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# SHINTANI FUNCTIONS ON $GL(2, \mathbf{C})$

MIKI HIRANO

Abstract. In this paper, in analogy to the real case, we give a formulation of the Shintani functions on $GL(2, \mathbf{C})$, which have been studied by Murase and Sugano within the theory of automorphic $L$-functions. Also, we obtain the multiplicity one theorem for these functions and an explicit formula in a special case.

## 1. Introduction

Various models for admissible representations of algebraic groups are closely connected with the theory of automorphic forms and automorphic representations. For example, the Whittaker model plays a central role in the theory of automorphic $L$-functions, and has been studied in the contexts of both number theory and representation theory.

The (Whittaker-)Shintani model studied in the papers of Murase and Sugano [8]–[10] is also interesting (see also [4], [7], [12] at the real prime). In [10], they defined the local and global Shintani functions on $GL(n)$ related to the subgroup $GL(n-1) \times GL(1)$ embedded diagonally in $GL(n)$, and obtained a new integral formula for the standard $L$-functions. Also, they proved the multiplicity one theorem for the local Shintani functions at the finite primes. However, they did not consider the multiplicity of the archimedean local Shintani functions, which are connected with the gamma factors of $L$-functions.

In this paper, in analogy to the real case [4], we define the Shintani functions on the real reductive group $G = GL(n, \mathbf{C})$ as follows. Let $H$ be a subgroup $\{\mathrm{diag}(g_0, g) \mid (g_0, g) \in GL(n-1, \mathbf{C}) \times GL(1, \mathbf{C})\}$ of $G$, and let $K \simeq U(n)$ be a maximal compact subgroup of $G$. Moreover, let $(\eta, \mathcal{F}_\eta)$ be an irreducible admissible representation of $H$, and let $\pi^*$ be the contragredient representation of an irreducible admissible representation $\pi$ of $G$. We consider the intertwining space $\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}^{\mathbb{C}},K)}(\pi^*, \mathrm{C}^\infty\mathrm{Ind}_H^G(\eta))$ between $(\mathfrak{g}^{\mathbb{C}}, K)$-modules and define a *Shintani function of type* $(\eta, \pi)$ to be a function in its image in the representation space $\mathrm{C}_\eta^\infty(H\backslash G)$ of $C^\infty$ induced representation $\mathrm{C}^\infty\mathrm{Ind}_H^G(\eta)$. Here $\mathrm{C}_\eta^\infty(H\backslash G)$ is the space of $C^\infty$-functions $F : G \to \mathcal{F}_\eta$ satisfying $F(hg) = \eta(h)F(g)$ for all $(h, g) \in H \times G$. Also we consider the restriction

$$\mathcal{I}_{\eta,\pi} \to \mathrm{Hom}_K(\tau^*, \mathrm{C}^\infty\mathrm{Ind}_H^G(\eta)) \cong \mathrm{C}_{\eta,\tau}^\infty(H\backslash G/K)$$

of $\mathcal{I}_{\eta,\pi}$ to the minimal $K$-type $(\tau^*, V_{\tau^*})$ of $\pi^*$, where $\mathrm{C}_{\eta,\tau}^\infty(H\backslash G/K)$ is the space of $C^\infty$-functions $F : G \to \mathcal{F}_\eta \otimes V_\tau$ satisfying $F(hgk) = (\eta(h) \otimes \tau(k)^{-1})F(g)$ for all

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$(h, g, k) \in H \times G \times K$. A *Shintani function of type* $(\eta, \pi; \tau)$ is defined to be a function in $\mathrm{C}^\infty_{\eta,\tau}(H\backslash G/K)$ that belongs to the image of the above map. Our definition coincides with that in [10, section 5.5] in the case where $\pi$ is a class 1 irreducible admissible representation of $G$ and $\eta$ is an irreducible admissible representation of $H$ which is compatible with the central character of $\pi$ and is trivial on $K \cap H$. Also, we can regard the Shintani functions for $n = 2$ as constituting a generalization of the $O_\xi$-model for $\pi$ considered in the paper of Waldspurger [13, Chapter IV].

In this paper we present a case study of the Shintani functions on $GL(2, \mathbf{C})$. The main theorem is the following multiplicity one theorem, which does not hold in the real case [4].

**Main Theorem** (see Theorem 5.6). *Let $\eta$ be a quasi character of $H \cong GL(1, \mathbf{C}) \times GL(1, \mathbf{C})$, and let $\pi$ be an irreducible non-unitary principal series representation of $G = GL(2, \mathbf{C})$. Then the space of the Shintani functions of type $(\eta, \pi)$ is not trivial if and only if $\pi$ coincides with $\eta$ on the center of $G$. Moreover, for such $(\eta, \pi)$ we have*

$$\dim \mathcal{I}_{\eta,\pi} = 1.$$

Our proof of this theorem consists of two parts, that of uniqueness (§4) and that of existence (§5). The former is a generalization of the proof by Waldspurger [13, Proposition 11] to the case of arbitrary quasi characters $\eta$ of $H$. To prove existence, we construct a non-zero element in $\mathcal{I}_{\eta,\pi}$ using a certain integral transformation which we call the Poisson integral (This is also sometimes referred to as the Poisson transformation; cf. [3], [11]). Moreover, using a system of differential equations that is derived from the action of the center of the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$, we obtain an expression for the Shintani functions of type $(\eta, \pi; \tau)$ in terms of Gauss's hypergeometric function in the case where the minimal $K$-type $\tau$ of $\pi$ is 1-dimensional (Theorem 6.6). Just as in the real case, we can prove that a particular set of differential equations characterizes the space of the Shintani functions (Theorem 6.7).

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## 2. Preliminaries

**2.1. Groups and algebras.** Let $G$ be the real reductive Lie group $GL(2, \mathbf{C})$, and let $\theta$ be an involution defined by $\theta(g) = {}^t\bar{g}^{-1}$ $(g \in G)$. Then the set of fixed points of $\theta$ is $K \simeq U(2)$, which is a maximal compact subgroup of $G$. Next, let us define an involutive automorphism $\sigma$ of $G$ by $\sigma(g) = JgJ$ $(g \in G)$, where $J = \mathrm{diag}(-1, 1)$. Then $\theta\sigma = \sigma\theta$, and the set of fixed points of $\sigma$ is $H \simeq GL(1, \mathbf{C}) \times GL(1, \mathbf{C})$; i.e.,

$$H = \{g \in G \,|\, \sigma(g) = g\} = \left\{\mathrm{diag}(h_1, h_2) \in G \,|\, h_i \in \mathbf{C}^\times\right\}.$$

Now, let $\mathfrak{g} = \mathfrak{gl}(2, \mathbf{C})$ be the Lie algebra of $G$. If we denote the differentials of $\theta$ and $\sigma$ by the same symbols $\theta$ and $\sigma$, then we have $\theta(X) = -{}^t\bar{X}$ and $\sigma(X) = JXJ$ for $X \in \mathfrak{g}$. Let us write the eigenspaces of $\theta$ and $\sigma$ in $\mathfrak{g}$ as

$$\mathfrak{k} = \{X \in \mathfrak{g} \,|\, \theta(X) = X\}, \qquad \mathfrak{p} = \{X \in \mathfrak{g} \,|\, \theta(X) = -X\},$$
$$\mathfrak{h} = \{X \in \mathfrak{g} \,|\, \sigma(X) = X\}, \qquad \mathfrak{q} = \{X \in \mathfrak{g} \,|\, \sigma(X) = -X\}.$$

Then

$$\mathfrak{k} = \mathbf{R}X_1 \oplus \mathbf{R}X_2 \oplus \mathbf{R}X_3 \oplus \mathbf{R}Z_{\mathfrak{k}}, \quad \mathfrak{p} = \mathbf{R}Y_1 \oplus \mathbf{R}Y_2 \oplus \mathbf{R}Y_3 \oplus \mathbf{R}Z_{\mathfrak{p}},$$
$$\mathfrak{h} = \mathbf{R}X_1 \oplus \mathbf{R}Y_1 \oplus \mathbf{R}Z_{\mathfrak{k}} \oplus \mathbf{R}Z_{\mathfrak{p}}, \quad \mathfrak{q} = \mathbf{R}X_2 \oplus \mathbf{R}X_3 \oplus \mathbf{R}Y_2 \oplus \mathbf{R}Y_3,$$

with

$$X_1 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \ X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ X_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad Z_{\mathfrak{k}} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \ Z_{\mathfrak{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $j^2 = -1$. Therefore we have the decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$. We note here that $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{h}$ is that of $H$.

