S_1/L) \uparrow_K^G = k_1H_L.$$

Put $\Delta_L = \mathrm{N}_\Gamma(H_L)$. Suppose now that the theorem has been proved in the special case indicated. As k_1 is a finite field, we then have $\mathrm{J}_{\Delta_L}(S/LS) = \mathrm{J}_{\Delta_L}(k_1H_L) = 0$.

[16, Theorem 4.2] shows that $\bigcap \max_f(S_1) = 0$. Since $S = S_1 \uparrow_K^G$, this implies that $\bigcap_{L \in \max_f(S_1)} LS = 0$. Lemma 2.3 shows that for each L the subgroup Δ_L has finite index in Γ; it follows by Lemma 2.2 that $\mathrm{J}_\Gamma(S) = 0$ (note that each of the ideals LS is Γ-invariant because Γ centralises K). □

From now on, we assume that *k is a finite field, that G is torsion-free and that $S = kG$.* Put

$$\pi = \operatorname{spec}(G), \quad q = \operatorname{char}(k), \quad R = \mathbb{Z}[p^{-1} \mid p \in \pi].$$

Recall that G is said to be a *plinth* for Γ if G is rationally irreducible as a Δ-module for every subgroup Δ of finite index in Γ.

Second reduction: to the special case where G is a plinth for Γ.

Suppose that G is not a plinth for Γ. Then there exist a subgroup Δ of finite index in Γ and a Δ-invariant subgroup A of G such that G/A is torsion-free and $\operatorname{rk}(A) < \operatorname{rk}(G)$, $\operatorname{rk}(G/A) < \operatorname{rk}(G)$. As usual, we may replace Γ by Δ and so assume that in fact $A = A^\Gamma$.

Now let $0 \neq \lambda \in S$. As $S = kG$, we can write $S = kA \uparrow_A^G$. Fixing a transversal T to G/A and writing $\lambda = \sum_{t \in T} \lambda_t t$ with each $\lambda_t \in kA$, we may suppose that $\lambda_1 \neq 0$. Arguing by induction on the rank of G, we may suppose that there exists $L \in \max_f(kA)$ such that $\lambda_1^\gamma \notin L$ for all $\gamma \in \Gamma$. This implies that $\lambda^\gamma \notin LS$ for all $\gamma \in \Gamma$.

Now put $k_1 = kA/L$. Then $L^\dagger \geq A^m$ for some positive integer m, so $|\Gamma : \mathrm{C}_\Gamma(A/L^\dagger)|$ is finite; the group $\mathrm{C}_\Gamma(A/L^\dagger)$ fixes L and acts trivially on k_1. As above, we can find a subgroup Δ_L of finite index in $\mathrm{C}_\Gamma(A/L^\dagger)$ and a torsion-free subgroup H_L of $G/L^\dagger$ such that $G/L^\dagger = A/L^\dagger \times H_L$ and Δ_L fixes H_L. Let T_1 be a transversal to the left cosets of Δ_L in Γ and put $\nu = \prod_{t \in T_1} \lambda^t$; then $\nu \notin LS$. Since $\operatorname{rk}(H_L) = \operatorname{rk}(G/A) < \operatorname{rk}(G)$, we may apply the inductive hypothesis now to H_L and suppose that $\mathrm{J}_{\Delta_L}(S/LS) = \mathrm{J}_{\Delta_L}(k_1 H_L) = 0$. Hence there exists $L_1/LS \in \max_f(S/LS)$ such that $\nu^\delta \notin L_1$ for all $\delta \in \Delta_L$. Then $\lambda^\gamma \notin L_1$ for all $\gamma \in \Gamma$, and we are done. □

Before completing the proof of 2.1 we need two simple lemmas:

2.4. Lemma. *Let $f(X) \in \mathbb{Z}[X]$ be a non-constant polynomial such that $f(1) \neq 0 \neq f(0)$. Then the π'-part of $f(q^{n!})$ tends to ∞ as $n \to \infty$.*

Proof. Suppose the conclusion fails. Since $\left|f(q^{n!})\right| \to \infty$ as $n \to \infty$, and π is finite, there exists $p \in \pi$ such that the p-part of $f(q^{n!})$ is unbounded as $n \to \infty$. We can choose e so that p^e divides neither $f(1)$ nor $f(0)$; the preceding sentence implies that $p^e \mid f(q^{n!})$ for infinitely many values of n. Let n be such a value which exceeds both e and $\phi(p^e)$. If $p = q$ then $q^{n!} \equiv 0 \pmod{p^e}$, while if $p \neq q$ then $q^{n!} \equiv 1 \pmod{p^e}$. The first case gives

$$0 \equiv f(q^{n!}) \equiv f(0) \not\equiv 0 \pmod{p^e},$$

and the second case gives

$$0 \equiv f(q^{n!}) \equiv f(1) \not\equiv 0 \pmod{p^e};$$

a contradiction either way. □

2.5. Lemma. *Let x be an automorphism of G and let χ be the characteristic polynomial of x (considered as an automorphism of $G \otimes R$). Let m be an integer which is not an eigenvalue of x. Then $|G : G^{x-m}|$ is finite and divisible by the π'-part of $\chi(m)$.*

