CENTRAL EXTENSIONS AND GENERALIZED PLUS-CONSTRUCTIONS

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Abstract. We describe the effect of homological plus-constructions on the homotopy groups of Eilenberg-MacLane spaces in terms of universal central extensions.

INTRODUCTION

Higher algebraic $K$-theory was introduced by Quillen [27] by means of the plus-construction (a precursor of which goes back to Varadarajan [32, p. 368]). When applied to a space $X$, it yields a map $X \to X^+$ which quotients out the maximal perfect subgroup of $\pi_1 X$ without changing the homology of $X$. In the case where $X = BGL(R)$ is the classifying space of the general linear group of a ring $R$, $K_n(R) := \pi_n(K_0(R) \times X^+)$. While this construction is readily described, its homotopy theoretic properties, especially its effect on homotopy groups, remain largely mysterious. General results in this direction are due to Kervaire [23]. He discovered the universal central extension $\pi_2 K(G, 1)^+ \to \overline{PG} \to PG$, where $PG$ is the maximal perfect subgroup of $G$, and $\overline{PG}$ is a perfect group satisfying $H_2(\overline{PG}; \mathbb{Z}) = 0$. From this one easily deduces the natural isomorphisms $\pi_2 K(G, 1)^+ \cong H_2(\overline{PG}; \mathbb{Z})$ and $\pi_3 K(G, 1)^+ \cong H_3(\overline{PG}; \mathbb{Z})$ which are special cases of our main theorem.

A homotopy theoretic environment which is suitable for the study of plus-constructions has been provided by the works of Bousfield [5, 6, 7, 8] and Dror Farjoun [14]. Given a homology theory $h$, there is a colocalizing functor $A_h X \to X$ which extracts from $X$ a universal $h$-acyclic cover of its $h$-acyclic essence: $A_h X$ is $h$-acyclic and $\text{map}_*(A, A_h X) \to \text{map}_*(A, X)$ is a weak homotopy equivalence for every $h$-acyclic space $A$. Dually, there is a localizing functor $X \to X^{+h}$ which strips $X$ of its $h$-acyclic essence: $\text{map}_*(A, X^{+h})$ is weakly equivalent to a point, for every $h$-acyclic space $A$.

The composite $A_h X \to X \to X^{+h}$ forms a homotopy fibration and, sometimes, also a homotopy cofibration—e.g. when $h = H(-; \mathbb{Z})$; see [20, 2.5]. Meier [25] first associated a plus-construction with a choice of an ordinary homology theory. This process was expanded in [11].

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The map $X \to X^{+h}$ induces an isomorphism in $h$-homology, but is usually not a homotopy equivalence. Thus the homotopy groups of $X$ must change to accommodate the reduction of $X$ by its $h$-acyclic essence. We endeavor to understand this change of homotopy groups, and this paper is a first step. Our approach builds on the following two pillars.

- The universal properties of the fibration $A_hX \to X \to X^{+h}$ should be reflected in the form of universal properties of appropriate segments within the long exact sequence of homotopy groups of this fibration.
- The fibration $A_hX \to X \to X^{+h}$ yields the $\Pi$-central fibration $\Omega X^{+h} \xrightarrow{\partial} A_hX \to X$; i.e. all Whitehead products of the form $[\partial \alpha, \beta]$ vanish, where $\alpha \in \pi_p\Omega X^{+h}$ and $\beta \in \pi_qA_hX$, $p, q \geq 1$; see (7.7).

Accordingly, appropriate segments within the long exact sequence of homotopy groups of the fibration $\Omega X^{+h} \xrightarrow{\partial} A_hX \to X$ are central extensions of groups with a certain universal property. This statement takes its purest form in the case where $X = K(G, n)$ and $G$ is a group, abelian if $n \geq 2$, such that $\pi_nK(G, n) \to \pi_nK(G, n)^{+h}$ is the 0-map. For in this case, $\pi_1\Omega X^{+h} \to \pi_1A_hX \to \pi_1K(G, n)$ is an extension of $\Pi$-algebras, in the sense of Dwyer-Kan [19], which is central.

Here, we use this platform to gain insight into the effect of $+h$-localization on the homotopy groups of higher Eilenberg-MacLane spaces and on the second and third homotopy groups of $K(G, 1)$'s. We summarize our main results in the following theorem.

**Theorem.** For $n \geq 1$ and a group $G$, abelian if $n \geq 2$, there is a unique maximal subgroup $P_n^hG$ of $G$ with the property that $\pi_nK(P_n^hG, n) \to \pi_nK(P_n^hG, n)^{+h}$ is the 0-map. Moreover, the following hold.

(i) The $(n+1)$-st homotopy group of $K(G, n)^{+h}$ fits into the central extension

$$\pi_{n+1}K(G, n)^{+h} \xrightarrow{\partial} P_n^hG \xrightarrow{\partial} P_n^hG,$$

which is universal in the sense explained in section 4. If $G$ is abelian, this identifies $\pi_{n+1}K(G, n)^{+h}$ as the representing object for the functor

$$\text{Ext}(P_n^hG, -) \cong H^2(P_n^hG; -)$$

on the category of all those groups $L$ for which $K(L, n+1)$ is $+h$-local.

(ii) If $n = 1$, we have

$$\pi_1K(G, 1)^{+h} \cong G/P_1^hG,$$

$$\pi_2K(G, 1)^{+h} \cong \pi_2K(P_1^hG, 1)^{+h} \cong H_2(K(P_1^hG, 1)^{+h}; \mathbb{Z}),$$

$$\pi_iK(G, 1)^{+h} \cong \pi_iK(P_1^hG, 1)^{+h}, \quad i \geq 2.$$

Further, if $h$ is $\pi_2$-compatible (see (2.18)), then

$$\pi_3K(G, 1)^{+h} \cong H_3\left(K(P_{-1}^hG, 1)^{+h}; \mathbb{Z}\right).$$

For abelian $G$, our results overlap with [29] and [12]. However, their approaches are derived from a different viewpoint.

This paper is organized as follows. Section 0 contains background material on (co)localization. Section 1 establishes key results on maps whose homotopy fiber is $h$-acyclic. In section 2, we introduce $hn$-perfect groups together with their duals,
hn-acyclically reduced groups, and present their basic properties. Section 3 introduces, for $n \geq 1$, a localization functor of abelian groups from the effect of $h$-localization on abelian $K(G,n)$’s. All of this is used in section 4 to formulate and prove our main results, stated above. Section 5 contains examples. Section 6 consists of an algebraic lemma. Section 7 presents prerequisites on II-central fibrations which motivate our approach but are also of independent interest.

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0. Preliminaries

Here we gather some concepts and facts about homological localization which are relevant in what follows. Most of the key insights here are work of Bousfield. A good exposition can also be found in [29].

Homotopical localization with respect to a based map $f: U \to V$ of CW-spaces is a continuous homotopically idempotent functor $L_f$ on the category of compactly generated Hausdorff spaces; see [15], [14] and compare [9], [21]. One may occasionally have need to resort to CW-substitutes to avoid pathologies in mapping spaces, or to ensure the existence of covering maps. Alternatively, one can work simpli-

The $L_f$-local objects $Z$, also called $f$-local objects, are characterized by the property that, between spaces of free maps, $f^*: \text{map}(V; Z) \to \text{map}(U; Z)$ is a weak homotopy equivalence. A function $u: A \to B$ is an $L_f$-equivalence if $L_f(u)$ is a (weak) homotopy equivalence. If $c: W \to *$ is the map to a 1-point space, we write $X//W$ for $L_c X$, and call it the $W$-reduction of $X$. Other notation in use includes $L_W X$ and $P_W X$. A $(W \to *)$-local space is also called $W$-reduced.

We consider only non-trivial homology theories ($\tilde{h}_* S^0 \neq 0$), which can be described by a CW-spectrum. Such theories are additive. Therefore $h$-acyclic CW-spaces ($h_*(X) \cong h_*(pt)$) are necessarily connected.

0.1. Proposition and Terminology. Given a homology theory $h$, there exists an $h$-equivalence $f$ between connected CW-spaces such that the following hold.

(i) $X \to L_f X =: X^h$ is Bousfield’s homology localization with respect to $h$; see [5]. $X$ is called $h$-local if $X$ is $L_f$-local.

(ii) If $A := \text{cofib}(f)$, then the $h$-localizing map $X \to X^h$ factors uniquely through the $h$-acyclic reduction $X^{+h} := L_A X$, which we call the $+h$-construction of $X$; compare [8, 4.4]. A space $Y$ is called $+h$-local or $h$-acyclically reduced, if $Y \simeq Y^{+h}$. A map $u$ is a $+h$-equivalence if $L_A(u)$ is a (weak) homotopy equivalence. The $+h$-localization of a disconnected CW-space $X$ is given by the disjoint union of the $+h$-localizations of the connected components of $X$.

(iii) The homotopical colocalization of $X$ with respect to $A$ is the functor $u : A_h X := \text{CW}_A X \to X$ of Dror Farjoun [13]. It is characterized by the property that $A_h X$ is $h$-acyclic and that $u_* : \text{map}_*(B, A_h X) \to \text{map}_*(B, X)$ is a weak homotopy equivalence for every $h$-acyclic space $B$.

(iv) For every connected space $X$, $A_h X \to X \to X^{+h}$ is a homotopy fibration.

Proof. Only (iv) remains to be shown. Let $F \to X \to X^{+h}$ be the homotopy fibration associated to $+h$-localization. By design, $\text{map}_*(B, X^{+h}) \simeq *$ for every
$h$-acyclic space $B$. Consequently, $\text{map}_*(B, F) \to \text{map}_*(B, X)$ is a weak homotopy equivalence. To see that $F$ is $h$-acyclic we consider the map of fibrations below [18, p. 74]:

\[
\begin{array}{ccc}
F & \to & F^{+h} \\
\downarrow & & \downarrow \\
X & \to & X^{+h} \\
\downarrow & & \downarrow \\
X^{+h} & \equiv & X^{+h}
\end{array}
\]

Visibly $F^{+h} \simeq \ast$, implying that $F$ is $h$-acyclic. From the universal property of $A_h$-colocalization we infer $F \simeq A_h X$.

The loop space operation relates $h$-localization and $+h$-localization as follows.