Next, let

$$A = \left\{ a_r = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} \in G \;\middle|\; r \in \mathbf{R} \right\}, \quad \mathfrak{a} = \mathbf{R}Y_2.$$

Then $\mathfrak{a}$ is the Lie algebra of $A$ and is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. For every $n \in \mathbf{Z}$, define $\mathfrak{g}_n = \{X \in \mathfrak{g} \mid [Y_2, X] = nX\}$. Then

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathbf{R}X_3 \oplus \mathbf{R}Z_{\mathfrak{k}} \oplus \mathbf{R}Z_{\mathfrak{p}}, \quad \mathfrak{g}_2 = \mathbf{R}(X_2 - Y_1) \oplus \mathbf{R}(X_1 - Y_3),$$

$\mathfrak{g}_{-2} = \theta\mathfrak{g}_2$ and $\mathfrak{g}_n = \{0\}$ for $n \neq 0, \ \pm 2$.

For a Lie algebra $\mathfrak{b}$, we denote by $\mathfrak{b}^{\mathbf{C}}$ the complexification $\mathfrak{b} \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{b}$ and by $U(\mathfrak{b}^{\mathbf{C}})$ the universal enveloping algebra of $\mathfrak{b}^{\mathbf{C}}$.

**2.2. Integral formula.** Here we present an integral formula which we need in §5.

Let $N_P A_P M_P$ be the Langlands decomposition of the upper triangular subgroup $P$ of $G$, and let us decompose $g \in G$ according to $G = N_P A_P M_P K$ as

$$g = n(g) \exp H(g) \mu(g) \kappa(g).$$

On $N_P$ and $\bar{N}_P = \theta(N_P)$ we fix Haar measures $dn$ and $d\bar{n}$ normalized by $\bar{n} = \theta(n)$ $(n \in N_P)$ and

$$\int_{\bar{N}_P} e^{2\rho H(\bar{n})} d\bar{n} = 1.$$

Here $\rho$ is the half sum of the roots of $(\mathfrak{a}_P, \mathfrak{g})$ $(\mathfrak{a}_P = \mathrm{Lie}(A_P))$ that are positive for $N_P$, and it is given by $\rho\,(\mathrm{diag}(t_1, t_2)) = t_1 - t_2$. Now, let $dk$ be the normalized Haar measure on $K$ with total measure 1. Then we have the following:

**Lemma 2.1.** *For $f \in C(K)$ left invariant under $M_P$, we have*

$$\begin{aligned} \int_K f(k)dk &= \int_{\bar{N}_P} f(\kappa(\bar{n})) e^{2\rho H(\bar{n})} d\bar{n} \\ &= \int_K f(\kappa(kg)) e^{2\rho H(kg)} dk. \end{aligned}$$

This formula is well known. See Knapp [6, pp. 140 and 170], for example.

**2.3. Representations.** In this subsection we recall the representation theory of $K$, $H$ and $G$.

Define $\mathfrak{t}_{\mathfrak{k}} = \mathbf{R}X_1 \oplus \mathbf{R}Z_{\mathfrak{k}}$. Then $\mathfrak{t}_{\mathfrak{k}}$ is a Cartan subalgebra of $\mathfrak{k}$. Next, define $T_1 = \frac{1}{2}(Z_{\mathfrak{k}} + X_1)$ and $T_2 = \frac{1}{2}(Z_{\mathfrak{k}} - X_1)$. For a linear form $\beta : \mathfrak{t}_{\mathfrak{k}}^{\mathbb{C}} \to \mathbf{C}$, we write $\beta(T_i) = \sqrt{-1}\beta_i \in \mathbf{C}$ and identify $\beta$ with $(\beta_1, \beta_2) \in \mathbf{C}^2$. Then the set $\Delta$ of roots of $(\mathfrak{t}_{\mathfrak{k}}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}})$ is given by $\Delta = \{\pm(1,-1)\}$. If we denote the root space for $\beta \in \Delta$ by $\mathfrak{k}_\beta$, then we can take $\frac{1}{2}(\pm X_2 - \sqrt{-1}X_3)$ as a basis of $\mathfrak{k}_{\pm(1,-1)}$. We fix a positive root system $\Delta^+ = \{(1,-1)\}$. The set $\Lambda = \{\lambda = (\lambda_1, \lambda_2) \,|\, \lambda_i \in \mathbf{Z}, \lambda_1 \geq \lambda_2\}$ parametrizes the $\Delta^+$-dominant integral weights, and thus it also parametrizes the equivalence classes of irreducible finite dimensional representations of $K$, as can be seen from the highest weight theory (cf. Knapp [6, Theorem 4.28]). For each $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, write $d_\lambda = \lambda_1 - \lambda_2$ and let $V_\lambda = \bigoplus_{i=0}^{i=d_\lambda} \mathbf{C}v_i^\lambda$ be a $(d_\lambda + 1)$-dimensional vector space with a basis $\{v_i^\lambda\}_{0 \leq i \leq d_\lambda}$. Now let us define the action $\tau_\lambda$ of $\mathfrak{k}^{\mathbb{C}}$ on $V_\lambda$ by

$$\begin{cases} \tau_\lambda(X_1)v_i^\lambda = (2i - d_\lambda)\sqrt{-1}v_i^\lambda, \\ \tau_\lambda(Z_{\mathfrak{k}})v_i^\lambda = (\lambda_1 + \lambda_2)\sqrt{-1}v_i^\lambda, \\ \tau_\lambda(X_2 - \sqrt{-1}X_3)v_i^\lambda = 2(i+1)v_{i+1}^\lambda, \\ \tau_\lambda(-X_2 - \sqrt{-1}X_3)v_i^\lambda = 2(d_\lambda - i + 1)v_{i-1}^\lambda, \end{cases} \qquad \text{for } 0 \leq i \leq d_\lambda.$$

Here it is understood that $v_{-1}^\lambda = v_{d_\lambda+1}^\lambda = 0$. Then $\tau_\lambda$ can be globalized to $K$. The basis $\{v_i^\lambda\}_{0 \leq i \leq d_\lambda}$ is called *the standard basis of* $V_\lambda$.

Let us consider the vector space $\mathfrak{p}^{\mathbb{C}}$ as a $K$-module with the adjoint representation. If we write $\mathfrak{p}_S = \mathbf{C}Y_1 \oplus \mathbf{C}Y_2 \oplus \mathbf{C}Y_3$, then $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}_S \oplus \mathbf{C}Z_{\mathfrak{p}}$ gives the irreducible decomposition of $\mathfrak{p}$. Clearly, we have the isomorphism $\mathfrak{p}_S \cong V_\lambda$ with $\lambda = (1,-1)$ through the correspondence $w_i \mapsto v_i^\lambda$ $(0 \leq i \leq 2)$ of the basis $\{w_i\}$ and $\{v_i^\lambda\}$, where

$$w_0 = \frac{Y_2 + \sqrt{-1}Y_3}{2}, \quad w_1 = Y_1, \quad w_2 = \frac{-Y_2 + \sqrt{-1}Y_3}{2}.$$

Also, we have $\mathbf{C}Z_{\mathfrak{p}} \cong V_{(0,0)}$. For a given irreducible $K$-module $V_\lambda$ with $\lambda \in \Lambda$, the tensor product $V_\lambda \otimes \mathfrak{p}_S$ has the following irreducible decomposition:

$$\begin{aligned} &V_\lambda \otimes \mathfrak{p}_S \cong V_{\lambda-(1,-1)} \oplus V_\lambda \oplus V_{\lambda+(1,-1)}, && \text{for } d_\lambda \geq 2, \\ &V_\lambda \otimes \mathfrak{p}_S \cong V_\lambda \oplus V_{\lambda+(1,-1)}, && \text{for } d_\lambda = 1, \\ &V_\lambda \otimes \mathfrak{p}_S \cong V_{\lambda+(1,-1)}, && \text{for } d_\lambda = 0. \end{aligned}$$

For each $s = (s_1, s_2) \in \mathbf{C}^2$ and $k = (k_1, k_2) \in \mathbf{Z}^2$, we define a quasi character $\eta_s^k$ of $H$ by

$$\eta_s^k\left(\operatorname{diag}(h_1, h_2)\right) = h_1^{k_1} h_2^{k_2} |h_1|^{s_1 - k_1} |h_2|^{s_2 - k_2}, \quad \operatorname{diag}(h_1, h_2) \in H.$$

Clearly, $\{\eta_s^k | s = (s_1, s_2) \in (\sqrt{-1}\mathbf{R})^2,\ k = (k_1, k_2) \in \mathbf{Z}^2\}$ constitutes the set of all unitary characters of $H$. The differential of $\eta_s^k$ is given by

$$\begin{aligned} \eta_s^k(X_1) &= (k_1 - k_2)\sqrt{-1}, & \eta_s^k(Y_1) &= s_1 - s_2, \\ \eta_s^k(Z_{\mathfrak{k}}) &= (k_1 + k_2)\sqrt{-1}, & \eta_s^k(Z_{\mathfrak{p}}) &= s_1 + s_2. \end{aligned}$$

