ESSENTIAL COHOMOLOGY AND EXTRASPECIAL $p$-GROUPS

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Abstract. Let $p$ be an odd prime number and let $G$ be an extraspecial $p$-group. The purpose of the paper is to show that $G$ has no non-zero essential mod-$p$ cohomology (and in fact that $H^*(G, F_p)$ is Cohen-Macaulay) if and only if $|G| = 27$ and $\exp(G) = 3$.

1. Introduction

Let $p$ be a prime number. For every $p$-group $K$, denote by $H^*(K)$ the mod-$p$ cohomology ring of $K$. A mod-$p$ cohomology class of $K$ is called essential if it vanishes on restriction to every proper subgroup of $K$. Let $\text{Ess}(K)$ be the ideal of $H^*(K)$ consisting of such classes of $H^*(K)$. As observed in [3], the study of $\text{Ess}(K)$ could provide interesting information for $H^*(K)$ (but, in contrast, it seems in general rather difficult to obtain elements of $\text{Ess}(K)$). For instance, $\text{Ess}(K) = \{0\}$ implies that the depth of $H^*(K)$ is just the rank of the center of $K$ (see [3] and [5]); furthermore, with the condition that $H^*(K)$ is Cohen-Macaulay, it follows from [1] that $\text{Ess}(K) = \{0\}$ if and only if every element of order $p$ of $K$ is central (a way to obtain some element of $\text{Ess}(K)$ in this case was shown there).

We are now interested in extraspecial $p$-groups $G$. For $p = 2$, it was proved by Quillen ([17]) that $H^*(G)$ is Cohen-Macaulay and $\text{Ess}(G) = \{0\}$, except for the case $G = Q_8$, the quaternion group of order 8 (this fact also follows from Adem and Karagueuzian’s result, as $Q_8$ is the unique group in which every element of order 2 is central). However, the situation is quite different for the case $p > 2$—which is assumed from now on. Consider first the case $|G| = p^3$; it follows from [3], [9], [10] that $\text{Ess}(G) = \{0\}$ (so $H^*(G)$ is not Cohen-Macaulay) if and only if $\exp(G) > 3$. In order to generalize this fact, in this note, we prove

Theorem. If $G$ is an extraspecial $p$-group, then $\text{Ess}(G) = \{0\}$ iff $\exp(G) = 3$ and $|G| = p^3$.

It follows that the unique extraspecial $p$-group which has no non-zero essential cohomology is the one of order 27 and of exponent 3. In each of the remaining cases, $H^*(G)$ is not Cohen-Macaulay and the depth of $H^*(G)$ is just 1; we also point out some non-zero essential classes of $G$ (it turns out that, if $|G| = p^5$ or $\exp(G) = p^2$, there exists such a class of $G$ belonging to $\text{Im Inf}^G_G/Z$ with $Z$ the center of $G$).

The note is organized as follows. In Section 2, given an extraspecial $p$-group $G$ of order $p^{2n+1}$, we shall consider $G$ as a subgroup of the central product $\Gamma_n = C_{p^2} \rtimes \Gamma_n$ and give a sufficient and necessary condition for the fact that $\text{Res}^G_{\Gamma_n}(\xi) \neq 0$ with

Received by the editors September 10, 1999.

2000 Mathematics Subject Classification. Primary 20J06; Secondary 20D15, 55R40.
\( \xi \in H^*(\Gamma_n) \). The proofs of the theorem for the cases \( \exp(G) > p \) or \( |G| = p^5 \), which are rather simple, will be given in Section 3. Section 4 is devoted to the case \( \exp(G) = p \).

2. The group \( \Gamma_n \)

Let us recall that an extraspecial \( p \)-group \( G \) is of order \( p^{2n+1} \) \( (n \in \mathbb{N}) \) and is isomorphic to one of the following central products of groups:
\[
\mathbb{E}_n = \mathbb{E} \bullet \cdots \bullet \mathbb{E} \ (n \text{ times}), \\
\mathcal{M}_n = \mathcal{M} \bullet \mathbb{E}_{n-1},
\]
where
\[
\mathcal{M} = (a, b| a^{p^2} = b_p = 1, b^{-1}ab = a^{1+p}), \\
\mathbb{E} = (a, b| a^p = b^p = [a, b]^p = [a, [a, b]] = [b, [a, b]] = 1)
\]
are extraspecial \( p \)-groups of order \( p^3 \). Note that
\[
\exp(G) = \begin{cases} 
p^2, & \text{for } G = \mathcal{M}_n, \\
p, & \text{for } G = \mathbb{E}_n,
\end{cases}
\]
and \( \mathcal{M}_n = \mathcal{M}_{n-1} \bullet \mathcal{M} \).

These groups can be obtained cohomologically as follows. Let \( V \) be a vector space of dimension \( 2n + 1 \) over the prime field \( \mathbb{F}_p \) with basis \( e, a_1, \ldots, a_{2n} \). Let \( x, x_1, \ldots, x_{2n} \) be a basis of \( H^1(V) \), dual to that of \( V \), and let \( y = \beta x, y_i = \beta x_i \) with \( \beta \) the Bockstein homomorphism, so
\[
H = H^*(V) = E[x, x_1, \ldots, x_{2n}] \otimes \mathbb{F}_p[y, y_1, \ldots, y_{2n}]
\]
with \( E[u, v, \ldots] \) (resp. \( \mathbb{F}_p[u, v, \ldots] \)) the exterior (resp. polynomial) algebra over \( \mathbb{F}_p \) with generators \( u, v, \ldots \) of degree 1 (resp. 2). Consider the central extension
\[
(\Gamma_n) \quad 0 \to \mathbb{F}_p \xrightarrow{i} \Gamma_n \to V \to 0,
\]
with factor set \( z = z_n = y + x_1 x_2 + \cdots + x_{2n-1} x_{2n} \). Via the inflation map, \( x \) and the \( x_i \)'s can be considered as elements of \( H^1(\Gamma_n) \). Given a subgroup \( K \) of \( \Gamma_n \), with some abuse of notation, we also denote by \( x \) (resp. \( x_i \)) the element \( \text{Res}_K^{\Gamma_n}(x) \) (resp. \( \text{Res}_K^{\Gamma_n}(x_i) \)).

It is easy to show

**Lemma 1.** (i) \( \Gamma_n = C_{p^2} \bullet \mathcal{M}_n = C_{p^2} \bullet \mathbb{E}_n = \Gamma_{n-1} \bullet \mathcal{M} \).

(ii) \( \mathcal{M}_n = \text{Ker} (x + \alpha), \mathbb{E}_n = \text{Ker} x \) and \( \Gamma_{n-1} \times C_p = \text{Ker} \alpha, \) with \( \alpha \) a non-zero linear combination of \( x_1, \ldots, x_{2n} \). \( \blacksquare \)

Then \( C_{p^2} = \cap_{i=1}^{2n} \text{Ker} x_i \) is a subgroup of \( \Gamma_n \). Let \( w \) be a generator of \( H^2(C_{p^2}) \), so
\[
H^*(C_{p^2}) = E[x] \otimes \mathbb{F}_p[w].
\]

Set \( G_n = C_{p^2} \times \mathbb{E}_n \). By the K"{u}nneth formula, we have
\[
H^*(G_n) = H^*(\mathbb{E}_n) \otimes E[x] \otimes \mathbb{F}_p[w].
\]
As \( \Gamma_n \) is the central product of \( C_{p^2} \) and \( \mathbb{E}_n \), there exists a central subgroup \( U_n \) of order \( p \) of \( G_n \) such that \( G_n / U_n = \Gamma_n \) and the factor set of the central extension
\[
1 \to U_n \to G_n \to \Gamma_n \to 1
\]
is just \( y \). Consider the following commutative diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & U_n & \longrightarrow & U_n \times E_n & \longrightarrow & E_n & \longrightarrow & 1 \\
| & & | & & | & & | & & \\
1 & \longrightarrow & U_n & \longrightarrow & G_n & \longrightarrow & \Gamma_n & \longrightarrow & 1
\end{array}
\]

(1)

whose rows are central extensions and whose vertical arrows are inclusion maps. Pick elements \( s, t \) of \( H^* (U_n) \) satisfying \( H^* (U_n) = E[s] \otimes P[t] \). It follows from [11] (see also [2]) that \( t \) can be chosen so that \( \text{Res}^E_n (w \times 1) = t \).

We now use the following notation. Given a ring \( R \) and elements \( r, s, \ldots \in R \), \( (r, s, \ldots) \) will denote the ideal of \( R \) generated by \( r, s, \ldots \). The main result of this section is the following.

**Proposition 1.** If \( \xi \in H^* (\Gamma_n) \), then \( \text{Res}^E_n (\xi) \neq 0 \) iff \( x \xi \notin (y) \).

**Proof.** Set \( X = \text{Inf}^E_n (\xi) \). As \( \text{Ker} \text{Inf}^E_n = (y) \), it follows that \( xX = 0 \) iff \( x \xi \in (y) \).

Write \( X = \sum w^i \otimes s_i + \sum w^i x \otimes t_i \) with \( s_i, t_i \in H^* (E_n) \). It is clear that \( \text{Res}^E_n (\xi) \neq 0 \) iff \( \text{Inf}^E_n \times \text{Res}^E_n (\xi) \neq 0 \). So, by the commutative diagram (1), \( \text{Res}^E_n (\xi) \neq 0 \) iff \( \text{Res}^E_n (X) \neq 0 \), which is equivalent to the fact that the \( s_i \)'s are not all equal to zero, or equivalently, \( x \xi \notin (y) \). The proposition follows.

For convenience, given a central extension of groups

\[
(K) \quad 1 \to A \to K \to C \to 1,
\]

denote by \( \{ E_r (K), d_r \} \) the Hochschild-Serre spectral sequence corresponding to the extension \( (K) \). We now recall some results given in [12], [13] (see also [2] for \( n = 1 \)) concerning \( \{ E_r (\Gamma_n), d_r \} \). As usual, denote by \( P^i \) the Steenrod operations. Set \( Z = i (\mathbb{F}_p) \subset C_{p^i} \subset \Gamma_n \). So \( v = \text{Res}^C_{C_{p^i}} (w) \) is a generator of \( H^2 (Z) \). Let

\[
X_n = x_1 x_2 \ldots x_{2n-1} x_{2n},
\]

\[
\eta_i = p^{p^{i-2}} \ldots p^1 \beta z
\]

\[
= \sum_{j=1}^n (x_{2j-1} y_{2j} - x_{2j} y_{2j-1}),
\]

\[
\xi_m = \beta p^{p^{m-1}} \ldots p^1 \beta z
\]

\[
= \sum_{j=1}^n (y_{2j-1} y_{2j}^m - y_{2j} y_{2j-1}^m),
\]

\( 1 \leq i \leq n + 1, 1 \leq m \leq n \), be elements of \( H^* (V) \). We have

**Theorem 1** ([2], as corrected in [13] Rk. 2.11(ii), [12]). (i) We have

\[
E_\infty (\Gamma_1) = H^* (V)/(z, \eta_1, \eta_2, \xi_1) \otimes \mathbb{F}_p [v^p]
\]

\[
\oplus (\mathbb{F}_p X_1 \oplus \mathbb{F}_p x_1) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p [v^p] v^i.
\]
(ii) For $n \geq 2$,
\[
E_{2p+1}(\Gamma_n) = H^*(V)/(z, \eta_1, \xi_1, A_n) \otimes \mathbb{F}_p[v^p] \\
+ (\mathbb{F}_p X_n \otimes \mathbb{F}_p x X_n) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i
\]

with $A_n = \sum_{i=1}^n x_1 x_2 \ldots \hat{x}_{2i-1} \hat{x}_{2i} \ldots x_{2n-1} x_{2n}$.