0.2. Lemma. If $X$ is $+h$-local and path connected, then $\Omega X$ is $h$-local.

Proof. Fiberwise $h$-localization gives the morphism of fibrations

\[
\begin{array}{ccc}
\Omega X & \to & (\Omega X)^h \\
\downarrow & & \downarrow \\
\ast & \to & Z \\
\downarrow & & \downarrow \\
X & \equiv & X
\end{array}
\]

The map $\ast \to Z$ is an $h$-equivalence. So $Z$ is $h$-acyclic. On the other hand, $Z$ is $+h$-local, being the total space of a fibration whose fiber and base are $+h$-local. Thus $Z \simeq \ast$, and the two fibrations are isomorphic.

For an arbitrary space $W$, the $W$-reduction map $X \to X//W$ commutes with the covering space operation in the following precise sense.

0.3. Lemma. Given connected CW-complexes $W$ and $X$, let $K$ be the kernel of $\pi_1X \to \pi_1(X//W)$. Then there is a natural homeomorphism

\[
\overline{X}/W \to \widetilde{X}/W,
\]

where $\overline{X}$ is the covering space of $X$ corresponding to $K < \pi_1X$ and $\widetilde{X}/W$ is the universal cover of $X//W$.

Proof sketch. Recall that $W$-reduction of a space $Y$ is constructed by repeating the pushout construction

\[
\begin{array}{ccc}
U & \to & V \\
\downarrow & & \downarrow \\
Y & \to & Y'
\end{array}
\]
where the top arrow is
\[
\prod_{k \geq 0} (\text{map}(S^k \times W, Y) \times (S^k \times W')) \to \prod_{k \geq 0} (\text{map}(S^k \times W, Y) \times C(S^k \times W')) ,
\]
and \(\varepsilon\) is evaluation. Thus \(\pi_1 Y \to \pi_1 Y'\) is onto. If \(\kappa\) is a normal subgroup of \(\pi_1 Y\) containing \(\ker(\pi_1 Y \to \pi_1 Y')\) let \(\tilde{Y}\) be the covering space corresponding to \(\kappa\). Then
\[
\prod_{k \geq 0} \text{map}(S^k \times W, \tilde{Y}) \to \prod_{k \geq 0} \text{map}(S^k \times W, Y)
\]
is a covering map with \(\pi_1 Y/\kappa\) sheets. The morphism of pushout diagrams
\[
\begin{array}{ccc}
U & \to & V \\
\downarrow & & \downarrow \\
\tilde{Y} & \to & \tilde{Y}' \\
\end{array}
\]
\[
\begin{array}{ccc}
U & \to & V \\
\downarrow & & \downarrow \\
Y & \to & Y' \\
\end{array}
\]
has \(\tilde{Y}'\) connected, which covers \(Y'\) with \(\pi_1 Y/\kappa\) sheets. Moreover,
\[
\pi_1 \tilde{Y}' \cong \kappa/\ker(\pi_1 Y \to \pi_1 Y').
\]
Thus \(\tilde{Y}' \to Y'\) is the covering map of \(Y'\) corresponding to \(\ker(\kappa \to \pi_1 Y')\). To infer the lemma, set \(\kappa := K\) and repeat this argument (transfinitely often if necessary).

0.4. Corollary. The higher homotopy groups of \(K(G,1)/W\) depend only on the kernel of \(\pi_1 K(G,1) \to \pi_1 K(G,1)/W\).

0.5. Corollary. Given a connected CW-space \(W\) and an arbitrary group \(G\), then \(K(G,1)\) is \(W\)-reduced if and only if \(\text{Hom}(\pi_1 W, G) = *\).

Proof. If \(K(G,1)\) is \(W\)-reduced, then \(\text{map}_*(W, K(G,1)) = *\) and so \(\text{Hom}(\pi_1 W, G) \cong \ast\). Conversely, if \(\text{Hom}(\pi_1 W, G) = \ast\), then \(\pi_1 K(G,1) \to \pi_1 K(G,1)/W\) is an isomorphism. But then, with the notation of 4.3, \(\ast \simeq K(G,1) \cong K(G,1)/W \cong K(G,1)/W\). Thus \(K(G,1) \to K(G,1)/W\) is a homotopy equivalence.

We will need to recognize \(h\)-acyclic spaces. If \(h\) is connective (i.e. \(h_n(pt) = 0\), for \(n\) sufficiently small), we are aided by the following result of Bousfield [4].

0.6. Theorem. Given a connective homology theory \(h\), let \(P\) denote the set of primes \(p\) for which \(h_*(pt)\) is not uniquely \(p\)-divisible. Let
\[
R(h) := \bigoplus_{p \in P} \mathbb{Z}/p \quad \text{if} \ h_*(pt) \text{ is torsion},
\]
\[
\mathbb{Z}_p \quad \text{if} \ h_*(pt) \text{ is not torsion}.
\]
Then a space \(X\) is \(h\)-acyclic if and only if \(\tilde{H}_*(X; R(h)) = 0\).
0.7. Lemma. For a homology theory $h$, define $R(h)$ as in (0.6). Then every $H(-; R(h))$-acyclic space is also $h$-acyclic. In particular, $+h$-localization factors through $+H(-; R(h))$-localization.

Proof. If $Z$ is $H(-; R(h))$-acyclic, then the second page of the Atiyah-Hirzebruch spectral sequence for $h_* Z$ consists only of the 0-th column. The claim follows.

A convenient tool for relating reduction and localizing functors of spaces is the following lemma of Zabrodsky; compare [31].

0.8. Lemma. Given a fibration $F \to E \xrightarrow{f} B$, if a space $Y$ is $F$-reduced, then $Y$ is $f$-local.

0.9. Corollary. Suppose $\Omega$ is an $h$-acyclic loop space. Then $B\Omega$ is $h$-acyclic.

Proof. We have the fibration $\Omega \to \ast \xrightarrow{f} B\Omega$. If $Y$ is arbitrary $+h$-local, then $Y$ is $\Omega$-reduced and, therefore, $f$-local; (0.8). Thus $f$ is a $+h$-equivalence, implying that $B\Omega$ is $h$-acyclic.

1. $h$-ACYCLIC MAPS

1.1. Definition. A map of path connected spaces $f : X \to Y$ is $h$-acyclic if $\text{fib}(f)$ is $h$-acyclic.

Our definition of “$h$-acyclic map” extends Quillen’s in [28]. We suggest thinking of an $h$-acyclic map as being $h$-acyclicity reducing, because the target of an $h$-acyclic map appears in a universal way between $X$ and the completely $h$-acyclicity reduced space $X^h$. This is one possible interpretation of the lemma below.

1.2. Lemma. Associated with an $h$-acyclic map $f : X \to Y$ there is the natural commutative diagram below whose rows and columns are fibrations:

\[
\begin{array}{ccc}
F & \xrightarrow{=} & A_h X \\
\downarrow & & \downarrow \\
\ast & \xrightarrow{=} & X^h
\end{array}
\quad
\begin{array}{ccc}
& \xrightarrow{f} & \\
A_h X & \xrightarrow{=} & A_h Y \\
\downarrow & & \downarrow \\
& \xrightarrow{f} & \\
X & \xrightarrow{=} & Y^h
\end{array}
\]

Proof. Universality of the operations $A_h$ and $^h$ yields the morphism of the two vertical fibrations on the right. From Zabrodsky’s lemma [18] or fiberwise localization [14, 1.H.1], we see that $f$ is a $+h$-equivalence. Therefore $X^h \to Y^h$ is a homotopy equivalence, and, hence, the homotopy fibers of the horizontal maps are as indicated.

1.3. Corollary. An $h$-acyclic map is a $+h$-equivalence and, hence, an $h$-equivalence.

Based on this information we expect

1.4. Corollary. If $X$ is path connected, then the universal map $X^h \to X$ is a homotopy equivalence if and only if $\text{fib}(X \to X^h)$ is $h$-acyclic, which is the case if and only if $\text{fib}(X^h \to X^h)$ is $h$-acyclic.
Proof. We verify one implication: Suppose \( F := \text{fib}(X \to X^h) \) is \( h \)-acyclic. By (1.2), \( X^{+h} \to (X^h)^+ = X^h \) is a homotopy equivalence. The rest of the argument is similar.

Thus an \( h \)-acyclic map is an \( h \)-equivalence with some additional property. We make this statement precise in the case where \( h \) is connective.

1.5. Proposition. Let \( h \) be a connective homology theory. Then a map \( f:X \to Y \) of path connected spaces is \( h \)-acyclic if and only if \( f \) induces an isomorphism in homology with twisted coefficients

\begin{align*}
\bullet f_\ast: \bigoplus_p \mathcal{H}_\ast(X; f^\ast\mathbb{Z}/p[\pi_1Y]) & \to \bigoplus_p \mathcal{H}_\ast(Y; \mathbb{Z}/p[\pi_1Y]), \text{ if } h_\ast(pt) \text{ is torsion}, \\
\bullet f_\ast: \mathcal{H}_\ast(X; f^\ast\mathbb{Z}/p[\pi_1Y]) & \to \mathcal{H}_\ast(Y; \mathbb{Z}/p[\pi_1Y]), \text{ if } h_\ast(pt) \text{ is not torsion}.
\end{align*}

In either case, \( P \) is the set of primes \( p \) for which \( h_\ast(pt) \) is not uniquely \( p \)-divisible.

Proof. This follows by combining Bousfield’s result (0.6) with \([20] \) or \([25, 1.1] \).

1.6. Proposition. If \( h \) is connective and \( X^h \) is simply connected, then the natural map \( u:X^{+h} \to X^h \) is a weak homotopy equivalence.

Proof. \( u \) is an \( h \)-equivalence and, hence, an \( H(\text{--; }R) \)-equivalence; see (0.6). Further, \( u \) is \( h \)-acyclic, by using (1.5) and the fact that \( X^h \) is simply connected. Thus \( \text{fib}(u) \) is \(+h\)-local and \( h \)-acyclic, hence is a point, which implies the claim.

1.7. Example. If \( h \) is connective, then \(+h\)-localization and \( h \)-localization agree on simply connected spaces. For, in this case, \( h \)-localization does not reduce connectivity; see \([7, 7.3] \).

The following example illustrates that \(+h\)-localization and \( h \)-localization can agree on Eilenberg-MacLane spaces, even if \( h \) is not connective.