Let $P = N_P A_P M_P$ be the upper triangular subgroup of $G$, as above. For each $l = (l_1, l_2) \in \mathbf{Z}^2$ and $z = (z_1, z_2) \in \mathbf{C}^2$, we define $\sigma_l$ on $M_P$ and $\nu_z$ on $\mathfrak{a}_P$ by

$$\begin{aligned} &\sigma_l\left(\operatorname{diag}(\varepsilon_1, \varepsilon_2)\right) = \varepsilon_1^{l_1}\varepsilon_2^{l_2}, && \operatorname{diag}(\varepsilon_1, \varepsilon_2) \in M_P, \quad \varepsilon_i \in \mathbf{C}, \ |\varepsilon_i| = 1, \\ &\nu_z\left(\operatorname{diag}(t_1, t_2)\right) = z_1 t_1 + z_2 t_2, && \operatorname{diag}(t_1, t_2) \in \mathfrak{a}_P, \quad t_i \in \mathbf{R}. \end{aligned}$$

Then we can construct a representation $\pi_z^l = \mathrm{Ind}_P^G(1_{N_P} \otimes \exp \nu_z \otimes \sigma_l)$ of $G$ which we call the non-unitary principal series representation. A dense subspace of the representation space is

$$\left\{ f \in \mathrm{C}^\infty(G) \;\middle|\; f(namx) = e^{(\nu_z+\rho)\log a}\sigma_l(m)f(x) \right\}$$

with norm

$$\|f\|^2 = \int_K |f(k)|^2 dk,$$

and $G$ acts by $\pi_z^l(g)f(x) = f(xg)$. From the Frobenius reciprocity theorem, we have the following $K$-types of $\pi_z^l$:

$$\pi_z^l|_K = \begin{cases} \sum_{j=0}^\infty \tau_{(l_1+j,\ \ l_2-j)}, & \text{if } l_1 \geq l_2, \\ \sum_{j=0}^\infty \tau_{(l_2+j,\ \ l_1-j)}, & \text{if } l_1 < l_2. \end{cases}$$

The representation $\pi_z^l$ is reducible if and only if the parameters $z$ and $l$ satisfy $|z_1 - z_2| = k+2$ and $|l_1 - l_2| \leq k$ for some non-negative integer $k$. If $\pi_z^l$ is irreducible, we have the equivalence $\pi_z^l \simeq \pi_{\bar{z}}^{\bar{l}}$ with $\bar{z} = (z_2, z_1)$ and $\bar{l} = (l_2, l_1)$. The representation $\pi_z^l$ with $z_i \in \sqrt{-1}\mathbf{R}$ is unitary and is usually called the unitary principal series representation of $G$. If the parameters $z$ and $l$ satisfy $z_1 + z_2 \in \sqrt{-1}\mathbf{R}$, $-2 < z_1 - z_2 < 0$ and $l_1 - l_2 = 0$, then $\pi_z^l$ is infinitesimally unitary, and its unitary version is called the complementary series representation. The irreducible unitary representations of $G$ are exhausted by these two series, together with the unitary characters. In particular, $G$ has no discrete series representations.

In the following sections, we use the same letter to denote a given representation and its underlying $(\mathfrak{g}^{\mathbb{C}}, K)$-module. Also, we define $s' = s_1 - s_2$, $k' = k_1 - k_2$, $z' = z_1 - z_2$ and $l' = l_1 - l_2$ for brevity.

## 3. Shintani functions and radial part

**3.1. Shintani functions.** Let $\eta$ be a quasi character of $H$. Consider the $C^\infty$-induced module $\mathrm{C}^\infty\mathrm{Ind}_H^G(\eta)$ with the representation space

$$\mathrm{C}_\eta^\infty(H\backslash G) = \left\{ F \in C^\infty(G) \;\middle|\; F(hg) = \eta(h)F(g), (h,g) \in H \times G \right\},$$

on which $G$ acts by right translation. Then $\mathrm{C}_\eta^\infty(H\backslash G)$ has the structure of a smooth $G$-module and a $(\mathfrak{g}^{\mathbb{C}}, K)$-module.

Then let us consider an irreducible Harish-Chandra module $\Pi$ of $G$ and the intertwining space

$$\mathcal{I}_{\eta,\Pi} = \mathrm{Hom}_{(\mathfrak{g}^{\mathbb{C}},K)}(\Pi^*, \mathrm{C}^\infty\mathrm{Ind}_H^G(\eta))$$

with its image

$$\mathcal{S}_{\eta,\Pi} = \bigcup_{T \in \mathcal{I}_{\eta,\Pi}} \mathrm{Image}(T).$$

Here $\Pi^*$ is the contragredient $(\mathfrak{g}^{\mathbb{C}}, K)$-module of $\Pi$. We call $\varphi \in \mathcal{S}_{\eta,\Pi}$ a *Shintani function of type* $(\eta, \Pi)$.

For any finite dimensional $K$-module $(\tau, V_\tau)$, we define $\mathrm{C}_{\eta,\tau}^\infty(H\backslash G/K)$ to be the space of smooth functions $F : G \to V_\tau$ with the property

$$F(hgk) = \eta(h)\tau(k)^{-1}F(g), \qquad (h,g,k) \in H \times G \times K.$$

Now, let us consider a finite dimensional $K$-module $(\tau,\ V_\tau)$ and a $K$-equivariant map $i : \tau^* \to \Pi^*|_K$. Here $\tau^*$ is the contragredient module of $\tau$. Moreover, let $i^*$ be the pullback of $i$. Then the map

$$\mathcal{I}_{\eta,\Pi} \xrightarrow{i^*} \mathrm{Hom}_K(\tau^*, \mathrm{C}^\infty \mathrm{Ind}_H^G(\eta)) \cong \mathrm{C}^\infty_{\eta,\tau}(H\backslash G/K)$$

gives the restriction of $T \in \mathcal{I}_{\eta,\Pi}$ to $\tau^*$, which we denote by $T_i \in \mathrm{C}^\infty_{\eta,\tau}(H\backslash G/K)$. Finally, define

$$\mathcal{S}_{\eta,\Pi}(\tau) = \bigcup_i T_i, \quad T \in \mathcal{I}_{\eta,\Pi}.$$

We refer to $\varphi \in \mathcal{S}_{\eta,\Pi}(\tau)$ as a *Shintani function of type* $(\eta, \Pi; \tau)$.

**3.2. Radial part.** Let us write the centralizer and the normalizer of $\mathfrak{a}$ in $K \cap H$ as $Z_{K\cap H}(\mathfrak{a})$ and $N_{K\cap H}(\mathfrak{a})$, respectively. Then

$$K \cap H = \left\{\mathrm{diag}(e^{j\theta_1}, e^{j\theta_2}) \,|\, \theta_i \in \mathbf{R}\right\}, \qquad Z_{K\cap H}(\mathfrak{a}) = \left\{\mathrm{diag}(e^{j\theta}, e^{j\theta}) \,|\, \theta \in \mathbf{R}\right\},$$

and $N_{K\cap H}(\mathfrak{a}) = Z_{K\cap H}(\mathfrak{a}) \cup w_0 Z_{K\cap H}(\mathfrak{a})$. Here $w_0 = \mathrm{diag}(1, -1)$, and $w_0 Z_{K\cap H}(\mathfrak{a})$ is the unique nontrivial element of the quotient group $W = N_{K\cap H}(\mathfrak{a})/Z_{K\cap H}(\mathfrak{a})$.

For each pair consisting of a quasi character $\eta$ of $H$ and a finite dimensional $K$-module $(\tau, V_\tau)$, let us denote by $\mathrm{C}^\infty_W(A; \eta, \tau)$ the space of smooth functions $\varphi : A \to V_\tau$ satisfying the following conditions:

$$\begin{cases} (1) \quad (\eta(m)\tau(m))\varphi(a) = \varphi(a), & m \in Z_{K\cap H}(\mathfrak{a}),\ \ a \in A, \\ (2) \quad (\eta(w_0)\tau(w_0))\varphi(a) = \varphi(a^{-1}), & a \in A, \\ (3) \quad (\eta(l)\tau(l))\varphi(1) = \varphi(1), & l \in K \cap H. \end{cases} \tag{3.1}$$

**Lemma 3.1** (Flensted and Jensen [2, Theorem 4.1]). (1) $G = HAK = HA^+K$, *where* $A^+ = \{a_r \in A \,|\, r > 0\}$.

(2) *The set* $\mathrm{C}^\infty_{\eta,\tau}(H\backslash G/K)$ *is in bijective correspondence, via the restriction* $A$, *with the set* $\mathrm{C}^\infty_W(A; \eta, \tau)$.