Let $W$ be the vector subspace of $V$ given by $W = \text{Ker}(x - x_1)$. We then have the central extension
\[
(M_n) \quad 1 \to Z \to M_n \to W \to 0
\]
with factor set $z' = z_n = y_1 + x_1 x_2 + \cdots + x_{2n-1} x_{2n}$. Following [10], [12], we also have

Proposition 2 ([10], [12]). (i) We have
\[
E_\infty(M_1) = E_{2p+1}(M_1) = H^*(W)/(z', \beta z') \otimes \mathbb{F}_p[v^p] \\
+ (\mathbb{F}_p[y_2] x_1 \otimes \mathbb{F}_p[y_2] x_2) \otimes \mathbb{F}_p[v^p]v^{p-1} \\
+ (\mathbb{F}_p x_1 \otimes \mathbb{F}_p x_2) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i.
\]

(ii) For $n \geq 2$,
\[
E_{2p+1}(M_n) = H^*(W)/(z', \beta z', \beta P^1 \beta z', B_n \cdot P^1 \beta z') \otimes \mathbb{F}_p[v^p] \\
+ (\mathbb{F}_p x_1 x_3 x_4 \ldots x_{2n-1} x_{2n} \otimes \mathbb{F}_p X_n) \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i
\]

with $B_n = \sum_{i=2}^n x_1 x_3 x_4 \ldots \hat{x}_{2i-1} \hat{x}_{2i} \ldots x_{2n-1} x_{2n}$.

We also prove

Proposition 3. If $\xi \in H^*(\Gamma_n)$ and $|\xi| < 2n + 2$, then $\xi \in \text{Im Inf}_{\Gamma_n}^V$.

The proof of the proposition is divided into the following lemmas. Set
\[
\mathcal{R} = E[x_1, \ldots, x_m] \otimes \mathbb{F}_p[t_1, \ldots, t_m],
\]
and let
\[
\alpha_j = \sum_{j=1}^{m} x_j t_{i_j}^{p^{j-1}}, \quad 1 \leq i \leq m,
\]
be elements of $\mathcal{R}$. Denote by $\mathcal{I}_{k, m}$ the set consisting of subsets of $k$ elements of $\{1, \ldots, m\}$. For every element $I = \{i_1, \ldots, i_k\}$ of $\mathcal{I}_{k, m}$ with $i_1 < \cdots < i_k$, set $x_I = x_{i_1} \ldots x_{i_k}$ and $x_{\emptyset} = 1$.

Lemma 2. For $X \in \mathcal{R}$ and $1 \leq k \leq m$,

(i) if $X \cdot \alpha_1 \ldots \alpha_k = 0$, then $X \in (\alpha_1, \ldots, \alpha_k, x_I \mid I \in \mathcal{I}_{m-k+1, m})$;

(ii) if $X \cdot \alpha_k = 0$, then $X \in (\alpha_k, x_1 \ldots x_m)$. 

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Hence we have
\[ \alpha_1 \ldots \alpha_m = \begin{pmatrix} t_1 & \ldots & t_m \\ t_1^p & \ldots & t_m^p \\ \vdots & \ddots & \vdots \\ t_1^{p^{m-1}} & \ldots & t_m^{p^{m-1}} \end{pmatrix} x_1 \ldots x_m; \]
so \( X \in (x_i | 1 \leq i \leq m) \). Suppose that \( k < m \). Write
\[ \alpha_i = \alpha'_i + x_m t_m^{p^{i-1}} \]
with \( \alpha'_i = \sum_{j=1}^{m-1} x_j t_j^{p^{i-1}}, 1 \leq i \leq m \), and
\[ X = X' + X'' x_m, \]
with \( X', X'' \) free of \( x_m \). Since \( X \alpha_1 \ldots \alpha_k = 0 \), we have
\[ 0 = X' \alpha'_1 \ldots \alpha'_k, \]
\[ 0 = (-1)^k X'' \alpha'_1 \ldots \alpha'_k + X' \sum_{i=1}^{k} (-1)^{k-i} t_m^{p^{i-1}} \alpha'_i \ldots \alpha'_k. \]
By writing
\[ X' = t_m^{r_1} f_1 + \ldots + t_m^{r_s} f_s, \]
\[ X'' = t_m^{s_i} g_i + \ldots + t_m^{s_i} g_i \]
with \( f_i, g_j \) free of \( t_m, r_1 < \ldots < r_s, s_1 < \ldots < s_i \), we have
\[ (-1)^k t_m^{s_i} g_i \alpha'_1 \ldots \alpha'_k + t_m^{p^{s_i-1}+r_j} f_j \alpha'_1 \ldots \alpha'_k = 0. \]

Consider the following cases:
\begin{itemize}
  \item \( r_j + p^{s_i} \geq s_i \): from (2), \( f_j \alpha'_1 \ldots \alpha'_{k-1} = 0 \). By the inductive hypothesis, \( f_j \in (\alpha'_1, \ldots, \alpha'_{k-1}, \mathcal{I}_{m-k+1,m-1}) \). Since \( \alpha'_i = \alpha_i - x_m t_m^{p^{i-1}}, \) we have \( X = t_m^{r_{i-1}} f_1 + \ldots + t_m^{r_{i-1}} f_{j-1} \mod (x_m, \alpha_1, \ldots, \alpha_k, \mathcal{I}_{m-k+1,m-1}). \) So we may suppose that \( f_j = 0 \).
  \item \( r_j + p^{s_i} < s_i \): from (2), \( g_i \alpha'_1 \ldots \alpha'_k = 0 \). By the inductive hypothesis, \( g_i \in (\alpha'_1, \ldots, \alpha'_k, \mathcal{I}_{m-k-1,m-1}). \) So \( x_m g_i \in (\alpha_1, \ldots, \alpha_k, \mathcal{I}_{m-k+1,m-1}). \) By the inductive hypothesis, \( f_j = g_i \alpha'_k \mod (\alpha'_1, \ldots, \alpha'_k, \mathcal{I}_{m-k+1,m-1}). \) Since \( \alpha'_i = \alpha_i - x_m t_m^{p^{i-1}}, \) we have \( f_j = g_i \alpha'_k \mod (x_m, \alpha_1, \ldots, \alpha_k, \mathcal{I}_{m-k+1,m-1}). \) So we may suppose that \( f_j = g_i \alpha'_k \). Since
\[ t_m^{r_{i-1}} f_j + t_m^{s_i} g_i x_m = t_m^{r_{i-1}} (f_j + t_m^{p^{s_i-1}} g_i x_m) = t_m^{r_{i-1}} g_i (\alpha'_k + t_m^{p^{s_i-1}} x_m) = t_m^{r_{i-1}} g_i \alpha_k, \]
we may then suppose that \( f_j = 0 \) and \( g_i = 0 \).
\end{itemize}
The above arguments show that we may reduce to the case \( X' = 0 \). It follows that \( X'' \alpha'_1 \ldots \alpha'_k = 0 \). By the inductive hypothesis, \( X'' \in (\alpha'_1, \ldots, \alpha'_k, \mathcal{I}_{m-k,m-1}). \) Hence \( x_m X'' \in (\alpha_1, \ldots, \alpha_k, \mathcal{I}_{m-k+1,m-1}). \) (i) is proved.
(ii) We again use induction on $m$. The case $m = 1$ is trivial. Assume that (ii) holds for $m - 1$. As above, write

\[ X = X' + X''x_m, \]

with $X', X''$ free of $x_m$. Arguing as above, we may reduce to the case $X' = 0$. It follows that $X''\alpha_k' = 0$. By the inductive hypothesis, $X'' \in (\alpha_k', x_1 \ldots x_{m-1})$. Hence $x_mX'' \in (\alpha_k, x_1 \ldots x_m)$.

The lemma is proved. \hfill \Box

Lemma 3. Let $1 \leq k \leq n$ and let $Y_1, \ldots, Y_k$ be elements of $H^*(V)$.

(i) If $Y_1\xi_1 + \cdots + Y_k\xi_k = 0$, then $Y_k \in (\xi_1, \ldots, \xi_{k-1})$.

(ii) Assume that

\[ Y_k = \sum_{I \subset \{1, \ldots, 2n\}} x_I f_I(y, y_1, \ldots, y_{2n}). \]

We have:

(ia) if $Y_1\eta_1 + \cdots + Y_k\eta_k = 0$, then $Y_k \in (\eta_1, \ldots, \eta_k)$;

(ib) if $Y_k \in \prod_{i=1}^k (\eta_i)$, then $Y_k \in (\eta_1 \ldots \eta_k)$;

(ic) if $Y_1\xi_1 + \cdots + Y_{k-1}\xi_{k-1} + Y_k\eta_k = 0$ with $1 \leq \ell \leq n$, then $Y_k \in (\eta_\ell, \xi_1, \ldots, \xi_{k-1})$.

Proof. (i) For $1 \leq i \leq k$, write

\[ Y_i = \sum_{I \subset \{1, \ldots, 2n\}} x_I f_I^{(i)}(y, y_1, \ldots, y_{2n}). \]

Then, for every $I$, we have

\[ \sum_{i=1}^k f_I^{(i)} \xi_i = 0. \]

According to [IK], $\xi_1, \ldots, \xi_k$ is a regular sequence in $P$. So the above equality implies $f_I^{(k)} \in (\xi_1, \ldots, \xi_{k-1})$. Therefore $Y_k \in (\xi_1, \ldots, \xi_{k-1})$.

(ia) It follows that $Y_k\eta_1 \ldots \eta_k = 0$. By Lemma 2, $Y_k \in (\eta_1, \ldots, \eta_k, I_{2n-k+1, 2n})$. So $Y_k \in (\eta_1, \ldots, \eta_k)$.

(ib) We use induction on $k$. For $k = 2$, $X\eta_1 + Y\eta_2 = 0$ implies $Y\eta_1\eta_2 = 0$. By Lemma 2, $Y \in (\eta_1, \eta_2, I_{2n-1, 2n})$. So $Y \in (\eta_1, \eta_2)$. Write $Y = a\eta_1 + b\eta_2$. Then $Y_2 = Y\eta_2 = a\eta_1\eta_2$.