1.8. Example. If \( h = K \) is real or complex \( K \)-theory, and \( G \) is an abelian group, then

\begin{align*}
(i) \quad K(G, 1)^{+h} & = K(G, 1)^h = K(G, 1), \\
(ii) \quad K(G, 2)^{+h} & = K(G, 2)^h = K(G/\text{torsion}, 2), \\
(iii) \quad K(G, i)^{+h} & = K(G, i)^h = K(G \otimes \mathbb{Q}, i), \text{ for } i \geq 3.
\end{align*}

Proof. \( K(G, n)^h \) has been computed by Mislin; see \([20] \). Thus the claims will follow once we have shown that, in each case, \( \text{fib}(K(G, n)^{+h}) \to K(G, n)^h \) is \( h \)-acyclic. Indeed, \( K(G, 1)^h = K(G, 1) \), implying (i). Moreover, \( K(G, 2)^h = K(G/\text{torsion}, 2) \) and, therefore, one has a fibration \( K(\text{torsion}(G), 2) \to K(G, 2) \to K(G, 2)^h = K(G/\text{torsion}, 2) \) with an \( h \)-acyclic fiber. Thus \( K(G, 2)^h = K(G, 2)^{+h} \). If \( i \geq 3 \), then

\[ F := \text{fib}(K(G, i) \to K(G \otimes \mathbb{Q}, i) = K(G, i)^h = K(B, i) \times K(C, i - 1) \]

with \( B \) and \( C \) torsion groups; see \([7, 2] \). Thus \( K(B, i) \) and \( K(C, i - 1) \) are \( h \)-acyclic. Localization commutes with finite products; \([13 \text{ p. 5}] \). Thus \( F \) is \( h \)-acyclic.

1.9. Remark. If \( h = H(\text{--; }\mathbb{Z}) \), then the fibration \( A_hX \to X \to X^{+h} \) is also a cofibration; \([20 \text{ 2.5}] \). This is not so in general, as the following example illustrates.

Take \( h := H(\text{--; }\mathbb{Z}/p) \) and \( X = K(\mathbb{Z}_{p^\infty}, 1) \). Then \( X^h = K(\mathbb{Z}_p^\wedge, 2) \), because of the short exact sequence \( \mathbb{Z}_p^\wedge \to \mathbb{Q}_p^\wedge \to \mathbb{Z}_{p^\infty} \), where \( \mathbb{Z}_p^\wedge \) denotes the \( p \)-adic integers. As \( X^h \) is 1-connected, \( X^{+h} \) is \( +h \)-acyclic by (1.6). The fiber of \( K(\mathbb{Z}_{p^\infty}, 1) \to K(\mathbb{Z}_p^\wedge, 2) \) is \( K(\mathbb{Q}_p^\wedge, 1) \), where \( \mathbb{Q}_p^\wedge \) denotes the \( p \)-adic numbers. But the cofiber \( C \) of \( K(\mathbb{Q}_p^\wedge, 1) \to...
Let $2.1$. Definition. see (3.7).

Thus, in general, $X \rightarrow X^{+h}$ factors through $\text{cofib}(A_h X \rightarrow X) =: C$. But $C$ need not be $h$-acyclically reduced.

2. $hn$-PERFECT AND $hn$-ACYCLICALLY REDUCED GROUPS

Fix a homology theory $h$. Given an integer $n$ and a group $G$, abelian if $n \geq 2$, we have the fibration

$$A_h K(G, n) \rightarrow K(G, n) \rightarrow K(G, n)^{+h}.$$ 

In order to explain the influence of the universal properties of these maps on the homotopy groups in the long exact sequence of this fibration we introduce the groups

$$P^n_h G := \ker(\pi_n K(G, n) \rightarrow \pi_n K(G, n)^{+h})$$

and

$$Q^n_h G := (\text{maximal quotient } Q \text{ of } G \text{ with } P^n_h Q \text{ trivial}).$$

As we shall see, the higher homotopy groups of $K(G, n)^{+h}$ depend only on $P^n_h G$; see (57).

2.1. Definition. Let $n \geq 1$. A group $G$, abelian if $n \geq 2$, is called $hn$-perfect if $P^n_h G = G$. Dually, $G$ is $hn$-acyclically reduced if $G = Q^n_h G$.

2.2. Example. Let us consider $h = H(\cdot, \mathbb{Z})$. Then $P^n_h G$ is just the unique maximal perfect subgroup of $G$. Moreover, $G$ is $h1$-perfect if and only if $G$ is perfect. After all, $G$ is perfect if and only if $K(G, 1)^+$ is simply connected. Dually, $G$ is $h1$-acyclically reduced exactly when its maximal perfect subgroup is trivial. Finally, every abelian group is $hn$-acyclically reduced for all $n \geq 1$.

In general, $G$ is $hn$-acyclically reduced if and only if $P^n_h G$ is trivial. We will discuss the case of $h = K$ (real or complex $K$-theory) below. We will show how to characterize $L := \pi_{n+1} K(G, n)^{+h}$ via a certain universal central extension

$$L \rightarrow A \rightarrow P^n_h G.$$

Dually, we show how to characterize $A' := \pi_{n-1} A_h K(G, n)$ via a certain short exact sequence

$$Q^n_h G \rightarrow L' \rightarrow A'$$

whose universal properties are dual to those of the $L$-determining extension. Hence we speak of a universal coextension here.

Here are some basic properties of $hn$-perfect and $hn$-acyclically reduced groups, followed by some lemmas needed to verify these properties.

2.3. Proposition (Basic properties of $hn$-perfect groups). The following hold.

(i) The class of $hn$-perfect groups is closed under quotients, arbitrary colimits and weak products.

(ii) If $A$ is an $(n-1)$-connected $h$-acyclic space, then $\pi_n A$ is $hn$-perfect.

(iii) For every space $X$, $\ker(\pi_1 X \rightarrow \pi_1 X^{+h})$ is $h1$-perfect.
(iv) Every group $G$ has a unique maximal $hn$-perfect subgroup $P^h_n G$.
(v) If $G$ is abelian and $n \geq 1$, then $P^h_n (G/P^h_n G) = 0$.
(vi) Let $h$ and $k$ be homology theories such that every $h$-equivalence is also a $k$-equivalence. Then, for each $n \geq 1$, $hn$-perfect groups are also $kn$-perfect.

2.4. Proposition. (Basic properties of $hn$-acyclically reduced groups) The following hold.

(i) The class of $hn$-acyclically reduced groups is closed under subgroups and arbitrary inverse limits.

(ii) Every group $G$ has a unique maximal $hn$-acyclically reduced quotient $\varinjlim_k G$. If $G$ is abelian then $\varinjlim_k G = \operatorname{coker}(\pi_n A_h K(G, n) \to \pi_n K(G, n))$. If $n = 1$, then there is the possibly transfinite sequence of epimorphisms with

$$Q^h_1 G = \varinjlim \{G \to G/P^h_1 G \to (G/P^h_1 G)/P^h_1 (G/P^h_1 G) \to \cdots\}.$$

(iii) Let $h$ and $k$ be homology theories such that every $h$-equivalence is also a $k$-equivalence. Then $kn$-acyclically reduced groups are also $hn$-acyclically reduced.

Here is the key lemma which is needed to understand the effect of $+h$-localization on $K(G, n)$’s with $G$ abelian.

2.5. Lemma. For $n \geq 1$ and an abelian group $G$, there is the natural fiber sequence

$$K(A', n - 1) \times K(A, n) \to K(G, n) \to K(L', n) \times K(L, n + 1)$$

whose long exact sequence of homotopy groups consists of

$$0 \to L \to A \to G \to L' \to A' \to 0.$$  

Moreover, if $n = 1$, then $A' = 0$.

Proof. According to [14, 4.12], $A_h K(G, n)$ is a GEM whose homotopy groups above dimension $n$ vanish. Further, $K(G, n)^{+h}$ is an $(n-1)$-connected GEM; see [15, 1.11]. The claim follows.

From (2.5) we deduce the following.

2.6. Corollary. For an arbitrary group $G$, abelian if $n \geq 2$, the canonical map $K(G, n) \to K(G, n)^{+h}$ is a homotopy equivalence if the induced map $\pi_n K(G, n) \to \pi_n K(G, n)^{+h}$ is an isomorphism.

Proof. Suppose $\pi_n K(G, n) \to \pi_n K(G, n)^{+h}$ is an isomorphism. If $n \geq 2$ then, in the notation of (2.5), $A' = 0$ and $L \cong A$. Thus the $h$-acyclic space $K(A, n)$ is homotopy equivalent to the $h$-local space $\Omega K(L, n + 1)$; see (0.2). Thus $A = L = 0$, which implies the claim.

If $n = 1$, we have the equivalence

$$K(G, 1) \cong K(\pi_1 K(G, 1)^{+h}, 1)$$
which exposes $K(G, 1)$ as a retract of the $+h$-local space $K(G, 1)^{+h}$. Thus $K(G, 1)$ is already $+h$-local, which implies the claim.

2.7. Corollary. A group $G$ is $h1$-acyclically reduced iff $K(G, 1)$ is $+h$-local.

2.8. Lemma. For an abelian group $G$ and $n \geq 1$ the following are equivalent.

(i) $G$ is $hn$-perfect.

(ii) $+h$-localization yields a fibration of the form:

\[
\begin{array}{ccc}
K(A, n) & \rightarrow & K(G, n) \\
\downarrow & & \downarrow \\
A_hK(G, n) & \rightarrow & K(G, n)^{+h}
\end{array}
\]

\[
\begin{array}{ccc}
\cong & & \cong \\
\end{array}
\]

\[
\begin{array}{ccc}
A_0K(G, n) & \rightarrow & K(G, n) \\
\end{array}
\]

Proof. (i) $\implies$ (ii) Since $G$ is $hn$-perfect, the exact sequence of homotopy groups from (2.5) takes the form $L \rightarrow A \rightarrow G \rightarrow L' \rightarrow A'$. Thus $L' \cong A'$, implying that the $h$-local space $K(L', n - 1)$, see (1.2), is a retract of the $h$-acyclic space $K(A', n)$. Therefore $K(L', n - 1)$ is $h$-acyclic as well. But then $K(L', n - 1)$ is a point, implying that $L' = 0$, as claimed.