Let $(\tau, V_\tau)$ and $(\tau', V_{\tau'})$ be two finite dimensional $K$-modules. Then for each $\mathbf{C}$-linear map $u : \mathrm{C}^\infty_{\eta,\tau}(H\backslash G/K) \to C^\infty_{\eta,\tau'}(H\backslash G/K)$, we have a unique $\mathbf{C}$-linear map $\mathcal{R}(u) : \mathrm{C}^\infty_W(A; \eta, \tau) \to C^\infty_W(A; \eta, \tau')$ with the property $(uf)|_A = \mathcal{R}(u)(f|_A)$ for $f \in \mathrm{C}^\infty_{\eta,\tau}(H\backslash G/K)$. We call $\mathcal{R}(u)$ *the radial part of* $u$.

## 4. Uniqueness of Shintani functions

Let $\Pi^* = \pi^l_z$ be an irreducible non-unitary principal series representation of $G$, and let $\eta = \eta^k_s$ be a quasi character of $H$. In this section, we prove the following proposition:

**Proposition 4.1.** *Let* $\Pi$ *and* $\eta$ *be as above. If the space of the Shintani functions is not trivial, then the parameters* $s$, $z$, $k$, *and* $l$ *satisfy the condition*

$$s_1 + s_2 = z_1 + z_2, \qquad k_1 + k_2 = l_1 + l_2. \tag{4.1}$$

*Moreover, we have* $\dim \mathcal{I}_{\eta,\Pi} \leq 1$.

*Proof.* Similarly, to the proof for the $O_\xi$-model in the paper of Waldspurger [13, Proposition 11], the following lemma is essential.

**Lemma 4.2.** *Let $\mathcal{H}_z^l$ be the representation space of $\pi_z^l$. Then, up to constants, there exists at most one linear map $L$ on $\mathcal{H}_z^l$ satisfying the conditions*

$$L(\pi_z^l(Y_1)v) = s'L(v), \qquad L(\pi_z^l(X_1)v) = k'L(v),$$
$$L(\pi_z^l(Z_{\mathfrak{p}})v) = (s_1+s_2)L(v), \quad L(\pi_z^l(Z_{\mathfrak{k}})v) = (k_1+k_2)L(v)$$

*for each $v \in \mathcal{H}_z^l$. Moreover, if such $L$ exists, then the condition (4.1) holds.*

*Proof of Lemma* 4.2. The validity of the latter assertion can be seen immediately by considering the action of the center of $\mathfrak{g}^{\mathbb{C}}$ on $\mathcal{H}_z^l$. With regard to the former assertion, for simplicity let us assume $l' \geq 0$. Then we have $\pi_z^l|_K = \sum_{m=0}^{\infty} \tau_{\lambda_m}$ with $\lambda_m = (l_1+m,\ l_2-m)$, and $\pi_z^l(X_1)v_i^{\lambda_m} = (2i-l'-2m)\sqrt{-1}v_i^{\lambda_m}$ for the standard basis $\{v_i^{\lambda_m}\}_{0\leq i\leq d_{\lambda_m}}$ of $V_{\lambda_m}$, from §2.3. Using the irreducible decomposition of $V_{\lambda_m} \otimes \mathfrak{p}_S$ and the fact that $H$ is abelian, we can prove the relation

$$\pi_z^l(Y_1)v_i^{\lambda_m} = \gamma_{i,-1}^{\lambda_m} v_{i-1}^{\lambda_m-(1,-1)} + \gamma_{i,0}^{\lambda_m} v_i^{\lambda_m} + \gamma_{i,1}^{\lambda_m} v_{i+1}^{\lambda_m+(1,-1)}$$

with some constants $\gamma_{i,j}^{\lambda_m}$ such that $\gamma_{i,1}^{\lambda_m} \neq 0$.

Now, let $L$ be a linear map satisfying the conditions in the lemma. Then we have $L(\pi_z^l(X_1)v_i^{\lambda_m}) = k'L(v_i^{\lambda_m}) = (2i-l'-2m)\sqrt{-1}L(v_i^{\lambda_m})$. Therefore, $L(v_i^{\lambda_m}) = 0$ for $i \neq i(m) = \frac{l'+k'+2m}{2}$, and $L$ is determined by its value $L(v_{i(m)}^{\lambda_m})$ for each $m \geq 0$. Furthermore, we have

$$\begin{aligned} L(\pi_z^l(Y_1)v_{i(m)}^{\lambda_m}) &= s'L(v_{i(m)}^{\lambda_m}) \\ &= \gamma_{i(m),-1}^{\lambda_m} L(v_{i(m)-1}^{\lambda_m-(1,-1)}) + \gamma_{i(m),0}^{\lambda_m} L(v_{i(m)}^{\lambda_m}) + \gamma_{i(m),1}^{\lambda_m} L(v_{i(m)+1}^{\lambda_m+(1,-1)}). \end{aligned}$$

Therefore, each $L(v_{i(m)}^{\lambda_m})$ is determined inductively from $L(v_{i(m_0)}^{\lambda_{m_0}})$, where $m_0$ is the minimum value of $m$ such that $l'+2m \geq |k'|$. Hence $L$ is unique up to constants. □

Now we return to the proof of the proposition. First, we note that for each $t \in \mathcal{I}_{\eta,\Pi}$, the evaluation map $v \mapsto t(v)(1)$ at the identity for $K$-finite $v \in \mathcal{H}_z^l$ is a linear map satisfying the conditions in Lemma 4.2. Next, let $t$ and $t'$ be two elements in $\mathcal{I}_{\eta,\Pi}$. Then there exist some constants $c$ and $c'$ with $(c,c') \neq (0,0)$ such that $ct(v)(1) = c't'(v)(1)$. In analogy to the argument in Knapp [6, §3 of Chapter 8], we have $ct(v)(g) = c't'(v)(g)$ for $g$ in a neighborhood of 1, and hence, for every $g \in G$, because $t(v)(g)$ is real analytic on $G$ and the formula $t(\pi_z^l(D)v)(g) = Dt(v)(g)$ holds for each $D \in U(\mathfrak{g}^{\mathbb{C}})$. The uniqueness of the Shintani functions is thus demonstrated. □

## 5. Existence of Shintani functions

**5.1. Poisson integral.** In this subsection, we introduce an integral transform, which we call the Poisson integral, in order to construct an element in the intertwining space $\mathcal{I}_{\eta,\Pi}$ for an irreducible non-unitary principal series representation $\Pi^* = \pi_z^l$ and a quasi-character $\eta = \eta_s^k$.

First, we have the following lemma, which provides a $H \times P$-double coset decomposition of $G$.

**Lemma 5.1.**

$$G = HP \sqcup H\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P \sqcup H\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} P.$$

*In particular,*

$$\Omega := H\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}P = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in G\middle| ac\neq 0\right\}$$

*is the unique open $H\times P$-double coset in $G$.*

*Proof.* This decomposition is a direct consequence of the Bruhat decomposition of $G$, since $\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. The second assertion in the lemma is clear from the relations

$$HP = P = \left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\in G\right\},\qquad H\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}P = \left\{\begin{pmatrix}0 & b\\ c & d\end{pmatrix}\in G\right\}.$$

□

For the parameters $s$, $z\in\mathbf{C}^2$ and $k$, $l\in\mathbf{Z}^2$ satisfying (4.1), we define a function $\xi(g)=\xi(s,k;z,l)(g)$ on $G$ as follows:

(1) The support of $\xi$ is $\Omega$.
(2) $\xi(hgp) = \eta_s^k(h)e^{(\nu_z-\rho)\log a}\sigma_l(m)\xi(g)$ with $h\in H$, $g\in G$ and $p=nam\in P$.
(3) $\xi\left(\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)\right)=1$.

