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SIMPLE HOLONOMIC MODULES OVER RINGS OF DIFFERENTIAL OPERATORS WITH REGULAR COEFFICIENTS OF KRULL DIMENSION 2

V. BAVULA AND F. VAN OYSTAEYEN

ABSTRACT. Let K be an algebraically closed field of characteristic zero. Let Λ be the ring of (K-linear) differential operators with coefficients from a regular commutative affine domain of Krull dimension 2 which is the tensor product of two regular commutative affine domains of Krull dimension 1. Simple holonomic Λ-modules are described. Let a K-algebra D be a regular affine commutative domain of Krull dimension 1 and $\mathcal{D}(D)$ be the ring of differential operators with coefficients from D. We classify (up to irreducible elements of a certain Euclidean domain) simple $\mathcal{D}(D)$-modules (the field K is not necessarily algebraically closed).

1. INTRODUCTION

Let K be an algebraically closed field of characteristic zero and let an algebra R be a regular commutative affine domain of Krull dimension $\mathcal{K}(R) = 2$. Let $\mathcal{D}(R)$ be the ring of (K-linear) differential operators with coefficients from R. The ring $\mathcal{D}(R)$ is a simple Noetherian affine algebra that coincides with its subalgebra, the *derivation ring*, $\Delta(R)$, this being the ring generated by R and its derivations. The Gelfand-Kirillov dimension of a simple $\mathcal{D}(R)$-module is either 2 or 3. In the first case such a $\mathcal{D}(R)$-module is called *holonomic* (a definition of holonomic module over rings of differential operators is given in Section 2).

The aim of the present paper is to describe the simple holonomic $\mathcal{D}(R)$-modules in case the algebra $R = D_1 \otimes D_2$ is the tensor product of regular commutative affine domains of Krull dimension 1. Observe that $\mathcal{D}(D_1 \otimes D_2) \simeq \mathcal{D}(D_1) \otimes \mathcal{D}(D_2)$ (Lemma 2.5) and that the second Weyl algebra $A_2 = A_1 \otimes A_1$ is an example of the ring $\mathcal{D}(D_1 \otimes D_2)$. The simple holonomic modules over the second Weyl algebra (and other popular simple generalized Weyl algebras of Gelfand-Kirillov dimension 4) were classified in [BVO2]. The present paper can be considered as a further development and extension of [BVO2], [Bl1, Bl2, Bl3], [Bav1, Bav2, Bav3] and [BVO1]. Stafford first gives examples of simple non-holonomic modules over the Weyl algebras [St]. Later Bernstein and Lunts [BeLu], [Lu], and Coutinho [Co] construct more sophisticated examples of simple non-holonomic modules over the Weyl algebras.

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Definitions and basic results on differential operators, Gelfand-Kirillov dimension and localizations are collected in Section 2 (for more details the reader is referred to [Bj], [Bor], [KL] and [MR]).

In Section 3 we provide a description of simple modules over the ring $\mathcal{D}(D)$ of differential operators with coefficients from a regular commutative affine domain D of Krull dimension 1. In geometric language simple modules are classified over the ring of differential operators on a smooth irreducible algebraic curve. We use the approach of Block, [Bl3], where the simple modules over the differential operator ring (the Ore extension) $T[X;\delta]$ were classified under a "geometrical" condition ((3.1.1), [Bl3]) on the Dedekind domain T and the derivation δ of T. Recently, these results were extended in [Bav3] to an arbitrary Ore extension $T[X;\sigma,\delta]$, where $\sigma \in \mathrm{Aut}\ T$ and δ is a σ-derivation of a Dedekind domain T. It is interesting to observe that the "nontrivial" (see below) holonomic $\mathcal{D}(R = D_1 \otimes D_2)$-modules arise *exactly* in the case where a noncommutative analog of the condition (3.1.1) of [Bl3] fails. Note that every simple $\mathcal{D}(D)$-module is holonomic (i.e. has Gelfand-Kirillov dimension 1).

Let M_i $(i = 1, 2)$ be a simple $\mathcal{D}(D_i)$-module. Then the tensor product $M_1 \otimes M_2$ is a simple holonomic $\mathcal{D}(D_1) \otimes \mathcal{D}(D_2)$-module and two such modules are isomorphic, $M_1 \otimes M_2 \simeq M_1' \otimes M_2'$, iff $M_1 \simeq M_1'$ and $M_2 \simeq M_2'$. Therefore, the set $\hat{\mathcal{D}}(D_1 \otimes D_2)$(holonomic) of isoclasses of simple holonomic modules contains the subset $\hat{\mathcal{D}}(D_1) \otimes \hat{\mathcal{D}}(D_2)$, the "*trivial*" holonimic modules. The "*nontrivial*" ones, i.e. $\hat{\mathcal{D}}(D_1 \otimes D_2)(\text{holonomic}) \backslash \hat{\mathcal{D}}(D_1) \otimes \hat{\mathcal{D}}(D_2)$, are described in Sections 4 and 5.

In Section 4, the "nontrivial" holonomic modules are described in a special case of the ring $\Lambda := \mathcal{D}(D_1) \otimes \mathcal{D}(D_2)$ where the second tensor term is isomorphic to a differential operator ring $\mathcal{D}(D_2) = D_2[X;\delta]$. Let a skew field k be the full quotient ring of $\mathcal{D}(D_1)$. Let $\mathcal{A}$ be the (two-sided) localization of Λ at $\mathcal{D}(D_1)_* := \mathcal{D}(D_1)\backslash\{0\}$ i.e.

$$\mathcal{A} = k \otimes \mathcal{D}(D_2) = k \otimes D_2[X;\delta].$$

The first main result (Theorem 4.9) states that *the map*

$$\hat{\Lambda}(\text{holonomic}) \backslash \hat{\mathcal{D}}(D_1) \otimes \hat{\mathcal{D}}(D_2) \to \hat{\mathcal{A}}(k\text{-fin.dim}),\ [M] \to [\mathcal{A} \otimes_\Lambda M],$$

is bijective with inverse $[N] \to [\mathrm{Soc}_\Lambda N]$, *where* $\hat{\mathcal{A}}(k\text{-fin.dim})$ *is the set of isoclasses of simple* $\mathcal{A}$*-modules which are a finite dimensional left vector space over the skew field* k.

The second main result of Section 4 (Corollary 4.11) provides a holonomicity criterion.

Let M *be a (nonzero) simple* Λ*-module and let* $\tilde{M} = \mathcal{A} \otimes_\Lambda M$. *Then*

1. $\tilde{M} = 0 \Leftrightarrow [M] \in \hat{\mathcal{D}}(D_1) \otimes \hat{\mathcal{D}}(D_2)$;
2. $1 \le \dim_k \tilde{M} < \infty \Leftrightarrow [M] \in \hat{\Lambda}(\text{holonomic}) \backslash \hat{\mathcal{D}}(D_1) \otimes \hat{\mathcal{D}}(D_2)$;
3. $\dim_k \tilde{M} = \infty \Leftrightarrow [M] \in \hat{\Lambda}(\text{non-holonomic})$.

Hence, M *is holonomic (resp. non-holonomic) iff* $\dim_k \tilde{M} < \infty$ *(resp.* M *contains a free* $\mathcal{D}(D_1) \otimes K[X]$*-module of rank 1).*

Corollary 4.12 yields a presentation of every element $[M] \in \hat{\Lambda}(\text{holonomic}) \backslash \hat{\mathcal{D}}(D_1) \otimes \hat{\mathcal{D}}(D_2)$ as a factor module of the Λ-module Λ.

The idea behind the description of $\hat{\mathcal{D}}(D_1 \otimes D_2)(\text{holonomic}) \backslash \hat{\mathcal{D}}(D_1) \otimes \hat{\mathcal{D}}(D_2)$ is to reduce to the case of Section 4.

In Section 5 the global results are obtained.

In Section 6, for the ring of differential operators $\mathcal{D}(R)$ with coefficients from a regular commutative affine domain R of Krull dimension 2 a holonomicity criterion (Theorem 6.1) is established.

In the paper, module means a left module.

2. Preliminary Results

Let K be a field of characteristic zero and R be a commutative K-algebra. A ring of (K-linear) *differential operators* $\mathcal{D}(R)$ on R is defined as $\mathcal{D}(R) = \bigcup_{i=0}^{\infty} \mathcal{D}^i(R)$ where $\mathcal{D}^0(R) = \{u \in \mathrm{End}_K(R) : ur - ru = 0, \text{ for all } r \in R\} = \mathrm{End}_R(R) \simeq R$,

$$\mathcal{D}^i(R) = \{u \in \mathrm{End}_K(R) : ur - ru \in \mathcal{D}^{i-1}(R), \text{ for all } r \in R\}.$$

Note that the $\{\mathcal{D}^i(R)\}$ defines a filtration for $\mathcal{D}(R)$. We say that an element $u \in \mathcal{D}^i(R) \backslash \mathcal{D}^{i-1}(R)$ has order i. The subalgebra $\Delta(R)$ of $\mathrm{End}_K(R)$ generated by $R \equiv \mathrm{End}_R(R)$ and by the set $\mathrm{Der}_K(R)$ of all K-derivations of R is called the *derivation ring* of R. The derivation ring $\Delta(R)$ is the subring of $\mathcal{D}(R)$.