(ii) $\implies$ (i) $G$ is $hn$-perfect, since $\pi_nK(G, n)^{+h} = 0$.

2.9. Corollary. If $G$ is $hn$-perfect, then $H_1(G; \mathbb{Z})$ is $h(n + 1)$-perfect.

Proof. We have the fibration $A_hK(G, n) \rightarrow K(G, n) \rightarrow K(G, n)^{+h}$, in which $A_hK(G, n)$ is $(n - 1)$-connected and $K(G, n)^{+h}$ is $n$-connected. Suspending the first map yields the diagram $\Sigma A_hK(G, n) \rightarrow \Sigma K(G, n) \rightarrow K(H_1(G; \mathbb{Z}), n + 1)$ in which all spaces are $n$-connected. $\Sigma A_hK(G, n)^{+h}$ is $h$-acyclic and $\pi_{n+1}\Sigma A_hK(G, n)$ maps onto $\pi_{n+1}K(H_1(G; \mathbb{Z}), n + 1)$. Thus $H_1(G; \mathbb{Z})$ is $h(n + 1)$-perfect.

2.10. Lemma. Let $n \geq 1$ and $G$ a group, abelian if $n \geq 2$. Then the following are equivalent.

(i) $G$ is $hn$-acyclically reduced.

(ii) $+h$-localization yields a fibration of the form:

\[
\begin{array}{ccc}
K(A', n - 1) & \rightarrow & K(G, n) \\
\downarrow & & \downarrow \\
A_hK(G, n) & \rightarrow & K(G, n)^{+h}
\end{array}
\]

Proof. If $n = 1$, the claim follows from (2.7). Thus suppose $n \geq 2$.

(i) $\implies$ (ii) According to (2.5) the $+h$-localization fibration of an abelian group $G$ yields the exact sequence of groups $L \rightarrow A \rightarrow G \rightarrow L' \rightarrow A'$. Considering that $P^n_{h}G = 0$ we find that the arrows in this sequence behave as indicated. Thus the $h$-local space $K(L, n)$ is a retract of the $h$-acyclic space $K(A, n) \times K(A', n - 1)$, implying that $L = 0 = A$.

(ii) $\implies$ (i) In this situation, $\pi_nK(G, n) \rightarrow \pi_nK(G, n)^{+h}$ is a monomorphism, implying that $G$ is $hn$-acyclically reduced.

2.11. Corollary. A group $G$ is $hn$-acyclically reduced if and only if $[Y, K(G, n)] = 0$, for every $(n - 1)$-connected $h$-acyclic space $Y$. 

Proof. The case $n = 1$ follows from (2.10). If $n > 1$, we have associated with every $h$-acyclic space $Y$ the isomorphism $[Y, A_h(K(G, n))] \cong [Y, K(G, n)]$. If $G$ is $hn$-acyclically reduced, then $A_h(K(G, n))$ is of the form $K(A', n - 1)$ (2.10), implying that $[Y, A_h(K(G, n))] = 0$ for $(n - 1)$-connected $Y$. Conversely, if $[Y, K(G, n)] = 0$ for $(n - 1)$-connected $h$-acyclic $Y$, then, in the notation of (2.10), $K(A, n) \to K(G, n)$ is null, implying that $A = L = 0$. The claim follows from (2.10).

2.12. Lemma. For $n \geq 2$, if an abelian group $G$ is $hn$-acyclically reduced, then $G$ is $h(n - 1)$-acyclically reduced.

Proof. The fibration $A_h(K(G, n)) \to K(G, n) \to K(G, n)^{+h}$ has the form

$$K(A', n - 1) \to K(G, n) \to K(L', n);$$

see (2.10). Looping it gives us a monomorphism

$$\pi_{n-1}K(G, n - 1) \to \pi_{n-1}K(L', n - 1).$$

But $K(L', n - 1)$ is $h$-local by (2.2). So $L'$ is $h(n - 1)$-acyclically reduced. $G$ is a subgroup of $L'$, and the argument is complete by (2.4).  

Proof of proposition (2.3). (i) Closure under quotients. If $n = 1$, we recall that the $+h$-construction is a coning construction, hence induces an epimorphism of fundamental groups, hence preserves epimorphisms of fundamental groups. Therefore the class of $h1$-perfect groups is closed under quotients.

If $n \geq 2$, consider the $+h$-localization fibration of $K(G, n)$, namely

$$K(A, n) \to K(G, n) \to K(L, n + 1).$$

Here we used the fact that $G$ is $hn$-perfect and (2.3), (i) $\implies$ (ii). If $G/H$ is a quotient of $G$, we find a map from the $h$-acyclic space $K(A, n)$ to $K(G/H, n)$ inducing an epimorphism on $\pi_n$. The $+h$-localization of $K(G/H, n)$ factors through the cofiber of this map, implying that $G/H = P^h_n(G/H)$.

Closure under colimits. Every colimit of groups is a quotient of the coproduct of those groups which occur in the colimit diagram. (If $n = 1$, “coproduct” means “free product”. If $n > 1$, “coproduct” means “direct sum”.) Thus it suffices to show that $\mathcal{C}_n$ is closed under coproducts. A coproduct $\bigsqcup G_\lambda$ of $hn$-perfect groups arises as $\pi_n$ of the wedge $\bigsqcup\bigvee K(G_\lambda, n)$. Now $\bigvee A_\lambda$ is $h$-acyclic and $\pi_n(\bigvee A_\lambda \to \bigvee K(G_\lambda, n))$ is onto. Thus $\pi_n K(\bigsqcup G_\lambda, n)^{+h} = 0$, implying that $\bigsqcup G_\lambda$ is $hn$-perfect.

Closure under weak products. The $+h$-construction commutes with finite products (17, Theorem 4), and a finite product of $h$-acyclic spaces is again $h$-acyclic. Therefore the class $\mathcal{C}_n$ of $hn$-perfect groups is closed under finite products. The weak product is a directed colimit of finite products. Now use the fact that $hn$-perfect groups are closed under colimits.

(ii) By hypothesis, $A^{+h} \simeq A$. Thus $P^h_n\pi_nA = \pi_nA$. If $n > 1$, the technique used in (i) shows that $K(\pi_nA, n)^{+h} \simeq K(L, n + 1)$. Thus $\pi_nA$ is $hn$-perfect.

(iii) $\ker(\pi_1X \to \pi_1X^{+h})$ is a quotient of the $h1$-perfect group $\pi_1A_hX$. The claim follows from (i).

(iv) $\ker(\pi_nK(G, n) \to \pi_nK(G, n)^{+h})$ is a quotient of $\pi_n$ of an $(n - 1)$-connected $h$-acyclic space. By (ii) this kernel is $hn$-perfect. On the other hand, every $hn$-perfect subgroup $H$ of $G$ is contained in $P^h_nG$. This can be read off the morphism
of fibrations below:

\[
\begin{align*}
A_hK(H, n) & \longrightarrow K(H, n) \longrightarrow K(H, n)^{+h} \\
A_hK(G, n) & \longrightarrow K(G, n) \longrightarrow K(G, n)^{+h}
\end{align*}
\]

(v) Let \( K(G, n) \rightarrow K(L', n) \times K(L, n + 1) \) be the \(+h\)-localizing map; see (2.5). In any case \( B := P^n_h(G/P^n_hG) < L' \). If \( B \neq 0 \), there exists an essential map from an \( h \)-acyclic space into \( K(G, n)^{+h} \) – a contradiction.

(vi) holds because \(+k\)-localization factors through \(+h\)-localization and does not decrease connectivity of Eilenberg-MacLane spaces.

The proof of (2.3) is complete. \( \square \)

Proof of proposition (2.7). (i) If \( G \) is \( hn \)-acyclically reduced, let \( G' < G \) be a subgroup. Then we have a short exact sequence \( \pi_n A_hK(G', n) \rightarrow G' \rightarrow \pi_nK(G', n)^{+h} \), where the arrow on the right is a monomorphism because \( \text{im}(\pi_n A_hK(G', n) \rightarrow G') < \ker(G \rightarrow Q^n_nK) = 0 \). Thus \( G' \) is \( hn \)-acyclically reduced.

Let \( G := \lim \{ G_\lambda \} \) be the inverse limit of a system of \( hn \)-acyclically reduced groups. If \( Y \) is an \( h \)-acyclic space, then \( [Y, K(G_\lambda, n)] = * \) for each \( \lambda \); see (2.11). Thus \( \prod G_\lambda \) is \( hn \)-acyclically reduced. \( G \) is a subgroup of \( \prod G_\lambda \). The claim follows.

(ii) Let \( Q := \text{im}(\pi_nK(G, n) \rightarrow \pi_nK(G, n)^{+h}) \). For an arbitrary \( hn \)-acyclically reduced quotient \( G' \) of \( G \), we have the commutative diagram below; see (2.10):

\[
\begin{align*}
\xymatrix{
G \ar[r] & \pi_nK(G, n)^{+h} \\
G'/\sim \ar[r] \ar[ru] & \pi_nK(G', n)^{+h}
}
\end{align*}
\]

Thus \( G' \) is a quotient of \( Q \). If \( n \geq 2 \), \( Q \) is \( hn \)-acyclically reduced by (2.10), in which case the claim follows. If \( n = 1 \), we have the tower of \( h_1 \)-acyclic reductions

\[
G \rightarrow G/P^n_1G \rightarrow (G/P^1_1G)/P^n_1(G/P^1_1G) \rightarrow \ldots,
\]

which stabilizes, as a possibly transfinite tower, at a group \( R \) which is \( h_1 \)-acyclically reduced. By design, every epimorphism \( G \rightarrow G' \), with \( G' \) \( h_1 \)-acyclically reduced, factors through \( R \). Thus \( Q^n_1G = R \) is the unique maximal \( h_1 \)-acyclically reduced quotient of \( G \).

