ESSENTIAL COHOMOLOGY AND EXTRASPECIAL $p$-GROUPS

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Abstract. Let $p$ be an odd prime number and let $G$ be an extraspecial $p$-group. The purpose of the paper is to show that $G$ has no non-zero essential mod-$p$ cohomology (and in fact that $H^*(G, F_p)$ is Cohen-Macaulay) if and only if $|G| = 27$ and $\text{exp}(G) = 3$.

1. Introduction

Let $p$ be a prime number. For every $p$-group $K$, denote by $H^*(K)$ the mod-$p$ cohomology ring of $K$. A mod-$p$ cohomology class of $K$ is called essential if it vanishes on restriction to every proper subgroup of $K$. Let $\text{Ess}(K)$ be the ideal of $H^*(K)$ consisting of such classes of $H^*(K)$. As observed in [3], the study of $\text{Ess}(K)$ could provide interesting information for $H^*(K)$ (but, in contrast, it seems in general rather difficult to obtain elements of $\text{Ess}(K)$). For instance, $\text{Ess}(K) \neq \{0\}$ implies that the depth of $H^*(K)$ is just the rank of the center of $K$ (see [3] and [5]); furthermore, with the condition that $H^*(K)$ is Cohen-Macaulay, it follows from [1] that $\text{Ess}(K) \neq \{0\}$ if and only if every element of order $p$ of $K$ is central (a way to obtain some element of $\text{Ess}(K)$ in this case was shown there).

We are now interested in extraspecial $p$-groups $G$. For $p = 2$, it was proved by Quillen ([17]) that $H^*(G)$ is Cohen-Macaulay and $\text{Ess}(G) = \{0\}$, except for the case $G = Q_8$, the quaternion group of order 8 (this fact also follows from Adem and Karagueuzian’s result, as $Q_8$ is the unique group in which every element of order 2 is central). However, the situation is quite different for the case $p > 2$—which is assumed from now on. Consider first the case $|G| = p^3$; it follows from [3], [9], [11], [16] that $\text{Ess}(G) \neq \{0\}$ (so $H^*(G)$ is not Cohen-Macaulay) if and only if $\text{exp}(G) > 3$. In order to generalize this fact, in this note, we prove

Theorem. If $G$ is an extraspecial $p$-group, then $\text{Ess}(G) = \{0\}$ iff $\text{exp}(G) = 3$ and $|G| = 3^3$.

It follows that the unique extraspecial $p$-group which has no non-zero essential cohomology is the one of order 27 and of exponent 3. In each of the remaining cases, $H^*(G)$ is not Cohen-Macaulay and the depth of $H^*(G)$ is just 1; we also point out some non-zero essential classes of $G$ (it turns out that, if $|G| = p^5$ or $\text{exp}(G) = p^2$, there exists such a class of $G$ belonging to $\text{Im} \text{Inf}^G/Z$ with $Z$ the center of $G$).

The note is organized as follows. In Section 2, given an extraspecial $p$-group $G$ of order $p^{2n+1}$, we shall consider $G$ as a subgroup of the central product $\Gamma_n = C_{p^2} \cdot G$ and give a sufficient and necessary condition for the fact that $\text{Res}^{\Gamma_n}_G(\xi) \neq 0$ with
\( \xi \in H^\ast(\Gamma_n) \). The proofs of the theorem for the cases \( \exp(G) > p \) or \( |G| = p^5 \), which are rather simple, will be given in Section 3. Section 4 is devoted to the case \( \exp(G) = p \).

2. The group \( \Gamma_n \)

Let us recall that an extraspecial \( p \)-group \( G \) is of order \( p^{2n+1} \) \((n \in \mathbb{N})\) and is isomorphic to one of the following central products of groups:

\[
\mathbb{E}_n = \mathbb{E} \cdots \mathbb{E} \quad \text{\( n \) times),}
\]

\[
\mathbb{M}_n = \mathbb{M} \cdot \mathbb{E}_{n-1},
\]

where

\[
\mathbb{M} = \langle a, b | a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p} \rangle,
\]

\[
\mathbb{E} = \langle a, b | a^p = b^p = [a, b]^p = [a, [a, b]] = [b, [a, b]] = 1 \rangle
\]

are extraspecial \( p \)-groups of order \( p^3 \). Note that

\[
\exp(G) = \begin{cases} p^2, & \text{for } G = \mathbb{M}_n, \\ p, & \text{for } G = \mathbb{E}_n, \end{cases}
\]

and \( \mathbb{M}_n = \mathbb{M}_{n-1} \cdot \mathbb{M} \).

These groups can be obtained cohomologically as follows. Let \( V \) be a vector space of dimension \( 2n + 1 \) over the prime field \( \mathbb{F}_p \) with basis \( e, a_1, \ldots, a_{2n} \). Let \( x, x_1, \ldots, x_{2n} \) be a basis of \( H^1(V) \), dual to that of \( V \), and let \( y = \beta x, y_i = \beta x_i \) with \( \beta \) the Bockstein homomorphism, so

\[
H = H^\ast(V) = E[x, x_1, \ldots, x_{2n}] \otimes \mathbb{F}_p[y, y_1, \ldots, y_{2n}]
\]

with \( E[u, v, \ldots] \) (resp. \( \mathbb{F}_p[u, v, \ldots] \)) the exterior (resp. polynomial) algebra over \( \mathbb{F}_p \) with generators \( u, v, \ldots \) of degree 1 (resp. 2). Consider the central extension

\[
(\Gamma_n, 0) \rightarrow \mathbb{F}_p \rightarrow \Gamma_n \rightarrow V \rightarrow 0,
\]

with factor set \( z = z_n = y + x_1 x_2 + \cdots + x_{2n-1} x_{2n} \). Via the inflation map, \( x \) and the \( x_i \)'s can be considered as elements of \( H^1(\Gamma_n) \). Given a subgroup \( K \) of \( \Gamma_n \), with some abuse of notation, we also denote by \( x \) (resp. \( x_i \)) the element Res\(^{\Gamma_n}_K(x) \) (resp. Res\(^{\Gamma_n}_K(x_i) \)).

It is easy to show

**Lemma 1.** (i) \( \Gamma_n = C_{p^2} \cdot \mathbb{M}_n = C_{p^2} \cdot \mathbb{E}_n = \Gamma_{n-1} \cdot \mathbb{M} \).

(ii) \( \mathbb{M}_n = \text{Ker}(x + \alpha), \mathbb{E}_n = \text{Ker} x \) and \( \Gamma_{n-1} \times C_p = \text{Ker} \alpha, \) with \( \alpha \) a non-zero linear combination of \( x_1, \ldots, x_{2n} \).

Then \( C_{p^2} = \bigcap_{i=1}^{2n} \text{Ker} x_i \) is a subgroup of \( \Gamma_n \). Let \( w \) be a generator of \( H^2(C_{p^2}) \), so

\[
H^\ast(C_{p^2}) = E[x] \otimes \mathbb{F}_p[w].
\]

Set \( \mathcal{G}_n = C_{p^2} \times \mathbb{E}_n \). By the K"{u}nneth formula, we have

\[
H^\ast(\mathcal{G}_n) = H^\ast(\mathbb{E}_n) \otimes E[x] \otimes \mathbb{F}_p[w].
\]

As \( \Gamma_n \) is the central product of \( C_{p^2} \) and \( \mathbb{E}_n \), there exists a central subgroup \( U_n \) of order \( p \) of \( \mathcal{G}_n \) such that \( \mathcal{G}_n/U_n = \Gamma_n \) and the factor set of the central extension

\[
1 \rightarrow U_n \rightarrow \mathcal{G}_n \rightarrow \Gamma_n \rightarrow 1
\]
is just $y$. Consider the following commutative diagram:

$$
\begin{array}{cccc}
1 & \longrightarrow & U_n & \longrightarrow & U_n \times E_n & \longrightarrow & E_n & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & U_n & \longrightarrow & G_n & \longrightarrow & \Gamma_n & \longrightarrow & 1 \\
\end{array}
$$

whose rows are central extensions and whose vertical arrows are inclusion maps. Pick elements $s, t$ of $H^*(U_n)$ satisfying $H^*(U_n) = E[s] \otimes P[t]$. It follows from [11] (see also [2]) that $t$ can be chosen so that $\text{Res}_{E_n}^E(w \times 1) = t$.

We now use the following notation. Given a ring $R$ and elements $r, s, \ldots \in R$, $(r, s, \ldots)$ will denote the ideal of $R$ generated by $r, s, \ldots$. The main result of this section is the following.

**Proposition 1.** If $\xi \in H^*(\Gamma_n)$, then $\text{Res}_{E_n}^E(\xi) \neq 0$ iff $x\xi \not\in (y)$.

**Proof.** Set $X = \text{Inf}_{E_n}^E(\xi)$. As $\text{Ker Inf}_{E_n}^E = (y)$, it follows that $xX = 0$ iff $x\xi \in (y)$.

Write $X = \sum w^i \otimes s_i + \sum w^i x \otimes t_i$ with $s_i, t_i \in H^*(E_n)$. It is clear that $\text{Res}_{E_n}^E(\xi) \neq 0$ iff $\text{Inf}_{E_n \times E_n}^E \text{Res}_{E_n}^E(\xi) \neq 0$. So, by the commutative diagram (1), $\text{Res}_{E_n}^E(\xi) \neq 0$ iff $\text{Res}_{E_n \times E_n}(X) \neq 0$, which is equivalent to the fact that the $s_i$’s are not all equal to zero, or equivalently, $x\xi \not\in (y)$. The proposition follows.