Let the algebra R be a *regular commutative affine domain of Krull dimension* $n < \infty$. In geometric terms, R is the coordinate ring $\mathcal{O}(X)$ of a smooth irreducible affine variety X of dimension n. Then

- $\mathrm{Der}_K(R)$ *is a finitely generated projective R-module of rank n*;
- $\mathcal{D}(R) = \Delta(R)$;
- $\mathcal{D}(R)$ *is a simple (left and right) Noetherian domain*;
- $\mathrm{gld}\, \mathcal{D}(R) = \mathcal{K}(\mathcal{D}(R)) = \mathrm{GK}(\mathcal{D}(R))/2 = \mathcal{K}(R) = n$ (*where* gld, $\mathcal{K}$ *and* GK *stand for the global, Krull and Gelfand-Kirillov dimension respectively*);
- $\mathrm{GK}(M) \geq n$ *for any nonzero finitely generated $\mathcal{D}(R)$-module M*;
- *if S is a multiplicatively closed subset of R, then S is a (left and right) Ore set of $\mathcal{D}(R)$ and $\mathcal{D}(S^{-1}R) = S^{-1}\mathcal{D}(R)$*;
- $\mathcal{D}(R) = \Delta(R)$ *is an almost centralizing extension of R*;
- $\Delta(R)$ *is a somewhat commutative algebra*;
- $\Delta(R)$ *satisfies the Nullstelenzatz over K. So,* $\mathrm{End}_{\Delta(R)}(M) = K$ *for every simple $\Delta(R)$-module M since K is an algebraically closed field*;
- *the associative graded ring* $\mathrm{gr}\, \mathcal{D}(R) = \bigoplus \mathcal{D}^i(R)/\mathcal{D}^{i-1}(R)$ *is a commutative domain.*

For the proofs the reader is referred to [MR], Chapter 15.

Definition. A finitely generated $\mathcal{D}(R)$-module M is called *holonomic* if it has Gelfand-Kirillov dimension

$$\mathrm{GK}(M) = \mathrm{GK}(\mathcal{D}(R))/2 = n.$$

Example. Let $P_n = K[X_1, \dots, X_n]$ be a polynomial ring in n indeterminates.

$$\mathcal{D}(P_n) = K[X_1, \dots, X_n, \partial/\partial X_1, \dots, \partial/\partial X_n]$$

is the nth *Weyl algebra*, $A_n = A_n(K)$. Clearly, $A_n = A_1 \otimes \cdots \otimes A_1$ (n times).

The following technical lemma will be used frequently in the paper (15.2.13 and 15.3.2, [MR]).

Lemma 2.1. *Let R be a regular commutative K-domain of Krull dimension n.*

1. *Let $\mathbf{m}$ be a maximal ideal of R. Then there exists $c = c(\mathbf{m}) \in R \backslash \mathbf{m}$ such that*

$$\mathcal{D}(R)_c = R_c[X_1; \partial/\partial Y_1] \cdots [X_n; \partial/\partial Y_n],$$

an iterated Ore extension for which $X_iX_j = X_jX_i$ *for* $1 \leq i, j \leq n$, *and* $\{Y_1, \dots, Y_n\}$ *is a transcendence basis for the quotient field of* R*; the algebra* $\mathcal{D}(R)_c$ *contains the* n*th Weyl algebra*

$$A_n = K[Y_1, \dots, Y_n][X_1; \partial/\partial Y_1] \cdots [X_n; \partial/\partial Y_n] = K[Y_1, \dots, Y_n, \partial/\partial Y_1, \dots, \partial/\partial Y_n]$$

and is a (left and right) finitely generated A_n*-module;*

2. *there is a finite subset* $\{c_1, \dots, c_s\}$ *of* $\{c(\mathbf{m}) \mid \mathbf{m}$ *is a maximal ideal of* $R\}$ *(of elements as above) such that the natural ring monomorphism* $\mathcal{D}(R) \to \prod_{i=1}^s \mathcal{D}(R)_{c_i}$ *is (left and right) faithfully flat.* □

The K-algebra R is affine, so let $\{R_i, i \geq 0\}$ ($R_0 = K$) be a finite dimensional filtration of R such that the associated graded algebra $\operatorname{gr} R = \bigoplus R_i/R_{i-1}$ is affine. In particular, every standard filtration of R satisfies this property. The $\mathcal{D}(R) = \Delta(R)$ is an *almost centralizing extension* of R (15.1.20, [MR], and $\mathrm{Der}_K(R)$ is a finitely generated R-module). By (8.6.7, [MR]), the algebra $\Delta(R)$ has a finite dimensional filtration $\mathcal{F} = \{\Delta_i(R), i \geq 0\}$ ($\Delta_0(R) = K$) such that $\operatorname{gr}_{\mathcal{F}} \Delta(R)$ is a commutative affine algebra (hence Noetherian), i.e., by definition, $\Delta(R)$ is a *somewhat commutative algebra*. The graded K-algebra $\operatorname{gr}_{\mathcal{F}} \Delta(R)$ is generated by homogeneous elements, say $x_1, \dots, x_t$, of positive graded degrees $k_1, \dots, k_t$, respectively. Let k be the least common multiple of $\{k_i\}$.

A filtration $\Gamma = \{\Gamma_i, i \geq 0\}$ of a $\Delta(R)$-module $M = \bigcup_{i=0}^\infty \Gamma_i$ is called *good* if the associated graded $\operatorname{gr}_{\mathcal{F}} \Delta(R)$-module $\operatorname{gr}_\Gamma M$ is finitely generated. A $\Delta(R)$-module M has a good filtration iff it is finitely generated; and if $\{\Gamma_i\}$ and $\{\Omega_i\}$ are two good filtrations on M, then there exists a natural number j such that $\Gamma_i \subseteq \Omega_{i+j}$ and $\Omega_i \subseteq \Gamma_{i+j}$ for all i. If a $\Delta(R)$-module M is finitely generated and M_0 is a finite dimensional generating subspace of M, then the standard filtration $\{\Gamma_i = \Delta_i(R)M_0\}$ is good (see [Bj] or [LVO] for details). The following lemma is well known for specialists (see, for example, [Bav4]).

Lemma 2.2. *Let* M *be a finitely generated* $\Delta(R)$*-module with good filtration* $\Gamma = \{\Gamma_i\}$*. Then*

1. *there exist* k *polynomials* $\gamma_0, \dots, \gamma_{k-1} \in \mathbf{Q}[t]$ *with coefficients from* $[k^{\mathrm{GK}(M)} \mathrm{GK}(M)!]^{-1}\mathbf{Z}$ *such that*

$$\dim \Gamma_i = \gamma_j(i) \quad \textit{for all } i >> 0 \textit{ and } j \equiv i (\operatorname{mod} k);$$

2. *the polynomials* γ_j *have the same degree* $\mathrm{GK}(M)$ *and the same leading coefficients* $e(M)/\mathrm{GK}(M)!$ *where* $e(M)$ *is called the multiplicity of* M*. The multiplicity* $e(M)$ *does not depend on the choice of good filtration* Γ*.* □

Let $0 \to N \to M \to L \to 0$ be an exact sequence of finitely generated $\Delta(R)$-modules and let $\Gamma = \{\Gamma_i\}$ be a good filtration on M. Then $\Gamma' = \{\Gamma_i{}' = \Gamma_i \cap N\}$ and $\Gamma'' = \{\Gamma_i'' = (\Gamma_i + N)/N\}$ are filtrations on N and L respectively such that the sequence of $\operatorname{gr}_{\mathcal{F}} \Delta(R)$-modules

$$0 \to \operatorname{gr}_{\Gamma'}(N) \to \operatorname{gr}_\Gamma(M) \to \operatorname{gr}_{\Gamma''}(L) \to 0$$

is exact. The ring $\operatorname{gr}_{\mathcal{F}} \Delta(R)$ is Noetherian and the $\operatorname{gr}_{\mathcal{F}} \Delta(R)$-module $\operatorname{gr}_\Gamma(M)$ is finitely generated, so the $\operatorname{gr}_{\mathcal{F}} \Delta(R)$-modules $\operatorname{gr}_{\Gamma'}(N)$ and $\operatorname{gr}_{\Gamma''}(L)$ are finitely generated, i.e. the filtrations Γ' and Γ'' are good and we have

$$\dim \Gamma_i = \dim \Gamma_i{}' + \dim \Gamma_i'', \tag{2.1}$$

hence, by Lemma 2.2,

$$\text{GK}(M) = \max\{\text{GK}(N), \text{GK}(L)\}, \tag{2.2}$$

and if $\text{GK}(M) = \text{GK}(N) = \text{GK}(L)$, then

$$e(M) = e(N) + e(L). \tag{2.3}$$

A finitely generated $\Delta(R)$-module M is called *holonomic* if $\text{GK}\, M = \text{GK}(\Delta(R))/2 = \mathcal{K}(R)$.

- *Every holonomic $\Delta(R)$-module has finite length.*

(For example, it follows from (2.3) and Lemma 2.2.) Let $0 \neq c \in R$ and M be an R-module. We denote by M_c the localization of M at the powers of the element c, i.e. $M_c = S^{-1}M$ where $S = \{c^i, i \geq 0\}$. The module M is called *c-torsionfree* (resp. *c-torsion*) if the map $c_M : M \to M$, $m \to cm$, is injective (resp. $S^{-1}M = 0$).