(iii) This follows from the fact that every \( h \)-acyclic space is also \( k \)-acyclic. \( \square \)

2.13. Remark. If \( X \) is an \( (n-1) \)-connected space, the morphism of fibrations

\[
\begin{align*}
A_hX & \longrightarrow X \longrightarrow X^{+h} \\
A_hK(\pi_nX, n) & \longrightarrow K(\pi_nX, n) \longrightarrow K(\pi_nX, n)^{+h}
\end{align*}
\]

shows that, always, \( \ker(\pi_nX \longrightarrow \pi_nX^{+h}) < P^n_n\pi_nX \). If \( h \) is connective and \( n \geq 2 \), then the two groups are equal by (1.10) and (10.10). In general, we have no accurate description of the relationship between the two groups.
If \( n = 1 \), we know that \( \ker(\pi_1 X \to \pi_1 X^{+h}) < P_1^h \pi_1 X \) is \( h \)-1-perfect; combine (2.3)i and ii). Should it happen that \( \ker(\pi_1 X \to \pi_1 X^{+h}) = P_1^h \pi_1 X \), for every \( X \), then \( \pi_1 X^{+h} \) depends only upon \( \pi_1 X \). Casacuberta-Rodriguez [3] call such a localizing functor \( \pi_1 \)-compatible. Tai [31] identifies the following homological plus-constructions as being \( \pi_1 \)-compatible.

(i) \( h_* (pt) \) has elements of infinite order and Bousfield’s transitional dimension \( d(h) \) for \( h \) is \( \geq 1 \).

(ii) \( h_* (pt) \) is torsion.

Bousfield’s transitional dimension is defined as follows. Let \( P(h) \) denote the set of primes \( p \) for which \( h_* (pt) \) is not uniquely \( p \)-divisible. For \( p \in P(h) \) set

\[
\begin{align*}
    d_p(h) & := \max \{ n \mid K(Z_p^n, n + 1) \text{ is } h\mathbb{Z}/p\text{-local} \} \leq \infty, \\
    d(h) & := \min \{ d_p(h) \mid p \in P(h) \}.
\end{align*}
\]

see [7].

2.14. Corollary. If \( +h \) is \( \pi_1 \)-compatible, then the following hold.

(i) For every space \( X \), \( P_1^h \pi_1 (X^{+h}) = 1 \).

(ii) For every group \( G \), \( P_1^h (G/P_1^h G) = 1 \).

Proof. (i) \( \iff \) (ii) is (2.13).

(ii) \( \implies \) (iii) Choose \( X = K(G, 1) \) in (ii).

(iii) \( \implies \) (ii) Choose a Kan-Thurston map \( K(V, 1) \to X \); see [22]. It follows that

\[
P_1^h \pi_1 X^{+h} \cong P_1^h (V/P_1^h V) = 1.
\]

(iii) \( \implies \) (iv) Given a space \( X \) with \( h \)-1-perfect fundamental group, choose a Kan-Thurston map \( K(U, 1) \to X \). It follows that \( K(U, 1)^{+hH} \cong X^{+hH} \) and, hence, \( K(U, 1)^{+h} \cong X^{+h} \). Thus \( U/P_1^h U = \pi_1 K(U, 1)^{+h} \cong \pi_1 X^{+h} \) is a quotient of the \( h \)-1-perfect group \( \pi_1 X \). By (2.3), \( U/P_1^h U \) is again \( h \)-1-perfect, hence is 1, by hypothesis (iii).

(iv) \( \implies \) (v) Suppose \( N \to G \to Q \) is an extension in which \( N \) and \( Q \) are \( h \)-1-perfect. Choose \( X := K(G, 1) \) to see that \( \pi_1 X^{+h} \cong G/P_1^h G \) is a quotient of \( Q \), hence is \( h \)-1-perfect; see (2.3). By hypothesis (iv), \( \pi_1 X^{+h} = 1 \), implying that \( G \) is \( h \)-1-perfect.

2.15. Example. Let \( K \) be real or complex \( K \)-homology. Then \( d_p(K) = 1 \), for all primes \( p \); see [13]. Thus \( d(K) = 1 \), implying that \( h \) is \( \pi_1 \)-compatible.

Here are some basic criteria for \( \pi_1 \)-compatibility.

2.16. Lemma. For a homology theory \( h \), the following are equivalent.

(i) \( +h \) is \( \pi_1 \)-compatible.

(ii) For every space \( X \), \( P_1^h (\pi_1 X^{+h}) = 1 \).

(iii) For every group \( G \), \( P_1^h (G/P_1^h G) = 1 \).

(iv) For every space \( X \), if \( \pi_1 X^{+h} \) is \( h \)-1-perfect, then \( X^{+h} \) is simply connected.

(v) If \( N \) and \( Q \) are \( h \)-1-perfect groups, then every extension of \( Q \) by \( N \) is \( h \)-1-perfect.

Proof. (i) \( \iff \) (ii) is (2.13).

(ii) \( \implies \) (iii) Choose \( X = K(G, 1) \) in (ii).

(iii) \( \implies \) (ii) Choose a Kan-Thurston map \( K(V, 1) \to X \); see [22]. It follows that

\[
P_1^h \pi_1 X^{+h} \cong P_1^h (V/P_1^h V) = 1.
\]

(iii) \( \implies \) (iv) Given a space \( X \) with \( h \)-1-perfect fundamental group, choose a Kan-Thurston map \( K(U, 1) \to X \). It follows that \( K(U, 1)^{+hH} \cong X^{+hH} \) and, hence, \( K(U, 1)^{+h} \cong X^{+h} \). Thus \( U/P_1^h U = \pi_1 K(U, 1)^{+h} \cong \pi_1 X^{+h} \) is a quotient of the \( h \)-1-perfect group \( \pi_1 X \). By (2.3), \( U/P_1^h U \) is again \( h \)-1-perfect, hence is 1, by hypothesis (iii).

(iv) \( \implies \) (v) Suppose \( N \to G \to Q \) is an extension in which \( N \) and \( Q \) are \( h \)-1-perfect. Choose \( X := K(G, 1) \) to see that \( \pi_1 X^{+h} \cong G/P_1^h G \) is a quotient of \( Q \), hence is \( h \)-1-perfect; see (2.3). By hypothesis (iv), \( \pi_1 X^{+h} = 1 \), implying that \( G \) is \( h \)-1-perfect.
A group $G$ gives rise to the following diagram of group extensions:

$$
\begin{array}{ccc}
P^h(G) & \rightarrow & P^h_1 G \\
\downarrow & & \downarrow \\
E & \rightarrow & G \\
\downarrow & & \downarrow \\
P^h_1(G/P^h_1 G) & \rightarrow & G/P^h_1 G \\
\end{array}
$$

By hypothesis, $E$ is $h_1$-perfect. Moreover, $E$ contains the unique maximal $h_1$-perfect subgroup $P^h_1 G$ of $G$; see (2.13.v). Thus $E = P^h_1 G$, implying that $P^h_1(G/P^h_1 G) = 1$. This completes the proof.

Every group $G$ is the colimit of the system of its finitely generated subgroups. Here is an analogue of this fact within the category of $hn$-perfect groups.

**2.17. Proposition.** Every $hn$-perfect group $G$ is the directed colimit of the system $G_\lambda$ of $hn$-perfect subgroups such that card($G_\lambda$) is less than or equal to the smallest infinite cardinal $c \geq \text{card}(h_*(pt))$.

**Proof.** We know that $G$ is a quotient of $\pi_n A_h K(G, n)$. From Bousfield [5, 11] we know that $A_h K(G, n)$ is the union of its acyclic subcomplexes $A_\lambda$ such that $A_\lambda$ has at most $c$ cells. Therefore each element of $A_h K(G, n)$ belongs to some $\pi_n A_\lambda$. Further $\pi_n A_\lambda$ is generated by at most $c$ elements. Consequently, card($\pi_n A_\lambda$) is less than or equal to $c$, because $c$ is infinite. We find that $\pi_n(A_h K(G, n) \rightarrow K(G, n))$ sends each $\pi_n A_\lambda$ to an $hn$-perfect subgroup $G_\lambda$ of $G$ with at most $c$ elements. By design, each element of $G$ belongs to some $G_\lambda$. The claim follows.

More generally, we make the following

2.18. **Definition.** A homology theory $h$ is $\pi_n$-compatible, $n \geq 1$, if, for every $(n-1)$-connected space $X$, $\pi_n X \rightarrow \pi_n X^{+h}$ depends only upon $\pi_n X$ (in particular, $\pi_n X^{+h} \cong \pi_n K(\pi_n X, n)^{+h}$).

By combining (6.10) with section 5 we obtain:

2.19. **Proposition.** Every connective $h$ is $\pi_n$-compatible, for $n \geq 1$.

2.20. **Remark.** We know of no homology theory which is not $\pi_1$-compatible. However, in general, homology theories need not be $\pi_n$-compatible if $n \geq 2$. To see this, we take $h = K$ (real or complex $K$-theory) and $X := BSO$. Now $\pi_3 BSO = \mathbb{Z}/2$, which is $K2$-perfect; see [1,8]. On the other hand, $X$ is $K$-local by Meier’s theorem [24], implying that $\text{ker}(\pi_2 BSO \rightarrow \pi_2 BSO^{+K}) = 0$. Thus $K$ is not $\pi_2$-compatible.

3. **$+hn$-LOCALIZATION OF ABELIAN GROUPS**

3.1. **Definition.** An abelian group $G$ is $+hn$-local if $K(G, n) = K(G, n)^{+h}$.

3.2. **Remark.** By (2.6), $G$ is $+hn$-local if and only if $\pi_n K(G, n) \rightarrow \pi_n K(G, n)^{+h}$ is an isomorphism. Therefore, if $h$ is $\pi_n$-compatible and $X$ is $(n-1)$-connected, then $\pi_n X^{+h}$ is $+hn$-local.
3.3. Corollary. For all abelian groups $G$ and $n \geq 1$, $\tau_n G = \tau_n (\tau_{n-1} G)$. \hfill \square

The following examples illustrate the effect of the functor $\tau_n$. The proofs follow directly from the works of Bousfield [6] and Mislin [26].

3.4. Example. Here $h = H(\cdot; \mathbb{Z}/p)$ for a prime $p$. If $n \geq 2$, then $\tau_n$ is the group $\text{Ext}(\mathbb{Z}_{p^n}, G)$, the “ext-$p$-completion” of the abelian group $G$. The maximal $h$-perfect subgroup of $G$ is $P^h_1 G = \bigcup \{ U < G \mid U/pU = 1 \}$. Then $\tau_1 G = G/P^h_1 G = \text{im}(G \to \text{Ext}(\mathbb{Z}_{p^n}, G))$.