Proof. We write G additively and identify G with $G \otimes \mathbb{Z} \leq G \otimes R = R^d$, so R^d/G is a divisible π-group (here $d = \mathrm{rk}(G)$). Let θ denote the endomorphism $x - m \cdot \mathrm{Id}$ of R^d, so $G^{x-m} = G\theta$. Since θ is non-singular, $G\theta \cong G$, which implies that $|G : G\theta|$ is finite as G contains no infinite chain of subgroups with infinite factors. Now

$$|R^d : R^d\theta| = |\det(\theta)|_{\pi'},$$

and $R^d\theta + G = R^d$ since R^d/G is divisible. The lemma follows, since $G\theta \leq G \cap R^d\theta$ and $\det(\theta) = \chi(m)$. □

Proof of Theorem 2.1: Conclusion. Now k is a finite field and G is a torsion-free plinth for Γ, where Γ is a soluble-by-finite group. We have to show that $\mathrm{J}_\Gamma(kG) = 0$. We may assume that Γ is soluble; then $\Gamma/\mathrm{C}_\Gamma(G)$ is an irreducible soluble linear group over R, hence finitely generated and abelian-by-finite. Replacing Γ by a suitable subgroup of finite index in $\Gamma/\mathrm{C}_\Gamma(G)$, we may therefore assume that Γ is in fact a free abelian group, acting faithfully on G.

If $\Gamma = 1$ then $\mathrm{J}_\Gamma(kG) = \bigcap \max_f(kG) = 0$, and there is no more to do. Assume henceforth that $\Gamma > 1$.

By Lemma 4 of [15], there exists $x \in \Gamma \setminus \{1\}$ such that G is a plinth for $\langle x \rangle$ (Roseblade's lemma is stated for the case where π is empty, but his proof yields the more general result). Note that $\mathrm{C}_G(x) = 1$; for if $\mathrm{C}_G(x) \neq 1$ then $\mathrm{C}_G(x) = G$, since G is rationally irreducible for Γ and Γ is abelian, and this is impossible since the action of Γ is faithful.

Let χ be the characteristic polynomial of x (as an endomorphism of $G \otimes R$), and for each integer $n \geq 0$ let θ_n denote the endomorphism of G given by

$$g \mapsto g^x g^{-q^{n!}}.$$

Lemma 2.5 shows that for almost all n, the index $|G : G\theta_n|$ is finite and divisible by the π'-part of $\chi(q^{n!})$; this holds for all $n \geq n_0$, say. Now neither 1 nor 0 is an eigenvalue of x, since $\mathrm{C}_G(x) = 1$ and x is invertible. So putting $f = h\chi$, where h is a π-number chosen so that $f \in \mathbb{Z}[X]$, we may infer from Lemma 2.4 that the π'-part of $\chi(q^{n!})$ tends to ∞ with n. It follows that $|G : G\theta_n| \to \infty$ as $n \to \infty$.

Let $n \geq n_0$ and put $G_n = G\theta_n$, $I_n = (G_n - 1)kG$. The k-algebra kG/I_n is finite, and x acts on it by $r \mapsto r^{q^{n!}}$. It follows that $\mathrm{Jac}(kG/I_n) = 0$, and that x fixes every ideal of kG containing I_n. Now put

$$\mathcal{L}_n = \left\{ L \in \max_f(kG) \mid I_n \subseteq L \text{ and } G^{x-1} - 1 \not\subseteq L \right\}.$$

If $L \in \mathcal{L}_n$, then x induces on kG/L the automorphism $\phi^{n!}$, where ϕ is the Frobenius automorphism $r \mapsto r^q$; as $g^x \not\equiv g \pmod{L}$ for some $g \in G$, it follows that the order of ϕ on the field kG/L is greater than n, and hence that $(kG/L : \mathbb{F}_q) > n$.

If $\mathcal{L}_n$ is empty, then $\left((G^{x-1} - 1)kG + I_n\right)/I_n \subseteq \mathrm{Jac}(kG/I_n) = 0$; this implies $G^{x-1} \leq G_n$. As $|G : G^{x-1}|$ is finite and $|G : G_n|$ tends to ∞, we see that $\mathcal{L}_n$ is nonempty for all large n, say for $n \geq n_1$, where $n_1 \geq n_0$.