For convenience, given a central extension of groups

$$(K) \quad 1 \to A \to K \to C \to 1,$$

denote by $\{E_r(K), d_r\}$ the Hochschild-Serre spectral sequence corresponding to the extension $(K)$. We now recall some results given in [12], [13] (see also [2] for $n = 1$) concerning $\{E_r(\Gamma_n), d_r\}$. As usual, denote by $P^i$ the Steenrod operations. Set $Z = i(\mathbb{F}_p) \subset C_{2p^2} \subset \Gamma_n$. So $v = \text{Res}_{C_{p^2}}(w)$ is a generator of $H^2(Z)$. Let

$$X_n = x_1x_2 \ldots x_{2n-1}x_{2n},$$

$$\eta_i = P^i \beta z = \sum_{j=1}^{n} (x_{2j-1}y_{2j}^{i-1} - x_{2j}y_{2j-1}^{i-1}),$$

$$\xi_m = \beta P^{m-1} \ldots P^1 \beta z = \sum_{j=1}^{n} (y_{2j-1}y_{2j}^{m} - y_{2j}y_{2j-1}^{m}),$$

$1 \leq i \leq n + 1, 1 \leq m \leq n$, be elements of $H^*(V)$. We have

**Theorem 1** ([2], as corrected in [13] Rk. 2.11(ii), [12]). (i) We have

$$E_\infty(\Gamma_1) = H^*(V)/\langle z, \eta_1, \eta_2, \xi_1 \rangle \otimes \mathbb{F}_p[v^p]$$

$$\oplus \langle \mathbb{F}_p X_1 \otimes \mathbb{F}_p x_1 \rangle \otimes \sum_{i=1}^{p-2} \mathbb{F}_p[v^p]v^i.$$
(ii) For $n \geq 2$,
\[
E_{2p+1}(\Gamma_n) = H^*(V)/(z, \eta_1, \xi_1, A_n, \eta_2) \otimes F_p[v^p]
\]
\[\oplus (F_pX_n \oplus F_pX_n) \otimes \sum_{i=1}^{p-2} F_p[v^p]v^i \]
\[\text{with } A_n = \sum_{i=1}^{n} x_1 x_2 \ldots \hat{x}_{2i-1} \hat{x}_{2i} \ldots x_{2n-1} x_{2n}. \]

Let $W$ be the vector subspace of $V$ given by $W = \text{Ker}(x - x_1)$. We then have the central extension
\[
(M_n) \quad 1 \to Z \to M_n \to W \to 0
\]
with factor set $z' = z'_n = x_1 x_2 + \cdots + x_{2n-1} x_{2n}$. Following [10], [12], we also have

**Proposition 2** ([10], [12]). (i) We have
\[
E_\infty(M_1) = E_{2p+1}(M_1) = H^*(W)/(z', \beta z') \otimes F_p[v^p]
\]
\[\oplus (F_p[y_2]x_1 \oplus F_p[y_2]x_1 x_2) \otimes F_p[v^p]v^{p-1} \]
\[\oplus (F_p x_1 \oplus F_p x_1 x_2) \otimes \sum_{i=1}^{p-2} F_p[v^p]v^i. \]

(ii) For $n \geq 2$,
\[
E_{2p+1}(M_n) = H^*(W)/(z', \beta z' + B_n \cdot \beta z') \otimes F_p[v^p]
\]
\[\oplus (F_p x_1 x_3 x_4 \ldots x_{2n-1} x_{2n} \oplus F_p X_n) \otimes \sum_{i=1}^{p-2} F_p[v^p]v^i \]
\[\text{with } B_n = \sum_{i=2}^{n} x_1 x_3 x_4 \ldots \hat{x}_{2i-1} \hat{x}_{2i} \ldots x_{2n-1} x_{2n}. \]

We also prove

**Proposition 3.** If $\xi \in H^*(\Gamma_n)$ and $|\xi| < 2n + 2$, then $\xi \in \text{Im Inf}_{\Gamma_n}^V$.

The proof of the proposition is divided into the following lemmas. Set
\[
R = E[x_1, \ldots, x_m] \otimes F_p[t_1, \ldots, t_m],
\]
and let
\[
\alpha_i = \sum_{j=1}^{m} x_j t_1^{p-1}, \quad 1 \leq i \leq m,
\]
be elements of $R$. Denote by $I_{k,m}$ the set consisting of subsets of $k$ elements of $\{1, \ldots, m\}$. For every element $I = \{i_1, \ldots, i_k\}$ of $I_{k,m}$ with $i_1 < \cdots < i_k$, set $x_I = x_{i_1} \ldots x_{i_k}$ and $x_{\emptyset} = 1$.

**Lemma 2.** For $X \in R$ and $1 \leq k \leq m$,
(i) if $X \cdot \alpha_1 \ldots \alpha_k = 0$, then $X \in (\alpha_1, \ldots, \alpha_k, x_I | I \in I_{m-k+1,m})$;
(ii) if $X \cdot \alpha_k = 0$, then $X \in (\alpha_k, x_1 \ldots x_m)$. 
Proof. (i) We argue by induction on $m$. The case $m = 2$ is obvious. Assume that (i) holds for $m - 1$.

If $k = m$, then

$$
\begin{align*}
\alpha_1 \cdots \alpha_m & = \begin{vmatrix}
  t_1 & \cdots & t_m \\
  t_1^p & \cdots & t_m^p \\
  \vdots & \ddots & \vdots \\
  t_1^{p^{m-1}} & \cdots & t_m^{p^{m-1}}
\end{vmatrix}
x_1 \cdots x_m;
\end{align*}
$$

so $X \in (x_i | 1 \leq i \leq m)$. Suppose that $k < m$. Write

$$
\alpha_i = \alpha_i' + x_m t_m^{p^{m-1}}
$$

with $\alpha_i' = \sum_{j=1}^{m-1} x_j t_j^{i-1}, 1 \leq i \leq m$, and

$$
X = X' + X'' x_m,
$$

with $X', X''$ free of $x_m$. Since $X \alpha_1 \cdots \alpha_k = 0$, we have

$$
0 = X' \alpha_1' \cdots \alpha_k',
$$

$$
0 = (-1)^k X'' \alpha_1' \cdots \alpha_k' + X' \sum_{i=1}^k (-1)^{k-i} t_m^{p^{i-1}} \alpha_1' \cdots \hat{\alpha_i}' \cdots \alpha_k'.
$$

By writing

$$
X' = t_m^{r_1} f_1 + \cdots + t_m^{r_j} f_j,
$$

$$
X'' = t_m^{s_1} g_1 + \cdots + t_m^{s_i} g_i
$$

with $f_i, g_j$ free of $t_m, r_1 < \cdots < r_j, s_1 < \cdots < s_i$, we have

$$
(2) \quad (-1)^k t_m^{r_i} g_i \alpha_1' \cdots \alpha_k' + t_m^{p^{k-1}+r_j} f_j \alpha_1' \cdots \alpha_k' = 0.
$$

Consider the following cases:

- $r_j + p^{k-1} > s_i$: from (2), $f_j \alpha_1' \cdots \alpha_k' = 0$. By the inductive hypothesis, $f_j \in (\alpha_1', \ldots, \alpha_{k-1}', I_{m-k+1,m-1})$. Since $\alpha_i' = \alpha_i - x_m t_m^{p^{i-1}}$, we have $X = t_m^{r_i} f_1 + \cdots + t_m^{r_j} f_j \bmod (x_m, \alpha_1, \ldots, \alpha_k, I_{m-k+1,m})$. So we may suppose that $f_j = 0$.

- $r_j + p^{k-1} < s_i$: from (2), $g_i \alpha_1' \cdots \alpha_k' = 0$. By the inductive hypothesis, $g_i \in (\alpha_1', \ldots, \alpha_k', I_{m-k+1,m})$. So $x_m g_i \in (\alpha_1, \ldots, \alpha_k, I_{m-k+1,m})$.

- $r_j + p^{k-1} = s_i$: from (2), $(-1)^k t_m^{r_i} g_i \alpha_1' + f_j) \alpha_1' \cdots \alpha_{k-1}' = 0$. By the inductive hypothesis, $f_j = g_i \alpha_k'$ mod $(\alpha_1', \ldots, \alpha_{k-1}', I_{m-k+1,m-1})$. Since $\alpha_i' = \alpha_i - x_m t_m^{p^{i-1}}$, we have $f_j = g_i \alpha_k'$ mod $(x_m, \alpha_1, \ldots, \alpha_{k-1}, I_{m-k+1,m})$. So we may suppose that $f_j = g_i \alpha_k'$. Since

$$
t_m^{r_i} f_j + t_m^{r_i} g_i x_m = t_m^{r_i} (f_j + g_i x_m) = t_m^{r_i} g_i (\alpha_k' + t_m^{p^{k-1}} x_m) = t_m^{r_i} g_i \alpha_k,
$$

we may then suppose that $f_j = 0$ and $g_i = 0$.

The above arguments show that we may reduce to the case $X' = 0$. It follows that $X'' \alpha_1' \cdots \alpha_k' = 0$. By the inductive hypothesis, $X'' \in (\alpha_1', \ldots, \alpha_k', I_{m-k,m-1})$. Hence $x_m X'' \in (\alpha_1, \ldots, \alpha_k, I_{m-k+1,m})$. (i) is proved.
(ii) We again use induction on $m$. The case $m = 1$ is trivial. Assume that (ii) holds for $m - 1$. As above, write

$$X = X' + X''x_m,$$

with $X', X''$ free of $x_m$. Arguing as above, we may reduce to the case $X' = 0$. It follows that $X''a_k^e = 0$. By the inductive hypothesis, $X'' \in (\alpha_k', x_1 \ldots x_{m-1})$. Hence $x_mX'' \in (\alpha_k, x_1 \ldots x_m)$.

The lemma is proved.

**Lemma 3.** Let $1 \leq k \leq n$ and let $Y_1, \ldots, Y_k$ be elements of $H^*(V)$.

(i) If $Y_1\xi_1 + \cdots + Y_k\xi_k = 0$, then $Y_k \in (\xi_1, \ldots, \xi_{k-1})$.

(ii) Assume that

$$Y_k = \sum_{\{I \subset \{1, \ldots, 2n\}, \#(I) < 2n-k+1} \prod_{i \in I} x_if_I(y, y_1, \ldots, y_{2n}).$$

We have:

(iia) if $Y_1\eta_1 + \cdots + Y_k\eta_k = 0$, then $Y_k \in (\eta_1, \ldots, \eta_k)$;

(iib) if $Y_k \in \bigcap_{i=1}^k (\eta_i)$, then $Y_k \in (\eta_1 \ldots \eta_k)$;

(iic) if $Y_1\xi_1 + \cdots + Y_{k-1}\xi_{k-1} + Y_k\eta_k = 0$ with $1 \leq \ell \leq n$, then $Y_k \in (\eta_\ell, \xi_1, \ldots, \xi_{k-1})$.

**Proof.** (i) For $1 \leq i \leq k$, write

$$Y_i = \sum_{\{I \subset \{1, \ldots, 2n\}\}} f_{I}^{(i)}(y, y_1, \ldots, y_{2n}).$$

Then, for every $I$, we have

$$\sum_{i=1}^k f_{I}^{(i)}\xi_i = 0.$$

According to [28], $\xi_1, \ldots, \xi_k$ is a regular sequence in $P$. So the above equality implies $f_{I}^{(k)} \in (\xi_1, \ldots, \xi_{k-1})$. Therefore $Y_k \in (\xi_1, \ldots, \xi_{k-1})$.

(iia) It follows that $Y_k\eta_1 \cdots \eta_{k} = 0$. By Lemma 2, $Y_k \in (\eta_1, \ldots, \eta_k, T_{2n-k+1, 2n})$.

So $Y_k \in (\eta_1, \ldots, \eta_k)$.

(iib) We use induction on $k$. For $k = 2$, $X\eta_1 + Y\eta_2 = 0$ implies $Y_2\eta_k = 0$. By Lemma 2, $Y \in (\eta_1, \eta_2, T_{2n-1, 2n})$. So $Y \in (\eta_1, \eta_2)$. Write $Y = a\eta_1 + b\eta_2$. Then $Y_2 = Y\eta_2 = a\eta_1\eta_2$.

Assume that (iib) holds for $k - 1 \geq 2$. As $Y_k \in \bigcap_{i=1}^{k-1} (\eta_i)$, it follows from the inductive hypothesis that $Y_k = X\eta_1 \cdots \eta_{k-1}$. Write $Y_k = X\eta_k$. Then $Y_1\eta_1 \cdots \eta_{k-1} = 0$. By Lemma 2, $Y_k = c_1\eta_1 + \cdots + c_k\eta_k$. So $Y_k = (-1)^{k-1}c_k\eta_1 \cdots \eta_k$.