3.5. Example. Here $h = K$, real or complex $K$-theory. For an abelian group $G$ we have $\tau_1 G = G$, $\tau_2 G = G/\text{torsion}(G)$ and $\tau_3 G = \mathbb{Q} \otimes \mathbb{Z}$ if $n \geq 3$.

3.6. Lemma. For $n \geq 1$, let $\varphi : L_1 \to L_2$ be a map between $+hn$-local abelian groups. Then $\ker(\varphi)$ is $+hn$-local and $\coker(\varphi)$ is $+h(n - 1)$-local.

Proof. By (1.3), we have $\text{fib}(\varphi, n) \simeq K(\coker(\varphi), n - 1) \times K(\ker(\varphi), n)$ and is $+hn$-local. The result follows. \hfill \square

3.7. Theorem. Let $n \geq 1$ and let $G$ be a group, abelian if $n \geq 2$. Then the group $K(P^h_n G, n)^{+h}$ is the $n$-connected cover of $K(G, n)^{+h}$.

Proof. For $n = 1$, this follows from the definition of $P^h_1 G$ and lemma (1.3). For $n > 1$, it suffices to show that $K(P^h_n G, n)^{+h} \to K(G, n)^{+h}$ induces an isomorphism in $\pi_{n+1}$; see (2.5). Indeed, using the notation of lemma (2.5), we have an exact sequence

$$0 \longrightarrow L \longrightarrow A \longrightarrow G \longrightarrow L' \longrightarrow A' \longrightarrow 0.$$

By classifying the short exact sequence on the left we obtain the fibration $K(A, n) \to K(P^h_n G, n) \to K(L, n + 1)$. Apply $+h$-localization to obtain the fibration $\ast \simeq K(A, n)^{+h} \to K(P^h_n G, n)^{+h} \to K(L, n + 1)$, from which the claim follows. \hfill \square

4. $hn$-CENTRAL EXTENSIONS

Given a homology theory $h$ and a group $G$, according to (7.7), we have the central extension $\pi_{n+1} K(G, n)^{+h} \to \pi_n A_h K(G, n) \to P^h_n G$. In addition, this sequence inherits universal properties from the universal properties of the colocalization/localization-fibration $A_h K(G, n) \to K(G, n) \to K(G, n)^{+h}$.

4.1. Definition. A central extension of groups $L \to H \to G$ is called $hn$-central if $L$ is $+h(n + 1)$-local.
4.2. Definition. A central extension

\[ (C) \quad A \rightarrowtail E \twoheadrightarrow G \]

is universal with respect to \( h^n \)-central extensions if every diagram

\[ (C) \quad A \rightarrowtail E \twoheadrightarrow G \]

\[ (\xi) \quad L \rightarrowtail H \twoheadrightarrow G \]

with \( h^n \)-central bottom row, can be filled uniquely as indicated. If, in addition, \( A \) is \( +h(n+1) \)-local, then we call \( (C) \) a universal \( h^n \)-central extension.

The main objective of this section is to establish the following statements.

4.3. Theorem. Every \( h^n \)-perfect group \( G, n \geq 1 \), gives rise to the central extension

\[ (C) \quad \pi_{n+1}K(G,n)^{+h} \rightarrowtail \pi_nA_hK(G,n) \twoheadrightarrow G \]

which is universal with respect to \( h^n \)-central extensions. Moreover, \( (C) \) is a universal \( h^n \)-central extension if \( G \) is abelian or if \( h \) is \( \pi_2 \)-compatible; see \( 2.18 \).

4.4. Proposition. Suppose \( G \) is abelian or \( h \) is \( \pi_2 \)-compatible. Then, for \( n \geq 1 \), the \( +h(n+1) \)-local group \( \pi_{n+1}K(G,n)^{+h} \) is a representing object for the functor

\[ \text{Ext}(P_n^{h}G,-) = H^2(P_n^{h}G;-) : +h(n+1)\mathbb{A}\mathbb{B} \rightarrow \mathbb{A}\mathbb{B}. \]

Proof. This follows from \( 4.12 \). \( \square \)

4.5. Remark. This is very much like in the classical situation considered by Kervaire: For any group \( G \), \( \pi_2K(G,1)^{+} \) is the representing object for the functor on \( \mathbb{Z} \)-modules

\[ H^2(PG; -) \cong \text{Hom}(H_2(PG;\mathbb{Z}), -) \cong \text{Hom}(\pi_2K(G,1)^{+}, -), \]

where \( PG \) denotes the maximal perfect subgroup of \( G \).

4.6. Theorem. Suppose \( h \) is \( \pi_2 \)-compatible. If \( G \) is an arbitrary group, then

\[ \pi_3K(G,1)^{+h} \cong H_3(K(P_1^{h}G,1)^{+h};\mathbb{Z}), \]

where \( P_1^{h}G \) is the universal \( h1 \)-central extension of \( P_1^{h}G \); see \( 4.13 \).

Finally, we analyze the situation where \( n = 1 \) and \( h \) is not necessarily \( \pi_2 \)-compatible. For an arbitrary abelian group \( H \) we have the natural isomorphism \( \text{Hom}(\pi_2H,L) \cong \text{Hom}(H,L) \) whenever \( L \) is \( +h2 \)-local. Moreover, if \( G \) is any \( h1 \)-perfect group, then \( H_1G \) is \( h1 \)-perfect too and, therefore, \( K(H_1G,1)^{+h} \cong K(\Lambda,2) \) for some \( +h2 \)-local group \( \Lambda \). Thus, for an arbitrary group \( G \),

\[ \pi_2H_2(H_1P_1^{h}G;\mathbb{Z}) = H_2(H_1P_1^{h}G;\mathbb{Z}). \]

As a result, we can compute the image of \( \pi_2K(G,1)^{+h} \cong \pi_2K(P_1^{h}G,1)^{+h} \) under the functor \( \tau_2 \) as follows.
4.7. Proposition. Associated to an $h1$-perfect group $G$ there is the natural short exact sequence

$$\tau_2 H_2G \xrightarrow{\varphi} \tau_2 H_2K(G,1)^{+h} \xrightarrow{\lambda} H_2K(H_1G,1)^{+h}$$

$$\tau_2\pi_2 K(G,1)^{+h} \quad \pi_2 K(H_1G,1)^{+h}$$

The sequence splits, but not naturally. Moreover, $\tau_2\pi_2 K(G,1)^{+h}$ is the kernel of the universal $h1$-central extension of $G$.

We proceed to prove these statements and establish some related facts.

Proof of theorem (4.7). From (4.4), we know that (C) is a central extension. To establish its universal property, let $(\zeta): L \to H \to G$ be an $hn$-central extension. We need to show that the identity map on $G$ lifts to a unique morphism $\beta : \pi_n A_h K(G, n) \to H$. To see that $\beta$ exists, consider the commutative diagram below:

$$\Omega K(G, n)^{+h} \xrightarrow{\Omega v} K(L, n)$$

$$A_h K(G, n) \xrightarrow{u} K(H, n)$$

$$K(G, n) \xrightarrow{q} K(L, n)$$

Here $\gamma$ classifies the principal fibration associated to $(\zeta)$. The universal property of the localizing map $K(G, n) \to K(G, n)^{+h}$ yields $v$ uniquely such that the bottom square commutes. Thus a map $u$ exists, and $\pi_n u$ is a candidate for $\beta$. To see that $\beta$ is unique, we argue as follows. Suppose $\beta' : \pi_n A_h K(G, n) \to H$ also makes the diagram commute. Let $\overline{\beta}$ denote the function obtained by following $\beta$ by the group inverse operation on $H$. Then $\overline{\beta} \overline{\beta} : \pi_n A_h K(G, n) \to H$ is a homomorphism because $L$ is central in $H$. Realize $\overline{\beta} \overline{\beta}$ by a map $u' \overline{u} : A_h K(G, n) \to K(H, n)$. This is possible since $A_h K(G, n)$ is $(n - 1)$-connected; see (2.8) if $n \geq 2$. By design, $q \circ (u' \overline{u}) \simeq 0$. Consequently, $u' \overline{u} : A_h K(G, n) \to K(L, n)$, which is 0 because $A_h K(G, n)$ is $h$-acyclic and $K(L, n)$ is $h$-local. Thus $\beta = \beta'$.

Finally, $\pi_{n+1} K(G, n)^{+h}$ is $+h(n + 1)$-local if $G$ is abelian, see (2.8), or if $h$ is $\pi_2$-compatible; see (2.18). The proof is complete.

4.8. Lemma. If $h$ is $\pi_2$-compatible and $G$ is $h1$-perfect, then $K(\pi_1 A_h K(G, 1), 1)^{+h}$ is 2-connected.

Proof. Write $U := \pi_1 A_h K(G, 1)$, the universal $h1$-central extension of $G$; see (4.3). The cofiber sequence $A_h K(G, 1) \to K(U, 1) \to \Gamma$ has $\Gamma 1$-connected and induces in $H(\Gamma; \mathbb{Z})$ the exact sequence

$$H_2 A_h K(G, 1) \to H_2 K(U, 1) \to H_2 \Gamma \to H_1 A_h K(G, 1) \xrightarrow{\pi} H_1 K(U, 1).$$
Thus $\Gamma$ is 2-connected. But $K(U, 1) \to \Gamma$ is a $+h$-equivalence, because $A_h K(G, 1) \to *$ is. Consequently, we get a homotopy equivalence $K(U, 1)^{+h} \to \Gamma^{+h}$, with $\Gamma^{+h}$
2-connected, since $h$ is $\pi_2$-compatible. \hfill \Box

As a corollary, we obtain the

\textbf{Proof of theorem (4.6).} We know from \cite{57} that $\pi_3 K(G, 1)^{+h} \cong \pi_3 K(P^h G, 1)^{+h}$. The universal $h_1$-central extension $C \to U \to P^h G$ yields the framed fibration

$$K(C, 1) \to K(U, 1) \to K(P^h G, 1) \xrightarrow{\gamma} K(C, 2),$$

which is classified by $\gamma$. $K(C, 2)$ is $+h$-local. Thus we obtain the fibration

$$K(U, 1)^{+h} \to K(P^h G, 1)^{+h} \to K(C, 2);$$

see \cite{13} p. 74. In the long exact sequence of homotopy groups of this fibration, we find the isomorphism $\pi_3 K(U, 1)^{+h} \to \pi_3 K(P^h G, 1)^{+h}$. The first group is isomorphic to $H_3 K(U, 1)^{+h}$, by \cite{13}. This completes the proof. \hfill \Box

\textbf{4.9. Lemma.} Let $(\zeta) : L \to U \to G$ be a universal $hn$-central extension, and let $(\zeta') : L' \to H \to G$ be an $hn$-central extension. If $H$ is $hn$-perfect, then the unique map $\beta : U \to H$ is onto.