Suppose now that $\lambda \in \mathrm{J}_\Gamma(kG)$. Then for each $n \geq n_1$ there exist $L_n \in \mathcal{L}_n$ and $\gamma \in \Gamma$ such that $\lambda \in L_n^\gamma$, and as $\mathcal{L}_n$ is clearly permuted by Γ we may assume that in fact $\lambda \in L_n$. Thus

$$\lambda \in I = \bigcap_{n \geq n_1} L_n.$$

Now I is an x-invariant ideal of kG. Suppose $I \neq 0$. Theorem C of [1] then shows that G contains an x-invariant subgroup $B \neq 1$ such that $(B - 1)^t \subseteq I$ for some

positive integer t. Since k has characteristic q, we then have $B^{q^{t-1}} - 1 \subseteq I$; as G is a plinth for $\langle x \rangle$, the quotient G/B is periodic. Consequently $G/I^\dagger$ is periodic. However, $G/I^\dagger$ is also residually finite, since each L_n has finite index in kG; hence $G/I^\dagger$ is finite. This is impossible, since for each $n \geq n_1$ we have

$$n < \dim_{\mathbb{F}_q}(kG/L_n) \leq \dim_{\mathbb{F}_q}(kG/I).$$

It follows that $I = 0$. Hence $\lambda = 0$, and the proof is complete. □

3. Upper ranks

Let R be a ring and M an R-module. For each prime p, we define the *upper p-rank* of M to be

$$\mathrm{ur}_p(M) = \sup\left\{\dim_{\mathbb{F}_p}(\overline{M}) \mid \overline{M} \text{ is a finite } R\text{-module image of } M/Mp\right\},$$

and the *upper rank* of M to be

$$\mathrm{ur}(M) = \sup_p \mathrm{ur}_p(M).$$

Throughout this section, Γ denotes *a minimax group* containing *an abelian normal subgroup* G such that Γ/G is *virtually polycyclic*. As before, k will be a finitely generated commutative ring. Our aim is to prove the following:

3.1. Theorem. *Let M be a finitely generated $k\Gamma$-module. If $\mathrm{ur}_p(M)$ is finite for every prime p, then M has finite upper rank.*

The proof will be completed in Subsection 3.3, where we establish a slightly more general result. The arguments depend heavily on the results of [16], and it is necessary to recall some definitions from that paper. A kG-module M is *non-singular* if (the underlying abelian group of) M has no p-torsion for $p \in \mathrm{spec}(G)$. We say that M is *qrf* ('quasi-residually finite') if $G/\mathrm{C}_G(a)$ is reduced for every $a \in M$ (this holds in particular if M is residually finite as a kG-module). The set of associated primes of M is denoted $\mathcal{P}(M)$; thus $P \in \mathcal{P}(M)$ if P is a prime ideal of kG and there exists $a \in M \setminus \{0\}$ such that $P = \mathrm{ann}_{kG}(a)$. The set of minimal (respectively, maximal) members of $\mathcal{P}(M)$ is denoted $\mathcal{Q}(M)$ (respectively, $\mathcal{M}(M)$). Finally, M is said to be *unmixed* if $\mathcal{P}(M) = \mathcal{M}(M) = \mathcal{Q}(M)$.

3.1. **Ring-theoretic preliminaries.** I shall call a prime ideal P of kG *quasi-maximal* if either

(a) P is maximal and has finite index in kG, or
(b) $\mathrm{char}(kG/P) = 0$, $\mathrm{Dim}(kG/P) = 1$, and every ideal of kG strictly containing P has non-zero intersection with $\mathbb{Z}$.

It follows from Theorem 4.2 of [16] that if P is regular and maximal and $G/P^\dagger$ is reduced, then P satisfies (a). The first lemma gives a sufficient condition for (b):

3.2. Lemma. *Let $R = kG/P$, where P is a prime ideal of kG. Suppose that $\mathrm{char}(R) = 0$ and that R has transcendence degree zero. Then:*

(i) *P is quasi-maximal.*
(ii) *There exist a finite set of primes ψ and a positive integer t such that $\mathrm{ur}(R_R) \leq t$, and $\mathrm{ur}_p(I) \leq t$ for every ideal I of R and every prime $p \notin \psi$.*

Proof. We may assume that P is faithful and that $P \cap k = 0$. By [16, Corollary 1.2], we have $R = S \uparrow_H^G$, where $S = (kH + P)/P$ and H is a finitely generated subgroup of G such that G/H is periodic and divisible. Then S is a finitely generated integral domain which is algebraic over $\mathbb{Z}$, so the field of fractions F of S is a finite extension of $\mathbb{Q}$. We take $t = (F : \mathbb{Q})$, and write $\overline{S}$ for the integral closure of S in F.

Since G/H is periodic, R is integral over S. Therefore $\mathrm{Dim}(R) = \mathrm{Dim}(S) = \mathrm{Dim}(\overline{S}) = 1$, since $\overline{S}$ is a Dedekind domain.