Assume that (ib) holds for $k - 1 \geq 2$. As $Y_k \in \prod_{i=1}^k (\eta_i)$, it follows from the inductive hypothesis that $Y_k = Y\eta_1 \ldots \eta_k$. Write $Y_k = X\eta_k$. Then $Y_1 \eta_1 \ldots \eta_k = 0$. By Lemma 2, $Y_1 = c_1\eta_1 + \cdots + c_k\eta_k$. So $Y_k = (-1)^{k-1}c_k\eta_1 \ldots \eta_k$.

(ic) Again, we use induction on $k$. $Y_1\xi_1 + Y_2\eta_k = 0$ implies $Y_1\eta_k = 0$. By Lemma 2, $Y_1 \in (\eta_k)$. Write $Y_1 = c_\eta_k$. Then $(\xi_1 + Y_2)\eta_k = 0$. By Lemma 2, $c_\eta_k + Y_2 \in (\eta_k)$; hence $Y_2 \in (\eta_k, \xi_1)$.

Assume that (ic) holds for $k - 1 \geq 2$. As $Y_1\eta_k\xi_1 + \cdots + Y_{k-1}\eta_k\xi_{k-1} = 0$, it follows from (i) that $Y_{k-1}\eta_k \in (\xi_1, \ldots, \xi_{k-2})$. By the inductive hypotheses, we may write $Y_{k-1} = c_1\xi_1 + \cdots + c_{k-2}\xi_{k-2} + c_{k-1}\eta_k$. Hence

\[ (Y_1 + c_1\xi_{k-1})\xi_1 + \cdots + (Y_2 + c_2\xi_{k-1})\xi_2 + (Y_k + c_{k-1}\xi_{k-1})\eta_k = 0. \]

By the inductive hypothesis, $Y_k + c_{k-1}\xi_{k-1} \in (\xi_1, \ldots, \xi_{k-2}, \eta_k)$, and hence $Y_k \in (\xi_1, \ldots, \xi_{k-1}, \eta_k)$. \hfill \Box
For $1 \leq i \leq n + 1, 0 \leq k \leq n$, denote by $\Delta_{i,k}$ the ideal of $H^*(V)$ given by
\[
\Delta_{i,k} = \begin{cases} 
(z, \eta_j, \xi_m | 1 \leq j \leq i, 1 \leq m \leq k) & \text{if } k \geq 1, \\
(z, \eta_j | 1 \leq j \leq i) & \text{if } k = 0.
\end{cases}
\]

Lemma 4. If $X = \sum_{t \leq i} x_i X_i(y_1, \ldots, y_{2n})$ and $X \eta_k \in \Delta_{k,j-1}$ with $1 \leq j \leq k \leq n$, then $X \in \Delta_{k,j-1}$.

Proof. Write
\[
X \eta_k = a_0 z + \sum_{i=1}^{k} a_i \eta_i + \sum_{i=1}^{k-1} b_i \xi_i
\]
with $a_i, b_i \in H^*(V)$. Since $y = z - x_1 x_2 - \cdots - x_{2n-1} x_{2n}$, we may suppose that $a_i, b_i, 1 \leq i \leq k, 1 \leq \ell \leq j - 1$, are free of $y$. It follows that $a_0 = 0$ and
\[
(3) \quad X \eta_1 \cdots \eta_k = \sum_{i=1}^{k-1} b_i \xi_i \eta_1 \cdots \eta_k.
\]

We now argue by induction on $j$. For $j = 1$, it follows that $X \eta_1 \cdots \eta_k = 0$. Hence $X \eta_1 \cdots \eta_k = 0$. By Lemma 2, $X \in (\eta_1, \ldots, \eta_k)$.

Assume that the lemma holds for $j - 1 \geq 1$. By Lemma 3 (i) and by (3), there exists $c_i \in H^*(V)$ such that
\[
X \eta_1 \cdots \eta_k = c_1 \xi_1 + \cdots + c_{j-1} \xi_{j-1}.
\]
Therefore, by Lemma 3 (i), $c_{j-1} \eta_i \in (\xi_1, \ldots, \xi_{j-2})$, for every $1 \leq i \leq k$; by Lemma 3 (iic), $c_{j-1} \in \bigcap_{i \leq k} (\xi_1, \ldots, \xi_{j-2}, \eta_i)$. By writing
\[
c_{j-1} = d_1 \xi_1 + \cdots + d_{j-2} \xi_{j-2} + d \eta_1 \cdots \eta_{j-1}
\]
we get
\[
[(e_1 - d_1) \xi_1 + \cdots + (e_{j-2} - d_{j-2}) \xi_{j-2}] \eta_1 \cdots \eta_{j-1} = 0.
\]
By Lemma 2, $(e_1 - d_1) \xi_1 + \cdots + (e_{j-2} - d_{j-2}) \xi_{j-2}$ contains $\eta_1 \cdots \eta_{j-1}$ as a factor. Hence $e \eta_i \in \bigcap_{l \leq i} (\eta_i)$. By Lemma 3 (iib), $c_{j-1} \in (\xi_1, \ldots, \xi_{j-2}, \eta_1, \ldots, \eta_k)$. So we may suppose that $c_{j-1} \in (\eta_1, \ldots, \eta_k)$. By writing $c_{j-1} = c \eta_1 \cdots \eta_k$, we have
\[
(X - c \xi_{j-1}) \eta_1 \cdots \eta_k = c_1 \xi_1 + \cdots + c_{j-2} \xi_{j-2}.
\]
By the inductive hypothesis, this implies $X - c \xi_{j-1} \in \Delta_{k,j-2}$. So $X \in \Delta_{k,j-1}$. The lemma follows.

Lemma 5. If $X = \sum_{t \leq i} x_i X_i(y_1, \ldots, y_{2n})$ and $X \eta_k \in \Delta_{k-1,k-1}$ with $1 \leq k \leq n + 1$, then $X \in \Delta_{k,k-1}$.

Proof. Write
\[
X \eta_k = a_0 z + \sum_{i=1}^{k-1} (a_i \eta_i + b_i \xi_i).
\]
Arguing as in the proof of Lemma 4, we may suppose that $a_i, b_i$, with $1 \leq i \leq k - 1$, are free of $y$. It follows that $a_0 = 0$ and
\[
X \eta_k = \sum_{i=1}^{k-1} (a_i \eta_i + b_i \xi_i).
\]
Furthermore, we may suppose that every $b_i$ is of form
$$b_i = \sum_{#(I)<2n-2k+1} x_I b_{i}^{I(i)}.$$ 
Therefore, applying Lemma 4 yields $b_{k-1} \in \Delta_{k,k-2}$. Hence, by induction, we need only consider the case
$$X \eta_k = b_1 \xi_1 + \sum_{i=1}^{k-1} a_i \eta_i.$$ 
This implies $b_1 \xi_1 \eta_1 \ldots \eta_k = 0$. So $b_1 \eta_1 \ldots \eta_k = 0$. By Lemma 2, $b_1 \in \eta_1, \ldots, \eta_k)$. The lemma follows.

Let us now consider the Hochschild-Serre spectral sequence $\{E_r(\Gamma_n), d_r\}$. It follows that, for $k < 2n + 2$,
$$\sum_{i+j=k} E_{2p+1}^{i,j} \subseteq E_{2p+1}^{k,0} \oplus \bigoplus_{r \geq 1} E_{2p+1}^{*2p^r}.$$ 
By Kudo’s transgression theorem, for $m \leq n$, $1 \otimes v^p^m$ (resp. $\eta_m \otimes v^{p^{n-1}(p-1)}$) survives to $E_{2p+1}^{*m+1}$ (resp. $E_{2p^{n-1}(p-1)+1}$) and
$$d_{2p^{m+1}}(1 \otimes v^p^m) = \eta_{m+1},$$ 
$$d_{2p^{m-1}(p-1)+1}(\eta_m \otimes v^{p^{n-1}(p-1)} = -\xi_m.$$ 

**Lemma 6.** For $k < 2n + 2$ and $1 \leq m \leq n$, we have
$$\sum_{i+j=k} E_{2p+1}^{i,j} \subseteq E_{2p+1}^{k,0} \oplus \bigoplus_{r \geq 1} E_{2p^{m}+1}^{*2p^m r}.$$ 

**Proof.** By the structure of $E_{2p+1}(\Gamma_n)$, the lemma holds for $m = 1$. Suppose that the lemma holds for $m = s \geq 1$. Let $\psi = X \otimes v^{p^s}$ be an element of $E_2(\Gamma_n)$ surviving to $E_{2p^{s+1}+1}$, with $1 \leq \ell \leq p-1$ and $|X| + 2\ell p^s = k < 2n + 2$. So $d_{2p^{s+1}}(\psi) = \ell X \eta_{s+1} \otimes v^{(\ell-1)p^s}$ must be hit by images under the differentials of elements of degrees less than $2n + 2$. By the inductive hypothesis and by Kudo’s theorem, these images belong to the ideal $\Delta_{s,s}$; hence so does $X \eta_{s+1}$. Since in $E_3(\Gamma_n)$ we have $y = -(x_1x_2 + \cdots + x_{2n-1}x_{2n})$, we may suppose that $X$ is free of $y$. As $|X| < 2n + 2 - 2p^s < 2n - 2s - 2$, by Lemma 5, this means that $X \in \Delta_{s+1,s}$. So $\psi = 0$ in $E_{2p^{s+2}}$ if $\ell < p-1$. If $\ell = p-1$, write $\psi = Y \eta_{s+1} \otimes v^{p^s(p-1)}$. Then $d_{2p^{s+1}+1}(\psi) = -Y \xi_{s+1} \in \Delta_{s+1,s}$. Arguing as above, we may suppose that $Y$ is free of $y$. By Lemma 4, as $|\psi| < 2n + 2$, we have $Y \in \Delta_{s+1,s}$. So $\psi = 0$ in $E_{2p^{s+1}+2}$. The lemma follows.

**Proof of Proposition 3.** It follows from Lemma 6 that $\xi$ either belongs to $\text{Im} \text{Inf}^Y_{\Gamma_n}$ or represents an element of $E^{\geq 2p^s r}$. As $2p^n > 2n + 2$, the fact that $|\xi| < 2n + 2$ implies $\xi \in \text{Im} \text{Inf}^Y_{\Gamma_n}$. The proposition follows.

3. The case $\exp(G) > p$ or $|G| = p^5$

We first consider the case $G = M_n$. Consider $G$ as a subgroup of $\Gamma_n$ by setting $G = \text{Ker}(x - x_1)$. If $n = 1$, it follows from [10] (see also [3]) that $H^*(M)$ contains
a non-zero essential element, namely $X_1$. Assume inductively that $0 \neq X_{n-1} \in \text{Ess}(M_{n-1})$. As $M_n = M_{n-1} \ast M$, we have the following central extension:

$$0 \to F_p \to M_{n-1} \times M \to M_n \to 1.$$ 

The fact that $H^*(M_n)$ contains non-zero essential elements follows from

**Proposition 4.** $0 \neq X_n \in \text{Ess}(M_n)$.

**Proof.** Let $K$ be a maximal subgroup of $M_n$. As $\dim_K H^1(K) = 2n - 1$, it follows that the product of any $2n$ elements of $H^1(K)$ vanishes. Hence $\text{Res}^M_K(X_n) = 0$, which implies that $X_n \in \text{Ess}(M_n)$. Furthermore, as $\text{Im}^M_{x_n-1 \times M}(X_n) = X_{n-1} \times x_{2n-1}x_{2n} \neq 0$ in $H^*(M_{n-1} \times M)$ by the inductive hypothesis, it follows that $X_n \neq 0$. The proposition is proved.