\textit{Proof.} The universal property of $(\zeta)$ yields uniquely the commutative diagram below:

$$\begin{array}{ccc} 
L & \xrightarrow{\alpha} & U \\
\downarrow{\alpha} & & \downarrow{\beta} \\
L' & \xrightarrow{\beta} & H \\
\end{array} \xrightarrow{\gamma} \begin{array}{ccc} 
& & G \\
& & \\
\end{array}$$

An arbitrary element of $H$ differs from $\text{im}(\beta)$ by an element of $L$, implying that $\text{im}(\beta)$ is normal in $H$. Therefore, $\text{coker}(\alpha) \cong \text{coker}(\beta)$. But $\text{coker}(\beta)$ is $hn$-perfect by \cite{23}, and $\text{coker}(\alpha)$ is $hn$-local by \cite{36}. Thus both groups are trivial, implying that $\beta$ is onto. \hfill \Box

The following facts are needed in order to bring methods from homological algebra to bear on $hn$-central extensions.

\textbf{4.10. Lemma.} If $G$ is an abelian $hn$-perfect group, $n \geq 1$, then every $hn$-central extension of $G$ is abelian.

\textit{Proof.} For a central extension $L \to E \to G$, the commutator map $\gamma : E \times E \to E$, $\gamma(t, x) := txt^{-1}x^{-1}$, factors to a bilinear map $\tilde{\gamma} : G \times G \to L$. Now $\tilde{\gamma}$ is trivial because $G \times G$ is $hn$-perfect, \cite{23}, and $L$ is $hn$-local; see \cite{12}. The claim follows. \hfill \Box

\textbf{4.11. Proposition.} To an $h_1$-perfect group $G$, there is associated the universal $h_1$-central extension $(\zeta_G)$, given by the pushout construction below:

$$\begin{array}{ccc} 
(CG) & \pi_2 (BG)^{+h} & \xrightarrow{\tau_2} \pi_1 A_h BG \\
& \xrightarrow{\pi_2} & \xrightarrow{G} \\
(\zeta_G) & C^h G & \xrightarrow{\text{pushout}} U^h G \xrightarrow{G} \\
\end{array}$$

Here $C^h G := \tau_2 (\pi_2 K(G, 1)^{+h})$; see section 3.
Proof. $(\zeta G)$ is $h_1$-central by design. To establish its universal property, consider the commutative diagram

\[
\begin{array}{c}
(CG) & \pi_2 K(G, 1)^{+h} & \rightarrow & \pi_1 A_h K(G, 1) & \rightarrow & G \\
\downarrow & & & & & \\
(\zeta) & \tau_2 & & & & \\
\downarrow & & & & & \\
(\zeta G) & C^h G & \rightarrow & U^h G & \rightarrow & G
\end{array}
\]

$(\zeta)$ is $hn$-central. Therefore we are entitled to the unique morphism $(CG) \rightarrow (\zeta)$, by (4.3). The unique map $\alpha'$ comes from the universal property of the localizing map $\tau_2$. The unique map $\beta'$ over $\text{Id}_G$ comes from the pushout construction of $U^h G$. This completes the proof.

4.12. Corollary. Given an abelian $+hn$-perfect group $G$, $n \geq 1$, let $(\zeta G) : L \rightarrow E \rightarrow G$ be the universal $hn$-central extension of $G$; see (4.3). If $L'$ is an arbitrary $+h(n+1)$-local group, then there is the natural equivalence $\text{Hom}(L, L') \rightarrow \text{Ext}(G, L')$.

4.13. Proposition. For an $hn$-perfect group $G$ and a $+h(n+1)$-local group $L$, there is a natural short exact sequence

\[
\text{Ext}(H_1 G, L) \xrightarrow{\alpha_L} H^2(G; L) \xrightarrow{\beta_L} \text{Hom}(H_2 G, L)
\]

Moreover, $\alpha_L$ is an isomorphism whenever $G$ is abelian.

Proof. The top row is the universal coefficient sequence. $H^2(G; L)$ classifies central extensions of $G$ by $L$. Such extensions are in natural bijective correspondence with $\text{Hom}(\pi_{n+1} K(H_1 G, n)^{+h}, L)$, using (4.3). Further, $\text{Ext}(H_1 G, L)$ consists of abelian extensions of the $h_1$-perfect group $H_1 G$ by $L$. According to (4.10), all central extensions of $H_1 G$ by $L$ are automatically abelian. The map

$\text{Hom}(\pi_{n+1} K(H_1 G, n)^{+h}, L) \rightarrow \text{Ext}(H_1 G, L)$

is a natural isomorphism by (4.3). The claim follows.

4.14. Corollary. If $G$ is abelian, then $\text{Hom}(H_2 G, L) = 0$, whenever $G$ is $hn$-perfect and $L$ is $+h(n+1)$-local.

4.15. Corollary. If $G$ is abelian and $hn$-perfect, then $H_2(G; \mathbb{Z})$ is $h(n+1)$-perfect.

Proof. From (4.14) we see that $\text{Hom}(H_2 G, L) = 0$, whenever $L$ is $+h(n+1)$-local. But

$\text{Hom}(H_2 G, L) = [K(H_2 G, n+1), K(L, n+1)]$

$= [K(H_2 G, n+1)^{+h}, K(L, n+1)]$

Choosing $L := \pi_{n+1} K(H_2 G, n+1)^{+h}$ shows that $H_2 G$ is $+h(n+1)$-perfect.
Proof of \((4.7)\). From \((4.13)\) we get, for each \(\pm h\)-2-local \(L\), the natural short exact sequence

\[
\text{Hom}(H_2K(H_1G,1)^{\pm h}, L) \xrightarrow{\alpha_L} \text{Hom}(\tau_2H_2K(G,1)^{\pm h}, L) \xrightarrow{\beta_L} \text{Hom}(\tau_2H_2G, L).
\]

This yields the split short exact sequence in question for formal reasons; see \((6.1)\). The identification of \(\tau_2\pi_2K(G,1)^{\pm h}\) as the kernel of the universal \(h\)-1-central extension of \(G\) follows from \((4.11)\).

5. Examples

Here we consider the two cases where \(h = H(\varnothing, \mathbb{Z}_p)\), \(P\) a set of primes, or \(h = H(\varnothing, \mathbb{Z}/p)\), \(p\) a prime.

5.1. Example. Let \(h = H(\varnothing, \mathbb{Z}_p)\). In this case, each Moore space \(M(\mathbb{Z}/p, 1)\), \(p \notin P\), is \(h\)-acyclic. Thus \(+h\)-localization factors through Anderson’s localization; see \([1]\), compare \([10]\). Consequently a \(\pm h\)-local space \(X\) has higher homotopy groups which are \(\mathbb{Z}_p\)-modules and, for \(p \notin P\), the \(p\)-th power function on \(\pi_1X\) is injective. If \(X\) is simply connected then \(X\) is \(\pm h\)-local if and only if \(X\) is \(h\)-local, which holds if and only if all homotopy groups of \(X\) are \(\mathbb{Z}_p\)-modules; compare \((1.6)\). In particular, \(h\) is \(\pi_n\)-compatible, for all \(n \geq 1\). Also, a group is \(h\)-1-perfect if and only if \(H_1(G; \mathbb{Z}_p) = 0\). Generalizing Kervaire’s results, see \([23]\), compare \([3, \text{Chap. 8}]\), we conclude at the level of fundamental groups that:

(i) For every group \(G\), \(P^h\pi_1K(G,1)^{\pm h} = 1\); i.e. \(h\) is \(\pi_1\)-compatible.
(ii) \(\pi_2K(G,1)^{\pm h} \rightarrow \pi_1A_1K(G,1) \rightarrow P^h_1G\) is the universal \(h\)-1-central extension of \(P^h_1G\).
(iii) In view of the discussion preceding \((4.7)\), we have

\[
\pi_2K(G,1)^{\pm h} \cong \pi_2K(P^h_1G,1)^{\pm h} \cong H_2(K(P^h_1G,1)^{\pm h}; \mathbb{Z}).
\]

The latter object is a \(\mathbb{Z}_p\)-module, hence is isomorphic to \(H_2(P^h_1G; \mathbb{Z}_p)\).
(iv) \(\pi_2K(G,1)^{\pm h} \cong H_3(K(P^h_1G,1)^{\pm h}; \mathbb{Z}) \cong H_3(P^h_1G; \mathbb{Z}_p)\), where \(P^h_1G\) denotes the universal \(h\)-1-central extension of \(P^h_1G\); see \((4.6)\).

At the level of higher Eilenberg-MacLane spaces we conclude that \(K(G,n)\) is \(+hn\)-perfect if and only if \(G\) is \(P^\prime\) torsion. In this case \(K(G,n)\) is \(h\)-acyclic. For an arbitrary abelian group \(G\), the \(+h\)-construction yields the fibration

\[
K(T_{p^\prime}G, n) \times K(\text{coker}(G \rightarrow \mathbb{Z}_p \otimes G), n - 1) \rightarrow K(G, n) \rightarrow K(\mathbb{Z}_p \otimes G, n),
\]

where \(T_{p^\prime}G\) denotes the \(P^\prime\)-torsion subgroup of \(G\).

5.2. Example. Let \(h = H(\varnothing, \mathbb{Z}/p)\), where \(p\) is a prime. On simply connected spaces we know that \(+h\)-localization agrees with \(h\)-localization; see \((1.7)\). Further, on \(p\)-good spaces, \(h\)-localization agrees with \(p\)-completion in the sense of Bousfield-Kan \([2]\).