Now there exists $m \in \mathbb{N}$ such that $S[\frac{1}{m}]$ is integral over $\mathbb{Z}[\frac{1}{m}]$, and so $R[\frac{1}{m}]$ is also integral over $\mathbb{Z}[\frac{1}{m}]$. It follows that if I is a non-zero ideal of R containing no power of m, then $I \cap \mathbb{Z}[\frac{1}{m}] \neq 0$. Thus every non-zero ideal of R meets $\mathbb{Z}$ non-trivially, and (i) follows.

Part (ii) is more delicate. Replacing m by a suitable multiple, we may assume that

(a) m is divisible by every prime in $\mathrm{spec}(G)$, and

(b) $S[\frac{1}{m}]$ is integrally closed in F.

We then define ψ to be the set of prime divisors of m. Now it follows from [16, Proposition 2.3] that $\mathrm{ur}(R_R) \leq \mathrm{ur}(S_S)$; since $S \leq F$, the rank of S as a $\mathbb{Z}$-module is at most t (in fact it equals t). Consequently $\mathrm{ur}(R_R) \leq t$. Thus to complete the proof it will suffice to show that if $p \in \psi'$ and $pI \leq J < I$ are ideals of R with I/J finite, then I/J is a principal ideal of R/J.

Since I/J is finite, we can find a finitely generated subgroup H_1 of G such that $I = J + (I \cap S_1)$, where $S_1 = (kH_1 + P)/P$. We choose H_1 so that $H_1 \geq H$; then H_1/H is a finite π-group, where $\pi = \mathrm{spec}(G)$. In this case, we have

$$S_1 = S[x_1, \dots, x_n],$$

where $x_i^{q_i} \in S[x_1, \dots, x_{i-1}]^*$ for each i, and $q_1, \dots, q_n \in \pi$. Let T_i be the integral closure of $S[x_1, \dots, x_i]$ in its field of fractions (so $T_0 = \overline{S}$, and T_n is the integral closure of S_1). Then T_i is also the integral closure of $T_{i-1}[x_i]$. By Theorem 10.18 of [12, Chapter 1], we have

$$q_i x_i^{q_i - 1} \cdot T_i \subseteq T_{i-1}[x_i]$$

for $i = 1, \dots, n$. As each x_i is a unit, it follows that $q_1 \dots q_n T_n \subseteq \overline{S}[x_1, \dots, x_n]$. With (a) and (b) this gives $T_n \subseteq S[\frac{1}{m}, x_1, \dots, x_n] = S_1[\frac{1}{m}]$, and it follows that $S_1[\frac{1}{m}] = T_n[\frac{1}{m}]$ is integrally closed.

Put $\widehat{I} = (I \cap S_1)S_1[\frac{1}{m}]$. Since $S_1[\frac{1}{m}]$ is a Dedekind domain, there exists $\lambda \in S_1[\frac{1}{m}]$ such that $\widehat{I} = p\widehat{I} + \lambda S_1[\frac{1}{m}]$. We may choose $\lambda \in S_1$, and then since $p \nmid m$ we find that $I \cap S_1 = p(I \cap S_1) + \lambda S_1$. Then $I = J + \lambda R$, so I/J is indeed a principal ideal of R/J. □

The next lemma will be needed as a step towards establishing the hypothesis of 3.2.

3.3. Lemma. *Let S be a finitely generated integral domain of characteristic zero and positive transcendence degree. Then there is a finite set of primes ρ such that if M is a maximal ideal of S and* $\mathrm{char}(S/M) = p \notin \rho$, *then $pS + M^n > pS + M^{n+1}$ for all n.*

Proof. By Noether's Normalization Lemma, there is a positive integer m such that $S[1/m] = S_1$, say, is an integral extension of $\mathbb{Z}[1/m][x_1, \dots, x_r] = S_2$, say, where

$r \geq 1$ and $x_1, \ldots, x_r$ are algebraically independent. We take ρ to be the set of prime divisors of m.

Now let M be as in the statement, and suppose that $pS + M^n = pS + M^{n+1}$ for some n. Krull's Intersection Theorem shows that then M must be a minimal prime above pS; hence by the Principal Ideal Theorem M has height 1. As $p \notin \rho$ it follows that MS_1 is a maximal ideal of height 1 in S_1, and then $MS_1 \cap S_2$ is a maximal ideal of height 1 in S_2. This is impossible, since every maximal ideal in S_2 has height $r + 1 \geq 2$. □

3.4. Lemma. *Let M be a kG-module and $P_1, \ldots, P_s$ pairwise incomparable quasi-maximal ideals of kG such that $MP_1^{e_1} \ldots P_s^{e_s} = 0$. Put $D_i = \prod_{j \neq i} P_j^{e_j}$ for each i. Then*

$$h({}^*D_1 \cap \ldots \cap {}^*D_s) = 0$$

for some non-zero $h \in \mathbb{Z}$, where ${}^*D_i = \{a \in M \mid aD_i = 0\}$.