The function $\xi$ is unique, since $H\cap P = H$ and the centralizer of $\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)$ in $H$ is $\{xI|x\in\mathbf{C}^\times\}$. We call this function $\xi$ *the Poisson kernel.*

**Lemma 5.2.** *Fix the parameter $s\in\mathbf{C}^2$, and choose the parameters $k$, $l\in\mathbf{Z}$ such that $l_1+l_2=k_1+k_2$. Also, define*

$$\mathfrak{X}_0 = \{z=(z_1,z_2)\in\mathbf{C}^2\,|\,\mathrm{Re}(z_1)>\max(\mathrm{Re}(s_1),\mathrm{Re}(s_2))+1,\ \ z_1+z_2=s_1+s_2\}.$$

*Then $\xi(g)=\xi(s,k;z,l)(g)$ is continuous on $\mathfrak{X}_0\times G$.*

*Proof.* To prove the assertion, it suffices to show that

$$\lim_{(z,g)\to(z',g')}\xi(s,k;z,l)(g)=0 \tag{5.1}$$

for each $(z',g')\in\mathfrak{X}_0\times G\backslash\Omega$, since $\mathfrak{X}_0\times\Omega$ is open and dense in $\mathfrak{X}_0\times G$ and $\xi(g)=0$ for any $g\in G\backslash\Omega$. To demonstrate (5.1), let $g=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Omega$ and decompose $g$ according to $H\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)N_PA_PM_P$ as

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}h_1 & 0\\ 0 & h_2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}\begin{pmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{pmatrix}.$$

Then we have

$$\begin{aligned}\xi(g) =& \left(\frac{h_1}{|h_1|}\right)^{k_1}\left(\frac{h_2}{|h_2|}\right)^{k_2}|h_1|^{s_1}|h_2|^{s_2}\varepsilon_1^{l_1}\varepsilon_2^{l_2}a_1^{z_1-1}a_2^{z_2+1}\\ =&|h_1|^{s_1}|h_2|^{s_2}a_1^{z_1-1}a_2^{z_2+1}\times\left\{\left(\frac{h_1}{|h_1|}\right)^{k_1}\left(\frac{h_2}{|h_2|}\right)^{k_2}\varepsilon_1^{l_1}\varepsilon_2^{l_2}\right\}\\ =&|a|^{z_1-s_2-1}|c|^{z_1-s_1-1}|\det g|^{z_2+1}\times\left\{\left(\frac{h_1}{|h_1|}\right)^{k_1}\left(\frac{h_2}{|h_2|}\right)^{k_2}\varepsilon_1^{l_1}\varepsilon_2^{l_2}\right\}.\end{aligned}$$

Thus (5.1) holds, since $G\backslash\Omega=\left\{\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\middle| ac=0\right\}$. □

For each $\varphi \in \pi_z^l$ and the Poisson kernel $\xi = \xi(s,k;z,l)$, we consider *the Poisson integral* $\mathcal{P}(s,k;z,l;\varphi)$, which is a function on $G$ defined by the integral

$$\mathcal{P}(s,k;z,l;\varphi)(g) = \int_K \xi(gk^{-1})\varphi(k)dk.$$

Lemma 5.2 implies that the Poisson integral $\mathcal{P}(s,k;z,l;\varphi)(g)$ converges absolutely for each $z \in \mathfrak{X}_0$. Let $C^\infty_{\sigma_l}(M_P\backslash K)$ be the set of $C^\infty$- functions $f$ on $K$ such that $f(mk) = \sigma_l(m)f(k)$ for all $(m,k) \in M_P \times K$, and fix $\tilde{\varphi} \in C^\infty_{\sigma_l}(M_P\backslash K)$. Then the function $z \mapsto \mathcal{P}(s,k;z,l;\varphi_z)(g)$ is holomorphic on $\mathfrak{X}_0$ for fixed $g \in G$, where $\varphi_z \in \pi_z^l$ $(z \in \mathbf{C}^2)$ is the extension of $\tilde{\varphi}$.

**5.2. Existence of Shintani functions.** Let us fix the parameter $s \in \mathbf{C}^2$, and choose the parameters $k$, $l \in \mathbf{Z}^2$ such that $l_1 + l_2 = k_1 + k_2$. In this subsection, we consider the analytic continuation of the Poisson integral $\mathcal{P}(s,k;z,l;\varphi)(g)$ with respect to the parameter $z \in \mathbf{C}^2$. To do this, we use the local functional equation for the Tate integral, as in the case of the Whittaker model (cf. Jacquet-Langlands [5, §6]).

**Lemma 5.3.** *Let $z \in \mathfrak{X}_0$ and $\varphi \in \pi_z^l$. Then we have*

$$\mathcal{P}(s,k;z,l;\varphi)(g) = \int_{\bar{N}_P} \xi(\bar{n}^{-1})\varphi(\bar{n}g)d\bar{n}.$$

*Proof.* This follows from the integral formula in Lemma 2.1. □

From the isomorphism $\bar{N}_P \simeq \mathbf{C}$ provided by the map $\bar{n} = \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \mapsto x$ and Lemma 5.3, it can be shown that the Poisson integral $\mathcal{P}(s,k;z,l;\varphi)(g)$ is a constant multiple of the integral

$$\int_{\mathbf{C}} \xi\left(\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}\right) \varphi\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} g\right) dx \tag{5.2}$$

over $\mathbf{C}$, where $dx$ is a Haar measure on $\mathbf{C}$. If $x \neq 0$, then $\left(\begin{smallmatrix} 1 & 0 \\ -x & 1 \end{smallmatrix}\right) \in \Omega$, and we have the decomposition

$$\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & |x|^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & |x| \cdot x^{-1} \end{pmatrix}.$$

Thus

$$\xi\left(\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}\right) = \begin{cases} |x|^{z_1-s_1-1}(-1)^{k_2}\left(\dfrac{x}{|x|}\right)^{k_2-l_2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and the integral (5.2) becomes

$$(-1)^{k_2} \int_{\mathbf{C}} |x|^{z_1-s_1-1} \left(\frac{x}{|x|}\right)^{k_2-l_2} \varphi\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} g\right) dx. \tag{5.3}$$

We are now in a position to consider the analytic continuation of the integral (5.3) with respect to the variable $z$.

**Proposition 5.4.** *Fix the parameter $s \in \mathbf{C}^2$ and choose the parameters $k$, $l \in \mathbf{Z}^2$ such that $l_1 + l_2 = k_1 + k_2$, and define*

$$\mathfrak{X} = \{z = (z_1, z_2) \in \mathbf{C}^2 \,|\, \mathrm{Re}(z_1) > \max(\mathrm{Re}(s_1), \mathrm{Re}(s_2)) - 1,\ \ z_1 + z_2 = s_1 + s_2\}.$$

*Then the Poisson integral $\mathcal{P}(s,k;z,l;\varphi)$ converges for $z \in \mathfrak{X}$ and has a meromorphic continuation to all $z \in \mathbf{C}^2$ satisfying $z_1 + z_2 = s_1 + s_2$.*

*Proof.* First, we demonstrate the equality

$$\mathcal{P}(s,k;z,l;\pi_z^l(g')\varphi)(g) = \mathcal{P}(s,k;z,l;\varphi)(gg'), \qquad g,g' \in G. \tag{5.4}$$

By definition, the left-hand side of this relation is equal to

$$\begin{aligned}\int_K \xi(gk^{-1})\pi_z^l(g')\varphi(k)dk &= \int_K \xi(gk^{-1})\varphi(kg')dk \\ &= \int_K \xi(gg'(kg')^{-1})\varphi(kg')dk \\ &= \int_K \xi(gg'\kappa(kg')^{-1})\varphi(\kappa(kg'))e^{2\rho H(kg')}dk.\end{aligned}$$

Since the integrand of the above integral is a left $M_P$-invariant function on $K$, this integral becomes

$$\mathcal{P}(s,k;z,l;\varphi)(gg') = \int_K \xi(gg'k^{-1})\varphi(k)dk,$$

as can be shown using Lemma 2.1.

Now we return to the proof of the proposition. In view of (5.4), we may assume that $g = 1$. We can think of the collection of the space $\pi_z^l$ with $z \in \mathbf{C}^2$ satisfying $z_1 + z_2 = s_1 + s_2$ as a fiber bundle with base $\mathbf{C}$. This fiber bundle is trivial, and we can define a holomorphic section through

$$\varphi_z^l(g) = \frac{1}{L(1,\omega_z^l)}|\det g|^{z_1+1}\left(\frac{\det g}{|\det g|}\right)^{l_1}\int_{\mathbf{C}^\times}\Phi((0,t)g)\omega_z^l(t)|t|_{\mathbf{C}}d^\times t, \quad g \in G. \tag{5.5}$$

Here $\Phi$ is a Schwartz function on $\mathbf{C}^2$, $|t|_{\mathbf{C}} = |t|^2$, $d^\times t = \frac{dt}{|t|_{\mathbf{C}}}$, $\omega_z^l(t) = |t|^{z'}\left(\frac{t}{|t|}\right)^{l'}$, and $L(s,\omega_z^l)$ $(s \in \mathbf{C})$ is the usual Euler factor attached to $\omega_z^l$ defined by

$$L(s,\omega_z^l) = 2\cdot(2\pi)^{-s-\frac{z'+|l'|}{2}}\Gamma\left(s + \frac{z'+|l'|}{2}\right).$$

The integral (5.5) converges only for $\mathrm{Re}(z') > -2$ but it has meaning for all $z \in \mathbf{C}^2$ satisfying $z_1 + z_2 = s_1 + s_2$ as can be seen from the local functional equation of the Tate integral.