Denote by $\mathbf{N} = \{0, 1, \dots\}$ and $\mathbf{R}$ the set of natural and real numbers. For a function $f : \mathbf{N} \to \mathbf{N}$ its *degree* is defined as follows:

$$\gamma(f) := \inf\{d \in \mathbf{R} : \ f(n) \leq n^d \text{ for sufficiently large } n >> 0\}.$$

Proposition 2.3. *Let c be a nonzero element of R and let M be a finitely generated c-torsionfree $\Delta(R)$-module. Then $\text{GK}_{\Delta(R)} M = \text{GK}_{\Delta(R)_c} M_c$.*

Proof. The $\Delta(R)$-module M is c-torsionfree, thus the map $M \to M_c$, $m \to m/1$, is a monomorphism of $\Delta(R)$-modules. So, $\text{GK}_{\Delta(R)} M \leq \text{GK}_{\Delta(R)_c} M_c$.

Conversely, fix a finite dimensional filtration $\mathcal{F} = \{\Delta_i(R), i \geq 0\}$ of the algebra $\Delta(R)$ as above, i.e. the associated graded algebra $\text{gr}_{\mathcal{F}}\, \Delta(R)$ is commutative affine. Let M_0 be a finite dimensional generating subspace of the $\Delta(R)$-module M. The M is equipped with the good filtration $\{M_i = \Delta_i(R)M_0\}$ and $\text{GK}_{\Delta(R)} M = \gamma(\dim M_i)$. Let $A_1 \ni 1$ be a finite dimensional subspace of algebra generators of $\Delta(R)$. Then $\Delta(R)_c$ has the standard filtration $\{B_i = B_1^i, i \geq 0\}$ where $B_1 = A_1 + Kc^{-1}$. The $\Delta(R)_c$-module M_c has the standard filtration $\{(M_c)_i = B_iM_0\}$ and $\text{GK}_{\Delta(R)_c} M_c = \gamma(\dim (M_c)_i)$. Using 15.1.17, [MR], we can find natural numbers α and β such that $c^{\alpha i}B_i \subseteq \Delta_{\beta i}$ for all i. Now, $c^{\alpha i}(M_c)_i = c^{\alpha i}B_iM_0 \subseteq \Delta_{\beta i}M_0 = M_{\beta i}$ and $\dim (M_c)_i \leq \dim M_{\beta i}$, hence $\text{GK}_{\Delta(R)_c} M_c \leq \text{GK}_{\Delta(R)} M$. $\square$

Theorem 2.4. *Let M be a finitely generated $\Delta(R)$-module and let $\Delta(R) \to \prod_{i=1}^s \Delta(R)_{c_i}$ be a faithfully flat extension from Lemma 2.1. Then*

1. $\text{GK}_{\Delta(R)} M = \max\{\text{GK}_{\Delta(R)_{c_i}} M_{c_i}\}$*;*
2. *M is a holonomic $\Delta(R)$-module iff each nonzero M_{c_i} is a holonomic $\Delta(R)_{c_i}$-module;*
3. *N is a holonomic $\Delta(R)_{c_i}$-module iff N is a holonomic module over the nth Weyl subalgebra $A_n^{(i)}$ from Lemma 2.1(1).*

Proof. 1. Denote by m the maximum in the statement of theorem. For a $\Delta(R)$-submodule N of M denote by $\bar{N}_i$ the image of N under the map $M \to M_{c_i}$. Then, by Proposition 2.3, $\text{GK}_{\Delta(R)} M \geq \text{GK}_{\Delta(R)} \bar{M}_i = \text{GK}_{\Delta(R)_{c_i}} (\bar{M}_i)_{c_i} = \text{GK}_{\Delta(R)_{c_i}} M_{c_i}$, hence $\text{GK}_{\Delta(R)} M \geq m$.

To prove the opposite inequality it suffices to prove the existence of a $\Delta(R)$-submodule N of M such that $\text{GK}_{\Delta(R)} \bar{N}_i = \text{GK}_{\Delta(R)} M$ for some i. Since then, by Proposition 2.3, $\text{GK}_{\Delta(R)} \bar{N}_i = \text{GK}_{\Delta(R)_{c_i}} (\bar{N}_i)_{c_i} \leq \text{GK}_{\Delta(R)_{c_i}} M_{c_i}$, i.e. $\text{GK}_{\Delta(R)} M \leq m$.

Let $n = n(M)$ be the number of nonzero maps $M \to M_{c_i}$. If $n = 1$, i.e. $M \simeq \bar{M}_i$ for a unique i, then we can take $N = M$. Suppose $n > 1$. Up to numeration we may suppose that first n maps $\{M \to M_{c_i},\ i = 1, \dots, n\}$ are nonzero. We have the exact sequences of $\Delta(R)$-modules: $0 \to K_i \to M \to \bar{M}_i \to 0$, $i = 1, \dots, n$, where K_i is the kernel of the map $M \to M_{c_i}$. If there exists i such that $\mathrm{GK}\, \bar{M}_i = \mathrm{GK}\, M$, then we take $N = M$. Otherwise, $\mathrm{GK}\, K_i = \mathrm{GK}\, M$ for all i. For the module K_1 the number $n(K_1)$ is less than n (since $\bar{K}_1 = 0$), so the induction completes the argument.

$1 \Rightarrow 2$. Evident.

3. The $A_n^{(i)}$ is the affine subalgebra of the affine algebra $\Delta(R)_{c_i}$ such that $\Delta(R)_{c_i}$ is a finitely generated $A_n^{(i)}$-module (Lemma 2.1.(1)). Now, by Lemma 4.11, [BVO2], $\mathrm{GK}_{\,\Delta(R)_{c_i}} N = \mathrm{GK}_{\,A_n^{(i)}} N$ and the result follows. □

Lemma 2.5. *Let R be a regular commutative affine domain of finite Krull dimension which is the tensor product $\bigotimes_{i=1}^n R_i$ of regular commutative affine domains R_i. Then $\mathcal{D}(R) = \bigotimes_{i=1}^n \mathcal{D}(R_i)$.*

Proof. Observe $\mathcal{D}(R) = \Delta(R)$. It is sufficient to prove the statement in case $n = 2$. Set $\Delta = \Delta(R)$, $\Delta_i = \Delta(R_i)$, $i = 1, 2$. Then $\Delta \supseteq \Delta_1 \otimes \Delta_2$. Choose faithfully flat extensions $\Delta_1 \to \prod_i \Delta_{1,c_i}$, $\Delta_2 \to \prod_j \Delta_{2,t_j}$ as in Lemma 2.1. Then $\Delta_1 \otimes \Delta_2 \to \prod_{i,j} (\Delta_1)_{c_i} \otimes (\Delta_2)_{t_j} = \prod_{i,j} (\Delta_1 \otimes \Delta_2)_{c_i t_j}$ is the faithfully flat extension (if M is a nonzero $\Delta_1 \otimes \Delta_2$-module, then there exists i such that $M_{c_i} \neq 0$ and then there exists $j = j(i)$ such that $M_{c_i t_j} \neq 0$). Evidently,

$$\Delta_{c_i t_j} = \Delta(R_{c_i t_j}) = \Delta((R_1)_{c_i}) \otimes \Delta((R_2)_{t_j}) = (\Delta_1 \otimes \Delta_2)_{c_i t_j}$$

for i, j. We have $\Delta \supseteq \Delta_1 \otimes \Delta_2$ with $\prod \Delta_{c_i t_j} = \prod (\Delta_1 \otimes \Delta_2)_{c_i t_j}$, since $\Delta_1 \otimes \Delta_2 \to \prod_{i,j} (\Delta_1 \otimes \Delta_2)_{c_i t_j}$ is a faithfully flat extension, we conclude that $\Delta = \Delta_1 \otimes \Delta_2$. □

Let A be a ring and let $B = S^{-1}A$ be the left (Ore) localization of the ring A at an Ore set $S \ni 1$ of A. We have the natural ring homomorphism $A \to B$, $a \to a/1$, which, in general, is not a monomorphism. For a left ideal $\mathbf{m}$ of B we denote by $A \cap \mathbf{m}$ the inverse image of $\mathbf{m}$ in A. The localization defines the localization functor

$$S^{-1} : A\text{-mod} \to B\text{-mod}, \quad M \to S^{-1}M \simeq B \otimes_A M,$$

from the category of A-modules to the category of B-modules. An A-module M contains the S-*torsion* submodule

$$\mathrm{tor}_S(M) = \{m \in M \ : \ sm = 0 \text{ for some } s = s(m) \in S\}.$$

If the A-module M is simple, then its localization $S^{-1}M$ is either zero ($\Leftrightarrow M = \mathrm{tor}_S(M)$) or not ($\Leftrightarrow \mathrm{tor}_S(M) = 0$), in the last case $S^{-1}M$ is a simple B-module. Correspondingly, we say that a simple A-module is either S-*torsion* or S-*torsionfree*, i.e.

$$\hat{A} = \hat{A}(S\text{-torsion}) \cup \hat{A}(S\text{-torsionfree}). \tag{2.4}$$

The sum of all simple submodules of an A-module M is called the *socle* $\mathrm{Soc}_A M$ of M. It is the largest semisimple submodule of M. A B-module N is called A-*socle* (or, *socle*, for short) provided $\mathrm{Soc}_A N \neq 0$. Denote by $\hat{B}(A\text{-socle})$ the set of isoclasses of simple A-socle B-modules. A submodule M' of M is called *essential* if it intersects nontrivially each nonzero submodule of M. The following two lemmas are evident (see [BVO2] for detail).

Lemma 2.6. 1. *The canonical map*

$$S^{-1} : \hat{A}(S\text{-}torsionfree) \to \hat{B}(A\text{-}socle),\ [M] \to [S^{-1}M],$$

is a bijection with inverse Soc : $[N] \to [Soc_A(N)]$.