Proof. The hypotheses imply that $D_i \not\subseteq P_i$ for each i, and hence that $D_i + P_i$ contains $h_i \cdot 1$ for some non-zero $h_i \in \mathbb{Z}$. It is clear that $h = \prod_{i=1}^{s} h_i^{e_i}$ then has the stated property. □

3.2. **Primary modules and unmixed modules.** In this section we consider some special cases of the main result. Unless otherwise indicated, when M is either a $k\Gamma$-module or a kG-module, I shall use $\mathrm{ur}_p(M)$ to denote the upper p-rank of M *considered as a kG-module.*

Let P be a prime ideal of kG. A kG-module M will be called *P-primary* if

(a) M is non-singular and qrf,
(b) $MP^m = 0$ for some m, and
(c) $\mathcal{P}(M) = \{P\}$.

If instead of (c) we have

(c′) M/MP is not a torsion module for kG/P,

then M is said to be *quasi-primary* for P. It is easy to see that *every P-primary module is quasi-primary for P.*

3.5. Proposition. *Let M be a finitely generated $k\Gamma$-module and P a Γ-invariant prime ideal of kG such that M is quasi-primary for P. Let σ be a finite set of primes, not containing* $\mathrm{char}(kG/P)$. *Suppose that* $\mathrm{ur}_p(M)$ *is finite for all $p \in \sigma'$. Then*

(i) $\mathrm{ur}_q(M) > 0$ *for some $q \in \sigma'$;*
(ii) *P is quasi-maximal;*
(iii) *there exist $r \leq \infty$ and a finite set of primes σ_1 such that $\mathrm{ur}_p(N) \leq r$ for all $p \in \sigma_1'$ and every kG-submodule N of M; and*
(iv) *$\pi(M)$ is finite.*

Proof. Let m be minimal such that $MP^m = 0$. Write $\overline{} : M \to M/MP$. By Theorem 1.7, there exist $\lambda \in kG \setminus P$ and a free kG/P-submodule $\overline{F}$ of $\overline{M}$ such that $\overline{M}/\overline{F}$ is $\langle \lambda^\Gamma \rangle$-torsion. Note that $\overline{F} \neq 0$, since M is quasi-primary for P.

Put $h = \prod_{p \in \sigma} p$. By Theorem 2.1, the ring kG/P contains a maximal ideal L/P of finite index such that $h\lambda^\gamma \notin L$ for all $\gamma \in \Gamma$. Then $\overline{M}L \cap \overline{F} = \overline{F}L$ and $\overline{M}L + \overline{F} = \overline{M}$, so $\overline{F}/\overline{F}L \cong \overline{M}/\overline{M}L$. Now let $q = \mathrm{char}(kG/L)$. Then $q \in \sigma'$, since $h \notin L$, so $\mathrm{ur}_q(M)$ is finite. As $\overline{M}/\overline{M}L$ is a vector space over the finite field kG/L,

it is residually finite as a kG-module; hence it is finite, of rank at most $\mathrm{ur}_q(M)$. It follows that $\overline{F}/\overline{F}L$ is finite; and this in turn implies that $\overline{F}$, as a kG/P-module, is free on finitely many, say f, generators . Note that $f \neq 0$. Arguing in the reverse direction, we see now that $\mathrm{ur}_q(M) \geq \dim_{\mathbb{F}_q}(\overline{M}/\overline{M}L) \geq f$; this establishes part (i).

Suppose next that $\ell = \mathrm{char}(kG/P) \neq 0$; then $\ell \in \sigma'$. Let $\mathcal{L}$ denote the set of all maximal ideals L of finite index in kG such that $L \geq P$ and $\lambda^\gamma \notin L$ for all $\gamma \in \Gamma$. It follows from (the remark below) Theorem 2.1 that $\bigcap \mathcal{L} = P$. Let $L_1, \ldots, L_n$ be finitely many members of $\mathcal{L}$, and put $D = L_1 \cap \ldots \cap L_n$. Then $D + \lambda^\gamma kG = kG$ for all $\gamma \in \Gamma$, so the argument of the previous paragraph shows that $\overline{M}/\overline{M}D \cong \overline{F}/\overline{F}L \cong (kG/D)^f$, which is finite. Therefore

$$f \dim_{\mathbb{F}_\ell}(kG/D) \leq \mathrm{ur}_\ell(M) < \infty.$$

It follows that $P = \bigcap \mathcal{L}$ is equal to the intersection of finitely many members of $\mathcal{L}$, and hence that kG/P is finite. This implies (ii); since $M\ell^m = 0$, (iv) is immediate, and (iii) follows on putting $\sigma_1 = \{\ell\}$.