For the section defined in (5.5), the integral (5.3) at $g = 1$ becomes

$$\begin{aligned}&(-1)^{k_2}\int_{\mathbf{C}}|x|^{z_1-s_1-1}\left(\frac{x}{|x|}\right)^{k_2-l_2} \\ &\quad\cdot\frac{1}{L(1,\omega_z^l)}\int_{\mathbf{C}^\times}\Phi((tx,t))\omega_z^l(t)|t|_{\mathbf{C}}d^\times t dx = \frac{(-1)^{k_2}}{L(1,\omega_z^l)} \\ &\quad\times\iint_{\mathbf{C}^\times\times\mathbf{C}^\times}\Phi(x,t)|x|^{z_1-s_1+1}\left(\frac{x}{|x|}\right)^{k_2-l_2}|t|^{z_1-s_2+1}\left(\frac{t}{|t|}\right)^{l_1-k_2}d^\times t d^\times x.\end{aligned}$$

Here we have changed the variable $x$ to $t^{-1}x$. This integral converges only for $z \in \mathfrak{X}$, but it extends meromorphically to all $z \in \mathbf{C}^2$ satisfying $z_1 + z_2 = s_1 + s_2$, as can be seen from the local functional equation of the Tate integral. □

**Corollary 5.5.** *Let $\eta = \eta_s^k$ be a quasi character of $H$, and let $\Pi^* = \pi_z^l$ be an irreducible non-unitary principal series representation of $G$. If the parameters $s$, $k$,*

*$z$, and $l$ satisfy (4.1), then we have*

$$\dim \mathcal{I}_{\eta,\Pi} \geq 1.$$

*Proof.* With the assumptions of the corollary and Proposition 5.4, the Poisson integral $\mathcal{P}(s,k;z,l;\varphi)$ converges for each $\varphi \in \Pi^*$.

Now we prove that the linear map $\varphi \mapsto \mathcal{P}(s,k;z,l;\varphi)$ gives a non-zero element in $\mathcal{I}_{\eta,\Pi}$. Clearly, the function $\mathcal{P}(s,k;z,l;\varphi)(g)$ on $G$ satisfies the relation

$$\mathcal{P}(s,k;z,l;\varphi)(hg) = \eta(h)\mathcal{P}(s,k;z,l;\varphi)(g)$$

for any $(h,g) \in H \times G$. If we fix $\varphi \in \Pi^*$, then the function $g \mapsto \Pi^*(g)\varphi$ on $G$ is smooth. Therefore the relation (5.4) shows that the right translation $\mathcal{P}(s,k;z,l;\varphi)(xg)$ by $g \in G$ is smooth and that the linear map $\varphi \mapsto \mathcal{P}(s,k;z,l;\varphi)$ is an element in the intertwining space $\mathrm{Hom}_G(\Pi^*, \mathrm{C}^\infty\mathrm{Ind}_H^G(\eta))$ between smooth $G$-modules. For a given $g \in G$, the linear form $\Lambda_g$ on $\Pi^*$ defined by $\Lambda_g(\varphi) = \mathcal{P}(s,k;z,l;\varphi)(g)$ is continuous with respect to the $C^\infty$-topology. In fact, the inequality

$$|\Lambda_g(\varphi)| = \left| \int_K \xi(gk^{-1})\varphi(k)dk \right| \leq C_g \sup_{k\in K} |\varphi(k)|$$

holds, where $C_g = \sup_{k\in K} |\xi(gk^{-1})|$. Therefore the restriction of an element in $\mathrm{Hom}_G(\Pi^*, \mathrm{C}^\infty\mathrm{Ind}_H^G(\eta))$ to the space of $K$-finite vectors gives an element in $\mathcal{I}_{\eta,\Pi}$.

Finally, we demonstrate the existence of $\varphi \in \Pi^*$ such that $\mathcal{P}(s,k;z,l;\varphi)(1) \neq 0$. First, note that since $G = PK$, the continuous function $k \mapsto \xi(k^{-1})$ on $K$ is not identically zero. Hence we can find a continuous function $f$ on $K$ such that $\int_K \xi(k^{-1})f(k)dk \neq 0$. Here we may assume that $f \in C^\infty_{\sigma_l}(M_P\backslash K)$ by averaging over $M_P$; $\int_{M_P} \sigma_l(m)^{-1}f(mk)dm$. If we consider the extension $\varphi \in \Pi^*$ of $f$, then

$$\begin{aligned}\mathcal{P}(s,k;z,l;\varphi)(1) &= \int_K \xi(k^{-1})\varphi(k)dk \\ &= \int_K \xi(k^{-1})f(k)dk \neq 0.\end{aligned}$$

□

From Proposition 4.1 and Corollary 5.5, we can state the multiplicity one theorem for the Shintani functions which is our main theorem in this paper.

**Theorem 5.6.** *Let $\eta = \eta_s^k$ be a quasi character of $H$, and let $\Pi^* = \pi_z^l$ be an irreducible non-unitary principal series representation of $G$. Then the space of the Shintani functions is not trivial if and only if the parameters $s$, $z \in \mathbf{C}^2$ and $k$, $l \in \mathbf{Z}^2$ satisfy (4.1). Moreover, for such parameters we have*

$$\dim \mathcal{I}_{\eta,\Pi} = 1.$$

*Remark.* From our construction of elements in $\mathcal{I}_{\eta,\Pi}$ given in the proofs of Corollary 5.5 and Proposition 4.1, we have the identity $\mathcal{I}_{\eta,\Pi} = \mathrm{Hom}_G(\Pi^*, \mathrm{C}^\infty\mathrm{Ind}_H^G(\eta))$.

## 6. Explicit formula

**6.1. Actions of elements in $Z(\mathfrak{g}^{\mathbb{C}})$.** Let $\eta = \eta_s^k$ be a quasi character of $H$, and adopt the standard basis $\{v_i^\lambda\}_{0\leq i\leq d_\lambda}$ of an irreducible $K$-module $(\tau_\lambda, V_\lambda)$. In this subsection, we calculate the radial parts of the elements of the center $Z(\mathfrak{g}^{\mathbb{C}})$ of the universal enveloping algebra $U(\mathfrak{g}^{\mathbb{C}})$ of $\mathfrak{g}^{\mathbb{C}}$ acting on $\mathrm{C}^\infty_W(A;\eta,\tau_\lambda)$.

Now, we express a $C^\infty$-function $\varphi : A \to V_\lambda$ as

$$\varphi(a_r) = \sum_{i=0}^{d_\lambda} c_i(r) v_i^\lambda, \qquad a_r \in A, \quad r \in \mathbf{R} \tag{6.1}$$

with $C^\infty$-functions $c_i(r)$. The following lemma is a rewritting of the condition (3.1) in terms of the expression (6.1).

**Lemma 6.1.** *A non-zero $C^\infty$-function $\varphi : A \to V_\lambda$ expressed as (6.1) belongs to the space $\mathrm{C}^\infty_W(A;\eta,\tau_\lambda)$ if and only if the following conditions are satisfied:*

(1) $k_1 + k_2 + \lambda_1 + \lambda_2 = 0$.
(2) $(-1)^{k_2+\lambda_1-i} c_i(r) = c_i(-r)$.
(3) $(k_2 + \lambda_1 - i) c_i(0) = 0$.