2. *Each simple* S*-torsionfree* A*-module has the form*

$$M_{\mathbf{m}} := A/A \cap \mathbf{m} \tag{2.5}$$

for some left maximal ideal $\mathbf{m}$ *of the ring* B. *Two such modules are isomorphic,* $M_{\mathbf{m}} \simeq M_{\mathbf{n}}$, *iff the* B*-modules* $B/\mathbf{m}$ *and* $B/\mathbf{n}$ *are isomorphic.* □

Write $LMAX(B)$ for the set of all left maximal ideals of B. A maximal left ideal $\mathbf{m}$ of the ring B is called *socle*, resp. *convenient*, provided $\mathrm{Soc}_A\, M_{\mathbf{m}} \neq 0$ resp. $M_{\mathbf{m}}$ is a simple A-module and the sets of all such ideals are denoted by $LMAX.soc(B)$ and $LMAX.con(B)$. Clearly, $LMAX.con(B) \subseteq LMAX.soc(B)$. In general, not every left maximal (resp. socle) ideal is socle (resp. convenient).

For a socle maximal left ideal $\mathbf{m}$ of B let $J(\mathbf{m})$ be the smallest of the left ideals of A strictly containing $A \cap \mathbf{m}$, then

$$J(\mathbf{m})/A \cap \mathbf{m} = \mathrm{Soc}_A M_{\mathbf{m}}.$$

Since $S^{-1}(J(\mathbf{m})/A \cap \mathbf{m}) = S^{-1}\mathrm{Soc}\ _A M_{\mathbf{m}} = B/\mathbf{m}$, the set

$$\mathbf{a}(\mathbf{m}) := J(\mathbf{m}) \cap S \tag{2.6}$$

is not empty.

Lemma 2.7. *Let* $\mathbf{m} \in LMAX.soc(B)$ *and* $\alpha \in S$. *The following are equivalent:*

1. $\alpha \in \mathbf{a}(\mathbf{m})$;
2. $J(\mathbf{m}) = A\alpha + A \cap \mathbf{m}$;
3. $M_{\mathbf{m}\alpha^{-1}}$ *is a simple* A*-module;*
4. $\mathbf{m}\alpha^{-1} \in LMAX.con(B)$. □

3. Classification of simple modules over rings of differential operators with regular coefficients of Krull dimension 1

Let a K-algebra D be a *regular affine commutative domain of Krull dimension* 1 over a field K of characteristic zero (not necessarily algebraically closed). The ring D is a Dedekind domain. The algebra D can be seen as the coordinate ring of a smooth irreducible algebraic curve. Let $\Delta = \Delta(D) = \mathcal{D}(D)$ be the ring of differential operators with coefficients from D. In geometric terms, Δ is the ring of differential operators on a smooth irreducible algebraic curve. Observe that the algebra Δ is a simple affine Noetherian domain with Gelfand-Kirillov dimension 2 and Krull dimension 1. Using results of [Bl3], in this section we classify (up to the irreducible elements of a noncommutative Euclidean domain B, see below) the simple Δ-modules (Theorems 3.4 and 3.6).

Let δ be a K-derivation of D such that the Ore extension (or the *differential operator ring*) $A = D[X;\delta]$ is a SIMPLE algebra. The first Weyl algebra $A_1 \simeq K[t][X; d/dt]$ gives an example of the ring A. Denote by l the field of fractions of D, i.e. $l = D_*^{-1}D$ where $D_* = D\backslash\{0\}$. Then the (two-sided) localization $B = D_*^{-1}A$ of A at D_* is the Ore extension $B = l[X;\delta]$ with coefficients from the field l. By the ring monomorphism $A \to B$, $a \to a/1$, we identify A with its image in B (the "new" δ is the unique extension of the δ from D to l). The ring B is a left and right *Euclidean* ring, hence, a left and right principal ideal domain. So, a B-module N

is simple iff $N \simeq B/Bb$ for some irreducible element b of B; and $B/Bb \simeq B/Bc$ iff b and c are similar (see [Jac] for details).

If M is a simple A-module, then the localization $D_*^{-1}M = B \otimes_A M$ of M at D_* is either 0 or a (nonzero) simple B-module. We say that M is *D-torsion* or *D-torsionfree* correspondingly. The set $\hat{A}$ is partitioned as

$$\hat{A} = \hat{A}(D\text{-torsion}) \cup \hat{A}(D\text{-torsionfree}). \tag{3.1}$$

The field K has characteristic zero, so the condition (3.1.1) of [Bl3] holds and there is no proper δ-invariant ideal of the ring D (since A is simple). Let us recall the description of $\hat{A}$ following [Bl3]. Denote by Specm D the set of prime ideals of D. Here "prime" means a nonzero prime (i.e. a maximal ideal).

Lemma 3.1 (4.1, [Bl3])**.** *The map*

$$\text{Specm } D \to \hat{A}(D\textit{-torsion}),\ \mathbf{p} \to [A/A\mathbf{p} \simeq A \otimes_D D/\mathbf{p}],$$

is a bijection. □

Suppose $\mathbf{p}$ is a prime ideal of D. Let $\nu_{\mathbf{p}}$ denote the valuation of l corresponding to $\mathbf{p}$, that is, if $d \in \mathbf{p}^i \backslash \mathbf{p}^{i-1}$, then $\nu_{\mathbf{p}} d = i$. The valuation ring $\{\alpha \in l \mid \nu_{\mathbf{p}}\alpha \geq 0\}$ coincides with the localization $D_{\mathbf{p}}$ of D at $\mathbf{p}$ which is a local Dedekind domain with the maximal ideal $D_{\mathbf{p}}\mathbf{p}$. We identify the residue class field $D_{\mathbf{p}}/D_{\mathbf{p}}\mathbf{p}$ with $l_{\mathbf{p}} = D/\mathbf{p} (\equiv (D/\mathbf{p})_{\mathbf{p}} = D_{\mathbf{p}}/D_{\mathbf{p}}\mathbf{p})$. We denote by $\eta_{\mathbf{p}}$ the canonical epimorphism $D_{\mathbf{p}} \to D_{\mathbf{p}}/D_{\mathbf{p}}\mathbf{p} = l_{\mathbf{p}}$.

The valuation $\nu_{\mathbf{p}}$ can be extended to a valuation on B, also denoted by $\nu_{\mathbf{p}}$ or by ν (for short), as follows: if $b = \sum b_i X^i \in B$, then (Lemma 3.1, [Bl3])

$$\nu_{\mathbf{p}} b = \min\{\nu_{\mathbf{p}} b_i - i \mid i \geq 0\}.$$

Suppose $\mathbf{p}$ is a prime ideal of D. Pick $g \in \mathbf{p}\backslash\mathbf{p}^2$. For $b = \sum b_i X^i \in B$, R. Block ([Bl3], (3.2.1)) defines a polynomial $Q_b = Q_b(t) = Q_{\mathbf{p},g,b}(t) \in l_{\mathbf{p}}[t]$ by

$$Q_{\mathbf{p},g,b}(t) = \sum_{i \geq 0} \eta_{\mathbf{p}}(g^{-\nu_{\mathbf{p}} b - i} b_i (\delta g)^i)\, t(t-1)\cdots(t-i+1).$$

The polynomial above is called the *indicial polynomial of b relative* to $\mathbf{p}$, g (or at $\mathbf{p}$, g). The *normalized indicial polynomial* $\bar{Q}_{\mathbf{p},b}$ of b relative to $\mathbf{p}$, which is obtained by dividing $Q_{\mathbf{p},g,b}$ by its leading coefficient, is independent of the choice of g (Lemma 3.2, [Bl3]). The roots of $Q_{\mathbf{p},g,b}$ are called the *indicial roots* of b relative to $\mathbf{p}$ (this being independent of g). The element b is *preserving relative to* $\mathbf{p}$ if there is no negative integer indicial root relative to $\mathbf{p}$. We shall also say that b is *preserving* if it is preserving relative to $\mathbf{p}$, for every prime $\mathbf{p}$ of D.

If $b = \sum b_i X^i \in B$ of degree $k > 0$, a prime $\mathbf{p}$ is called the *special prime* of b if $\nu_{\mathbf{p}} b = \nu_{\mathbf{p}} b_i - i$ for some $i < k$. The element b has only finitely many special primes. If $\mathbf{p}$ is not a special prime of b, then b is preserving relative to $\mathbf{p}$. Hence the property of b being preserving depends on only finitely many primes (see 3.4, [Bl3] for details).

Theorem 3.2 (4.4, [Bl3])**.** *Let $b = \sum b_i X^i \in B$ be an irreducible preserving element. Then $A/A \cap Bb$ is a simple D-torsionfree A-module. Up to isomorphism every simple D-torsionfree A-module arises in this way and two such A-modules are isomorphic, $A/A \cap Bb \simeq A/A \cap Bc$, iff the B-modules B/Bb and B/Bc are isomorphic.* □

Fix a faithfully flat extension $\Delta \to \prod_{i=1}^{s} \Delta_{c_i}$ as in Lemma 2.1, where $\Delta_i = \Delta_{c_i} = \Delta(D_{c_i})$ is the localization of Δ at the powers of $c_i \in D$. Let $B = D_*^{-1}\Delta$ be the localization of Δ at $D_* = D\backslash\{0\}$. Then

$$\hat{\Delta} = \hat{\Delta}(D\text{-torsion}) \cup \hat{\Delta}(D\text{-torsionfree}). \tag{3.2}$$

The ring Δ_i is the Ore extension $D_{c_i}[X_i;\delta_i]$, it can be considered as the subring of B. Moreover, $B = B_i := (D_{c_i})_*^{-1}\Delta_i = l[X_i;\delta_i]$ is the localization of Δ_i at $(D_{c_i})_* = D_{c_i}\backslash\{0\}$. Then $X_i = \alpha_{ij}X_j + \beta_{ij}$ for some $0 \neq \alpha_{ij}, \beta_{ij} \in l$.