(iic) Again, we use induction on $k$. $Y_1\xi_1 + Y_2\eta_k = 0$ implies $Y_1\eta_k = 0$. By Lemma 2, $Y_1 \in (\eta_k)$. Write $Y_1 = c\eta_k$. Then $(\xi_1 + Y_2)\eta_k = 0$. By Lemma 2, $c\xi_1 + Y_2 \in (\eta_k)$; hence $Y_2 \in (\eta_k, \xi_1)$.

Assume that (iic) holds for $k - 1 \geq 2$. As $Y_1\eta_k\xi_1 + \cdots + Y_1\eta_k\xi_{k-1} = 0$, it follows from (i) that $Y_{k-1}\eta_k \in (\xi_1, \ldots, \xi_{k-2})$. By the inductive hypotheses, we may write $Y_{k-1} = c_1\xi_1 + \cdots + c_{k-2}\xi_{k-2} + c_{k-1}\eta_k$. Hence

$$(Y_1 + c_1\xi_{k-1})\xi_1 + \cdots + (Y_{k-2} + c_{k-2}\xi_{k-2})\xi_{k-2} + (Y_{k-1} + c_{k-1}\xi_{k-1})\eta_k = 0.$$  

By the inductive hypothesis, $Y_k + c_{k-1}\xi_{k-1} \in (\xi_1, \ldots, \xi_{k-2}, \eta_k)$, and hence $Y_k \in (\xi_1, \ldots, \xi_{k-1}, \eta_k)$. \qed
For $1 \leq i \leq n+1, 0 \leq k \leq n$, denote by $\Delta_{i,k}$ the ideal of $H^*(V)$ given by

$$
\Delta_{i,k} = \begin{cases} 
(z,\eta_j,\xi_m | 1 \leq j \leq i, 1 \leq m \leq k) & \text{if } k \geq 1, \\
(z,\eta_j | 1 \leq j \leq i) & \text{if } k = 0.
\end{cases}
$$

**Lemma 4.** If $X = \sum_{\#(I)<2n-2k+1} x_I X_I(y_1,\ldots,y_{2n})$ and $X_\xi_j \in \Delta_{k,j-1}$ with $1 \leq j \leq k \leq n$, then $X \in \Delta_{k,j-1}$.

**Proof.** Write

$$
X_\xi_j = a_0 z + \sum_{i=1}^k a_i \eta_i + \sum_{i=1}^{j-1} b_i \xi_i
$$

with $a_i, b_i \in H^*(V)$. Since $y = z - x_1 x_2 - \cdots - x_{2n-1} x_{2n}$, we may suppose that $a_i, b_t, 1 \leq i \leq k, 1 \leq t \leq j - 1$, are free of $y$. It follows that $a_0 = 0$ and

$$(3) \quad X_\xi_j \eta_1 \ldots \eta_k = \sum_{i=1}^{j-1} b_i \xi_i \eta_1 \ldots \eta_k.$$  

We now argue by induction on $j$. If $j = 1$, it follows that $X_\xi_j \eta_1 \ldots \eta_k = 0$. Hence $X \eta_1 \ldots \eta_k = 0$. By Lemma 2, $X \in (\eta_1, \ldots, \eta_k)$.

Assume that the lemma holds for $j - 1 \geq 1$. By Lemma 3 (i) and by (3), there exists $c_i \in H^*(V)$ such that

$$
X \eta_1 \ldots \eta_k = c_1 \xi_1 + \cdots + c_{j-1} \xi_{j-1}.
$$

Therefore, by Lemma 3 (i), $c_{j-1} \eta_i \in (\xi_1, \ldots, \xi_{j-2})$, for every $1 \leq i \leq k$; by Lemma 3 (ii), $c_{j-1} \in \bigcap_{i \leq k} (\xi_1, \ldots, \xi_{j-2}, \eta_i)$. By writing

$$
c_{j-1} = d_1 \xi_1 + \cdots + d_{j-2} \xi_{j-2} + d \eta_1 \ldots \eta_{j-1}
$$

we get

$$
[(e_1 - d_1) \xi_1 + \cdots + (e_{j-2} - d_{j-2}) \xi_{j-2}] \eta_1 \ldots \eta_i = 0.
$$

By Lemma 2, $(e_1 - d_1) \xi_1 + \cdots + (e_{j-2} - d_{j-2}) \xi_{j-2}$ contains $\eta_1 \ldots \eta_i$ as a factor. Hence $d \eta_1 \in \bigcap_{1 \leq i} (\eta_i)$. By Lemma 3 (ii), $c_{j-1} \in (\xi_1, \ldots, \xi_{j-2}, \eta_1 \ldots \eta_k)$. So we may suppose that $c_{j-1} \in (\eta_1 \ldots \eta_k)$. By writing $c_{j-1} = c \eta_1 \ldots \eta_k$, we have

$$(X - c \xi_{j-1}) \eta_1 \ldots \eta_k = c_1 \xi_1 + \cdots + c_{j-2} \xi_{j-2}.$$  

By the inductive hypothesis, this implies $X - c \xi_{j-1} \in \Delta_{k,j-2}$. So $X \in \Delta_{k,j-1}$. The lemma follows.  

**Lemma 5.** If $X = \sum_{\#(I)<2n-2k} x_I X_I(y_1,\ldots,y_{2n})$ and $X \eta_k \in \Delta_{k-1,k-1}$ with $1 \leq k \leq n+1$, then $X \in \Delta_{k,k-1}$.

**Proof.** Write

$$
X \eta_k = a_0 z + \sum_{i=1}^{k-1} (a_i \eta_i + b_i \xi_i).
$$

Arguing as in the proof of Lemma 4, we may suppose that $a_i, b_i, 1 \leq i \leq k - 1$, are free of $y$. It follows that $a_0 = 0$ and

$$
X \eta_k = \sum_{i=1}^{k-1} (a_i \eta_i + b_i \xi_i).$$
Furthermore, we may suppose that every $b_i$ is of form

$$b_i = \sum_{#(I)<2n-2k+1} x_I b_i^{(j)}.$$ 

Therefore, applying Lemma 4 yields $b_{k-1} \in \Delta_{k,k-2}$. Hence, by induction, we need only consider the case

$$X \eta_k = b_1 \xi_1 + \sum_{i=1}^{k-1} a_i \eta_i.$$ 

This implies $b_1 \xi_1 \ldots \xi_k = 0$. So $b_1 \eta_1 \ldots \eta_k = 0$. By Lemma 2, $b_1 \in (\eta_1, \ldots, \eta_k)$. The lemma follows.

Let us now consider the Hochschild-Serre spectral sequence $\{E_r(\Gamma_n), d_r\}$. It follows that, for $k < 2n + 2$,

$$\sum_{i+j=k} E^{i,j}_{2p+1} \subset E^{k,0}_{2p+1} \oplus \bigoplus_{r \geq 1} E^{s,2p^r}_{2p+1}.$$ 

By Kudo’s transgression theorem, for $m \leq n$, $1 \otimes v^{p^m}$ (resp. $\eta_m \otimes v^{p^{m-1}(p-1)}$) survives to $E_{2p^m+1}$ (resp. $E_{2p^{m-1}(p-1)+1}$) and

$$d_{2p^{m-1}(p-1)+1}(\eta_m \otimes v^{p^{m-1}(p-1)}) = -\xi_m.$$ 

Let us consider the Hochschild-Serre spectral sequence $\{E_r(\Gamma_n), d_r\}$. It follows that, for $k < 2n + 2$,

$$\sum_{i+j=k} E^{i,j}_{2p+1} \subset E^{k,0}_{2p^m+1} \oplus \bigoplus_{r \geq 1} E^{s,2p^m+r}_{2p^m+1}.$$ 

**Lemma 6.** For $k < 2n + 2$ and $1 \leq m \leq n$, we have

$$\sum_{i+j=k} E^{i,j}_{2p^m+1} \subset E^{k,0}_{2p^m+1} \oplus \bigoplus_{r \geq 1} E^{s,2p^m+r}_{2p^m+1}.$$ 

**Proof.** By the structure of $E_{2p^m+1}(\Gamma_n)$, the lemma holds for $m = 1$. Suppose that the lemma holds for $m = s \geq 1$. Let $\psi = X \otimes v^{p^s}$ be an element of $E_2(\Gamma_n)$ surviving to $E_{2p^s+1}$, with $1 \leq \ell \leq p - 1$ and $|X| + 2\ell p^s = k < 2n + 2$. So $d_{2p^s+1}(\psi) = \int X \eta_{s+1} \otimes v^{(\ell-1)p^s}$ must be hit by images under the differentials of elements of degrees less than $2n + 2$. By the inductive hypothesis and by Kudo’s theorem, these images belong to the ideal $\Delta_{s,s}$; hence so does $X \eta_{s+1}$. Since in $E_3(\Gamma_n)$ we have $y = -(x_1x_2 + \cdots + x_{2n-1}x_{2n})$, we may suppose that $X$ is free of $y$. As $|X| < 2n + 2 - 2p^s < 2n - 2s - 2$, by Lemma 5, this means that $X \in \Delta_{s+1,s}$. So $\psi = 0$ in $E_{2p^s+2}$ if $\ell < p - 1$. If $\ell = p - 1$, write $\psi = Y \eta_{s+1} \otimes v^{p^s(p-1)}$. Then $d_{2p^s(p-1)+1}(\psi) = -Y \xi_{s+1} \in \Delta_{s+1,s}$. Arguing as above, we may suppose that $Y$ is free of $y$. By Lemma 4, as $|\psi| < 2n + 2$, we have $Y \in \Delta_{s+1,s}$. So $\psi = 0$ in $E_{2p^s(p-1)+2}$. The lemma follows.

**Proof of Proposition 3.** It follows from Lemma 6 that $\xi$ either belongs to $\text{Im} \text{Inf}^{Y}_{1,n}$ or represents an element of $E^{*,2n^r}_{2n^r}$. As $2p^s > 2n + 2$, the fact that $|\xi| < 2n + 2$ implies $\xi \in \text{Im} \text{Inf}^{Y}_{1,n}$. The proposition follows.

3. The case $exp(G) > p$ or $|G| = p^5$

We first consider the case $G = \mathbb{M}_n$. Consider $G$ as a subgroup of $\Gamma_n$ by setting $G = \text{Ker} \ (x - x_1)$. If $n = 1$, it follows from [10] (see also [3]) that $H^*(\mathbb{M})$ contains
a non-zero essential element, namely \( X_1 \). Assume inductively that \( 0 \neq X_{n-1} \in \text{Ess}(\mathbb{M}_{n-1}) \). As \( \mathbb{M}_n = \mathbb{M}_{n-1} \cdot \mathbb{M} \), we have the following central extension:

\[
0 \to \mathbb{F}_p \to \mathbb{M}_{n-1} \times \mathbb{M} \to \mathbb{M}_n \to 1.
\]

The fact that \( H^*(\mathbb{M}_n) \) contains non-zero essential elements follows from

**Proposition 4.** \( 0 \neq X_n \in \text{Ess}(\mathbb{M}_n) \).

**Proof.** Let \( K \) be a maximal subgroup of \( \mathbb{M}_n \). As \( \dim_K H^1(K) = 2n - 1 \), it follows that the product of any \( 2n \) elements of \( H^1(K) \) vanishes. Hence \( \text{Res}^{M_n}_{K}(X_n) = 0 \), which implies that \( X_n \in \text{Ess}(\mathbb{M}_n) \). Furthermore, as \( \text{Inf}^{\mathbb{M}_n}_{\mathbb{M}_{n-1}}(X_n) = X_{n-1} \times x_{2n-1}x_{2n} \neq 0 \) in \( H^*(\mathbb{M}_{n-1} \times \mathbb{M}) \) by the inductive hypothesis, it follows that \( X_n \neq 0 \). The proposition is proved.