By Theorem 1 (i), $X_1$ and $xX_1$ are non-zero elements of $H^*(\Gamma_1)$. By considering the central extension $0 \to F_p \to \Gamma_{n-1} \times M \to \Gamma_n \to 1$, and by using the same argument given in the proof of Proposition 4, we also have

**Proposition 5.** The elements $xX_n$ and $X_n$ are non-zero elements of $H^*(\Gamma_n)$.

Our next task is to prove that the theorem holds for the extraspecial $p$-group $G = E_2$. Consider $E_2$ as a subgroup of $\Gamma_2$ as in Lemma 1. Let $Q$ be the element of $H^*(V)$ defined by $Q = Q_{1,2}^{1,2} - Q_{2,1}^{3,4}$ with

$$Q_{2,1}^{i,j} = Q_{2,1}(y_i, y_j) = \frac{y_i^p y_j - y_j^p y_i}{y_i^p y_j - y_j^p y_i},$$

(so $Q_{2,1}^{i,j}$ is nothing but the Dickson invariant of order $2(p^2 - p)$ with variables $y_i, y_j$), and set $\eta = x_1 x_2 Q$. It follows from [19] Th. 8.25] that $0 \neq \eta \in H^*(E_2)$. The case $G = E_2$ is then proved by the following:

**Proposition 6.** $\eta \in \text{Ess}(E_2)$.

**Proof.** Let $K$ be a maximal subgroup of $E_2$, so $K \cong \mathbb{E} \times C_p$. If $\text{Res}^E_K(x_3x_4) = 0$, it is clear that $\text{Res}^E_K(x_1x_2Q_{1,2}) = 0$; we can then assume that $\text{Res}^E_K(x_3x_4) \neq 0$. Choose a basis $u_1, u_2, u_3, u_4$ of $H^1(E_2/\mathbb{Z})$ such that $K = \text{Ker} u_4$, $x_1x_2 + x_3x_4 = u_1u_2 + u_3u_4$, $\text{Res}^E_K(x_1x_2) = u_1u_2 + u_1u_3$, and $\text{Res}^E_K(x_3x_4) = -u_1u_3$. This implies that $u_1u_2 = 0$ in $H^*(K)$. By setting $v_1 = \beta u_1$, we have

$$\text{Res}^E_K(\eta) = (u_1u_2 + u_1u_3)(Q_{2,1}(v_1, v_2 + v_3) - Q_{2,1}(v_3, v_1))$$

$$= u_1u_3(Q_{2,1}(v_1, v_2 + v_3) - Q_{2,1}(v_3, v_1)) \quad \text{as } u_1u_2 = 0 \text{ in } H^*(K).$$

Set $Y = Q_{2,1}(v_1, v_2 + v_3) - Q_{2,1}(v_3, v_1)$. Following [13] Proof of Lemma 1.10], $Y$ contains $v_2^p - v_2v_3^{p-1}$ as a factor. As $u_1(v_2^p - v_2v_3^{p-1}) = -p\beta(u_1u_2) + v_1^{p-1}\beta(u_1u_2)$, we have $\text{Res}^E_K(\eta) = 0$. So $\eta \in \text{Ess}(E_2)$. The proposition is proved.

For $\exp(G) > p$ or $|G| = p^5$, Propositions 4 and 6 tell us that there exist non-zero essential cohomology classes of $G$ which belong to $\text{Im} \text{Inf}_G^{V'}$. Furthermore, if $G = M_2$, then [12] Proposition 1.9] and [13] Theorem 3.10] tell us that

$$x_3x_4N \quad \text{and} \quad (y_3x_4 - y_4x_3)N$$

are also non-zero elements of $\text{Ess}(M_2)$ with $N = (y_2^{p-1} - y_3^{p-1})(y_2^{p-1} - y_4^{p-1})$. We can then end the section by the following

**Question.** For $G \neq \mathbb{E}$, is it true that $\text{Ess}(G) \cap \text{Im} \text{Inf}_G^{V'} \neq \{0\}$?
4. The Case $\exp(G) = p$

We first point out some mod-$p$ cohomology classes of $\Gamma_n$, by using the following argument given by D.J. Green [3]. Let $K$ be a $p$-group containing $C$ as a central subgroup. We have the central extension

$$(K) \quad 1 \to C \to K \overset{pr}{\to} K/C \to 1,$$

On the other hand, by considering the extension

$$(K \times C) \quad 1 \to C \overset{\ell}{\to} K \times C \overset{j}{\to} K \to 1$$

with $\ell(c) = (1, c), j(k, c) = k, c \in C, k \in K$, we have the commutative diagram

$$
\begin{array}{c}
1 \longrightarrow C \overset{\ell}{\longrightarrow} K \times C \overset{j}{\longrightarrow} K \longrightarrow 1 \\
\mu \downarrow \quad \downarrow pr \\
1 \longrightarrow C \longrightarrow K \overset{pr}{\longrightarrow} K/C \longrightarrow 1
\end{array}
$$

with $\mu(k, c) = kc, k \in K, c \in C$. The Hochschild-Serre spectral sequences corresponding to these extensions are of the forms

$$E_2(K) = H^*(K/C) \otimes H^*(C) \Rightarrow H^*(K),$$

$$E_2(K \times C) = E_{\infty}(K \times C) = H^*(K) \otimes H^*(C).$$

Furthermore, vertical arrows in (4) also induce a map $\{\mu_r : E_r(K) \to E_r(K \times C)\}$ between spectral sequences with $\mu_2 = (\text{Im}^{K/C}_1, 1_{H^*(C)})$.

The following is due to D.J. Green.

**Proposition 7.** For $r \geq 2$,

$$\text{Im}(d_r : E_r(K) \to E_{r+1}(K)) \subset \text{Ker} \text{Inf}^{K/C}_r \otimes H^*(C).$$

**Proof.** Let $\xi \in E_r(K)$ and write $d_r(\xi) = \sum \phi_j \otimes \psi_j, \phi_j \in H^*(K/C), \psi_j \in H^*(C)$. We can suppose that the $\psi_j$’s are linearly independent in $H^*(C)$. From the commutative diagram (4) and from the fact that $d_r : E_r(K \times C) \to E_r(K \times C)$ vanishes, we have

$$\sum \text{Inf}^{K/C}_r(\phi_j) \otimes \psi_j = \mu_r(d_r(\xi)) = d_r(\mu_r(\xi)) = 0.$$

So $\phi_j \in \text{Ker} \text{Inf}^{K/C}_r$. The proposition follows. \qed

Since $d_{2p+1}(v^p) = \eta_2$ in $E_{2p+1}(\Gamma_n)$ (resp. $P^1\beta z'$ in $E_{2p+1}(\mathbb{M}_n)$), it follows from Theorem 1 and Proposition 2 that $A_n \otimes v^p \in E_{2p+2}(\Gamma_n)$ and $B_n \otimes v^p \in E_{2p+2}(\mathbb{M}_n)$. We then get

**Proposition 8.** For $1 \leq i \leq p-2$,

(i) if $n \geq 2$, then $x_1x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^i, X_n \otimes v^i, x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^p$ and $B_n \otimes v^p$ represent non-zero elements of $E_{2p+2}(\mathbb{M}_n)$;

(ii) $X_n \otimes v^i, x X_n \otimes v^i$ and $A_n \otimes v^p$ represent non-zero elements of $E_{\infty}(\Gamma_n)$.

**Proof.** Note that, in $H^*(W)$, we have

$$x_3x_4 \ldots x_{2n-1}x_{2n} \cdot P^1\beta z' = (x_3x_4 \ldots x_{2n-1}x_{2n})(y_1^2x_2 - y_2^2x_1)$$

$$= (x_3x_4 \ldots x_{2n-1}x_{2n})(z'p x_2 + y_2^{-1}y_1 x_2)$$

$$= (x_3x_4 \ldots x_{2n-1}x_{2n})(z'p x_2 + y_2^{-1}y_1 x_2)$$

$$\in (z', \beta z').$$
So $d_{2p+1}(x_3x_4 \ldots x_{2n-1} x_{2n} \otimes v^p) = 0$. Therefore $x_3x_4 \ldots x_{2n-1} x_{2n} \otimes v^p$ survives to $E_\infty(M_n)$.

By Proposition 4, $X_n \neq 0$ in $H^*(M_n)$ implies that $X_n, x_1 x_3 x_4 \ldots x_{2n-1} x_{2n}$ and $B_n$ are not elements of $\text{Ker Inf}_{M_n}^W$. Similarly, Proposition 5 shows that $X_n, x X_n$ and $A_n$ are not elements of $\text{Ker Inf}_{M_n}^V$. The proposition follows from Proposition 7.

For $n \geq 1$ and for $1 \leq i \leq p - 2$, let us pick elements $X_{n,i} \in H^{2(n+i)-1}(M_n)$ and $Y_{n,i} \in H^{2(n+i)}(\Gamma_n)$ which represent respectively $x_1 x_3 x_4 \ldots x_{2n-1} x_{2n} \otimes v^i \in E_\infty(M_n)$ and $X_n \otimes v^i \in E_\infty(\Gamma_n)$; for $n \geq 2$, pick elements $X_{n,p-1} \in H^{2(n+p)-3}(M_n)$, $Z_{n,p-1} \in H^{2(n+p)-2}(M_n)$ and $Y_{n,p-1} \in H^{2(n+p)-2}(\Gamma_n)$ which represent respectively $B_n \otimes v^p \in E_\infty(M_n), x_3 x_4 \ldots x_{2n-1} x_{2n} \otimes v^p \in E_\infty(M_n)$ and $A_n \otimes v^p \in E_\infty(\Gamma_n)$ (the existence of such elements follows from Propositions 2 and 8). In particular, define $Y_{1,p-1}$ by

$$Y_{1,p-1} = N_{\text{Ker } x_2 - \Gamma_1}(w)$$

with $N$ the Evens norm map (note that $\text{Ker } x_2 \cong C_p^2 \times C_p \subset \Gamma_1$, so, by the Künneth formula, $w$ can be considered as an element of $H^2(\text{Ker } x_2)$).