Finally, suppose that $\mathrm{char}(kG/P) = 0$. Write $\overline{} : kG \to kG/P$. By [16, Corollary 1.2], we have $\overline{kG} = S \uparrow_H^G$, where $H/P^\dagger$ is a finitely generated subgroup of $G/P^\dagger$ with G/H divisible, and $S = \overline{kH}$, so S is a finitely generated domain. Suppose the transcendence degree of S is positive, and let ρ be the finite set of primes given in Lemma 3.3. Defining $\mathcal{L}$ as above, we can find $L \in \mathcal{L}$ such that $\mathrm{char}(kG/L) = p \in \rho' \cap \sigma' \cap \pi'$, where $\pi = \mathrm{spec}(G)$. Let $n > \mathrm{ur}_p(M)$. By [16, Proposition 2.3] we have $\overline{L}^n \cap S = (\overline{L} \cap S)^n$ and $\overline{kG}/\overline{L}^n \cong S/(\overline{L} \cap S)^n$. As $\overline{kG}$ is integral over S, the ideal $\overline{L} \cap S$ is maximal in S, and it follows by Lemma 3.3 that $S/((\overline{L} \cap S)^n + pS)$ has (finite) dimension at least n over $\mathbb{F}_p$. Therefore so does $\overline{kG}/(\overline{L}^n + p\overline{kG})$, in contradiction to our choice of n. The conclusion is that S, and hence also $\overline{kG}$, has transcendence degree zero.

We may therefore apply Lemma 3.2 to the ring $R = \overline{kG}$. This shows at once that P is quasi-maximal, and it remains to establish (iii). Let t and ψ be as given in 3.2, so we have $\mathrm{ur}_p(I) \leq t$ for every $p \in \psi'$ and every ideal I of R. As P is quasi-maximal, the ideal $\lambda kG + P$ contains some non-zero integer b. Now Proposition 1.8 of [16] shows that P contains a finitely generated ideal $P^\natural$ such that $P/(P^\natural + P^m)$ is a π-group; let us suppose that $P^\natural$ can be generated by s elements. Putting $E_i = F(P^\natural)^{i-1} + MP^i$ for each i, we see that $\widetilde{E_i} = E_i/MP^i$ is an R-module generated by fs^{i-1} elements. From this follow

(a) every R-submodule V of $\widetilde{E_i}$ satisfies $\mathrm{ur}_p(V) \leq tfs^{i-1}$ for all $p \in \psi'$, and
(b) $\pi(\widetilde{E_i}) = \kappa_i$, say, is finite;

for $\widetilde{E_i}$ has a filtration of length fs^{i-1} in which each factor is a cyclic R-module, so (a) follows from the definition of t and ψ, and (b) holds because $R = kG/P$ and P is quasi-maximal.

Consider the successive factors in the chain of kG-modules

$$MP^{i-1} \geq FP^{i-1} + MP^i \geq E_i \geq MP^i.$$

The first is $\langle b \rangle$-torsion, the second is π-torsion, and the third is $\widetilde{E_i}$. It follows that

$$\pi(MP^{i-1}/MP^i) \subseteq \beta \cup \pi \cup \kappa_i,$$

where β denotes the set of prime divisors of b. Therefore $\pi(M) \subseteq \beta \cup \pi \cup \bigcup_{i=1}^m \kappa_i$, and (iv) follows. If N is a kG-submodule of M, then N has a filtration

$$\ldots N \cap MP^{i-1} \geq N \cap E_i \geq N \cap MP^i \ldots .$$

For each i, the factor $(N \cap MP^{i-1})/(N \cap E_i)$ is a $(\beta \cup \pi)$-group, while the factor $(N \cap E_i)/(N \cap MP^i)$ is isomorphic to an R-submodule of $\widetilde{E_i}$. It follows that for each $p \in (\beta \cup \pi \cup \psi)'$ we have $\mathrm{ur}_p(N) \leq \sum_{i=1}^m tfs^{i-1}$, giving (iii). □

We recall some more notation from [16]. For any set $\mathcal{X}$ of ideals of kG, the symbol $\langle \mathcal{X} \rangle$ denotes the set of all ideals that are products of finitely many members of $\mathcal{X}$; and for a kG-module M we write

$$M(\mathcal{X}) = \{a \in M \ \mid \ aI = 0 \text{ for some } I \in \langle \mathcal{X} \rangle\}.$$

Lemma 6.5(iv) of [16] shows that if M is qrf then $M = M(\mathcal{P}(M))$.