By Theorem 1 (i), \( X_1 \) and \( xX_1 \) are non-zero elements of \( H^*(\Gamma_1) \). By considering the central extension \( 0 \to \mathbb{F}_p \to \Gamma_{n-1} \times \mathbb{M} \to \Gamma_n \to 1 \), and by using the same argument given in the proof of Proposition 4, we also have

**Proposition 5.** The elements \( xX_n \) and \( X_n \) are non-zero elements of \( H^*(\Gamma_n) \).

Our next task is to prove that the theorem holds for the extraspecial \( p \)-group \( G = \mathbb{E}_2 \). Consider \( \mathbb{E}_2 \) as a subgroup of \( \Gamma_2 \) as in Lemma 1. Let \( Q \) be the element of \( H^*(V) \) defined by \( Q = Q_{2,1}^{1,2} - Q_{2,1}^{3,4} \) with

\[
Q_{2,1}^{i,j} = Q_{2,1}(y_i, y_j) = \frac{y_i^p y_j - y_j^p y_i}{y_i^p y_j - y_j^p y_i}
\]

(so \( Q_{2,1}^{i,j} \) is nothing but the Dickson invariant of order \( 2(p^2 - p) \) with variables \( y_i, y_j \)), and set \( \eta = x_1 x_2 Q \). It follows from [19] Th. 8.25] that \( 0 \neq \eta \in H^*(\mathbb{E}_2) \). The case \( G = \mathbb{E}_2 \) is then proved by the following:

**Proposition 6.** \( \eta \in \text{Ess}(\mathbb{E}_2) \).

**Proof.** Let \( K \) be a maximal subgroup of \( \mathbb{E}_2 \), so \( K \cong \mathbb{E} \times C_p \). If \( \text{Res}^{\mathbb{E}_2}_{K}(x_3x_4) = 0 \), it is clear that \( \text{Res}^{\mathbb{E}_2}_{K}(x_1x_2Q_{2,1}^{1,2}) = 0 \); we can then assume that \( \text{Res}^{\mathbb{E}_2}_{K}(x_3x_4) \neq 0 \). Choose a basis \( u_1, u_2, u_3, u_4 \) of \( H^1(\mathbb{E}_2/\mathbb{Z}) \) such that \( K = \text{Ker} u_1, x_1x_2 + x_3x_4 = u_1u_2 + u_3u_4 \), \( \text{Res}^{\mathbb{E}_2}_{K}(x_1x_2) = u_1u_2 + u_1u_3 \) and \( \text{Res}^{\mathbb{E}_2}_{K}(x_3x_4) = -u_1u_3 \). This implies that \( u_1u_2 = 0 \) in \( H^*(K) \). By setting \( v_1 = \beta u_1 \), we have

\[
\text{Res}^{\mathbb{E}_2}_{K}(\eta) = (u_1u_2 + u_1u_3)(Q_{2,1}(v_1, v_2 + v_3) - Q_{2,1}(v_3, v_1)) = u_1u_3(Q_{2,1}(v_1, v_2 + v_3) - Q_{2,1}(v_3, v_1)) \quad \text{as} \quad u_1u_2 = 0 \quad \text{in} \quad H^*(K).
\]

Set \( Y = Q_{2,1}(v_1, v_2 + v_3) - Q_{2,1}(v_3, v_1) \). Following [13] Proof of Lemma 1.10, \( Y \) contains \( \nu_2^p - \nu_2\nu_1^{p-1} \) as a factor. As \( u_1(\nu_2^p - \nu_2\nu_1^{p-1}) = -\nu_1^p \beta(u_1u_2) + \nu_1^{p-1} \beta(u_1u_2) \), we have \( \text{Res}^{\mathbb{E}_2}_{K}(\eta) = 0 \). So \( \eta \in \text{Ess}(\mathbb{E}_2) \). The proposition is proved.

For \( \exp(G) > p \) or \( |G| = p^5 \), Propositions 4 and 6 tell us that there exist non-zero essential cohomology classes of \( G \) which belong to \( \text{Inf}^r_G \). Furthermore, if \( G = \mathbb{M}_2 \), then [12] Proposition 1.9 and [13] Theorem 3.10 tell us that

\[
x_3x_4 N \quad \text{and} \quad (y_3x_4 - y_4x_3) N
\]

are also non-zero elements of \( \text{Ess}(\mathbb{M}_2) \) with \( N = (y_2^{p-1} - y_3^{p-1})(y_2^{p-1} - y_4^{p-1}) \). We can then end the section by the following

**Question.** For \( G \neq \mathbb{E} \), is it true that \( \text{Ess}(G) \cap \text{Inf}^r_G \neq \{0\} \)?
4. The case $exp(G) = p$

We first point out some mod-$p$ cohomology classes of $\Gamma_n$, by using the following argument given by D.J. Green \cite{Green}. Let $K$ be a $p$-group containing $C$ as a central subgroup. We have the central extension

\[(K) \quad 1 \to C \to K \xrightarrow{pr} K/C \to 1,\]

On the other hand, by considering the extension

\[(K \times C) \quad 1 \to C \xrightarrow{\ell} K \times C \xrightarrow{\mu} K \to 1\]

with $\ell(c) = (1, c), j(k, c) = k, c \in C, k \in K$, we have the commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & C \\
\downarrow & & \downarrow \mu \\
K \times C & \longrightarrow & K \\
\downarrow & & \downarrow pr \\
1 & \longrightarrow & K/C \\
\end{array}
\]

(4)

with $\mu(k, c) = kc, k \in K, c \in C$. The Hochschild-Serre spectral sequences corresponding to these extensions are of the forms

\[
E_2(K) = H^*(K/C) \otimes H^*(C) \Rightarrow H^*(K),
\]

\[
E_2(K \times C) = E_\infty(K \times C) = H^*(K) \otimes H^*(C).
\]

Furthermore, vertical arrows in (4) also induce a map $\{\mu_r : E_r(K) \to E_r(K \times C)\}$ between spectral sequences with $\mu_2 = (\text{Im}^{K/C}_K, 1_{H^*(C)})$.

The following is due to D.J. Green.

**Proposition 7.** For $r \geq 2$,

\[
\text{Im}(d_r : E_r(K) \to E_r(K)) \subset \text{Ker} \text{Inf}^{K/C}_K \otimes H^*(C).
\]

**Proof.** Let $\xi \in E_r(K)$ and write $d_r(\xi) = \sum \phi_j \otimes \psi_j, \phi_j \in H^*(K/C), \psi_j \in H^*(C)$.

We can suppose that the $\psi_j$’s are linearly independent in $H^*(C)$. From the commutative diagram (4) and from the fact that $d_r : E_r(K \times C) \to E_r(K \times C)$ vanishes, we have

\[
\sum \text{Inf}^{K/C}_K(\phi_j) \otimes \psi_j = \mu_r(d_r(\xi)) = d_r(\mu_r(\xi)) = 0.
\]

So $\phi_j \in \text{Ker} \text{Inf}^{K/C}_K$. The proposition follows. $\square$

Since $d_{2p+1}(v^p) = \eta_2$ in $E_{2p+1}(\Gamma_n)$ (resp. $P^1(\beta z')$ in $E_{2p+1}(M_n)$), it follows from Theorem 1 and Proposition 2 that $A_n \otimes v^p \in E_{2p+2}(\Gamma_n)$ and $B_n \otimes v^p \in E_{2p+2}(M_n)$.

We then get

**Proposition 8.** For $1 \leq i \leq p - 2$,

(i) if $n \geq 2$ then $x_1x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^i, x_n \otimes v^i, x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^p$ and $B_n \otimes v^p$ represent non-zero elements of $E_\infty(M_n)$;

(ii) $x_n \otimes v^i, x x_n \otimes v^i$ and $A_n \otimes v^p$ represent non-zero elements of $E_\infty(\Gamma_n)$.

**Proof.** Note that, in $H^*(W)$, we have

\[
x_3x_4 \ldots x_{2n-1}x_{2n} \cdot P^1(\beta z') = (x_3x_4 \ldots x_{2n-1}x_{2n})(y_1^p x_2 - y_2^p x_1)
= (x_3x_4 \ldots x_{2n-1}x_{2n})(z^px_2 + y_2^p - \beta z' - y_2^p y_1 x_2)
= (x_3x_4 \ldots x_{2n-1}x_{2n})(z^px_2 + y_2^p \beta z' - y_2^p y_1 x_2)
\in (z', \beta z').
\]
So $d_{2p+1}(x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^p) = 0$. Therefore $x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^p$ survives to $E_{\infty}(M_n)$.

By Proposition 4, $X_n \neq 0$ in $H^\ast(M_n)$ implies that $X_n, x_1x_3x_4 \ldots x_{2n-1}x_{2n}$ and $B_n$ are not elements of $\text{Ker Inf}^V_{M_n}$. Similarly, Proposition 5 shows that $X_n, xx_n$ and $A_n$ are not elements of $\text{Ker Inf}^V_{\Gamma_n}$. The proposition follows from Proposition 7.

For $n \geq 1$ and for $1 \leq i \leq p - 2$, let us pick elements $X_{n,i} \in H^{2(n+i)-1}(M_n)$ and $Y_{n,i} \in H^{2(n+i)}(\Gamma_n)$ which represent respectively $x_1x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^i \in E_{\infty}(M_n)$ and $X_n \otimes v^i \in E_{\infty}(\Gamma_n)$; for $n \geq 2$, pick elements $X_{n,p-1} \in H^{2(n+p)-3}(M_n)$, $Z_{n,p-1} \in H^{2(n+p)-2}(M_n)$ and $Y_{n,p-1} \in H^{2(n+p)-2}(\Gamma_n)$ which represent respectively $B_n \otimes v^p \in E_{\infty}(M_n)$, $x_3x_4 \ldots x_{2n-1}x_{2n} \otimes v^p \in E_{\infty}(M_n)$ and $A_n \otimes v^p \in E_{\infty}(\Gamma_n)$ (the existence of such elements follows from Propositions 2 and 8). In particular, define $Y_{1,p-1}$ by

$$Y_{1,p-1} = \mathcal{N}\text{Ker} x_2 - \Gamma_1(w)$$

with $\mathcal{N}$ the Evens norm map (note that $\text{Ker} x_2 \cong C_{p^2} \times C_p \subset \Gamma_1$, so, by the Künneth formula, $w$ can be considered as an element of $H^2(\text{Ker} x_2)$).