We now define the following subgroups of $\Gamma_n$:

$M_n = \text{Ker}(x - x_1),$

$\Gamma'_{n-1} = \text{Ker } x_2n \cong \Gamma_{n-1} \times C_p,$

$M'_{n-1} = \text{Ker } x_2n \cap \text{Ker}(x - x_1)$ (so $M'_{n-1} \cong \Gamma_{n-1} \times C_p$ for $n > 1$),

$\Gamma_{n-1} = \text{Ker } x_2 \cap \text{Ker}(x - x_1) \cong \Gamma_{n-1},$

$\Gamma_{n-2} = \text{Ker } x_2 \cap \text{Ker}(x - x_1) \cong \Gamma_{n-2} \times C_p$ (for $n \geq 2$),

with the convention that $\Gamma_0 = C_p^2$. Therefore $\Gamma_{n-2} = M'_{n-1} \cap \Gamma_{n-1}$ and $\Gamma_0 = C_p^2 \times C_p$. If $K$ is one of the above subgroups, then $K$ contains $Z$ as a central subgroup and we have the central extension

$$(K) \quad 1 \to Z \to K \to K/Z \to 1.$$ 

For convenience, we also define the elements $Y_{0,i} \in H^{2i}(\Gamma_0), 1 \leq i \leq p - 1$, by $Y_{0,i} = w^i$. With some abuse of notation, by the Künneth formula, the $Y_{n-1,i}$ (resp. $X_{n-1,i}, Z_{n-1,i}$’s) are considered as elements of $H^*(\Gamma'_{n-1}), H^*(\Gamma_{n-1})$ and $H^*(\Gamma_{n-1})$ (resp. $H^*(M'_{n-1})$). We have

**Lemma 7.** For $n \geq 1$ and $1 \leq i \leq p - 1, 1 \leq j \leq p - 2$,

(i) $\text{Res}_{M_n}^{\Gamma_n}(Y_{n,j}) + x_2X_{n,j} \in \text{Im } \text{Inf}_{M_n}^W$; if $n > 1$ then $\text{Res}_{M_n}^{\Gamma_n}(Y_{n,p-1}) + x_2X_{n,p-1} - Z_{n,p-1} \in \text{Im } \text{Inf}_{M_n}^W$;

(ii) if $n > 1$ then $\text{Res}_{M_n}^{\Gamma_n}(X_{n,i}) - xY_{n-1,i} \in \text{Im } \text{Inf}_{M_{n-1}/Z}^{\Gamma_{n-1}/Z}$;

(iii) $\text{Res}_{\Gamma_{n-1}}^{\Gamma_n}(Y_{n,j})$ belongs to $\text{Im } \text{Inf}_{\Gamma_{n-1}}^{\Gamma_{n-1}/Z}$; and

(iv) $xY_{n,i} \notin (y)$; furthermore, there exists no element $\xi \in H^*(\Gamma_n)$ satisfying $xY_{n,i} = y \xi \mod \text{Im } \text{Inf}_{\Gamma_n}^V$.

**Proof.** (i) and (ii) follow by considering the restriction in spectral sequences and by the structures of $E_{2p+1}(M_n)$ and $E_{2p+1}(\Gamma_n)$ given in Theorem 1 (ii) and Proposition 2.
(iii) Set $T_{n,i} = \text{Res}^\Gamma_{\Gamma_1}^{n-1}(Y_{n,i})$. Since $y_{2n-1}Y_{n,i} \in \text{Im Inf}_n^Y$, it follows that

$$y_{2n-1}T_{n,i} = \text{Res}^\Gamma_{\Gamma_1}^{n-1}(y_{2n-1}Y_{n,i})$$

belongs to $\text{Im Inf}_n^{\Gamma_1}$. As $H^*(\Gamma_1) = H^*(\Gamma) \otimes E[x_{2n-1}] \otimes F_p[y_{2n-1}], T_{n,i}$ also belongs to $\text{Im Inf}_n^{\Gamma_1}$. 

(iv) Assume that there exists $\xi \in H^{2n+1}(\Gamma_n)$ such that

$$y^\xi = xY_{n,1} \mod \text{Im Inf}_n^Y.$$ 

By Proposition 3, $\xi \in \text{Im Inf}_n^Y$. So $y^\xi \in \text{Im Inf}_n^Y$. Hence $xY_{n,1} \in \text{Im Inf}_n^Y$, a contradiction.

Assume inductively that (iv) holds for $i - 1$. For $i \geq 2$, we will prove in Lemmas 10, 11, 16 and 17 that $\text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(Y_{n,i}) = \lambda_i Y_{n+1,i-1} \mod \text{Im Inf}_{n+1}^Y$ with $0 \neq \lambda_i \in F_p$. Let $\phi$ be the element of $\text{Im Inf}_n^Y$ satisfying $\text{tr}_{\Gamma_n^{i+1}}^{\Gamma_n}(Y_{n,i}) = \lambda_i Y_{n+1,i-1} + \phi$. Suppose that $xY_{n,i} + \eta = y^\xi$, with $\xi \in H^*(\Gamma_n)$ and $\eta \in \text{Im Inf}_n^Y$. So

$$\lambda_i xY_{n,i} + x\phi = x\text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(Y_{n,i})$$

$$= \text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(xY_{n,i} + \eta) \quad \text{since} \quad \text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(\eta) = 0$$

$$= \text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(y^\xi)$$

$$= y\text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(\xi).$$

Hence $\lambda_i xY_{n,i} + x\phi = y\text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(\xi) - x\phi$, which contradicts the inductive hypothesis. (iv) is then proved.

The lemma follows.

Further properties of $X_{n,i}$ and $Y_{n,i}$ are given by the following lemmas. The first one follows from Theorem 1, Proposition 2 and [14, Theorem 1.1].

**Lemma 8.** $\text{tr}_{M_n^{i-1}}^{\Gamma_n}(X_{n-1,i})$ (resp. $\text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(Y_{n-2,i})$) represents an element of $E_n^{*;2j}(M_n)$ (resp. $E_n^{*;2j}(\Gamma_n)$), with $j < i$. 

**Lemma 9.** For $2 \leq i \leq p - 1$ we have $\text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(Y_{0,i}) = \lambda_i Y_{1,i-1} \mod \text{Im Inf}_1^Y$, with $0 \neq \lambda_i \in F_p$.

**Proof.** Consider the commutative diagram

$$\begin{array}{ccc}
H^*(C_p^x \times C_p) & \xrightarrow{\text{tr}} & H^*(\Gamma_1) \\
\text{Res} \downarrow & & \text{Res} \downarrow \\
H^*(Z \times C_p) & \xrightarrow{\text{tr}} & H^*(E)
\end{array}$$

We have $\text{Res}_E^\Gamma \text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(Y_{0,i}) = \text{tr}_E^{Z \times C_p}(v^i)$. Following [8] (see also [19]), $\text{tr}_E^{Z \times C_p}(v^i)$ is a non-zero element of $H^*(E) \setminus \text{Im Inf}_2^{E/Z}$. So, by Theorem 1, $\text{tr}_{\Gamma_n^{i-1}}^{\Gamma_n}(Y_{0,i})$ represents an element of the form $\lambda_i x_1 x_2 \otimes v^{i-1} \in E_2^{*;2(i-1)}(\Gamma_1)$, with $0 \neq \lambda_i \in F_p$. The lemma follows.

In the following two lemmas, $p$ is assumed to be greater than 3.
Lemma 10. For $2 \leq i \leq p - 2$,
\[ \text{tr}^{M'}_{M_2}(X_{1,i}) = \lambda_i X_{2,i-1} \mod \text{Im} \text{Inf}^W_{M_2} \]
and
\[ \text{tr}^{\Gamma'}_{\Gamma_2}(Y_{1,i}) = \lambda_i Y_{2,i-1} \mod \text{Im} \text{Inf}^V_{\Gamma_2}, \]
with $\lambda_i$ given in Lemma 9.

Proof. Set $Z_i = \text{Res}^{M_2}_{\Gamma_1} \text{tr}^{M'}_{M_2}(X_{1,i})$. By the double coset formula and by Lemma 7 (ii), we have
\[ Z_i = \text{tr}^{\Gamma'}_{\Gamma_1} \text{Res}^{M'}_{M_1}(X_{1,i}) = \text{tr}^{\Gamma'}_{\Gamma_1}(x_1 Y_{0,i}) = x_1 \text{tr}^{\Gamma'}_{\Gamma_1}(Y_{0,i}). \]
By Lemma 9, $Z_i$ represents $\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E^{3,2(i-1)}_{\infty}(\Gamma_1)$.

By Lemma 8 and Proposition 2, this means that $\text{tr}^{M'}_{M_2}(X_{1,i})$ represents $\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E^{3,2(i-1)}_{\infty}(M_2)$.

The first part of the lemma follows from the definition of $X_{2,i-1}$.

On the other hand, by setting $Y_i = \text{Res}^{\Gamma_2}_{M_2} \text{tr}^{\Gamma'}_{\Gamma_2}(Y_{1,i})$, by the double coset formula, we have
\[ Y_i = \text{tr}^{\Gamma'}_{\Gamma_2} \text{Res}^{\Gamma'}_{\Gamma_1}(Y_{1,i}) \]
\[ = \text{tr}^{\Gamma'}_{\Gamma_2}(-x_2 X_{1,i}) \quad \text{by Lemma 7 (i)} \]
\[ = -x_2 \text{tr}^{M'}_{M_2}(X_{1,i}). \]
As shown above, $\text{tr}^{M'}_{M_2}(X_{1,i})$ represents $\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E^{3,2(i-1)}_{\infty}(M_2)$, so $Y_i$ represents $\lambda_i x_1 x_2 x_3 x_4 \otimes v^{i-1} \in E^{3,2(i-1)}_{\infty}(M_2)$. By Lemma 8 and Theorem 1 (ii), this means that $\text{tr}^{\Gamma'}_{\Gamma_2}(Y_{1,i})$ represents $\lambda_i x_1 x_2 x_3 x_4 \otimes v^{i-1} \in E^{3,2(i-1)}_{\infty}(\Gamma_2)$. The last part follows from the definition of $Y_{2,i-1}$. The lemma is proved. 

In general, we have

Lemma 11. For $2 \leq i \leq p - 2$ and $n \geq 2$,
\[ \text{tr}^{M'}_{M_n}(X_{n-1,i}) = \lambda_i X_{n,i-1} \mod \text{Im} \text{Inf}^W_{M_n} \]
and
\[ \text{tr}^{\Gamma'}_{\Gamma_n}(Y_{n-1,i}) = \lambda_i Y_{n,i-1} \mod \text{Im} \text{Inf}^V_{\Gamma_n}, \]
with $\lambda_i$ given in Lemma 9.

Proof. We argue by induction on $n$. The case $n = 2$ follows from the above lemma. Assume that the lemma holds for $n - 1$. Set $Z_i = \text{Res}^{M_n}_{\Gamma_{n-1}} \text{tr}^{M'}_{M_n}(X_{n-1,i})$. By the double coset formula, we have
\[ Z_i = \text{tr}^{\Gamma_{n-2}}_{\Gamma_{n-1}} \text{Res}^{M'}_{M_{n-1}}(X_{n-1,i}) \]
\[ = \text{tr}^{\Gamma_{n-2}}_{\Gamma_{n-1}}(x Y_{n-2,i}) \quad \text{by Lemma 7 (ii)} \]
\[ = x \text{tr}^{\Gamma_{n-2}}_{\Gamma_{n-1}}(Y_{n-2,i}). \]
By the inductive hypothesis, $\text{tr}^n_{\Gamma_{n-1}}(Y_{n-2},i) = \lambda_i Y_{n-1,i-1} \mod \text{Im} \Gamma_{n-1}^{n-1}/Z$. So $Z_i$ and $\lambda_i x Y_{n-1,i-1}$ represent the same element of $E_{2n-1,i-1}^{2n+1,2(i-1)-(\Gamma_{n-1})}$. The first part follows from Lemma 8 and Proposition 2.