The ring D is a commutative domain of Krull dimension 1, thus, for $0 \neq c \in D$, the set $V(c)$ of prime ideals in D containing c is finite. Clearly, for $\mathbf{p} \in \text{Specm } D$, the Δ-module $\Delta/\Delta\mathbf{p}$ is holonomic (since Δ is a simple domain of Gelfand-Kirillov dimension 2 and in a view of Lemma 2.2 and (2.3)), hence, of finite length. Moreover,

$$\Delta/\Delta\mathbf{p} = \bigcup_{i=1}^{\infty} \text{ann } \mathbf{p}^i,$$

where ann $\mathbf{p}^i = \{u \in \Delta/\Delta\mathbf{p} : \mathbf{p}^i u = 0\}$.

Lemma 3.3. *Let M be a nonzero Δ-module satisfying the following property: if N is a nonzero submodule of M, then $N_{c_i} \neq 0$ for every i such that $M_{c_i} \neq 0$ (*e.g.*, M is D-torsionfree; M is an epimorphic image of $\Delta/\Delta\mathbf{p}$ for some $\mathbf{p} \in$ Specm D). Then the Δ-module M is simple iff for each i either $M_{c_i} = 0$ or M_{c_i} is a nonzero simple Δ_i-module.*

Proof. Evident. □

Theorem 3.4. *The map*

$$\text{Specm } D \to \hat{\Delta}(D\textit{-torsion}),\ \mathbf{p} \to [\Delta/\Delta\mathbf{p} \simeq \Delta \otimes_D D/\mathbf{p}],$$

is a bijection.

Proof. For a commutative Noetherian domain of Krull dimension 1 the restricted minimum condition holds (i.e., every proper factor ring is Artinian). Thus a simple D-torsion Δ-module is an epimorphic image of the Δ-module $\Delta/\Delta\mathbf{p}$ for some prime $\mathbf{p}$ of D. If $\mathbf{p}_{c_i} \neq D_{c_i}$, then it is the maximal ideal of D_{c_i}. The localization $(\Delta/\Delta\mathbf{p})_{c_i} \simeq \Delta_i/\Delta_i\mathbf{p}_{c_i}$ is either 0 or a simple Δ_i-module (by Lemma 3.1). Hence, by Lemma 3.3, every Δ-module $\Delta/\Delta\mathbf{p}$ is simple. It means that a simple D-torsion Δ-module is isomorphic to some $\Delta/\Delta\mathbf{p}$.

Since $\Delta/\Delta\mathbf{p} = \bigcup_{i=1}^{\infty}$ ann $\mathbf{p}^i$, where ann $\mathbf{p}^i = \{u \in \Delta/\Delta\mathbf{p} : \mathbf{p}^i u = 0\}$, the Δ-modules $\Delta/\Delta\mathbf{p}$ and $\Delta/\Delta\mathbf{p}'$ are isomorphic iff $\mathbf{p} = \mathbf{p}'$ ($\mathbf{p}$ and $\mathbf{p}'$ are primes). □

Evidently, Specm $D_{c_i} \subseteq$ Specm D and Specm $D = \bigcup_{i=1}^{s}$ Specm D_{c_i}. For every $i = 1, \dots, s$ we have the situation

$$\Delta_i = D_{c_i}[X_i;\delta_i] \to B_i = (D_{c_i})_*^{-1}\Delta_i = l[X_i;\delta_i] = B$$

as at the beginning of this section.

Definition. We say that an element $b \in B$ is *preserving with respect to the faithfully flat extension* $\Delta \to \prod_{i=1}^{s} \Delta_i$ (or *ext-preserving* for short) if for every $i = 1, \dots, s$ the element $b \in B_i = B$ is preserving in the case $\Delta_i \to B_i = B$.

Lemma 3.5. *Suppose* $0 \neq d_1, \dots, d_j \in B$. *Then there exists* $0 \neq \xi \in D$ *such that all elements* $d_1\xi^{-1}, \dots, d_j\xi^{-1}$ *are preserving with respect to the faithfully flat extension* $\Delta \to \prod_{i=1}^s \Delta_i$. *More precisely, let* $\{\mathbf{p}_1, \dots, \mathbf{p}_k\}$ *be those maximal ideals which are a special prime of at least one of* $d_1, \dots, d_j$ *in at least one of the situations* $\Delta_i \to B_i = B$, $i = 1, \dots, j$. *Let* $g_l \in \mathbf{p}_l \backslash \mathbf{p}_l^2$ $(l = 1, \dots, k)$. *Also take a natural number* v *with* $v \geq v_{l,i}$ *for every* $v_{l,i} \in \mathbf{N}$ *(if any) such that* $-v_{l,i}$ *is a root of some* $Q^{(i)}_{\mathbf{p}_l, d_m}$ *in the case* $\Delta_i \to B_i = B$. *Then all elements* $d_m(g_1 \cdots g_k)^{-v}$ $(m = 1, \dots, j)$ *are preserving with respect to the faithfully flat extension* $\Delta \to \prod_{i=1}^s \Delta_i$.

Proof. It follows immediately from Lemma 3.4, [Bl3]. □

Theorem 3.6. *Let* b *be an irreducible preserving with respect to the faithfully flat extension* $\Delta \to \prod_{i=1}^s \Delta_i$ *element of* B. *Then* $M_b := \Delta/\Delta \cap Bb$ *is a simple* D-*torsionfree* Δ-*module. Up to isomorphism every simple* D-*torsionfree* Δ-*module arises in this way and two such* Δ-*modules are isomorphic,* $M_b \simeq M_c$, *iff the simple* B-*modules* B/Bb *and* B/Bc *are isomorphic.*

Proof. Let M be a simple D-torsionfree Δ-module. Then the localization $D_*^{-1}M$ of M at D_* is a simple B-module, thus the module $D_*^{-1}M$ is isomorphic to the B-module B/Bb for some irreducible element b of B. The B-modules B/Bb and B/Bbs^{-1} are isomorphic for every $s \in D_*$. By Lemma 3.5 we can suppose b to be preserving (with respect to $\Delta \to \prod_{i=1}^s \Delta_i$). The M is the Δ-submodule of its localization $D_*^{-1}M \simeq B/Bb$ (via $M \to D_*^{-1}M$, $m \to m/1$). Moreover, $M = \mathrm{Soc}_\Delta\, B/Bb$. Every nonzero Δ-submodule of $D_*^{-1}M$ is essential, so $M \subseteq M_b$ and $0 \neq M_{c_i} \subseteq (M_b)_{c_i}$ for every i. The Δ_i-modules M_{c_i} and $(M_b)_{c_i}$ are simple (Theorem 3.2), hence $M_{c_i} = (M_b)_{c_i}$ for $i = 1, \dots, s$. The extension $\Delta \to \prod_{i=1}^s \Delta_i$ is faithfully flat, so $M = M_b$.

Let b be as in the theorem. Then each $(M_b)_{c_i} \simeq \Delta_i/\Delta_i \cap Bb$ is a nonzero simple D_{c_i}-torsionfree Δ_i-module (by Theorem 3.2). By Lemma 3.3, the Δ-module M_b is simple. The other claims are evident. □

4. Simple holonomic modules

Let K be an algebraically closed field of characteristic zero and let the algebra

$$\Lambda = C \otimes A$$

be the tensor product of rings of differential operators with coefficients from a regular commutative affine domain of Krull dimension 1: $C = \Delta(D_1)$ and $A = \Delta(D_2)$. Moreover, let

$$A = D[X; \delta],\ D = D_2,$$

be an *Ore extension*. Observe, that Λ is isomorphic to the ring of differential operators $\Delta(D_1 \otimes D_2)$ (Lemma 2.5).

Example. The second Weyl algebra $A_2 = A_1 \otimes A_1$ is an example of the ring Λ ($A_1 = K[t][X; d/dt]$).

Denote by k the full quotient ring of C:

$$k = C_*^{-1}C,\ \ C_* = C\backslash\{0\},$$

Then k is a skew field (division ring) with center $Z(k) = K$. The ring

$$\mathcal{A} := C_*^{-1}\Lambda = k \otimes A = k \otimes D[X; \delta] = \mathcal{D}[X; \delta],\ \ \mathcal{D} = k \otimes D,$$

is a ring of differential operators with $\operatorname{Ker}\delta \supseteq k$. The ring $\mathcal{A} = \bigcup_{i\geq 0} \mathcal{A}_i$ is filtered by the powers of X: $\mathcal{A}_i = \bigoplus_{j\leq i} \mathcal{D}X^j = \bigoplus_{j\leq i} X^j\mathcal{D}$. The associated graded ring $\operatorname{gr}\mathcal{A} = \bigoplus \mathcal{A}_i/\mathcal{A}_{i-1}$ is the polynomial ring $\mathcal{D}[x]$ with coefficients from $\mathcal{D}$ where $x = X + \mathcal{D} \in \mathcal{A}_1/\mathcal{A}_0$.