We now define the following subgroups of $\Gamma_n$:

$$M_n = \text{Ker}(x - x_1),$$
$$\Gamma'_n = \text{Ker} x_2n \cong \Gamma_{n-1} \times C_p,$$
$$M'_n = \text{Ker} x_2n \cap \text{Ker}(x - x_1) \quad (\text{so } M'_n \cong M_{n-1} \times C_p \text{ for } n > 1),$$
$$\Gamma_{n-1} = \text{Ker} x_2 \cap \text{Ker}(x - x_1) \cong \Gamma_{n-1},$$
$$\Gamma_{n-2} = \text{Ker} x_2 \cap \text{Ker}(x - x_1) \cong \Gamma_{n-2} \times C_p \quad (\text{for } n \geq 2),$$

with the convention that $\Gamma_0 = C_{p^2}$. Therefore $\Gamma_{n-2} = M'_{n-1} \cap \Gamma_{n-1}$ and $\Gamma_0 = C_{p^2} \times C_p$. If $K$ is one of the above subgroups, then $K$ contains $Z$ as a central subgroup and we have the central extension

$$(K) \quad 1 \rightarrow Z \rightarrow K \rightarrow K/Z \rightarrow 1.$$
(iii) Set $T_{n,i} = \text{Res}_{Y_{n+1}}^{Y_n}(Y_{n,i})$. Since $y_{2n-1}Y_{n,i} \in \text{Im Inf}^Y_{n+1}$, it follows that

$$y_{2n-1}T_{n,i} = \text{Res}_{Y_{n+1}}^{Y_n}(y_{2n-1}Y_{n,i})$$

belongs to $\text{Im Inf}_{n+1}^{Y_n}$. As $H^*(\Gamma_n) = H^*(\Gamma_{n-1}) \otimes E[x_{2n-1}] \otimes F_p[y_{2n-1}]$, $T_{n,i}$ also belongs to $\text{Im Inf}_{n+1}^{Y_n}$.

(iv) Assume that there exists $\xi \in H^{2n+1}(\Gamma_n)$ such that

$$y_\xi = xY_{n,1} \mod \text{Im Inf}^Y_{n+1}.$$  

By Proposition 3, $\xi \in \text{Im Inf}_{n}^V$. So $y_\xi \in \text{Im Inf}_{n}^V$. Hence $xY_{n,1} \in \text{Im Inf}_{n}^V$, a contradiction.

Assume inductively that (iv) holds for $n - 1$. For $i \geq 2$, we will prove in Lemmas 10, 11, 16 and 17 that $\text{tr}_{Y_{n+1}}^{Y_n}(Y_{n,i}) = \lambda_i Y_{n+1,i-1} \mod \text{Im Inf}_{n+1}^{Y_n}$ with $0 \neq \lambda_i \in F_p$.

Let $\phi$ be the element of $\text{Im Inf}_{n+1}^{Y_n}$ satisfying $\text{tr}_{Y_{n+1}}^{Y_n}(Y_{n,i}) = \lambda_i Y_{n+1,i-1} + \phi$. Suppose that $xY_{n,i} + \eta = y_\xi$, with $\xi \in H^*(\Gamma_n)$ and $\eta \in \text{Im Inf}_{n}^V$. So

$$\lambda_i xY_{n,i-1} + x\phi = \text{tr}_{Y_{n+1}}^{Y_n}(Y_{n,i})$$

$$= \text{tr}_{Y_{n+1}}^{Y_n}(xY_{n,i} + \eta) \quad \text{since} \quad \text{tr}_{Y_{n+1}}^{Y_n}(\eta) = 0$$

$$= \text{tr}_{Y_{n+1}}^{Y_n}(y_\xi)$$

$$= y_\xi \text{tr}_{Y_{n+1}}^{Y_n}(\xi).$$

Hence $\lambda_i xY_{n,i-1} = y_\xi \text{tr}_{Y_{n+1}}^{Y_n}(\xi) - x\phi$, which contradicts the inductive hypothesis. (iv) is then proved.

The lemma follows.

Further properties of $X_{n,i}$ and $Y_{n,i}$ are given by the following lemmas. The first one follows from Theorem 1, Proposition 2 and [14, Theorem 1.1].

**Lemma 8.** $\text{tr}_{M_{n+1}}^{Y_n}(X_{n-1,1})$ (resp. $\text{tr}_{Y_{n+1}}^{Y_n}(Y_{n-1,1})$) represents an element of $E_{\infty,2j}(M_n)$ (resp. $E_{\infty,2j}(\Gamma_n)$), with $j < i$. □

**Lemma 9.** For $2 \leq i \leq p - 1$ we have $\text{tr}_{Y_{n+1}}^{Y_n}(Y_{0,i}) = \lambda_i Y_{1,i-1} \mod \text{Im Inf}_{n+1}^{Y_n}$, with $0 \neq \lambda_i \in F_p$.

**Proof.** Consider the commutative diagram

$$
\begin{array}{ccc}
H^*(C_p \times C_p) & \xrightarrow{\text{tr}} & H^*(\Gamma_1) \\
\text{Res} & & \text{Res} \\
H^*(Z \times C_p) & \xrightarrow{\text{tr}} & H^*(E)
\end{array}
$$

We have $\text{Res}_{E}^{\Gamma_1} \text{tr}_{Y_{n+1}}^{Y_n}(Y_{0,i}) = \text{tr}_{E}^{Z \times C_p}(v^i)$. Following [8] (see also [19]), $\text{tr}_{E}^{Z \times C_p}(v^i)$ is a non-zero element of $H^*(E) \setminus \text{Im Inf}_{E}^{Z \times C_p}$. So, by Theorem 1, $\text{tr}_{Y_{n+1}}^{Y_n}(Y_{0,i})$ represents an element of the form $\lambda_i x_1 x_2 \otimes v^{i-1} \in E_{\infty,2j}^{2,i-1}(\Gamma_1)$, with $0 \neq \lambda_i \in F_p$. The lemma follows. □

In the following two lemmas, $p$ is assumed to be greater than 3.
Lemma 10. For $2 \leq i \leq p - 2$,
\[ \text{tr}_{M_2}^{M_1'}(X_{1,i}) = \lambda_i X_{2,i-1} \mod \text{Im} \text{Inf}_{M_2}^W \]
and
\[ \text{tr}_{\Gamma_2'}(Y_{1,i}) = \lambda_i Y_{2,i-1} \mod \text{Im} \text{Inf}_{\Gamma_2}^V, \]
with $\lambda_i$ given in Lemma 9.

Proof. Set $Z_i = \text{Res}_{\Gamma_1}^{\Gamma_2} \text{tr}_{M_2}^{M_1'}(X_{1,i})$. By the double coset formula and by Lemma 7 (ii), we have
\[ Z_i = \text{tr}_{\Gamma_1}^{\Gamma_1'} \text{Res}_{M_1}^{M_2'}(X_{1,i}) = \text{tr}_{\Gamma_1}^{\Gamma_1'}(x_1 Y_{0,i}) = x_1 \text{tr}_{\Gamma_1}^{\Gamma_0}(Y_{0,i}). \]
By Lemma 9, $Z_i$ represents
\[ \lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E_{\infty}^{3,2(i-1)}(\Gamma_1). \]
By Lemma 8 and Proposition 2, this means that $\text{tr}_{M_2}^{M_1'}(X_{1,i})$ represents
\[ \lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E_{\infty}^{3,2(i-1)}(M_2). \]
The first part of the lemma follows from the definition of $X_{2,i-1}$.

On the other hand, by setting $Y_i = \text{Res}_{\Gamma_1}^{\Gamma_2} \text{tr}_{\Gamma_2'}(Y_{1,i})$, by the double coset formula, we have
\[ Y_i = \text{tr}_{\Gamma_2}^{\Gamma_1} \text{Res}_{M_1}^{M_2'}(Y_{1,i}) = \text{tr}_{\Gamma_2}^{\Gamma_1}(-x_2 X_{1,i}) \text{ by Lemma 7 (i)} = -x_2 \text{tr}_{M_2}^{M_1'}(X_{1,i}). \]
As shown above, $\text{tr}_{M_2}^{M_1'}(X_{1,i})$ represents $\lambda_i x_1 x_3 x_4 \otimes v^{i-1} \in E_{\infty}^{3,2(i-1)}(M_2)$, so $Y_i$ represents $\lambda_i x_1 x_2 x_3 x_4 \otimes v^{i-1} \in E_{\infty}^{4,2(i-1)}(M_2)$. By Lemma 8 and Theorem 1 (ii), this means that $\text{tr}_{\Gamma_2'}(Y_{1,i})$ represents $\lambda_i x_1 x_2 x_3 x_4 \otimes v^{i-1} \in E_{\infty}^{3,2(i-1)}(\Gamma_2)$. The last part follows from the definition of $Y_{2,i-1}$. The lemma is proved.

In general, we have

Lemma 11. For $2 \leq i \leq p - 2$ and $n \geq 2$,
\[ \text{tr}_{M_n}^{M_1} (X_{n-1,i}) = \lambda_i X_{n,i-1} \mod \text{Im} \text{Inf}_{M_n}^W \]
and
\[ \text{tr}_{\Gamma_n}^{\Gamma_1} (Y_{n-1,i}) = \lambda_i Y_{n,i-1} \mod \text{Im} \text{Inf}_{\Gamma_n}^V, \]
with $\lambda_i$ given in Lemma 9.

Proof. We argue by induction on $n$. The case $n = 2$ follows from the above lemma. Assume that the lemma holds for $n - 1$. Set $Z_i = \text{Res}_{\Gamma_1}^{\Gamma_n} \text{tr}_{M_n}^{M_1'}(X_{n-1,i})$. By the double coset formula, we have
\[ Z_i = \text{tr}_{\Gamma_n}^{\Gamma_n} \text{Res}_{\Gamma_n}^{\Gamma_1} (X_{n-1,i}) = \text{tr}_{\Gamma_n}^{\Gamma_1} (x Y_{n-1,i}) \text{ by Lemma 7 (ii)} = x \text{tr}_{\Gamma_n}^{\Gamma_n} (Y_{n-2,i}). \]
By the inductive hypothesis, $\text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-2}}(Y_{n-2,i}) = \lambda_i Y_{n-1,i-1} \mod \text{Im} \text{Inf}_{\Gamma_{n-1}}^{\Gamma_{n-1}/Z}$. So $Z_i$ and $\lambda_i x Y_{n-1,i-1}$ represent the same element of $E_2^{2n+1,2(i-1)}(\Gamma_{n-1})$. The first part follows from Lemma 8 and Proposition 2.