Finally, by setting $Y_i = \text{Res}^n_{\Gamma_{n-1}} \Lambda^1_{\Gamma_{n-1}}(Y_{n-1,i})$, we have

$$Y_i = \text{tr}^n_{\Gamma_{n-1}} \text{Res}^n_{\Gamma_{n-1}}(Y_{n-1,i}) = \text{tr}^n_{\Gamma_{n-1}}(-x_2 X_{n-1,i}) \text{ by Lemma 7 (i)} = -x_2 \text{tr}^n_{\Gamma_{n-1}}(X_{n-1,i}).$$

As shown above, $\text{tr}^n_{\Gamma_{n-1}}(X_{n-1,i}) = \lambda_i x Y_{n,i-1} \mod \text{Im} \Gamma_{n-1}^{dW}$. So $Y_i$ and $-\lambda_i x_2 Y_{n,i-1}$ represent the same element of $E_{2n}^{2n,2(i-1)}(M_n)$. The last part follows from Lemma 8 and Theorem 1 (ii). The lemma is proved. \Box

We now calculate $\text{tr}^n_{\Gamma_{n-1}}(Y_{n-1,1})$. In so doing, let us recall the determination of the transfer map on bar cochain levels. Let $L, K$ be subgroups of $\Gamma_n$ with $Z \subset L \subset K$ and let $D = \{d\}$ be the set of cosets of $L$ in $K$. For each $d$, specify a representative $d'$ of $d$ such that $L = 1$ and $d' d d' d' \in Z$. The transfer map $\text{tr}^L_K : C^*(L) \rightarrow C^*(K)$ is determined in [20] as follows:

$$\text{tr}^L_K f(\ell) = \sum_{d \in D} f(d' \ell),$$

$$\text{tr}^L_K f(\ell_1, \ldots, \ell_n) = \sum_{d \in D} f(d' \ell_1 \ell_2 \cdots \ell_{n-1} \ell_n d' \ell_1 \cdots \ell_{n-1} \ell_n).$$

for $f \in C^*(L), \ell_i \in K$.

Some properties of $\text{tr}^L_K$ were given in [14]. Note that, if $L$ is a direct factor of $K$, then $\text{tr}^L_K$ is the zero map. Furthermore, if $M$ is also a subgroup of $\Gamma_n$ containing $Z$, we can choose representatives of the cosets of $M$ in $KM$, and those of $K \cap M$ in $K$, so that the double coset formula

$$(5) \quad \text{Res}^K_M \text{tr}^M_{KM} = \text{tr}^K_{\Gamma \cap M} \text{Res}^M_{K \cap M}$$

holds at the cochain level.

Since $v \in E_2(\Gamma_{n-1})$ is transgressive, there exists a 2-cochain $\bar{v}$ of $\Gamma_{n-1}$ satisfying $\bar{v}|_Z = v$, $d\bar{v} = \beta z_{n-1}$ (see e.g. [15] for a determination of such a cochain). It follows from [14] Lemma 1.4 that $\text{tr}^n_{\Gamma_{n-1}}(\beta z_{n-1}) = 0$, hence $\hat{\delta} \text{tr}^n_{\Gamma_{n-1}}(\bar{v}) = 0$; in other words, $\text{tr}^n_{\Gamma_{n-1}}(\bar{v})$ is a 2-cocycle of $\Gamma_n$. Set $v = [\text{tr}^n_{\Gamma_{n-1}}(\bar{v})] \in H^2(\Gamma_n)$ and let $e, a_1, \ldots, a_{2n}$ be elements of $\Gamma_n$ satisfying $eZ = e, a_i Z = a_i$ (recall that $e, a_1, \ldots, a_{2n}$ was defined in Section 2 as a basis of $V$ of which the dual is $X, x_1, \ldots, x_{2n}$). We have

Lemma 12. $v = -x_2 x_{n-1} x_{n-2}$.\)

The proof of Lemma 12 is straightforward and is omitted.

Proof. Write

$$v = \sum_{1 \leq i \leq 2n} \mu_i x_i + \sum_{1 \leq i < j \leq 2n} \mu_{ij} x_i x_j + \sum_{1 \leq i \leq 2n} \nu_i y_i$$

with $\mu_i, \mu_{ij}, \nu_i \in F_p$ (note that, in $H^2(\Gamma_n), y = -(x_1 x_2 + \cdots + x_{2n-1} x_{2n})$). Consider the double coset formula (5) with $M = \Gamma_{n-1}$ and $KM = \Gamma_n$ (this means that
A cocycle representing $H^1(K; M) = 0$, so $\mu_i = \mu_{2n} = \nu_i = \nu_{2n} = 0$ for $K = \langle \tilde{e}, \tilde{a}_i, \tilde{a}_j, \tilde{a}_{2n} \rangle$ with $1 \leq i, j \leq 2n - 2$, as $\tilde{a}_{2n}$ commutes with every element of $K \cap M$. For $K = \langle \tilde{e}, \tilde{a}_i, \tilde{a}_{2n-1}, \tilde{a}_{2n} \rangle$ with $1 \leq i \leq 2n - 2$, we have $K \cong \Gamma_1 \times C_p, K \cap M = C_p \times C_p$, and $\text{Res}^M_{K/M}(\tilde{v}) = w$; by a direct verification, we can show that $\text{tr}_{K/M}(w) = y$, therefore $[\text{tr}_K(w)] = y = -x_{2n-1}x_{2n}$, so $\mu_{2n-1} = \mu_{2n-1} = \nu_{2n-1} = 0$ and $\mu_{2n-2} = -1$. The lemma follows.

Lemma 13. For $n \geq 1$, $\text{tr}^\Gamma_{\Gamma_1}(Y_{n-1}) = -X_n$; hence $\text{Res}^\Gamma_{\Gamma_1} \text{tr}^\Gamma_{\Gamma_1}(Y_{n-1}) = 0$.

Proof. A cocycle representing $Y_{n-1}$ can be chosen as follows. Since $x_1x_2 \cdots x_{2n-3}$, $x_{2n-2} \beta z_{n-1} = 0$ in $\text{H}^{*}(\Gamma^\prime_{n-1}/Z)$, there exists a cochain $f$ of $\Gamma^\prime_{n-1}/Z$ (considered as a cocycle of $\Gamma^\prime_{n-1}$ via the inflation map on cochains) satisfying $\delta f = x_1x_2 \cdots x_{2n-3} \cdot x_{2n-2} \beta z_{n-1}$. Furthermore, it follows from the definition of $\tilde{v}$ that

$$\delta(x_1x_2 \cdots x_{2n-3}x_{2n-2} \cdot \tilde{v}) = x_1x_2 \cdots x_{2n-3}x_{2n-2} \beta z_{n-1};$$

hence $\delta(x_1x_2 \cdots x_{2n-3}x_{2n-2} \cdot \tilde{v} - f) = 0$. Clearly $g = x_1x_2 \cdots x_{2n-3}x_{2n-2} \cdot \tilde{v} - f$ is a cocycle representing $X_n \otimes v \in E_\infty(\Gamma^\prime_{n-1})$. Hence

$$Y_{n-1} - [g] \in \text{Im } \text{Inf}^{\Gamma^\prime_{n-1}/Z}_{\Gamma_1};$$

which implies that $\text{tr}^\Gamma_{\Gamma_1}(Y_{n-1})$ is represented by $\text{tr}^\Gamma_{\Gamma_1}(g)$. By [13] Lemma 1.4, $\text{tr}^\Gamma_{\Gamma_1}(g) = x_1x_2 \cdots x_{2n-3}x_{2n-2} \cdot \text{tr}^{\Gamma^\prime_{n-1}}_{\Gamma_1}(\tilde{v})$. So $[\text{tr}^\Gamma_{\Gamma_1}(g)] = x_1x_2 \cdots x_{2n-3}x_{2n-2} \cdot \tilde{v}$.

The lemma now follows from Lemma 12.

Arguing as in the above proof, we can also choose a cocycle representing $X_{n-1} \otimes v^p$ (which is non-zero in $E_\infty(\Gamma^\prime_{n-1})$, by Theorem 1, Propositions 5 and 7), as follows. As $v^p \in E_2(\Gamma^\prime_{n-1})$ is transgressive and $d_{2p+1}(v^p) = P^1 \beta z_{n-1}$, there exists a cochain $\tilde{v}^p$ of $\Gamma^\prime_{n-1}$ such that $\tilde{v}^p|Z = v^p$, and $\delta \tilde{v}^p = P^1 \beta z_{n-1}$. Let $h$ be a cochain of $\Gamma^\prime_{n-1}/Z$ satisfying $\delta h = P^1 \beta z_{n-1} \cdot X_{n-1}$. We have

Lemma 14. $k = k_n = \tilde{v}^p \cdot x_1x_2 \cdots x_{2n-3}x_{2n-2} - h$ is a cocycle representing $X_{n-1} \otimes v^p$ and $\text{Res}^\Gamma_{\Gamma_1} \text{tr}^\Gamma_{\Gamma_1}([k]) = 0$.

Proof. It follows from the definitions of $\tilde{v}^p$ and $h$ that $k$ is a cocycle representing $X_{n-1} \otimes v^p$. Set $X = \text{Res}^\Gamma_{\Gamma_1} \text{tr}^\Gamma_{\Gamma_1}([k])$; then $X = \text{tr}^{E_{\Gamma_1} \times C_p}_{E_{\Gamma_1}} \text{Res}^{\Gamma^\prime_{n-1}}_{E_{\Gamma_1} \times C_p}(k)$ by the double coset formula. Denote also by $\tilde{v}^p$ (resp. $h$) the restriction of the cochain $\tilde{v}^p$ (resp. $h$) to $E_{\Gamma_1} \times C_p$. By [13] Lemma 1.4, $\text{tr}^{E_{\Gamma_1} \times C_p}_{E_{\Gamma_1}}(h) = 0$; hence $X = \text{tr}^{E_{\Gamma_1} \times C_p}_{E_{\Gamma_1}}(\tilde{v}^p \cdot x_1x_2 \cdots x_{2n-3}x_{2n-2})$. Note that, in $H^*(E_{\Gamma_1} \times C_p)$ we have $X_{n-1} = X_{n-2}(x_1x_2 + \cdots + x_{2n-3}x_{2n-2})$, and $x_1x_2 + \cdots + x_{2n-3}x_{2n-2} = 0$, so there exist $c$ of $(E_{\Gamma_1} \times C_p)/Z$ and $b$ of $E_{\Gamma_1} \times C_p$ satisfying

$$\delta b = x_1x_2 + \cdots + x_{2n-3}x_{2n-2},$$

$$x_1x_2 \cdots x_{2n-3}x_{2n-2} = x_1x_2 \cdots x_{2n-5}x_{2n-4} \cdot \delta b + \delta c.$$
So $X = -[\text{tr}^2_{E_{n-1} \times C_p}(\delta \overline{v} \cdot x_1 x_2 \ldots x_{2n-5} x_{2n-4} \cdot b + \delta \overline{v} \cdot c)]$. Following [14] Lemma 1.4, $\text{tr}^2_{E_{n-1} \times C_p}(\delta \overline{v} \cdot c) = 0$ and $\text{tr}^2_{E_{n-1} \times C_p}(\delta b) = 0$. This implies that $\text{tr}^2_{E_{n-1} \times C_p}(b)$ is a cocycle of $E_n$ and $X = -\mathcal{P}^1 \beta z_{n-1} \cdot X_{n-2} \cdot [\text{tr}^2_{E_{n-1} \times C_p}(b)]$.