3.6. Proposition. *Let M be a finitely generated $k\Gamma$-module, and suppose that M is unmixed as a kG-module. Let σ be a finite set of primes, disjoint from $\pi(M)$, and suppose that $\mathrm{ur}_p(M)$ is finite for all $p \in \sigma'$. Then*

(i) *every member of $\mathcal{P}(M)$ is quasi-maximal;*
(ii) *$\mathcal{P}(M)$ is finite;*
(iii) *there exist $r < \infty$ and a finite set of primes σ_1 such that $\mathrm{ur}_p(N) \leq r$ for all $p \in \sigma_1'$ and every kG-submodule N of M.*

Proof. We apply Proposition 7.3 of [16]. This shows that $\mathcal{P}(M)$ is the union of finitely many Γ-orbits P_1^Γ, ..., P_t^Γ say; put $M_j = M(\mathcal{P}(M) \setminus \{P_j\})$, $U_j = M/M_j$ and $\Delta_j = \mathrm{N}_\Gamma(P_j)$. Then for each j, U_j is finitely generated as a $k\Delta_j$-module and P_j-primary as a kG-module.

Now fix $j \in \{1, \ldots, t\}$. Since $\ell_j = \mathrm{char}(kG/P_j) \in \pi(M) \cup \{0\}$, we have $\ell_j \notin \sigma$, and may therefore apply Proposition 3.5 with U_j, Δ_j in place of M, Γ. This shows that P_j is quasi-maximal. It also shows that $\mathrm{ur}_p(U_j) > 0$ for some $p \in \sigma'$, so M contains a proper kG-submodule N of finite index with $pM + M_j \leq N$. Put $N_0 = \bigcap_{\gamma \in \Gamma} N\gamma$. Then M/N_0 is residually finite as a kG-module, and has exponent p, hence is finite (of $\mathbb{F}_p$-dimension at most $\mathrm{ur}_p(M)$). Thus $M = N_0 + A$ for some finitely generated kG-submodule A of M. Now let T be a transversal to the right cosets of Δ_j in Γ, and let T_0 be the set of those $t \in T$ such that $A \not\leq M_j t$. It follows from [16, Lemma 7.2] that T_0 is finite. Suppose now that $T \neq T_0$, and let $t \in T \setminus T_0$. Then

$$M = N_0 t^{-1} + At^{-1} \leq N + M_j = N,$$

a contradiction since N is a proper submodule of M. It follows that $T = T_0$ is finite.

Thus Δ_j has finite index in Γ, for each j, and (ii) follows. We have also established (i), as every member of $\mathcal{P}(M)$ is Γ-conjugate to some P_j.

It remains to prove (iii). As $|\Gamma : \Delta_1 \cap \ldots \cap \Delta_t|$ is finite, M is still finitely generated as a module for $k(\Delta_1 \cap \ldots \cap \Delta_t)$. Replacing Γ by $\Delta_1 \cap \ldots \cap \Delta_t$, we may therefore assume that $\mathcal{P}(M) = \{P_1, \ldots, P_t\}$, where each P_j is Γ-invariant; each U_j is now a finitely generated $k\Gamma$-module. Applying 3.5(iii), we find for each j a finite set of primes $\sigma(j)$ and a positive integer r_j such that $\mathrm{ur}_p(N) \leq r_j$ for every kG-submodule N of U_j and every $p \in \sigma(j)'$.

Now according to Proposition 7.3 of [16], M can be embedded in $\bigoplus_{j=1}^t U_j$. Setting $\sigma_1 = \bigcup_{j=1}^t \sigma(j)$ and $r_0 = \sum_{j=1}^t r_j$, we deduce (iii). □

3.3. **The main theorem.** We begin with a crucial lemma:

3.7. Lemma. *Let M be a finitely generated $k\Gamma$-module. Then, for every prime p, $\mathrm{ur}_p(M_{k\Gamma}) = \mathrm{ur}_p(M_{kG})$.*

Proof. It is clear that $\mathrm{ur}_p(M_{k\Gamma}) \leq \mathrm{ur}_p(M_{kG})$. Now suppose N is a kG-submodule of finite index in M, with $pM \leq N$. Then $G^m \leq \mathrm{C}_G(M/N)$ for some finite m, and the ideal $J = \mathrm{ann}_k(M/N)$ has finite index in k and contains p. Put $\overline{M} = M/(MJ + M(G^m - 1))$. Then $\overline{M}$ is a finitely generated module for the group ring $R = (k/J)(\Gamma/G^m)$. Since G/G^m is finite, the group Γ/G^m is virtually polycyclic; it follows by the theorem of Roseblade and Jategaonkar [15, 4] that $\overline{M}$ is residually finite as an R-module. This implies that $\dim_{\mathbb{F}_p}(\overline{M}) \leq \mathrm{ur}_p(\overline{M}_R) \leq \mathrm{ur}_p(M_{k\Gamma})$, and as M/N is a quotient of $\overline{M}$ it follows that $\dim_{\mathbb{F}_p}(M/N) \leq \mathrm{ur}_p(M_{k\Gamma})$. Thus $\mathrm{ur}_p(M_{kG}) \leq \mathrm{ur}_p(M_{k\Gamma})$. □