Finally, by setting $Y_i = \text{Res}_{M_n}^{\Gamma_n} \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(Y_{n-1,i})$, we have

$$Y_i = \text{tr}_{M_n}^{\Gamma_n} \text{Res}_{M_n}^{\Gamma_n} \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(Y_{n-1,i})$$

$$= \text{tr}_{M_n}^{\Gamma_n}(-x_2 X_{n-1,i}) \quad \text{by Lemma 7 (i)}$$

$$= -x_2 \text{tr}_{M_n}^{\Gamma_n}(X_{n-1,i}).$$

As shown above, $\text{tr}_{M_n}^{\Gamma_n}(X_{n-1,i}) = \lambda_i x X_{n,i-1} \mod \text{Im} \text{Inf}_{M_n}^{M_n}$. So $Y_i$ and $-\lambda_i x X_{n,i-1}$ represent the same element of $E_2^{2n,2(i-1)}(M_n)$. The last part follows from Lemma 8 and Theorem 1 (ii). The lemma is proved.

We now calculate $\text{tr}_{\Gamma_n}^{\Gamma_n}(Y_{n-1,1})$. In so doing, let us recall the determination of the transfer map on bar cochain levels. Let $L, K$ be subgroups of $\Gamma_n$ with $Z \subseteq L \subseteq K$ and let $D = \{d\}$ be the set of cosets of $L$ in $K$. For each $d$, specify a representative $\overline{d}$ of $d$ such that $\overline{1} = 1$ and $\overline{d} \overline{d'} \overline{d''}^{-1} \in Z$. The transfer map $\text{tr}_L^K: C^*(L) \rightarrow C^*(K)$ is determined in [20] as follows:

$$\text{tr}_L^K f(\ell_1, \ldots, \ell_n) = \frac{1}{|D|} \sum_{d \in D} f(\ell_1 \overline{d} \ell_1^{-1}, \ldots, \ell_{n-1} \overline{d} \ell_{n-1}^{-1})$$

for $f \in C^*(L)$, $\ell_i \in K$.

Some properties of $\text{tr}_L^K$ were given in [14]. Note that, if $L$ is a direct factor of $K$, then $\text{tr}_L^K$ is the zero map. Furthermore, if $M$ is also a subgroup of $\Gamma_n$ containing $Z$, we can choose representatives of the cosets of $M$ in $K M$, and of those of $K \cap M$ in $K$, so that the double coset formula

$$\text{Res}_{K M}^{K M} \text{tr}_K^M = \text{tr}_{K \cap M}^K \text{Res}_{K \cap M}^M (5)$$

holds at the cochain level.

Since $v \in E_2(\Gamma_{n-1})$ is transgressive, there exists a 2-cochain $\tilde{v}$ of $\Gamma_{n-1}$ satisfying $\tilde{v}|_Z = v$, $\delta \tilde{v} = \beta_{z_{n-1}}$ (see e.g. [15] for a determination of such a cochain). It follows from [14] Lemma 1.4 that $\text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(\beta_{z_{n-1}}) = 0$, hence $\delta \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(\tilde{v}) = \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(\delta \tilde{v}) = 0$; in other words, $\text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(\tilde{v})$ is a 2-cocycle of $\Gamma_n$. Set $\nu = [\text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(\tilde{v})] \in H^2(\Gamma_n)$ and let $\tilde{e}, \tilde{a}_1, \ldots, \tilde{a}_{2n}$ be elements of $\Gamma_n$ satisfying $\tilde{e} Z = e, \tilde{a}_i Z = a_i$ (recall that $e, a_1, \ldots, a_{2n}$ was defined in Section 2 as a basis of $V$ of which the dual is $x, x_1, \ldots, x_{2n}$). We have

**Lemma 12.** $\nu = -x_2 x_1 \cdots x_{2n}$.

**Proof.** Write

$$\tilde{v} = \sum_{1 \leq i \leq 2n} \mu_i x_i x_i + \sum_{1 \leq i < j \leq 2n} \mu_{ij} x_i x_j + \sum_{1 \leq i \leq 2n} \nu_i y_i$$

with $\mu_i, \mu_{ij}, \nu_i \in F_p$ (note that, in $H^2(\Gamma_n)$, $y = -(x_1 x_2 + \cdots + x_{2n-1} x_{2n})$). Consider the double coset formula (5) with $M = \Gamma_{n-1}$ and $K M = \Gamma_n$ (this means that
For \( K = \langle \bar{e}, \bar{a}_i, \bar{a}_j, \bar{a}_{2n} \rangle \) with \( 1 \leq i, j \leq 2n - 2 \), as \( \bar{a}_{2n} \) commutes with every element of \( K \cap M \), we have \( \text{tr}_{K^{(M)}} \) = 0, so \( \mu_i = \mu_{2n} = \nu_i = \nu_{2n} = \mu_{ij} = \mu_{12n} = 0 \). For \( K = \langle \bar{e}, \bar{a}_i, \bar{a}_{2n-1}, \bar{a}_{2n} \rangle \) with \( 1 \leq i \leq 2n - 2 \), we have \( K \cong \Gamma_1 \times C_p, K \cap M = C_p \times C_p^{2} \) and \( \text{Res}_{K^{(M)}}^{M}(\bar{v}) = \bar{w} \); by a direct verification, we can show that \( \text{tr}_{K}^{(M)}(\bar{w}) = y \), therefore \( [\text{tr}_{K}^{(M)}(\bar{w})] = y = -x_{2n-1}x_{2n} \), so \( \mu_{2n-1} = \mu_{12n-1} = \nu_{2n-1} = 0 \) and \( \mu_{2n-1} = -1 \). The lemma follows.

**Lemma 13.** For \( n \geq 1 \), \( \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(Y_{n-1}, 1) = -X_n \); hence \( \text{Res}_{\Gamma_{n}}^{\Gamma_{n-1}} \text{tr}_{\Gamma_{n}}^{\Gamma_{n-1}}(Y_{n-1}, 1) = 0 \).

**Proof.** A cocycle representing \( Y_{n-1, 1} \) can be chosen as follows. Since \( x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} \cdot \beta_{z_{n-1}} = 0 \) in \( H^{1}((\Gamma_{n-1})/\mathbb{Z}) \), there exists a cochain \( f \) of \( \Gamma_{n-1}/\mathbb{Z} \) (considered as a cochain of \( \Gamma_{n-1} \) via the inflation map on cochains) satisfying \( \delta f = x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} \cdot \beta_{z_{n-1}} \). Furthermore, it follows from the definition of \( \bar{v} \) that

\[
\delta(x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} \cdot \bar{v}) = x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} \cdot \beta_{z_{n-1}}; \\
\text{hence } \delta(x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} \cdot \bar{v} - f) = 0.
\]

Clearly \( g = x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} \cdot \bar{v} - f \) is a cocycle representing \( X_n \otimes v \in E_{\infty}((\Gamma_{n-1})/\mathbb{Z}). \)

Hence

\[
Y_{n-1, 1} - [g] \in \text{Im} \text{Inf}_{\Gamma_{n-1}/\mathbb{Z}}^{\Gamma_{n-1}},
\]

which implies that \( \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(Y_{n-1}, 1) \) is represented by \( \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(g) \). By [14] Lemma 1.4, \( \text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(g) = x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} \cdot \beta_{z_{n-1}} \). So \( [\text{tr}_{\Gamma_{n-1}}^{\Gamma_{n-1}}(g)] = x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} \cdot \bar{v} \).

The lemma now follows from Lemma 12.

Arguing as in the above proof, we can also choose a cocycle representing \( X_{n-1} \otimes v \) (which is non-zero in \( E_{\infty}((\Gamma_{n-1})/\mathbb{Z}) \), by Theorem 1, Propositions 5 and 7), as follows.

As \( v \in E_{2}((\Gamma_{n-1})/\mathbb{Z}) \) is transgressive and \( d_{2n}^{2}(v) = \mathcal{P}^{1}k_{2n-1} \), there exists a cochain \( \bar{v} \) of \( \Gamma_{n-1} \) such that \( \bar{v}^{2} = v \), and \( \delta \bar{v} = \mathcal{P}^{1}k_{2n-1} \). Let \( h \) be a cochain of \( \Gamma_{n-1}/\mathbb{Z} \) satisfying \( \delta h = \mathcal{P}^{1}k_{2n-1} \cdot X_{n-1} \). We have

**Lemma 14.** \( k = k_{n} = \bar{v} \cdot x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} - h \) is a cocycle representing \( X_{n-1} \otimes v \) and \( \text{Res}_{\Gamma_{n}}^{\Gamma_{n-1}} \text{tr}_{\Gamma_{n}}^{\Gamma_{n-1}}([k]) = 0 \).

**Proof.** It follows from the definitions of \( \bar{v} \) and \( h \) that \( k \) is a cocycle representing \( X_{n-1} \otimes v \). Set \( X = \text{Res}_{\Gamma_{n}}^{\Gamma_{n-1}} \text{tr}_{\Gamma_{n}}^{\Gamma_{n-1}}([k]); \) then \( X = [\text{tr}_{\Gamma_{n}}^{\Gamma_{n-1} \times C_p} \text{Res}_{\Gamma_{n-1} \times C_p}^{\Gamma_{n-1}}(k)] \) by the double coset formula. Denote also by \( \bar{v} \) (resp. \( h \)) the restriction of the cochain \( \bar{v} \) (resp. \( h \)) to \( \mathbb{E}_{n-1} \times C_p \). By [14] Lemma 1.4, \( [\text{tr}_{\Gamma_{n}}^{\Gamma_{n-1} \times C_p}(\bar{v} \cdot x_{1}x_{2} \ldots x_{2n-3}x_{2n-2})] = 0 \); hence \( X = \left[ \text{tr}_{\Gamma_{n}}^{\Gamma_{n-1} \times C_p}(\bar{v} \cdot x_{1}x_{2} \ldots x_{2n-3}x_{2n-2}) \right] \). Note that, in \( H^{1}(\mathbb{E}_{n-1} \times C_p) \) we have \( X_{n-1} - X_{n-2} = x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} = 0 \), so there exist cochains \( c \) of \( (\mathbb{E}_{n-1} \times C_p)/\mathbb{Z} \) and \( b \) of \( \mathbb{E}_{n-1} \times C_p \) satisfying

\[
\delta b = x_{1}x_{2} \ldots x_{2n-3}x_{2n-2}, \\
x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} = x_{1}x_{2} \ldots x_{2n-5}x_{2n-4} \cdot \delta b + \delta c.
\]

Hence

\[
\bar{v} \cdot x_{1}x_{2} \ldots x_{2n-3}x_{2n-2} = \bar{v} \cdot x_{1}x_{2} \ldots x_{2n-5}x_{2n-4} \cdot \delta b + \bar{v} \cdot \delta c = -\delta \bar{v} \cdot x_{1}x_{2} \ldots x_{2n-5}x_{2n-4} \cdot b - \delta \bar{v} \cdot c \text{ mod } \text{Im } \delta.
\]
So \( X = -[\text{tr}_{E_n}^{E_{n-1} \times C_p'}(\delta \bar{v} \cdot x_1 x_2 \ldots x_{2n-5} x_{2n-4} \cdot b + \delta \bar{v} \cdot c)]. \) Following [14] Lemma 1.4, \( \text{tr}_{E_n}^{E_{n-1} \times C_p'}(\delta \bar{v} \cdot c) = 0 \) and \( \text{tr}_{E_n}^{E_{n-1} \times C_p'}(\delta b) = 0. \) This implies that \( \text{tr}_{E_n}^{E_{n-1} \times C_p'}(b) \) is a cocycle of \( E_n \) and

\[
X = -\mathcal{P}^1 \beta z_{n-1} \cdot X_{n-2} \cdot [\text{tr}_{E_n}^{E_{n-1} \times C_p'}(b)].
\]

Arguing as in the proof of Lemma 12, we can show that \( [\text{tr}_{E_n}^{E_{n-1} \times C_p'}(b)] = 0. \) Hence \( X = 0. \) The lemma follows.