Let k be a skew field. We say that a ring T is an *affine algebra over* k if it is a factor ring of a polynomial ring $k[X_1, \dots , X_n]$ with coefficients from k. In that case T is generated over k by the images x_i of X_i, $T = k\langle x_1, \dots , x_n\rangle$. The ring T is Noetherian and equipped with a standard filtration $\mathcal{T} = \{T_i, i \geq 0\}$ by the degree of the generators $T_0 = k$, $T_1 = k + \sum_{i=1}^n kx_i$, $T_m = (T_1)^m = \sum_{s\leq m} kx_{i_1} \cdots x_{i_s}$, $m \geq 2$. Note that the dimension $\dim_k T_m$ of T_m as the left k-vector space is finite for every m. The associated graded ring $\operatorname{gr}_{\mathcal{T}} T = \bigoplus T_i/T_{i-1}$ is an affine k-algebra with generators of graded degree 1 (i.e. from R_1/R_0). Thus $\operatorname{gr}_{\mathcal{T}}$ is a Noetherian ring.

A filtration $\Gamma = \{\Gamma_i\}$ on a T-module M ($M = \bigcup \Gamma_i$, $T_i\Gamma_j \subseteq \Gamma_{i+j}$) is called *good* if the associated graded $\operatorname{gr}_{\mathcal{T}} T$-module $\operatorname{gr}_\Gamma M = \bigoplus \Gamma_i/\Gamma_{i-1}$ is finitely generated. A T-module M has a good filtration iff it is finitely generated (Noetherian). If $\{\Gamma_i\}$ and $\{\Omega_i\}$ are two good filtrations on M, then there exists a natural number s such that $\Gamma_i \subseteq \Omega_{i+s}$ and $\Omega_i \subseteq \Gamma_{i+s}$ for all i. Let M be a finitely generated T-module and let M_0 be a finitely generated k-module of generators of M. The filtration $\Gamma_i = \{T_iM_0,\ i \geq 0\}$ is called *standard*, if $\dim_k \Gamma_i < \infty$ for all $i \geq 0$. Every standard filtration is good.

A function $f : \mathbf{N} \to \mathbf{N}$ has polynomial growth if, for some $d \in \mathbf{R}$, $f(n) \leq n^d$ for $n >> 0$; and then

$$\gamma(f) := \inf\{d \in \mathbf{R} :\ f(n) \leq n^d \text{ for sufficiently large } n >> 0\}$$

is called the *degree* of f.

Let $T = \bigcup_{i=0}^{\infty} T_i$ be as above and M be a finitely generated T-module with a standard filtration $\Gamma = \{\Gamma_i\}$. The *Gelfand-Kirillov* dimension (with respect to the base division ring k): $\mathrm{GK}_k(T) = \gamma(\dim_k T_i)$ and $\mathrm{GK}_k(M) = \gamma(\dim_k M_i)$. The number $\mathrm{GK}_k(T)$ and $\mathrm{GK}_k(M)$ does not depend on the choice of standard filtration $\mathcal{T}$ and Γ. In case the division ring is the ground field K we write GK for GK_K.

Let M be a T-module with a good filtration $\Gamma = \{\Gamma_i\}$. The integer valued function $\chi_{M,\Gamma}(i) = \dim_k \Gamma_i$ is called the *Hilbert function* of the module M with respect to Γ. Standard arguments show that there exists a polynomial $H(t) = H_{M,\Gamma}(t) = a_dt^d + \cdots + a_0 \in \mathbf{Q}[t]$ with rational coefficients (see, for example, [Bav4] and Remark 2.5 there):

$$\chi_{M,\Gamma}(i) = H_{M,\Gamma}(i) \ \text{ for all } \ i >> 0.$$

The polynomial $H_{M,\Gamma}$ is called the *Hilbert polynomial* of M with respect to the filtration Γ. The degree $d = d(M)$ of $H_{M,\Gamma}$ coincides with $\mathrm{GK}_k(M)$. The positive integer $e(M) = d!a_d$ is called the *multiplicity* of M. The degree $d(M)$ and the multiplicity $e(M)$ of M do not depend on the choice of good filtration Γ.

Let $0 \to N \to M \to L \to 0$ be an exact sequence of finitely generated T-modules. Then

$$\mathrm{GK}_k(M) = \max\{\mathrm{GK}_k(N), \mathrm{GK}_k(L)\} \tag{4.1}$$

and if $\mathrm{GK}_k(N) = \mathrm{GK}_k(M) = \mathrm{GK}_k(L)$, then

$$e(M) = e(N) + e(L). \tag{4.2}$$

In fact, let Γ be a good filtration on M. Then $\Gamma_i' = \{\Gamma_i \cap N\}$ and $\{\Gamma_i'' = (\Gamma_i + N)/N\}$ are filtrations on N and L respectively, such that the sequence of graded $\mathrm{gr}_{\mathcal{T}}\, T$-modules

$$0 \to \mathrm{gr}_{\Gamma'} N \to \mathrm{gr}_\Gamma M \to \mathrm{gr}_{\Gamma''} L \to 0 \tag{4.3}$$

is exact. The ring $\mathrm{gr}_{\mathcal{T}}\, T$ is Noetherian and the $\mathrm{gr}_{\mathcal{T}}\, T$-module $\mathrm{gr}_\Gamma M$ is finitely generated, hence, Noetherian, thus the $\mathrm{gr}_{\mathcal{T}}\, T$-modules $\mathrm{gr}_{\Gamma'} N$ and $\mathrm{gr}_{\Gamma''} L$ are finitely generated. Consequently, the filtrations Γ' and Γ'' are good and

$$\chi_{M,\Gamma}(i) = \chi_{N,\Gamma'}(i) + \chi_{L,\Gamma''}(i), \tag{4.4}$$

hence,

$$H_{M,\Gamma}(i) = H_{N,\Gamma'}(i) + H_{L,\Gamma''}(i), \tag{4.5}$$

and (4.1), (4.2) follow.

A non-left (resp. non-right, resp. non-) Artinian ring satisfies the *left (resp. right) restricted minimum condition* (l.r.m.c., resp. r.r.m.c., resp. r.m.c.) provided every proper left (resp. right, resp. left and right) factor module of it is Artinian. The ring $\mathcal{D}$ is the tensor product of the central simple algebra k and the Noetherian algebra D, hence, the ring $\mathcal{D}$ is a Noetherian k-affine algebra. Moreover, $\mathcal{D}$ is a Noetherian domain as the localization of the Noetherian domain $C \otimes D$.

Lemma 4.1. 1. *For the ring $\mathcal{D}$ the restricted minimum condition holds;*
2. $\mathrm{GK}_k\ \mathcal{D} = 1$*;*
3. *any proper (left or right) factor module of $\mathcal{D}$ is finitely generated over k. In particular, every simple $\mathcal{D}$-module is finitely generated over k.*

Proof. 2. The K-algebra D is a finitely generated module over a polynomial ring $K[H]$, so $\mathcal{D}$ is a finitely generated module over $k \otimes K[H] = k[H]$. Hence,

$$1 = \mathrm{GK}_k(k[H]) \leq \mathrm{GK}_k(\mathcal{D}) \leq \mathrm{GK}_k(k[H]) = 1.$$

3. Let us prove the statement for left modules. Let x be a nonzero element of $\mathcal{D}$. The left ideal $\mathcal{D}x$ is isomorphic to $\mathcal{D}$ as a left module. For the exact sequence of $\mathcal{D}$-modules: $0 \to \mathcal{D}x \to \mathcal{D} \to \mathcal{D}/\mathcal{D}x \to 0$, let Γ, Γ', and Γ'' be good filtrations from (4.4). The Hilbert polynomials of $\mathcal{D}$ and $\mathcal{D}x \simeq \mathcal{D}$ are equal to $ai + b$ and $ai + c$, respectively for some integers a, b, c. It follows from (4.5) that the Hilbert polynomial of $\mathcal{D}/\mathcal{D}x$ is constant, i.e. $\dim_k\ \mathcal{D}/\mathcal{D}x < \infty$.

$3 \Rightarrow 1$. Evident. □

Lemma 4.2. 1. *Any proper factor module of ${}_\Lambda\Lambda$ has Gelfand-Kirillov dimension ≤ 3;*
2. *any nonzero submodule of ${}_\Lambda\Lambda$ has Gelfand-Kirillov dimension* 4*;*
3. *for the algebra $C = \Delta(D_1)$ the restricted minimum condition holds and every simple C-module has Gelfand-Kirillov dimension* 1.