We can identify $\mathfrak{g}^{\mathbb{C}}$ with $\mathfrak{g} \oplus \mathfrak{g}$ in such way that $X \in \mathfrak{g}^{\mathbb{C}}$ corresponds to the element $X \oplus \bar{X}$, where $\bar{X}$ is the complex conjugate of $X$. Hence $U(\mathfrak{g}^{\mathbb{C}})$ is isomorphic to $U(\mathfrak{g}) \otimes_{\mathbf{C}} U(\mathfrak{g})$. It is known that $Z(\mathfrak{g}^{\mathbb{C}})$ is generated by

$$\Omega_1 = \Omega_{\mathbf{R}} \otimes 1, \;\; \Omega_2 = 1 \otimes \Omega_{\mathbf{R}}, \;\; Z_1 = Z_{\mathfrak{p}} \otimes 1, \;\; Z_2 = 1 \otimes Z_{\mathfrak{p}},$$

when it is considered as a subalgebra of $U(\mathfrak{g}) \otimes_{\mathbf{C}} U(\mathfrak{g})$, where $\Omega_{\mathbf{R}}$ is the Casimir operator of $U(\mathfrak{g})$ given by $\Omega_{\mathbf{R}} = \frac{1}{2}(-X_2^2 + Y_1^2 + Y_2^2)$. In $U(\mathfrak{g}^{\mathbb{C}})$, these generators correspond to

$$\begin{aligned}
\Omega_1 &= \tfrac{1}{8}\big\{-X_1^2 - X_2^2 - X_3^2 + Y_1^2 + Y_2^2 + Y_3^2 + 2\sqrt{-1}(X_2Y_3 - Y_1X_1 - Y_2X_3)\big\},\\
\Omega_2 &= \tfrac{1}{8}\big\{-X_1^2 - X_2^2 - X_3^2 + Y_1^2 + Y_2^2 + Y_3^2 - 2\sqrt{-1}(X_2Y_3 - Y_1X_1 - Y_2X_3)\big\},\\
Z_1 &= \tfrac{1}{2}(Z_{\mathfrak{p}} - \sqrt{-1}Z_{\mathfrak{k}}), \qquad\qquad Z_2 = \tfrac{1}{2}(Z_{\mathfrak{p}} + \sqrt{-1}Z_{\mathfrak{k}}).
\end{aligned}$$

**Proposition 6.2.** *Set $\xi = e^{2r}$ ($r \in \mathbf{R}$), and let $\eta$ and $(\tau_\lambda, V_\lambda)$ be as above. Then for $\varphi \in \mathrm{C}^\infty_W(A;\eta,\tau_\lambda)$ expressed as in (6.1) and $a_r \in A$, we have*

$$\begin{aligned}
4\mathcal{R}(\Omega_1 + \Omega_2)\varphi(a_r) = \sum_{i=0}^{d_\lambda} \Big[ & A_i(\xi) c_i(r) v_{i+2}^\lambda + B_i(\xi) c_i(r) v_{i+1}^\lambda \\
& + \Big\{ 4\xi^2 \tfrac{d^2}{d\xi^2} + 4\xi \tfrac{3\xi^4+1}{\xi^4-1} \tfrac{d}{d\xi} + C_i(\xi) \Big\} c_i(r) v_i^\lambda \\
& + D_i(\xi) c_i(r) v_{i-1}^\lambda + E_i(\xi) c_i(r) v_{i-2}^\lambda \Big],
\end{aligned}$$

$$\begin{aligned}
-2\mathcal{R}(\Omega_1 - \Omega_2)\varphi(a_r) = \sum_{i=0}^{d_\lambda} \Big[ & \Big\{ (i+1) 2\xi \tfrac{d}{d\xi} + F_i(\xi) \Big\} c_i(r) v_{i+1}^\lambda + G_i(\xi) c_i(r) v_i^\lambda \\
& + \Big\{ (d_\lambda - i + 1) 2\xi \tfrac{d}{d\xi} + H_i(\xi) \Big\} c_i(r) v_{i-1}^\lambda \Big],
\end{aligned}$$

$$\mathcal{R}(Z_1 + Z_2)\varphi(a_r) = (s_1 + s_2) \sum_{i=0}^{d_\lambda} c_i(r) v_i^\lambda,$$

$$\mathcal{R}(Z_1 - Z_2)\varphi(a_r) = (k_1 + k_2) \sum_{i=0}^{d_\lambda} c_i(r) v_i^\lambda = -(\lambda_1 + \lambda_2) \sum_{i=0}^{d_\lambda} c_i(r) v_i^\lambda,$$

*where*

$$A_i(\xi) = \left(\tfrac{\xi^2-1}{\xi^2+1}\right)^2 (i+1)(i+2), \quad B_i(\xi) = \tfrac{4\xi(\xi^2-1)}{(\xi^2+1)^2} s'(i+1),$$

$$C_i(\xi) = \left(\tfrac{2\xi}{\xi^2+1}\right)^2 (s'^2 + 2d_\lambda i - 2i^2 + d_\lambda) + (2d_\lambda i - 2i^2 + d_\lambda)$$
$$- \tfrac{1}{(\xi^2-1)^2}\left\{4\xi^2 k'^2 + 4\xi(\xi^2+1)k'(2i-d_\lambda) + 4\xi^2(2i-d_\lambda)^2\right\},$$

$$D_i(\xi) = \tfrac{-4\xi(\xi^2-1)}{(\xi^2+1)^2} s'(d_\lambda - i + 1), \quad E_i(\xi) = \left(\tfrac{\xi^2-1}{\xi^2+1}\right)^2 (d_\lambda - i + 1)(d_\lambda - i + 2),$$

$$F_i(\xi) = (i+1)\left\{-\tfrac{2\xi}{\xi^2-1}k' + 2\cdot\tfrac{\xi^2+1}{\xi^2-1} - \tfrac{4\xi^2}{\xi^4-1}(2i+2-d_\lambda)\right\},$$

$$G_i(\xi) = \tfrac{2\xi}{\xi^2+1} s'(2i - d_\lambda),$$

$$H_i(\xi) = (d_\lambda - i + 1)\left\{\tfrac{2\xi}{\xi^2-1}k' + 2\cdot\tfrac{\xi^2+1}{\xi^2-1} + \tfrac{4\xi^2}{\xi^4-1}(2i-2-d_\lambda)\right\}.$$

*Proof.* First, note that the following two lemmas are obvious.

**Lemma 6.3.** *For $a_r \in A$ $(r \neq 0)$, we have $\mathfrak{g} = \mathrm{Ad}(a_r^{-1})\mathfrak{h} + \mathfrak{a} + \mathfrak{k}$.*

**Lemma 6.4.** *Let $f \in \mathrm{C}^\infty_{\eta,\tau}(H\backslash G/K)$. For $X \in U(\mathfrak{k}^{\mathbb{C}})$, $Y \in U(\mathfrak{h}^{\mathbb{C}})$, $Z \in U(\mathfrak{a}^{\mathbb{C}})$ and $a_r \in A$, we have $\left(\mathrm{Ad}(a_r^{-1})Y\right) ZXf(a_r) = \eta(Y)\tau(-X)(Zf)(a_r)$.*

Now, the assertions in the proposition involving $Z_1 \pm Z_2$ are obvious. Then, to prove the assertions involving $\Omega_1 \pm \Omega_2$, we express $\Omega_1 \pm \Omega_2$ as follows:

$$\begin{aligned}
4(\Omega_1 + \Omega_2) &= -X_1^2 - X_2^2 - X_3^2 + Y_1^2 + Y_2^2 + Y_3^2 \\
&= Y_2^2 + 4\cdot\tfrac{\xi^4+1}{\xi^4-1}Y_2 \\
&\quad + \tfrac{1}{(\xi^2+1)^2}\left\{4\xi^2(\mathrm{Ad}(a_r^{-1})Y_1)^2 - 4\xi(\xi^2-1)(\mathrm{Ad}(a_r^{-1})Y_1)X_2 - 4\xi^2X_2^2\right\} \\
&\quad + \tfrac{1}{(\xi^2-1)^2}\left\{4\xi^2(\mathrm{Ad}(a_r^{-1})X_1)^2 - 4\xi(\xi^2+1)(\mathrm{Ad}(a_r^{-1})X_1)X_1 + 4\xi^2X_1^2\right\}, \\
-2\sqrt{-1}(\Omega_1 - \Omega_2) &= X_2Y_3 - Y_1X_1 - Y_2X_3 \\
&= -Y_2X_3 - 2\cdot\tfrac{\xi^2+1}{\xi^2-1}X_3 \\
&\quad + \tfrac{2\xi}{\xi^2-1}\left(\mathrm{Ad}(a_r^{-1})X_1\right)X_2 - \tfrac{2\xi}{\xi^2+1}\left(\mathrm{Ad}(a_r^{-1})Y_1\right)X_1 - \tfrac{4\xi^2}{\xi^4-1}X_2X_1.
\end{aligned}$$

Here we have used the relations

$$2\mathrm{Ad}(a_r^{-1})Y_1 = (\xi + \xi^{-1})Y_1 + (\xi - \xi^{-1})X_2,$$
$$2\mathrm{Ad}(a_r^{-1})X_1 = (\xi + \xi^{-1})X_1 + (\xi - \xi^{-1})Y_3.$$

Then, since $Y_2\varphi(a_r) = 2\xi\frac{d}{d\xi}\varphi(a_r)$, we obtain the proposition through direct computation from Lemma 6.4 and the actions $\eta_s^k$ and $\tau_\lambda$ given in §2.3. □

**6.2. Explicit formula.** Let $\Pi^* = \pi_z^l$ be an irreducible non-unitary principal series representation of $G$, and let $(\tau_\lambda, V_\lambda)$ be the minimal $K$-type of $\Pi$; i.e.