With some abuse of notation, we also denote by \( \bar{v} \) (resp. \( \bar{v}' \)) the restriction of \( \tilde{v} \) (resp. \( \tilde{v}' \)) to \( M'_{n-1}. \) So \( \delta(\bar{v}) = \beta z'_{n-1} \) and \( \delta(\bar{v}') = \mathcal{P}^1 \beta z'_{n-1} \) in \( C^*(M'_{n-1}). \) Let \( \bar{u} \) be a 1-cocochain of \( M'_{n-1} \) satisfying \( \delta(\bar{u}) = z'_{n-1}. \) It follows from the proof of Proposition 8 that there exists a cocochain \( d \) of \( M'_{n-1}/Z \) such that

\[
\delta d = x_1 x_2 \ldots x_{2n-3} x_{2n-2} \left( \mathcal{P}^1 \beta z'_{n-1} - x_2 z'_{n-2} - y_2^{-1} \beta z'_{n-1} + y_2^{-1} x_2 z'_{n-1} \right)
= \delta(x_1 x_2 \ldots x_{2n-3} x_{2n-2} (\bar{v} + x_2 z'_{n-1} \bar{u} - y_2^{-1} \bar{v} - y_2^{-1} x_2 \bar{u})).
\]

So, for \( n \geq 3, q = x_1 x_2 \ldots x_{2n-3} x_{2n-2} (\bar{v} + x_2 z'_{n-1} \bar{u} - y_2^{-1} \bar{v} - y_2^{-1} x_2 \bar{u}) - d \) is a cocycle of \( M'_{n-1} \) representing \( Z_{n-1}. \) We have

**Lemma 15.** For \( n \geq 3, \)

\[
\text{tr}_{M'_{n-1}}^{M'_{n-1}}(Z_{n-1,p-1}) \in \text{Im Inf}_{bl}^W.
\]

**Proof.** It follows that \( \text{tr}_{M'_{n-1}}^{M'_{n-1}}(Z_{n-1,p-1}) = [\text{tr}_{M_{n-1}}^{M'_{n-1}}(q)]. \) By [14] Lemma 1.4,

\[
\text{tr}_{M'_{n-1}}^{M'_{n-1}}(Z_{n-1,p-1}) = x_1 x_2 \ldots x_{2n-3} x_{2n-2} \left( [\text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{v})] + x_2 z'_{n-1} [\text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{u})] \right)
- y_2^{-1} [\text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{v})] - y_2^{-1} x_2 [\text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{u})]
\]

(note that \( \text{tr}_{M_{n-1}}^{M'_{n-1}} \) maps each of \( \bar{v}, \bar{u}, \bar{v}, \bar{u} \) to a cocycle). Since each of \( \text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{v}), \text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{v}), \text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{u}), \) \( \text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{u}) \) is of degree \( \leq 2p \), it follows from the structure of \( E_{2p+1}(M_n) \) that \( \text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{v}), \text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{u}) \) and the cup-product of \( x_1 x_2 \ldots x_{2n-3} x_{2n-2} \) with \( [\text{tr}_{M_{n-1}}^{M'_{n-1}}(\bar{v})] \) belong to \( \text{Im Inf}_{bl}^W. \) The lemma follows.

**Lemma 16.** There exists a non-zero \( \lambda \in \mathbb{F}_p \) such that \( \text{tr}_{1_2}^{1_2}(Y_{1,p-1}) - \lambda Y_{2,p-2} \in \text{Im Inf}_{1_2}^{1_2}. \)

**Proof.** Set \( K = \text{Ker} x_2 \cap \text{Ker}(x - x_4) \subset \Gamma_2 \) and \( X = \text{tr}_{1_2}^{1_2}(Y_{1,p-1}). \) So \( K \cong M \times C_p, K \cap \Gamma_1' \cong C_3^p \) and \( \text{Res}_{K}^{K}(X) = \text{tr}_{K}^{K} \cap K \text{Res}_{1_2}^{1_2}(Y_{1,p-1}). \) As \( \text{Res}_{1_2}^{1_2}(Y_{1,p-1}) = \nu p - v_1 p^{-1}, \) we have

\[
\text{Res}_{K}^{K}(X) = \text{tr}_{K}^{K} \cap K (\nu p - v_1 p^{-1}) = \text{tr}_{K}^{K} \cap K (\nu p) - \text{tr}_{K}^{K} \cap K (v_1 p^{-1}).
\]

A direct verification shows that \( \text{tr}_{K}^{K} \cap K (v) = y_4, \) so \( \text{Res}_{K}^{K}(X) = -y_4 v_1 p^{-1} \neq 0. \) Hence \( X \neq 0. \)

Suppose that \( X \in \text{Im Inf}_{1_2}^{1_2}. \) Since \( y_4 X = 0, y_4 X \) must belong to \( (z, \eta_1, \eta_2, \xi_1). \) Write

\[
y_4 X = az + b\eta_1 + c\eta_2 + \mu \xi_1
\]
with \(a, b, c \in H^*(V)\) and \(\mu \in \mathbb{F}_p\). Multiplying (6) by \(x_1 x_2 x_3 x_4\) yields \(\mu \xi_1 \in (y, y_4)\). Hence \(\mu = 0\). Multiplying (6) by \(\eta_2\) yields \(y_4 X \eta_2 \in (z, \eta_1)\). So, by [13] Lemma 2.4, \(X \eta_2 \in (z, \eta_1, X_2)\). Since \(X \eta_2\) is of degree \(> 4\), it follows that \(X \eta_2 \in (z, \eta_1)\). By [13] Lemma 2.14], \(X = ey \mod (z, \eta_1)\) with \(e \in H^{2p-2}(V)\). Write

\[
e_{\mu} y_4 = a_1 z + b_1 \eta_1 + c_1 \eta_2.
\]

Multiplying (7) by \(\eta_1 \eta_2\) yields

\[
e_{\mu} y_4 \eta_1 \eta_2 = a_1 z \eta_1 \eta_2 = a_1 y \eta_1 \eta_2 - a_1 X_2 \xi_1.
\]

So \(a_1 \in (y, x_1, \ldots, x_4)\). Therefore \(b_1 \in (y, x, x_j)\) and \(c_1 = 0\). By [13] Lemma 2.4, we have \(e y \in (z, \eta_1, X_2)\). Since \(e y\) is of degree \(> 4\), it follows that \(e y \in (z, \eta_1)\). So \(X \in (z, \eta_1)\), and hence \(X = 0\) in \(H^*(\Gamma_2)\), a contradiction. The lemma follows. \(\square\)

**Lemma 17.** For \(n \geq 3\),

\[
\text{tr}^{M_{n-1}}(X_{n-1,p-1}) = \lambda X_{n,p-2} \mod \text{Im Inf}^{W}_{M_n}
\]

and

\[
\text{tr}^{\Gamma_{n-1}}(Y_{n-1,p-1}) = \lambda Y_{n,p-2} \mod \text{Im Inf}^V_{\Gamma_n},
\]

with \(\lambda\) given in Lemma 16.

**Proof.** Consider the case \(n = 3\). Set \(X = \text{Res}_{\Gamma_2}^{M_3} \text{tr}^{M_2}_{\Gamma_1}(X_{2,p-1})\). By the double coset formula, we have

\[
X = \text{tr}^{\Gamma_1}_{\Gamma_2} \text{Res}_{\Gamma_1}^{M_2}(X_{2,p-1})
\]

\[
= \text{tr}^{\Gamma_1}_{\Gamma_3}(x Y_{1,p-1}) \quad \text{by Lemma 7 (ii)}
\]

\[
= x \text{tr}^{\Gamma_1}_{\Gamma_2}(Y_{1,p-1}).
\]

It follows from Lemma 16 that \(X\) and \(\lambda y Z_{p-2}\) represent the same element of \(E_{5,2(p-2)}(\Gamma_2)\). By Lemma 8 and Proposition 2, it follows that \(\text{tr}^{M_2}_{\Gamma_3}(X_{2,p-1}) = \lambda X_{3,p-2} \mod \text{Im Inf}^V_{\Gamma_3}\) and \(- \lambda y X_{3,p-2} \mod \text{Im Inf}^V_{\Gamma_3}\). Similarly, by setting \(Y = \text{Res}_{\Gamma_3}^{M_3} \text{tr}^{M_2}_{\Gamma_3}(Y_{2,p-1})\), we have

\[
Y = \text{tr}^{M_3}_{\Gamma_3} \text{Res}_{\Gamma_2}^{M_2}(Y_{2,p-1})
\]

\[
= \text{tr}^{M_3}_{\Gamma_3}(-x_2 X_{2,p-1} + Z_{2,p-1}) \quad \text{by Lemma 7 (i)}
\]

\[
= -x_2 \text{tr}^{M_3}_{\Gamma_3}(X_{2,p-1}) + \text{tr}^{M_3}_{\Gamma_3}(Z_{2,p-1}).
\]

As shown above, \(\text{tr}^{M_2}_{\Gamma_3}(X_{2,p-1}) = \lambda X_{3,p-2} \mod \text{Im Inf}^V_{\Gamma_3}\). So, by Lemma 15, \(Y\) and \(- \lambda y X_{3,p-2}\) represent the same element of \(E_{5,2(p-2)}(\Gamma_3)\). By Lemma 8 and Proposition 2, it follows that \(\text{tr}^{\Gamma_1}_{\Gamma_3}(Y_{2,p-1}) = \lambda Y_{3,p-2} \mod \text{Im Inf}^V_{\Gamma_3}\).

Assume that the lemma holds for \(n - 1\). Set \(Z = \text{Res}_{\Gamma_{n-1}}^{M_n} \text{tr}^{M_{n-1}}_{\Gamma_{n-1}}(X_{n-1,p-1})\). By the double coset formula, we have

\[
Z = \text{tr}^{\Gamma_{n-2}}_{\Gamma_{n-1}} \text{Res}_{\Gamma_{n-2}}^{M_{n-1}}(X_{n-1,p-1})
\]

\[
= \text{tr}^{\Gamma_{n-2}}_{\Gamma_{n-1}}(x Y_{n-2,p-1}) \quad \text{by Lemma 7 (ii)}
\]

\[
= x \text{tr}^{\Gamma_{n-2}}_{\Gamma_{n-1}}(Y_{n-2,p-1}).
\]
By the inductive hypothesis, \( \text{tr}^{\Gamma_{n-1}}_{\Gamma_{n-1}}(Y_{n-2,p-1}) = \lambda Y_{n-1,p-2} \mod \text{Im} \text{Inf}^{\Gamma_{n-1}/Z}_{\Gamma_{n-1}}. \) So \( Z \) and \( \lambda x Y_{n-1,p-2} \) represent the same element of \( E_{\infty}^{2n-1,2(p-2)}(\Gamma_{n-1}) \). The first part follows from Lemma 8 and Proposition 2.