Proof. Let I be a nonzero left ideal of Λ (resp. C) and let $0 \neq x \in I$. The Λ (resp. C) is the domain, so $\Lambda x \simeq \Lambda$ and $4 = \mathrm{GK}\,\Lambda x \leq \mathrm{GK}\, I \leq \mathrm{GK}\,\Lambda = 4$, i.e. $\mathrm{GK}\, I = 4$. By [MR], 8.3.5, $\mathrm{GK}(\Lambda/\Lambda x) < \mathrm{GK}\,\Lambda = 4$ (resp. $\mathrm{GK}(C/Cx) < \mathrm{GK}\, C = 2$), hence $\mathrm{GK}(\Lambda/I) \leq \mathrm{GK}(\Lambda/\Lambda x) \leq 3$ (resp. $\mathrm{GK}(C/I) \leq \mathrm{GK}(C/Cx) \leq 1$, i.e. any proper factor module of ${}_C C$ is holonomic). □

Let M be a nonzero simple Λ-module, the localization

$$C_*^{-1}M = \mathcal{A} \otimes_\Lambda M$$

of the module M at C_* is either zero or not. The latter is a simple $\mathcal{A}$-module. With respect to these two possibilities we say that the module M is either *C-torsion* or *C-torsionfree*, so

$$\hat{\Lambda} = \hat{\Lambda}(C\text{-torsion}) \cup \hat{\Lambda}(C\text{-torsionfree}). \tag{4.6}$$

Let R be a K-algebra and let M be a simple R-module such that the endomorphism ring $\mathrm{End}_R(M) = K$. Let S be a K-algebra and N be an S-module. Then the tensor product $M \otimes N$ is an $R \otimes S$-module. By [Bav5], *any submodule of $M \otimes N$ is equal to $M \otimes N'$ for some S-submodule N' of N. In particular, $M \otimes N$ is a simple $R \otimes S$-module if ${}_SN$ is simple*, i.e.

$$\hat{R} \otimes \hat{S} \subseteq (R \otimes S)\hat{}. \tag{4.7}$$

$$\hat{\Delta}(D_1) \otimes \hat{\Delta}(D_2) \subseteq \hat{\Delta}(D_1 \otimes D_2). \tag{4.7.1}$$

Proposition 4.3.

$$\hat{\Lambda}(C\textit{-torsion}) = \hat{C} \otimes \hat{A} \subseteq \hat{\Lambda}(\textit{holonomic}).$$

Proof. The inclusion $\hat{C} \otimes \hat{A} \subseteq \hat{\Lambda}(\text{holonomic})$ follows from (4.7) and Lemma 4.2.(3). The inclusion $\hat{\Lambda}(C\text{-torsion}) \supseteq \hat{C} \otimes \hat{A}$ is evident (Lemma 4.2.(3)).

Let M be a nonzero simple C-torsion Λ-module. The algebra C satisfies the restricted minimum condition, so M contains a simple C-submodule, say N. Then the Λ-module M is an epimorphic image of the Λ-module $N \otimes A$. Hence, $M \simeq N \otimes A/J$ for some maximal left ideal J of A, i.e. $\hat{\Lambda}(C\text{-torsion}) \subseteq \hat{C} \otimes \hat{A}$. □

Denote by h the skew field $\mathcal{D}_*^{-1}\mathcal{D}$ where $\mathcal{D}_* = \mathcal{D}\backslash\{0\}$ and by

$$\mathcal{B} := \mathcal{D}_*^{-1}\mathcal{A} = h[X;\delta] \tag{4.8}$$

the localization of the algebra $\mathcal{A}$ at $\mathcal{D}_*$. Observe that h is the full quotient ring of the Noetherian domain $C \otimes D$.

We identify $\mathcal{A}$ with its image in $\mathcal{B}$ via the algebra monomorphism $\mathcal{A} \to \mathcal{B}$, $x \to x/1$. So, we have

$$\hat{\mathcal{A}} = \hat{\mathcal{A}}(\mathcal{D}\text{-torsion}) \cup \hat{\mathcal{A}}(\mathcal{D}\text{-torsionfree}). \tag{4.9}$$

Lemma 4.4. *If $[N] \in \hat{\mathcal{A}}(\mathcal{D}\text{-torsionfree}, \Lambda\text{-socle})$, then* $\mathrm{GK}\,\mathrm{Soc}_\Lambda\, N = 3$.

Proof. Since $\mathrm{Soc}_\Lambda\, N \neq 0$ and N is D-torsionfree, the socle $\mathrm{Soc}_\Lambda\, N$ is a simple Λ-module that contains a free $C \otimes D$-module. Hence $4 = \mathrm{GK}\,\Lambda > \mathrm{GK}(\mathrm{Soc}_\Lambda\, N)$ and

$$3 \geq \mathrm{GK}(\mathrm{Soc}_\Lambda\, N) \geq \mathrm{GK}(C \otimes D) = \mathrm{GK}\, C + \mathrm{GK}\, D = 2 + 1 = 3.$$

□

Let $V = \mathcal{D}/I$ be a simple $\mathcal{D}$-module for some maximal left ideal I of $\mathcal{D}$. The endomorphism ring

$$\varepsilon \equiv \varepsilon(V) \equiv \mathrm{End}_{\mathcal{D}}(V)$$

is a skew field and $V = {}_{\mathcal{D}}V_{\varepsilon}$ is a $(\mathcal{D}, \varepsilon)$-bimodule (in this situation we write endomorphisms from ε on the right). The induced $(\mathcal{A}, \varepsilon)$-bimodule

$$\mathcal{A}(V) := \mathcal{A} \otimes_{\mathcal{D}} V = \bigoplus_{i \geq 0} X^i \otimes V \tag{4.10}$$

is a filtered $\mathcal{A}$-module

$$\mathcal{A}(V) = \bigcup_{i \geq 0} \mathcal{A}(V)_i, \;\; \mathcal{A}(V)_i = \bigoplus_{j \leq i} X^j \otimes V.$$

Note that every $\mathcal{A}(V)_i$ is a $(\mathcal{D}, \varepsilon)$-bimodule. A nonzero element $u \in \mathcal{A}(V)$ can be uniquely written as a sum

$$u = 1 \otimes u_0 + X \otimes u_1 + \cdots + X^n \otimes u_n, \; u_i \in V, \; u_n \neq 0.$$

The number $n := \deg u$ is called the *degree* of u. Clearly, $u \in \mathcal{A}(V)_{\deg u} \backslash \mathcal{A}(V)_{\deg u-1}$ and $\deg u$ is defined by this property. The elements $X^n \otimes u_n$ and u_n are called the *leading term* and the *leading coefficient* of u respectively. An element from $\mathcal{A}(V)$ is called *monic* if its leading coefficient is $\bar{1} := 1 + I$.

The right $\mathcal{D}$-module $\mathcal{A}$ is free, so applying the exact functor $\mathcal{A} \otimes_{\mathcal{D}} -$ to the exact sequence of $\mathcal{D}$-modules: $0 \to I \to \mathcal{D} \to V \to 0$ we obtain the canonical isomorphism of $\mathcal{A}$-modules:

(4.11)
$$\mathcal{A}/\mathcal{A}I \to \mathcal{A}(V), \;\; \sum X^i d_i + \mathcal{A}I \to \sum X^i \otimes (d_i + I), \;\; d_i \in \mathcal{D}.$$

We identify the $\mathcal{A}$-modules $\mathcal{A}/\mathcal{A}I$ and $\mathcal{A}(V)$ via (4.11).

Lemma 4.5. *Let $[M] \in \hat{\mathcal{A}}(\mathcal{D}$-torsion). Then M is an epimorphic image of the $\mathcal{A}$-module $\mathcal{A}(V)$ for some $[V] \in \hat{\mathcal{D}}$.*

Proof. Follows immediately from Lemma 4.1.(1). □

Lemma 4.6. *Suppose $\mathcal{A}(V)$ is a simple Λ-socle $\mathcal{A}$-module for some $[V] \in \hat{\mathcal{D}}$. Then* $\mathrm{GK}\,\mathrm{Soc}_\Lambda\; \mathcal{A}(V) = 3$*, i.e. it is a simple non-holonomic Λ-module.*

Proof. Observe that $\mathrm{Soc}_\Lambda\; \mathcal{A}(V)$ is a simple Λ-module and, for any nonzero $u \in \mathcal{A}(V)$, the $C \otimes K[X]$-submodule $C \otimes K[X]u$ of $\mathcal{A}(V)$ is free. Choose u from $\mathrm{Soc}_\Lambda\; \mathcal{A}(V)$. Then $3 \geq \mathrm{GK}\,\mathrm{Soc}_\Lambda\; \mathcal{A}(V) \geq \mathrm{GK}({}_{C\otimes K[X]}C\otimes K[X]u) = \mathrm{GK}\, C\otimes K[X] = \mathrm{GK}\, C + \mathrm{GK}\, K[X] = 2 + 1 = 3$, i.e. $\mathrm{GK}\,\mathrm{Soc}_\Lambda\; \mathcal{A}(V) = 3$. □

Let R be a ring and let J be a left ideal of R. The ring $\mathcal{I}(J) = \{r \in R \mid Jr \subseteq J\}$ is called the *idealizer* of J, and is easily seen to be the largest subring of R containing J as an ideal. The ring $\bar{\mathcal{I}}(J) = \mathcal{I}(J)/J$ is called the *eigenring* of J. This acts, by right multiplication, on the module R/J and it can be checked that $\bar{\mathcal{I}}(J)$ is canonically isomorphic to the endomorphism ring $\mathrm{End}_R(R/J)$ of the R-module R/J: $(v \leftrightarrow (f_v : 1 + J \to v))$. In this case we write endomorphisms from $\mathrm{End}_R(R/J)$ on the right, i.e. $(u)f_v = uv$. We identify $\bar{\mathcal{I}}(J)$ and $\mathrm{End}_R(R/J)$ via the isomorphism above. We have

$$\mathrm{End}_{\mathcal{D}}(\mathcal{D}/I) \equiv \bar{\mathcal{I}}(I) = \mathrm{ann}_{\;\mathcal{D}/I}(I) \subseteq \mathcal{D}/I, \tag{4.12}$$

$$\mathrm{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{A}I) \equiv \bar{\mathcal{I}}(\mathcal{A}I) = \mathrm{ann}_{\;\mathcal{A}/\mathcal{A}I}(I) \subseteq \mathcal{A}/\mathcal{A}I \equiv \mathcal{A}(V), \tag{4.13}$$

where $\mathrm{ann}_{\;\mathcal{D}/I}(I) = \{u \in \mathcal{D}/I : Iu = 0\}$, etc. Set $\mathcal{E} = \mathcal{E}(\mathcal{A}/\mathcal{A}I)$ for the endomorphism ring $\mathrm{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{A}I)$. The ring $\mathcal{E}$ is a filtered ring, $\mathcal{E} = \bigcup_{i\geq 0} \mathcal{E}_i$, where $\mathcal{E}_i = \mathcal{E} \cap \mathcal{A}(V)_i$ ($\mathcal{E}_i\mathcal{E}_j \subseteq \mathcal{E}_{i+j}$ for all $i, j \geq 0$). Clearly, $\mathcal{E}_0 = \varepsilon$.