Arguing as in the proof of Lemma 12, we can show that $[\text{tr}^2_{E_{n-1} \times C_p}(b)] = 0$. Hence $X = 0$. The lemma follows.

With some abuse of notation, we also denote by $\bar{v}$ (resp. $\overline{v}$) the restriction of $\bar{v}$ (resp. $\overline{v}$) to $M_{n-1}^p$. So $\delta(\bar{v}) = \beta z_{n-1}^p$ and $\delta(\overline{v}) = \mathcal{P}^1 \beta z_{n-1}^p$ in $C^*(M_{n-1}^p)$. Let $\bar{u}$ be a 1-cochain of $M_{n-1}^p$ satisfying $\delta(\bar{u}) = z_{n-1}^p$. It follows from the proof of Proposition 8 that there exists a cocycle $d$ of $M_{n-1}^p/Z$ such that

$$\delta d = x_3 x_4 \ldots x_{2n-3} x_{2n-2} (\mathcal{P}^1 \beta z_{n-1}^p - x_2 z_{n-1}^p) - y_2^{-1} \beta z_{n-1}^p + y_2^{-1} x_2 z_{n-1}^p = \delta (x_3 x_4 \ldots x_{2n-3} x_{2n-2} (\delta v + x_2 z_{n-1}^p - y_2^{-1} v - y_2^{-1} x_2 u)).$$

So, for $n \geq 3$, $q = x_3 x_4 \ldots x_{2n-3} x_{2n-2} (\delta v + x_2 z_{n-1}^p - y_2^{-1} v - y_2^{-1} x_2 u) - d$ is a cocycle of $M_{n-1}^p$ representing $Z_{n-1,p-1}$. We have

**Lemma 15.** For $n \geq 3$,

$$\text{tr}^p_{M_{n}^p}(Z_{n-1,p-1}) \in \text{Im Inf}^{W}_{M_{n}^p}.$$  

**Proof.** It follows that $\text{tr}^p_{M_{n}^p}(Z_{n-1,p-1}) = [\text{tr}^p_{M_{n}^p}(q)]$. By [14] Lemma 1.4,

$$\text{tr}^p_{M_{n}^p}(Z_{n-1,p-1}) = x_3 x_4 \ldots x_{2n-3} x_{2n-2} (\text{tr}^p_{M_{n}^p}(\delta v) + x_2 z_{n-1}^p - y_2^{-1} \text{tr}^p_{M_{n}^p}(\bar{u}))$$

(note that $\text{tr}^p_{M_{n}^p}$ maps each of $\delta v, \bar{u}, \bar{v}$ to a cocycle). Since each of $\text{tr}^p_{M_{n}^p}(\delta v), \text{tr}^p_{M_{n}^p}(\bar{v}), \text{tr}^p_{M_{n}^p}(\bar{u})$ is of degree $\leq 2p$, it follows from the structure of $E_{2p+1}(M_{n})$ that $\text{tr}^p_{M_{n}^p}(\delta v), \text{tr}^p_{M_{n}^p}(\bar{u})$ and the cup-product of $x_3 x_4 \ldots x_{2n-3} x_{2n-2}$ with $\text{tr}^p_{M_{n}^p}(\delta v)$ belong to $\text{Im Inf}^{W}_{M_{n}^p}$. The lemma follows.

**Lemma 16.** There exists a non-zero $\lambda \in \mathbb{F}_p$ such that $\text{tr}^p_{\Gamma_2}(Y_{1, p-1}) - \lambda Y_{2, p-2} \in \text{Im Inf}^{V}_{\Gamma_2}$.

**Proof.** Set $K = \text{Ker} x_2 \cap \text{Ker}(x - x_1) \subset \Gamma_2$ and $X = \text{tr}^p_{\Gamma_2}(Y_{1, p-1})$. So $K \cong M \times C_p, K \cap \Gamma_1 \cong C_p$ and $\text{Res}_{K}^p(X) = \text{tr}^p_{K} \cap K \text{Res}_{\Gamma_1 \cap K}(Y_{1, p-1}) - \text{Res}_{\Gamma_1 \cap K}(Y_{1, p-1}) - v^p - vy_1^{p-1}$, we have

$$\text{Res}_{K}^p(X) = \text{tr}^p_{K} \cap K (v^p - vy_1^{p-1}) = \text{tr}^p_{K} \cap K (v^p) - \text{tr}^p_{K} \cap K (vy_1^{p-1}).$$

A direct verification shows that $\text{tr}^p_{K} \cap K (v^p) = y_4$, so $\text{Res}_{K}^p(X) = -y_4 y_1^{p-1} \neq 0$. Hence $X \neq 0$.

Suppose that $X \in \text{Im Inf}^{V}_{\Gamma_2}$. Since $y_4 X = 0$, $y_4 X$ must belong to $(z, \eta_1, \eta_2, \xi_1)$. Write

$$(6) \quad y_4 X = az + bn_1 + cn_2 + \mu \xi_1$$
with \( a, b, c \in H^*(V) \) and \( \mu \in \mathbb{F}_p \). Multiplying (6) by \( x_1 x_2 x_3 x_4 \) yields \( \mu \xi_1 \in (y, y_4) \).

Hence \( \mu = 0 \). Multiplying (6) by \( \eta_2 \) yields \( y_4 X \eta_2 \in (z, \eta_1) \). So, by [13, Lemma 2.4], \( X \eta_2 \in (z, \eta_1, X_2) \). Since \( X \eta_2 \) is of degree 4, it follows that \( X \eta_2 \in (z, \eta_1) \). By [13, Lemma 2.14], \( X = ey \bmod (z, \eta_1) \) with \( e \in H^{2p-2}(V) \). Write

\[
(7) \quad eyy_4 = a_1 z + b_1 \eta_1 + c_1 \eta_2.
\]

Multiplying (7) by \( \eta_1 \eta_2 \) yields

\[
eyy_4 \eta_1 \eta_2 = a_1 z \eta_1 \eta_2 = a_1 y \eta_1 \eta_2 - a_1 X_2 \xi_1.
\]

So \( a_1 \in (y, x_1, \ldots, x_4) \). Therefore \( b_1 \in (y, x, x_j) \) and \( c_1 = 0 \). By [13, Lemma 2.4], we have \( ey \in (z, \eta_1, X_2) \). Since \( ey \) is of degree 4, it follows that \( ey \in (z, \eta_1) \). So \( X \in (z, \eta_1) \), and hence \( X = 0 \) in \( H^*(\Gamma_2) \), a contradiction. The lemma follows. \( \square \)

**Lemma 17.** For \( n \geq 3 \),

\[
\text{tr}_{M_n}^{M_{n-1}}(X_{n-1,p-1}) = \lambda X_{n,p-2} \bmod \text{Im} \text{Inf}_{M_n}^V
\]

and

\[
\text{tr}_{\Gamma_n}^{\Gamma_{n-1}}(Y_{n-1,p-1}) = \lambda Y_{n,p-2} \bmod \text{Im} \text{Inf}_{\Gamma_n}^V,
\]

with \( \lambda \) given in Lemma 16.

**Proof.** Consider the case \( n = 3 \). Set \( X = \text{Res}_{\Gamma_2}^{\Gamma_3} \text{tr}_{M_3}^{M_2}(X_{2,p-1}) \). By the double coset formula, we have

\[
X = \text{tr}_{\Gamma_2}^{\Gamma_3} \text{Res}_{\Gamma_1}^{\Gamma_2} (X_{2,p-1})
= \text{tr}_{\Gamma_2}^{\Gamma_3} (xY_{1,p-1}) \quad \text{by Lemma 7 (ii)}
= x \text{tr}_{\Gamma_2}^{\Gamma_3} (Y_{1,p-1}).
\]

It follows from Lemma 16 that \( X \) and \( \lambda y X_{2,p-2} \) represent the same element of \( E_{\infty}^{5,2(p-2)}(\Gamma_2) \). By Lemma 8 and Proposition 2, it follows that \( \text{tr}_{M_2}^{M_1}(X_{2,p-1}) = \lambda X_{3,p-2} \bmod \text{Im} \text{Inf}_{M_2}^V \). Similarly, by setting \( Y = \text{Res}_{\Gamma_3}^{\Gamma_2} \text{tr}_{\Gamma_2}^{\Gamma_3}(Y_{2,p-1}) \), we have

\[
Y = \text{tr}_{\Gamma_3}^{\Gamma_2} \text{Res}_{\Gamma_3}^{\Gamma_2} (Y_{2,p-1})
= \text{tr}_{M_3}^{M_2} (-x_2 X_{2,p-1} + Z_{2,p-1}) \quad \text{by Lemma 7 (i)}
= -x_2 \text{tr}_{M_3}^{M_2} (X_{2,p-1}) + \text{tr}_{M_3}^{M_2} (Z_{2,p-1}).
\]

As shown above, \( \text{tr}_{M_2}^{M_1}(X_{2,p-1}) = \lambda X_{3,p-2} \bmod \text{Im} \text{Inf}_{M_2}^V \). So, by Lemma 15, \( Y \) and \( -\lambda x_2 X_{3,p-2} \) represent the same element of \( E_{\infty}^{5,2(p-2)}(M_3) \). By Lemma 8 and Proposition 2, it follows that \( \text{tr}_{\Gamma_3}^{\Gamma_2}(Y_{2,p-1}) = \lambda Y_{3,p-2} \bmod \text{Im} \text{Inf}_{\Gamma_3}^V \).

Assume that the lemma holds for \( n - 1 \). Set \( Z = \text{Res}_{\Gamma_{n-1}}^{\Gamma_n} \text{tr}_{M_n}^{M_{n-1}}(X_{n-1,p-1}) \). By the double coset formula, we have

\[
Z = \text{tr}_{\Gamma_{n-1}}^{\Gamma_n} \text{Res}_{\Gamma_{n-2}}^{\Gamma_{n-1}} (X_{n-1,p-1})
= \text{tr}_{\Gamma_{n-2}}^{\Gamma_{n-1}} (xY_{n-2,p-1}) \quad \text{by Lemma 7 (ii)}
= x \text{tr}_{\Gamma_{n-2}}^{\Gamma_{n-1}} (Y_{n-2,p-1}).
\]
By the inductive hypothesis, \( \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(Y_{n-2,p-1}) = \lambda Y_{n-1,p-2} \mod \text{Im} \text{Inf}_{\Gamma_{n-1}}^{\Gamma_{n-1}/Z} \).
So \( Z \) and \( \lambda x Y_{n-1,p-2} \) represent the same element of \( E_{2n-1,2(p-2)}(\Gamma_{n-1}) \). The first part follows from Lemma 8 and Proposition 2.