Thus, for an abelian group \(G\) and \(n \geq 2\), \([5, 4.3]\) yields the short exact sequence

\[
1 \longrightarrow \text{Ext}(\mathbb{Z}_{p^n}, \pi_1K(G,n)) \longrightarrow \pi_1K(G,n)^{\pm h} \longrightarrow \text{Hom}(\mathbb{Z}_{p^n}, \pi_{1-1}K(G,n)).
\]

Therefore \(G\) is \(hn\)-perfect if and only if \(\text{Ext}(\mathbb{Z}_{p^n},G) = 0\). According to \([9, \text{p. 166}]\), this happens exactly when \(\lim \frac{1}{1} \text{Hom}(\mathbb{Z}/p^n, G) = 0 = \lim \{G/p^nG\}\).

Turning to the effect of \(+h\)-localization on \(K(G,1)\)’s, we begin by identifying \(h\)-1-perfect groups.
5.3. Lemma. For \( h = H(-; \mathbb{Z}/p) \), a group \( G \) is \( h \)-1-perfect if and only if \( H_1(G; \mathbb{Z}/p) = 0 \).

Proof. If \( G \) is \( h \)-1-perfect, then \( K(G, 1)^{+h} \) is 1-connected. Thus \( H_1(G; \mathbb{Z}/p) = 0 \) because \( K(G, 1) \rightarrow K(G, 1)^{+h} \) induces an \( h \)-isomorphism. Conversely, if \( H_1(G; \mathbb{Z}/p) = 0 \), then \( K(G, 1)^{+h} \) is 1-connected. To see this, recall that \( \pi_1 K(G, 1)^{+h} \) is an \( h \)-local group, and such a group \( \Gamma \) vanishes exactly when \( H_1(\Gamma; \mathbb{Z}/p) = 0 \). By (1,6), \( K(G, 1)^{+h} \simeq K(G, 1)^{+h} \), implying that \( G \) is \( h \)-1-perfect.

We know from (2,3v) that every group \( G \) has a unique maximal \( H\mathbb{Z}/p \)-perfect subgroup \( P^hG \); i.e. \( P^hG \) is maximal in \( G \) with \( \mathbb{Z}/p \otimes H_1(P^hG; \mathbb{Z}) = 0 \). \( H\mathbb{Z}/p \) is seen to be \( \pi_1 \)-compatible, either by [30, 6.1], or by showing that \( h \)-1-perfect groups are closed under extensions (Serre spectral sequence) and invoking (2,16). Also, if \( X \) is \((n-1)\)-connected and \( n \geq 2 \), then (cf. Bousfield [5, 4.3]) \( \pi_nX \rightarrow \pi_nX^h \) agrees with \( \pi_nX \rightarrow \text{Ext}(\mathbb{Z}^p, \pi_nX) \) and \( h \) is thus \( \pi_n \)-compatible for \( n \geq 2 \) as well. This, in conjunction with the discussion in (5.1), establishes the claim (2,19).

5.4. Proposition. For \( h = H(-; \mathbb{Z}/p) \), the group \( \pi_2 K(G, 1)^{+h} \) fits into the natural short exact sequence

\[
\text{Ext}(\mathbb{Z}^p, H_2(P^hG; \mathbb{Z})) \rightarrow _{} \pi_2 K(G, 1)^{+h} \rightarrow _{} \text{Hom}(\mathbb{Z}^p, H_1(P^hG; \mathbb{Z})).
\]

The sequence splits, but not naturally.

Proof. From (4,7) we get the natural short exact sequence

\[
\tau_2 H_2(P^hG; \mathbb{Z}) \rightarrow _{} \pi_2 K(G, 1)^{+h} \rightarrow _{} \pi_2 K(H_1(P^hG; \mathbb{Z}), 1)^{+h},
\]

which splits. The terms at the end are homotopy groups of \( h \)-localizations of abelian Eilenberg-MacLane spaces. Further, [9, p. 183],

\[
\tau_2 H_2(P^hG; \mathbb{Z}) \cong \text{Ext}(\mathbb{Z}^p, H_2(P^hG; \mathbb{Z}))
\]

and

\[
\pi_2 K(H_1 P^hG, 1)^{+h} \cong \text{Hom}(\mathbb{Z}^p, H_1(P^hG; \mathbb{Z})).
\]

6. AN ALGEBRAIC LEMMA

Let \( \mathcal{C} \) be a class of abelian groups with the following properties:

(i) \( \mathcal{C} \) is closed under products.
(ii) If \( A \in \mathcal{C} \) and \( A \cong B \times C \), then \( B, C \in \mathcal{C} \).

The sole purpose of this section is to formulate the following lemma, which we need in the proof of (4,7).

6.1. Lemma. Suppose for groups \( A, B, C \in \mathcal{C} \) there is, for each \( L \in \mathcal{C} \), a natural short exact sequence

\[
(H_{\text{Hom-S}}) \quad \quad \quad \text{Hom}(A, L) \xrightarrow{\alpha_L} \text{Hom}(B, L) \xrightarrow{\beta_L} \text{Hom}(C, L).
\]

Then there is a natural short exact sequence

\[
(S) \quad \quad \quad A \xleftarrow{\lambda} B \xrightarrow{\phi} \text{Hom}(C, L) \xrightarrow{\psi} C,
\]

which induces (Hom-S). Further, (S) splits, but not naturally.
Proof. We obtain maps $\lambda$, $\phi$ and $\psi$ as follows: Choose $L := B$ to find the map $\beta_B(\text{Id}_B) := \varphi: C \rightarrow B$. It induces $\beta_L$ for arbitrary $L$. Next choose $L := C$. Since $\beta_L$ is onto, there exists some $\psi: B \rightarrow C$ such that $\beta_L(\psi) = \text{Id}_C$. It follows that $\psi \circ \varphi = \text{Id}_C$ and, therefore, $\phi$ is a split monomorphism and $\psi$ is onto. Next choose $L := A$ to find the map $\lambda := \alpha_A(\text{Id}_A): B \rightarrow A$. The proof that these maps satisfy the required properties is a bit tedious but entirely elementary. We omit it. 

7. Some fiber lemmas

The lemma and its corollary below are presumably known, even though we are unable to find a reference. They are included here because the corollary is needed in the proof of (3.6).

7.1. Lemma. Given a based continuous map $f: E \rightarrow B$, let $E_0 = \{ (e, \alpha) \in E \times B^I \mid f(e) = \alpha(0) \}$ and let $f': E' \rightarrow B$, $(e, \alpha) \mapsto \alpha(1)$, be the homotopy theoretical replacement of $f$ by the fibration $f'$. Then the following hold. If $f$ is a homomorphism of topological monoids (topological groups, abelian topological groups), then $\text{fib}(f)$ is a topological monoid (topological group, abelian topological group) of $E'$.

Proof. Since $E$ and $B$ are topological monoids (topological groups, topological abelian groups), so is $E'$ via the operation $E' \times E' \rightarrow E'$, $(e, \alpha) \cdot (e_1, \alpha_1) = (ee_1, \alpha \alpha_1)$, and from this the claim can be read. 

7.2. Corollary. Let $f: K(A, n) \rightarrow K(B, n)$ be a continuous map between abelian Eilenberg-MacLane spaces, $n \geq 1$. Then

$$\text{fib}(f) \simeq K(\ker(\pi_n f), n) \times K(\coker(\pi_n f), n - 1).$$

Proof. We may assume that $f$ is a homomorphism of abelian topological groups. By (7.1), $\text{fib}(f)$ is an abelian topological group. Thus $\text{fib}(f)$ is a retract of the infinite symmetric product $\text{SP}^\infty \text{fib}(f)$ and, hence, is a product of Eilenberg-MacLane spaces

$$\text{fib}(f) \simeq \prod_{k \geq 1} K(\pi_k \text{fib}(f), k);$$

see [13], compare [14, p. 88f]. The claim follows. 

The proposition below is the key to all centrality phenomena which are associated with fibrations of the form $A_hX \rightarrow X \rightarrow X^{+h}$. It constitutes a strengthened version of [20, A.2].

7.3. Proposition. Let $f: E \rightarrow B$ be a morphism of topological monoids which have a homotopy inverse. Then $\Omega B \xrightarrow{\gamma} \text{fib}(f) \rightarrow E$ is a homotopy central extension; i.e. the commutator map

$$\gamma: \Omega B \times \text{fib}(f) \xrightarrow{j \times \text{Id}} \text{fib}(f) \times \text{fib}(f) \xrightarrow{[\cdot, \cdot]} \text{fib}(f)$$

is null homotopic.

It is possible to give a direct proof of this claim. It follows, in spirit, the classical argument that higher homotopy groups are commutative; see e.g. [33, p.125]. An alternate argument can be based on unpublished work of M. Arkowitz [2]. However, for our present purposes, it will suffice to establish the following corollary (7.7) which only depends upon the much simpler self-contained development below.
7.4. **Definition.** A based fibration \( F \overset{i}{\to} E \to X \) is called \( \Pi \)-central if all Whitehead products \([i, \alpha, \beta]\) vanish, where \( \alpha \in \pi_p F, \beta \in \pi_q E \) and \( p, q \geq 1 \).

7.5. **Example.** The path fibration over any based space \( X \) is \( \Pi \)-central.

7.6. **Lemma.** Any pullback of a \( \Pi \)-central fibration along a based map is \( \Pi \)-central.

**Proof.** Given a pullback diagram of a \( \Pi \)-central fibration,

\[
\begin{array}{ccc}
F & \overset{i}{\to} & E \\
\downarrow & & \downarrow u \\
F' & \overset{i'}{\to} & W \\
\downarrow & & \downarrow u' \\
\downarrow f & & \downarrow \text{pullback} f \\
\downarrow f' & & \downarrow \\
X & \overset{}{\to} & Y
\end{array}
\]

we need to check that \([i', \alpha, \beta]\) = 0 whenever \( \alpha \in \pi_p F \) and \( \beta \in \pi_q W \). Writing \( W \) as the appropriate subspace of \( E \times X \), we find that \( i' \alpha = (f' \alpha, u' \alpha) = (i \alpha, 0) \) and \( \beta = (f' \beta, u' \beta) \). Therefore, \([i, \alpha, \beta]\) = \((i \alpha, f' \beta, 0)\) = \((0, 0)\), since \( F \to E \to Y \) is a \( \Pi \)-central fibration.

7.7. **Corollary.** If \( f : Y \to X \) is a based map, then \( \Omega X \to \text{fib}(f) \to Y \) is a \( \Pi \)-central fibration.

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