Finally, by setting \( Y = \text{Res}_{M_n}^{\Gamma_{n}} \text{tr}^{\Gamma_{n-1}}_{\Gamma_{n}}(Y_{n-1,p-1}) \), we have
\[
Y = \text{tr}^{\Gamma_{n-1}}_{M_n}(Y_{n-1,p-1}) = \text{tr}^{\Gamma_{n-1}}_{M_n}(-x_2 X_{n-1,p-1}) + \text{tr}^{\Gamma_{n-1}}_{M_n}(Z_{n-1,p-1}) \quad \text{by Lemma 7 (i)}
\]
\[
= -x_2 \text{tr}^{\Gamma_{n-1}}_{M_n}(X_{n-1,p-1}) + \text{tr}^{\Gamma_{n-1}}_{M_n}(Z_{n-1,p-1}).
\]
As shown above, \( \text{tr}^{\Gamma_{n-1}}_{M_n}(X_{n-1,p-1}) = \lambda X_{n,p-2} \mod \text{Im} \text{Inf}^{\Gamma_{n-1}}_{M_n} \). So, by Lemma 15, \( Y \) and \( -\lambda x_2 X_{n-1,p-2} \) represent the same element of \( E_{\infty}^{2n,2(p-2)}(\Gamma_{n-1}) \). The last part follows from Lemma 8 and Theorem 1 (ii). The lemma is proved. \( \square \)

Let
\[
\cdots \supset F^i C^e(\Gamma'_{n-1}) \supset F^{i+1} C^e(\Gamma'_{n-1}) \supset \cdots
\]
be the filtration of \( C^e(\Gamma'_{n-1}) \) introduced by Hochschild and Serre (\( \mathbb{Z} \)) corresponding to the central extension \( (\Gamma'_{n-1}) \). Let us recall that
\[
F^i C^e(\Gamma'_{n-1}) = \begin{cases} C^e(\Gamma'_{n-1}) & \text{for } i \leq 0, \\ \sum_{m=0}^{\infty} F^m C^e(\Gamma'_{n-1}) & \text{for } i > 0,
\end{cases}
\]
where \( F^i C^m(\Gamma'_{n-1}) = 0 \) for \( i > m \); and for \( 0 < i \leq m \), \( F^i C^m(\Gamma'_{n-1}) \) is the group of all \( m \)-cochains \( f \) for which \( f(g_1, \ldots, g_m) = 0 \) whenever \( m - i + 1 \) of the arguments \( g_k \) belong to \( Z \). It is clear that the conjugation by \( a = a_{2n} \) on \( C^e(\Gamma'_{n-1}) \) is compatible with the Hochschild-Serre filtration. We then have the induced conjugation on the Hochschild-Serre spectral sequence \( \{ E_r(\Gamma'_{n-1}) \} \). As the action of \( a \) on \( E_2^{i,e}(\Gamma'_{n-1}) \) satisfies \( a x_k = x_k, 1 \leq k \leq 2n-1, \) and \( a v = v + y_{2n-1} \), it follows from the structure of \( E_{2p+1}(\Gamma'_{n-1}) \) that \( Y_{n-1,i} \) and \( a Y_{n-1,i} \) represent the same element of \( E_{\infty}(\Gamma'_{n-1}) \).

Hence
\[
Y_{n-1,i} = a Y_{n-1,i} = \sum_{0 < j < i} \mu_j Y_{n-1,j} y_{2n-1}^{i-j} + \sum_{0 < j < i} \nu_j Y_{n-1,j} y_{2n-1}^{i-j} x y_{2n-1} \mod \text{Im} \text{Inf}^{\Gamma'_{n-1}/Z}_{\Gamma_{n-1}}
\]
with \( \mu_j, \nu_j \in \mathbb{F}_p \). We have

**Lemma 18.** For \( n \geq 2 \) we have \( Y_{n-1,1} = a Y_{n-1,1} = 0. \)

**Proof.** Set \( K = \text{Ker} x_{2n-2} \cap \Gamma'_{n-1} \). Since the transfer commutes with the conjugation and \( \text{Im} \text{Inf}^{\Gamma'_{n-1}/Z}_{\Gamma_{n-1}} \) is invariant under the action of \( a \), by Lemmas 9, 10, 11, 16 and 17, we have
\[
Y_{n-1,1} = a Y_{n-1,1} = \text{tr}^K_{\Gamma'_{n-1}}(Y_{n-2,2} - a Y_{n-2,2})
\]
up to a non-zero constant multiple. By Lemma 13, \( \text{tr}^K_{\Gamma'_{n-1}}(Y_{n-2,1}) = -X_{n-1}; \) hence \( \text{tr}^K_{\Gamma'_{n-1}}(Y_{n-2,2}) = 0 \) and \( \text{tr}^K_{\Gamma'_{n-1}}(Y_{n-2,1} x_{2n-3}) = 0 \) in \( H^*(\Gamma'_{n-1}) \). The lemma follows from (8) and from the fact that \( \text{tr}^K_{\Gamma'_{n-1}} \text{Inf}^K_{\Gamma'_{n-1}} = 0. \)

We now have
Lemma 19. For \( n \geq 2 \) and \( 1 \leq i \leq p - 1 \),
\[
Y_{n-1,i} + a Y_{n-1,i} + \cdots + a^{p-1} Y_{n-1,i} = 0;
\]
hence
\[
\text{Res}_{\Gamma_n}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i}) = 0.
\]

Proof. Since \( 1 + a + \cdots + a^{p-1} = (1 - a)^{p-1} \), we need prove that \((1-a)^{p-1} Y_{n-1,i} = 0\). For \( 1 \leq k \leq p - 1 \), by (8) and by Lemma 18, \((1-a)^{p-1} Y_{n-1,k} = 0\). Since \( \text{Res}_{\Gamma_n}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i}) = (1-a)^{p-1} Y_{n-1,i} \), the lemma follows. \( \Box \)

For \( n \geq 2 \) and \( 1 \leq i \leq p - 2 \), set \( \kappa_{n,i} = \text{Res}_{\Gamma_n}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i+1}) \). The proof of the theorem is completed by the following fact.

Proposition 9. \( 0 \neq \kappa_{n,i} \in \text{Ess}(E_n) \) with \( 1 \leq i < p - 2 \) for \( p > 3 \), and \( i = 1 \) for \( p = 3 \).

Proof. It follows from Proposition 1, Lemmas 7 (iv), 9, 11 and 17 that \( \kappa_{n,i} \neq 0 \) in \( H^*(E_n) \). Let \( K \) be a maximal subgroup of \( E_n \). \( K \) is then of the form \( E_n - x \). Let \( L \) be the central product of \( K \) and \( C_{p^2} = \bigcap_{j=1}^{2n} \text{Ker} x_j \). It follows that \( L \) is a subgroup of \( \Gamma_n \) containing \( K \) and \( L \cong \Gamma_{n-1} \times C_{p^2} \). Therefore
\[
\text{Res}_{\Gamma_n}^{\Gamma_n} (\kappa_{n,i}) = \text{Res}_{\Gamma_n}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i+1})
\]
\[
= \text{Res}_{\Gamma_n}^{\Gamma_n} \text{Res}_{\Gamma_n}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i+1}).
\]

Hence, if \( \Gamma_n = \Gamma_{n-1} \times L \), it follows from the double coset formula that
\[
\text{Res}_{\Gamma_n}^{\Gamma_n} (\kappa_{n,i}) = \text{Res}_{\Gamma_n}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i+1}).
\]

By Lemma 1, there exists a non-zero linear combination \( \alpha \) of \( x_1, \ldots, x_{2n} \) such that \( L = \text{Ker} \alpha \). Consider the following cases:

- \( \alpha = x_{2n-1} + \gamma \) with \( \gamma \) a linear combination of \( x_1, \ldots, x_{2n-2}, x_{2n} \): it follows that \( \Gamma_n = \Gamma_{n-1} \times L \) and \( L \cap \Gamma_{n-1} = \Gamma_{n-1} \times L \) is a direct factor of \( L \). Hence \( \text{tr}_{\Gamma_n}^{\Gamma_n-1} \) is the zero map. We have
\[
\text{Res}_{\Gamma_n}^{\Gamma_n} (\kappa_{n,i}) = \text{Res}_{\Gamma_n}^{\Gamma_n} \text{tr}_{\Gamma_n}^{\Gamma_n-1}(Y_{n-1,i+1}) = 0;
\]

- \( \alpha = \mu x_{2n} + \gamma \) with \( \gamma \) a non-zero linear combination of \( x_1, \ldots, x_{2n-2} \) and \( \mu \in \mathbb{F}_p \); it follows that \( L \cap \Gamma_{n-1} = H \times \langle a_{2n-1} \rangle \) for a subgroup \( H \) of \( \Gamma_{n-1} \) with \( H \cong \Gamma_{n-2} \). If \( p > 3 \), it follows from the proof of Lemma 7(iii) that \( \text{Res}_{\Gamma_n}^{\Gamma_n-1} (Y_{n-1,i+1}) \) belongs to the ideal generated by \( \text{Im} \text{Inf}_{L \cap \Gamma_{n-1}}^{L \cap \Gamma_{n-1}} / Z \); since
\[
\text{Im} \text{Inf}_{L \cap \Gamma_{n-1}}^{L \cap \Gamma_{n-1}} / Z \subset \text{Ker} \text{tr}_{\Gamma_n}^{\Gamma_n-1},
\]
it follows that
\[
\text{Res}_{\Gamma_n}^{\Gamma_n} (\kappa_{n,i}) = 0.
\]
If \( p = 3 \), by Lemma 14, there exist \( \phi \in H^2(L \cap \Gamma_{n-1}') \), \( \psi \in H^1(L \cap \Gamma_{n-1}') \) such that 
\[
\Res_{L \cap \Gamma_{n-1}'}(Y_{n-1,2}) \text{ is a linear combination of } 
[\kappa_{n-1}], 
Y_{n-2,1} \cdot \phi, 
Y_{n-2,1} \cdot x \psi
\]
and an element of \( \text{Im} \, \text{In}_{L \cap \Gamma_{n-1}'}^{L \cap \Gamma_{n-1}'}/\mathbb{Z} \), since \( \phi, \psi \) belong to \( \text{Im} \, \text{In}_{L \cap \Gamma_{n-1}'}^{L \cap \Gamma_{n-1}'}/\mathbb{Z} \), by Lemmas 13 and 14, it follows that 
\[
\Res_{L}^{S_{n}}(\kappa_{n,1}) = 0.
\]
Finally, the case \( \alpha = x_{2n} \) follows from Lemma 19. The proposition is proved. 

Acknowledgments

Most of the results of this paper were obtained during a stay at the University of Essen and the ETH-Zentrum in Autumn 1997. I would like to thank Eckart Viehweg, Hélène Esnault and Urs Stammbach for making the visits possible. Many thanks to David John Green for valuable comments.

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