Finally, by setting \( Y = \text{Res}_{M_n}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_{n-1}/Z}(Y_{n-1,p-1}) \), we have

\[
Y = \text{tr}_{M_n}^{\Gamma_n} \text{Res}_{M_n}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_{n-1}/Z}(Y_{n-1,p-1})
\]

\[
= \text{tr}_{M_n}^{\Gamma_n}(-x_2 X_{n-1,p-1}) + \text{tr}_{M_n}^{\Gamma_n}(Z_{n-1,p-1}) \quad \text{by Lemma 7 (i)}
\]

As shown above, \( \text{tr}_{M_n}^{\Gamma_n}(X_{n-1,p-1}) = \lambda X_{n,p-2} \mod \text{Im} \text{Inf}_{M_n}^{\Gamma_n} \). So, by Lemma 15, \( Y \) and \( -\lambda x_2 X_{n,p-2} \) represent the same element of \( E_{2n,2(p-2)}(M_n) \). The last part follows from Lemma 8 and Theorem 1 (ii). The lemma is proved.

Let

\[
\cdots \supset F^i C^* (\Gamma'_{n-1}) \supset F^{i+1} C^* (\Gamma'_{n-1}) \supset \cdots
\]

be the filtration of \( C^* (\Gamma'_{n-1}) \) introduced by Hochschild and Serre \((\mathbb{F})\) corresponding to the central extension \((\Gamma'_{n-1})\). Let us recall that

\[
F^i C^* (\Gamma'_{n-1}) = \begin{cases} C^* (\Gamma'_{n-1}) & \text{for } i \leq 0, \\ \sum_{m=0}^{\infty} F^i C^m (\Gamma'_{n-1}) & \text{for } i > 0, \end{cases}
\]

where \( F^i C^m (\Gamma'_{n-1}) = 0 \) for \( i > m \); and for \( 0 < i < m \), \( F^i C^m (\Gamma'_{n-1}) \) is the group of all \( m \)-cochains \( f \) for which \( f(g_1, \ldots, g_m) = 0 \) whenever \( m-i+1 \) of the arguments \( g_k \) belong to \( Z \). It is clear that the conjugation by \( a = a_{2n} \) on \( C^* (\Gamma'_{n-1}) \) is compatible with the Hochschild-Serre filtration. We then have the induced conjugation on the Hochschild-Serre spectral sequence \( E_r (\Gamma'_{n-1}) \).

As the action of \( a \) on \( E_2^{*,*} (\Gamma'_{n-1}) \) satisfies \( a x_k = x_k, 1 \leq k \leq 2n-1 \), and \( a v = v + y_{2n-1} \), it follows from the structure of \( E_{2p+1} (\Gamma'_{n-1}) \) that \( Y_{n-1, i} \) and \( a Y_{n-1, i} \) represent the same element of \( E_{\infty} (\Gamma'_{n-1}) \).

Hence

\[
(8) \quad Y_{n-1, i} = a Y_{n-1, i} = \sum_{0 < j < i} \mu_j Y_{n-1, j} y_{2n-1}^{i-j} + \sum_{0 < j < i} \nu_j Y_{n-1, j} y_{2n-1}^{i-j} x x_{2n-1} \mod \text{Im} \text{Inf}_{\Gamma_{n-1}}^{\Gamma_{n-1}/Z},
\]

with \( \mu_j, \nu_j \in \mathbb{F}_p \). We have

**Lemma 18.** For \( n \geq 2 \) we have \( Y_{n-1, 1} = a Y_{n-1, 1} = 0 \).

**Proof.** Set \( K = \text{Ker} x_{2n-2} \cap \Gamma'_{n-1} \). Since the transfer commutes with the conjugation and \( \text{Im} \text{inf}_{\Gamma_{n-1}}^{\Gamma_{n-1}/Z} \) is invariant under the action of \( a \), by Lemmas 9, 10, 11, 16 and 17, we have

\[
Y_{n-1, 1} = a Y_{n-1, 1} = \text{tr}_{\Gamma_{n-1}}^{K} (Y_{n-2, 2} - a Y_{n-2, 2})
\]

up to a non-zero constant multiple. By Lemma 13, \( \text{tr}_{\Gamma_{n-1}}^{K} (Y_{n-2, 1}) = -X_{n-1} \); hence \( \text{tr}_{\Gamma_{n-1}}^{K} (Y_{n-2, 2}) = 0 \) and \( \text{tr}_{\Gamma_{n-1}}^{K} (X_{n-2, 1} x_{2n-3}) = 0 \) in \( H^*(\Gamma_{n-1}) \). The lemma follows from (8) and from the fact that \( \text{tr}_{\Gamma_{n-1}}^{K} \text{Inf}_{K}^{\Gamma_{n-1}/Z} = 0 \). 

We now have
Lemma 19. For $n \geq 2$ and $1 \leq i \leq p - 1$,
\[ Y_{n-1,i} + a Y_{n-1,i} + \cdots + a^{p-1} Y_{n-1,i} = 0; \]

hence
\[ \text{Res}_{n-1}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i}) = 0. \]

Proof. Since $1 + a + \cdots + a^{p-1} = (1 - a)^{p-1}$, we need prove that $(1 - a)^{p-1} Y_{n-1,i} = 0$. For $1 \leq k \leq p - 1$, by (8) and by Lemma 18, $(1 - a)^{p-1} Y_{n-1,i} = 0$. Since $\text{Res}_{n-1}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i}) = (1 - a)^{p-1} Y_{n-1,i}$, the lemma follows. \qed

For $n \geq 2$ and $1 \leq i \leq p - 2$, set $\kappa_{n,i} = \text{Res}_{E_n}^{E_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i+1})$. The proof of the theorem is completed by the following fact.

Proposition 9. $0 \neq \kappa_{n,i} \in \text{Ess}(E_n)$ with $1 \leq i < p - 2$ for $p > 3$, and $i = 1$ for $p = 3$.

Proof. It follows from Proposition 1, Lemmas 7 (iv), 9, 11 and 17 that $\kappa_{n,i} \neq 0$ in $H^*(E_n)$. Let $K$ be a maximal subgroup of $E_n$. $K$ is then of the form $E_n \times C_p$. Let $L$ be the central product of $K$ and $C_p$.

Let $n = 2^n - 1$, it follows from the double coset formula that
\[ \text{Res}_{K}^{E_n}(\kappa_{n,i}) = \text{Res}_{K}^{L} \text{tr}_{L}^{\Gamma_{n-1}}(Y_{n-1,i+1}) \]
\[ = \text{Res}_{K}^{L} \text{Res}_{L}^{\Gamma_{n-1}} \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(Y_{n-1,i+1}). \]

Hence, if $\Gamma_n = \Gamma_{n-1} L$, it follows from the double coset formula that
\[ \text{Res}_{K}^{E_n}(\kappa_{n,i}) = \text{Res}_{K}^{L} \text{tr}_{L}^{L \cap \Gamma_{n-1}} \text{Res}_{L}^{\Gamma_{n-1}} \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(Y_{n-1,i+1}). \]

By Lemma 1, there exists a non-zero linear combination $\alpha$ of $x_1, \ldots, x_{2n}$ such that $L = \text{Ker} \alpha$. Consider the following cases:

- $\alpha = x_{2n-1} + \gamma$ with $\gamma$ a linear combination of $x_1, \ldots, x_{2n-2}, x_{2n}$: it follows that $\Gamma_n = \Gamma_{n-1} L$ and $L \cap \Gamma_{n-1} \cong \Gamma_{n-1}$ is a direct factor of $L$. Hence $\text{tr}_{L}^{L \cap \Gamma_{n-1}}$ is the zero map. We have
\[ \text{Res}_{K}^{E_n}(\kappa_{n,i}) = \text{Res}_{K}^{L} \text{tr}_{L}^{L \cap \Gamma_{n-1}} \text{Res}_{L}^{\Gamma_{n-1}} \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(Y_{n-1,i+1}) = 0; \]

- $\alpha = \mu x_{2n} + \gamma$ with $\gamma$ a non-zero linear combination of $x_1, \ldots, x_{2n-2}$ and $\mu \in \mathbb{F}_p$: it follows that $L \cap \Gamma_{n-1} = H \times \langle a_{2n-1} \rangle$ for a subgroup $H$ of $\Gamma_{n-1}$ with $H \cong \Gamma_{n-2}$. If $p > 3$, it follows from the proof of Lemma 7(iii) that $\text{Res}_{L}^{\Gamma_{n-1}}(Y_{n-1,i+1})$ belongs to the ideal generated by $\text{Im} \text{Inf}_{L \cap \Gamma_{n-1}}^{(L \cap \Gamma_{n-1})/\langle \mu \rangle}$; since
\[ \text{Im} \text{Inf}_{L \cap \Gamma_{n-1}}^{(L \cap \Gamma_{n-1})/\langle \mu \rangle} \subset \text{Ker} \text{tr}_{L}^{L \cap \Gamma_{n-1}}, \]

it follows that
\[ \text{Res}_{K}^{E_n}(\kappa_{n,i}) = 0. \]
If \( p = 3 \), by Lemma 14, there exist \( \phi \in H^2(L \cap \Gamma'_{n-1}) \), \( \psi \in H^1(L \cap \Gamma'_{n-1}) \) such that

\[
\text{Res}^{\Gamma'_{n-1}}_{L \cap \Gamma'_{n-1}} (Y_{n-1,2}) \text{ is a linear combination of } \]

\[
[k_{n-1}], \quad Y_{n-2,1} \cdot \phi, \quad Y_{n-2,1} \cdot x \psi
\]

and an element of \( \text{Im} \text{ Im}^{(L \cap \Gamma'_{n-1})/Z} \), since \( \phi, \psi \) belong to \( \text{Im} \text{ Im}^{(L \cap \Gamma'_{n-1})/Z} \), by Lemmas 13 and 14, it follows that

\[
\text{Res}^{\Gamma'_{n-1}}_{L \cap \Gamma'_{n-1}} \text{Res}^{\Gamma'_{n-1}}_{L \cap \Gamma'_{n-1}} (Y_{n-1,2}) = 0,
\]

so \( \text{Res}^{\Gamma'_{n-1}}_{K}(\kappa_{n,1}) = 0 \).

Finally, the case \( \alpha = x_{2n} \) follows from Lemma 19. The proposition is proved. \( \square \)

ACKNOWLEDGMENTS

Most of the results of this paper were obtained during a stay at the University of Essen and the ETH-Zentrum in Autumn 1997. I would like to thank Eckart Viehweg, Hélène Esnault and Urs Stammbach for making the visits possible. Many thanks to David John Green for valuable comments.

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