$$\lambda = \begin{cases} (-l_2, -l_1), & \text{if } l_1 \geq l_2, \\ (-l_1, -l_2), & \text{if } l_1 < l_2, \end{cases} \qquad d_\lambda = |l'|.$$

Moreover, let $\eta = \eta_s^k$ be a quasi character of $H$. If $u \in Z(\mathfrak{g}^{\mathbb{C}})$, then $u$ acts on $\Pi^*$, and hence on $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)|_A$, as a scalar operator $\chi_u$. Therefore, for each $\varphi \in \mathcal{S}_{\eta,\Pi}(\tau_\lambda)|_A$ and $u \in Z(\mathfrak{g}^{\mathbb{C}})$, we have the differential equation

$$\mathcal{R}(u)\varphi(a_r) = \chi_u\varphi(a_r). \tag{6.2}$$

**Lemma 6.5.**

$$\chi_{\Omega_1+\Omega_2} = \tfrac{z'^2+l'^2-4}{4}, \qquad \chi_{\Omega_1-\Omega_2} = \tfrac{z'l'}{2},$$
$$\chi_{Z_1+Z_2} = z_1+z_2, \qquad \chi_{Z_1-Z_2} = l_1+l_2.$$

*Proof.* See Jacquet-Langlands [5, Lemma 6.1]. □

Now we solve the system of the differential equations represented by (6.2) for the case $d_\lambda = 0$, i.e. $l' = 0$, explicitly. Equation (6.2) for $u = Z_1 \pm Z_2$ simply yields the condition (4.1) on the parameters necessary for the existence of non-zero Shintani functions. Thus we only consider equation (6.2) for $u = \Omega_1 + \Omega_2$, since it makes no sense for $u = \Omega_1 - \Omega_2$ in the case we consider presently. Proceeding, if we write $\varphi(a_r)$ as $c_0(r)v_0^\lambda$, then we have the equation

$$-\left(4\xi^2\tfrac{d^2}{d\xi^2} + 4\xi\tfrac{3\xi^4+1}{\xi^4-1}\tfrac{d}{d\xi}\right)c_0(r) = (C_0(\xi) - z'^2 + 4)c_0(r) \tag{6.3}$$

with

$$C_0(\xi) = \left(\tfrac{2\xi}{\xi^2+1}\right)s'^2 - \left(\tfrac{2\xi}{\xi^2-1}\right)^2 k'^2.$$

Setting $x = \left(\frac{\xi^2-1}{\xi^2+1}\right)^2 = \tanh^2 2r$ and $c_0(r) = x^{\frac{|k'|}{4}}(1-x)^{\frac{z'+2}{4}}u(r)$, from (6.3), we find that the function $u(r)$ satisfies the hypergeometric equation (cf. [1, Chapter 2])

$$x(1-x)\tfrac{d^2u}{dx^2} + \{c-(a+b+1)x\}\tfrac{du}{dx} - abu = 0,$$

with

$$a = \tfrac{z'+s'+2+|k'|}{4}, \quad b = \tfrac{z'-s'+2+|k'|}{4}, \quad c = \tfrac{2+|k'|}{2}.$$

Then, since $k' \equiv 0 \pmod 2$, the parameter $c$ is an integer, and hence we have the unique $C^\infty$-solution $u(r) = {}_2F_1(a,b;c;x)$ in the neighborhood of $x = 0$, up to constants.

Now, we have the following result.

**Theorem 6.6.** *Let $\eta = \eta_s^k$ be a quasi character of $H$, and let $\Pi^* = \pi_z^l$ be an irreducible non-unitary principal series representation of $G$ with the 1-dimensional minimal $K$-type $(\tau_\lambda, V_\lambda)$. If the parameters $s$, $z$, $k$, and $l$ satisfy (4.1), the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ has a base whose radial part is given by*

$$x^{\frac{|k'|}{4}}(1-x)^{\frac{z'+2}{4}}{}_2F_1\left(\tfrac{z'+s'+2+|k'|}{4}, \tfrac{z'-s'+2+|k'|}{4}; \tfrac{2+|k'|}{2}; x\right)v_0^\lambda,$$

*with $x = \tanh^2 2r$. Here ${}_2F_1(a,b;c;x)$ is Gauss's hypergeometric function.*

*Remark.* We can demonstrate that the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)$ is characterized by a system of differential equations. The proof is similar to that used in the real case [4, Proposition 6.1]. We define the Schmid operator $\nabla^S_{\eta,\tau_\lambda} : \mathrm{C}^\infty_{\eta,\tau_\lambda}(H\backslash G/K) \to \mathrm{C}^\infty_{\eta,\tau_\lambda\otimes\mathrm{Ad}_{\mathfrak{p}_S}}(H\backslash G/K)$ by

$$\nabla^S_{\eta,\tau_\lambda} f = \tfrac{1}{2}\sum_{i=1}^{3} R_{Y_i} f \otimes Y_i = \tfrac{1}{2}R_{w_1} f \otimes w_1 - R_{w_2} f \otimes w_0 - R_{w_0} f \otimes w_2$$

for $f \in \mathrm{C}^\infty_{\eta,\tau_\lambda}(H\backslash G/K)$, where $R_X f(g) = \frac{d}{dt} f(g\cdot\exp(tX))\big|_{t=0}$ for $X \in \mathfrak{g}^{\mathbb{C}}$, $g \in G$.

Moreover, we define *the minus shift operator* $\nabla^-_{\eta,\tau_\lambda}$ as the composition of $\nabla^S_{\eta,\tau_\lambda}$ with the projector from $V_\lambda \otimes \mathfrak{p}_S$ into its irreducible component $V_{\lambda-(1,-1)}$. For $\varphi \in \mathrm{C}^\infty_W(A;\eta^k_s,\tau_\lambda)$ as given in (6.1), the radial part of $\nabla^-_{\eta^k_s,\tau_\lambda}$ is given by

$$\begin{aligned}2\mathcal{R}(\nabla^-_{\eta^k_s,\tau_\lambda})\varphi(a_r) = \sum_{i=0}^{d_\lambda}\Big[&\Big\{-2\xi\tfrac{d}{d\xi} + I_i(\xi)\Big\}c_i(r)v_i^{\lambda-(1,-1)}\\ &+ J_i(\xi)c_i(r)v_{i-1}^{\lambda-(1,-1)}\\ &+ \Big\{2\xi\tfrac{d}{d\xi} + K_i(\xi)\Big\}c_i(r)v_{i-2}^{\lambda-(1,-1)}\Big],\end{aligned}$$

where

$$\begin{aligned}I_i(\xi) &= \tfrac{2\xi}{\xi^2-1}k' + \tfrac{\xi^2+1}{\xi^2-1}(2i-d_\lambda) - 2\cdot\tfrac{\xi^2-1}{\xi^2+1}(i+1), \quad J_i(\xi) = -\tfrac{4\xi}{\xi^2+1}s',\\ K_i(\xi) &= \tfrac{2\xi}{\xi^2-1}k' + \tfrac{\xi^2+1}{\xi^2-1}(2i-d_\lambda) + 2\cdot\tfrac{\xi^2-1}{\xi^2+1}(d_\lambda-i+1).\end{aligned}$$

**Theorem 6.7.** *Let $\eta = \eta^k_s$ be a quasi character of $H$, $\Pi^* = \pi^l_z$ be an irreducible non-unitary principal series representation of $G$, and $(\tau_\lambda, V_\lambda)$ be the minimal $K$-type of $\Pi$. Then the following differential equations characterize the space $\mathcal{S}_{\eta,\Pi}(\tau_\lambda)|_A \subset \mathrm{C}^\infty_W(A;\eta,\tau_\lambda)$ of the Shintani functions:*

(1) *If $d_\lambda = 0$, the equations represented by (6.2) with $u = \Omega_1+\Omega_2$ and $u = Z_1 \pm Z_2$.*
(2) *If $d_\lambda = 1$, the equations represented by (6.2) with $u = \Omega_1-\Omega_2$ and $u = Z_1 \pm Z_2$.*
(3) *If $d_\lambda \geq 2$, the equations represented by (6.2) with $u = \Omega_1-\Omega_2$ and $u = Z_1 \pm Z_2$, and the equation $\mathcal{R}(\nabla^-_{\eta,\tau_\lambda})\varphi(a_r) = 0$.*

Using this characterization, we can also prove the uniqueness of the Shintani functions.

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Department of Mathematical Sciences, Faculty of Science, Ehime University, Ehime, 790-8577, Japan

*E-mail address*: hirano@math.sci.ehime-u.ac.jp