The main result is

3.8. Theorem. *Let M be a finitely generated residually finite $k\Gamma$-module. Let σ be a finite set of primes, and suppose that $\mathrm{ur}_p(M_{k\Gamma})$ is finite for each $p \in \sigma'$. Then*

(i) *there exists $r' < \infty$ such that $\mathrm{ur}_p(M_{k\Gamma}) \leq r'$ for all $p \in \sigma'$;*
(ii) *$\pi(M)$ is finite;*
(iii) *there exist $r < \infty$ and a finite set of primes σ_1 such that $\mathrm{ur}_p(N) \leq r$ for all $p \in \sigma_1'$ and every kG-submodule N of M.*

Theorem 3.1 is just the case $\sigma = \emptyset$ of part (i).

Proof. Note to begin with that (i) follows from (iii), on putting

$$r' = \max\{r,\ \mathrm{ur}_p(M) : p \in \sigma' \cap \sigma_1\}.$$

In view of Lemma 3.7, we may replace $\mathrm{ur}_p(M_{k\Gamma})$ by $\mathrm{ur}_p(M_{kG})$ in both hypothesis and conclusion. Write $\pi = \mathrm{spec}(G)$, and put $B = \tau_{\pi\cup\sigma}(M)$. Now M is qrf as a kG-module, and Corollary 6.4 and Lemma 6.5 of [16] show that $B = M(\mathcal{W})$, where $\mathcal{W} = \{P \in \mathcal{P}(M) \mid \mathrm{char}(kG/P) \in \pi \cup \sigma\}$. It follows by [16, 6.5] that M/B is again a qrf module. If (ii) and (iii) hold with M/B in place of M, they hold for M. So replacing M by M/B, we may assume that M is non-singular and qrf for kG and that $\pi(M) \cap \sigma = \emptyset$.

Let $\mathcal{N} = \mathcal{Q}(M)$. Then $M = M(\mathcal{N})$ by [16, 6.5(iv)]. Since M is finitely generated as a $k\Gamma$-module, there is a finite subset $\mathcal{N}_0$ of $\mathcal{N}$ such that every element of M is annihilated by some product of Γ-conjugates of members of $\mathcal{N}_0$. It follows that each member of $\mathcal{N}$ contains such a product, and hence that in fact $\mathcal{N} = \mathcal{N}_0^\Gamma$.

Put $\mathcal{Y} = \mathcal{P}(M) \setminus \mathcal{N}$ and $N = M(\mathcal{Y})$. Lemma 6.5(iii) of [16] shows that $\mathcal{P}(M/N) = \mathcal{M}(M/N) = \mathcal{N}$, so M/N is an unmixed module. It follows by Proposition 3.6 that $\mathcal{N}$ is finite and that P is quasi-maximal for every $P \in \mathcal{N}$. Say $\mathcal{N} = \{P_1, \ldots .P_s\}$. Since $M = M(\mathcal{N})$ is finitely generated as a $k\Gamma$-module, there exists e such that $M \prod_{i=1}^s P^e = 0$. Put $D_i = \prod_{j\neq i} P_j^e$ and $N_i = {}^*D_i$ for each i. By Lemma 3.4, we have $h(N_1 \cap \ldots \cap N_s) = 0$ for some positive integer h. Hence if (ii) and (iii) hold with M/N_i in place of M, for each $i = 1, \ldots, s$, then they hold for M.

Since $\mathcal{N}$ is finite, the subgroup $\Gamma_0 = \bigcap_{i=1}^s \mathrm{N}_\Gamma(P_i)$ has finite index in Γ. Now fix i and put $\overline{M} = M/N_i$, $P = P_i$. Then $\overline{M}$ is a finitely generated $k\Gamma_0$-module, and $\overline{M}P^e = 0$. [16, Lemma 6.2] shows that $\overline{M}$ is non-singular and qrf. There are now two possibilities:

(a) $\overline{M}c = 0$ for some non-zero integer c. In this case, both (ii) and (iii) hold trivially for $\overline{M}$.

(b) $\overline{M}c \neq 0$ for all non-zero integers c. In this case, $\overline{M}/\overline{M}P$ is not a torsion module for kG/P. Indeed, suppose that $\overline{M}/\overline{M}P$ is a torsion module for kG/P. Let X be a finite set of $k\Gamma_0$-module generators for $\overline{M}/\overline{M}P$; then $\mathrm{ann}_{kG}(X)$ strictly contains P. As P is quasi-maximal, it follows that $\mathrm{ann}_{kG}(X)$ contains some non-zero integer z; then $\overline{M}z \leq \overline{M}P$, so $\overline{M}z^e \leq \overline{M}P^e = 0$.

Thus in case (b), $\overline{M}$ is quasi-primary; that (ii) and (iii) hold for $\overline{M}$ therefore follows from Proposition 3.5. □

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