Define $H^0 = \{i : \mathcal{E}_{i-1} \neq \mathcal{E}_i\}$. Then $0 \in H^0$ $(I \cdot 1 \otimes \bar{1} = 0)$. Let $u_i = \sum_{j=0}^{i} X^j \otimes \lambda_{ij} \in \mathcal{E}_i \backslash \mathcal{E}_{i-1}$ $(\lambda_{ij} \in V,\ \lambda_{ii} \neq 0)$; then

$$0 = Iu_i = \sum_{k=0}^{i} X^k \otimes \sum_{k \leq j}^{i} (-1)^{j-k} \binom{j}{k} \delta^{j-k}(I)\lambda_{ij},$$

or, equivalently,

$$\sum_{k \leq j}^{i} (-1)^{j-k} \binom{j}{k} \delta^{j-k}(I)\lambda_{ij} = 0, \quad k = 0, \dots, i. \tag{4.14}$$

If $k = i$, then $I\lambda_{ii} = 0$, so $0 \neq \lambda_{ii} \in \varepsilon$. Therefore, for every $i \in H^0$ there exists

$$u_i = \sum_{j=0}^{i} X^j \otimes \lambda_{ij} \in \mathcal{E}_i \backslash \mathcal{E}_{i-1} \text{ with } \lambda_{ii} = \bar{1} = 1 + I. \tag{4.15}$$

The set $\{u_i, i \in H^0\}$ from (4.15) is a left and right ε-basis of $\mathcal{E}$:

$$\mathcal{E} = \bigoplus_{i \in H^0} \varepsilon u_i = \bigoplus_{i \in H^0} u_i \varepsilon, \quad \mathcal{E}_i = \bigoplus_{j \leq i} \varepsilon u_j = \bigoplus_{j \leq i} u_j \varepsilon.$$

By (4.15), $\mathcal{E} = \bigoplus_{j \leq i} \mathcal{E}_i$ is an H^0-filtered domain and H^0 is an additive submonoid of $\mathbf{N}_0$.

Let $i \in H^0$ and let j be the maximal positive element of H^0 with $j < i$. For $\lambda \in \varepsilon$, $\lambda u_i - u_i \lambda \in \mathcal{E}_j$. Suppose $H^0 \neq 0$. Let m be the minimal positive element of H^0 and let $g > 0$ be the greatest common divisor of the elements of H^0. Evidently, $\mathbf{Z} \geq \mathbf{Z}H^0 = g\mathbf{Z}$. Choose $0 < i, j \in H^0$ such that $g = j - i$. All elements

$$i^2 + (k + li)g = lgi + (i-k)i + kj, \quad k = 0, \dots, i-1,\ l \geq 0,$$

belong to H^0, thus the following definition is correct. The *starting point* $sp(H^0)$ of H^0 is the minimal nonzero $h \in H^0$ such that $ig \in H^0$ for all $i \geq hg^{-1}$. Thus for any $k \in H^0$ the set $H^0 \backslash \{k + H^0\}$ is finite. The existence of the starting point guarantees that for each $j = 0, \dots, g^{-1}m - 1$ there exists a minimal element m_j of H^0 with $m_j \equiv jg \,(\mathrm{mod}\ m)$. We have a partition

$$H^0 = \bigcup_{j=0}^{g^{-1}m-1} (m_j + m\mathbf{N}_0). \tag{4.16}$$

The subring R of $\mathcal{E}$ generated by ε and u_m is the differential operator ring

$$R = \varepsilon[u_m; \partial], \quad \partial\lambda = u_m\lambda - \lambda u_m, \qquad \lambda \in \varepsilon. \tag{4.17}$$

By (4.15),

$$\partial\bar{\mu} = \lambda_{m0}\mu - \mu\lambda_{m0} - \sum_{j=1}^{m} (-1)^j \delta^j(\mu)\lambda_{mj} + I, \quad \bar{\mu} = \mu + I \in \varepsilon. \tag{4.18}$$

For $u, v \in \mathcal{E}$, $\deg(u + v) \leq \max\{\deg u, \deg v\}$ and $\deg(uv) = \deg u + \deg v$.

Lemma 4.7. *Suppose that $H^0 \neq 0$. Then*

1. *Each proper left (right) factor module of $\mathcal{E}$ is a finite dimensional left (right) ε-vector space, so the ring $\mathcal{E}$ satisfies the restricted minimum condition and is a Noetherian domain;*

2. *the ring $\mathcal{E}$ is a finitely generated free left and right R-module*

$$\mathcal{E} = \bigoplus_{j=0}^{g^{-1}m-1} Ru_{m_j} = \bigoplus_{j=0}^{g^{-1}m-1} u_{m_j}R.$$

Proof. Evident. □

Denote by $\mathrm{Mon}\,\mathcal{E}$ (respectively $\mathrm{Irr.mon}\,\mathcal{E}$) the set of all (respectively monic and irreducible) elements of $\mathcal{E}$; and by $\mathrm{Submod}\,\mathcal{A}(V)$ (resp. $\mathrm{Max.submod}\,\mathcal{A}(V)$) the set of all (resp. maximal A-submodules) of $\mathcal{A}(V)$. Denote by $\mathrm{Sim.fac}\,\mathcal{A}(V)$ the set of isoclasses of simple epimorphic images of the $\mathcal{A}$-module $\mathcal{A}(V)$.

Proposition 4.8. *Let $[V = \mathcal{D}/I] \in \hat{\mathcal{D}}$.*

1. *Any proper factor module (resp. submodule) of $\mathcal{A}(V)$ has finite (resp. infinite) length as a left $\mathcal{D}$-module;*
2. *any nonzero homomorphism of the $\mathcal{A}$-module $\mathcal{A}(V)$ is a monomorphism;*
3. *the map*

$$\mathrm{Mon}\,\mathcal{E} \to \mathrm{Submod}\,\mathcal{A}(V)\backslash\{0\}, \;\; v \to \mathcal{A}v, \tag{4.19}$$

is a bijection with inverse $0 \neq N \to \{0 \neq v = X^i \otimes \bar{1} + \cdots \in N$ has minimal degree $i\}$, where $\bar{1} = 1 + I \in V$; and $\mathcal{A}(V) = \mathcal{A}(V)_{i-1} \oplus \mathcal{A}v$, where $i = \deg v$;

4. *let $v, u \in \mathrm{Mon}\,\mathcal{E}$. Then $Av \subseteq Au$ iff $v = wu$ for some $w \in \mathrm{Mon}\,\mathcal{E}$. Hence, the map*

$$\mathrm{Irr.mon}\,\mathcal{E} \to \mathrm{Max.submod}\,\mathcal{A}(V), \;\; v \to \mathcal{A}v, \tag{4.19.1}$$

is a bijection;

5. *any nonzero submodule of $\mathcal{A}(V)$ is isomorphic to $\mathcal{A}(V)$;*
6. *the $\mathcal{A}$-module $\mathcal{A}(V)$ is simple iff $(\mathrm{ann}\,_{\mathcal{A}(V)} I \equiv \bar{\mathcal{I}}(\mathcal{A}I) \equiv)\ \mathcal{E} = \varepsilon$.*
7. *The map*

$$\mathrm{Irr.mon}\,\mathcal{E} \to \mathrm{Sim.fac}\,\mathcal{A}(V), \;\; v \to [\mathcal{A}(V)/\mathcal{A}v], \tag{4.20}$$

is surjective; and the $\mathcal{D}$-length $l_{\mathcal{D}}(\mathcal{A}(V)/\mathcal{A}v) = \deg v < \infty$.

8. *every nonzero Λ-submodule ($\mathcal{A}$-submodule) of $\mathcal{A}(V)$ is essential, hence the $\mathcal{A}$-module $\mathcal{A}(V)$ is indecomposable;*
9. *if the $\mathcal{A}$-module $\mathcal{A}(V)$ is not simple, then $\mathrm{Soc}_{\Lambda}\,\mathcal{A}(V) = 0$ and any nonzero finitely generated Λ-submodule of $\mathcal{A}(V)$ has Gelfand-Kirillov dimension 3;*
10. *for any $v \in \mathcal{E}\backslash\varepsilon$: $\bigcap_{i=1}^{\infty} \mathcal{A}v^i = 0$.*

Proof. 1. Let $0 \neq u \in \mathcal{A}(V)_i \backslash \mathcal{A}(V)_{i-1}$ $(i \geq 0)$. Then $\mathcal{A}(V)_{i-1} + \mathcal{A}u = \mathcal{A}(V)$ and the result follows $l_{\mathcal{D}}(\mathcal{A}(V)/\mathcal{A}(V)_{i-1}) = l_{\mathcal{D}}\,\mathcal{A}(V)_i = i + 1 < \infty$ and $l_{\mathcal{D}}\,\mathcal{A}(V) = \infty$, where $l_{\mathcal{D}}\,M$ is the length of a $\mathcal{D}$-module M.

$1 \Rightarrow 2$. Evident.

3. Let N be a nonzero $\mathcal{A}$-submodule of $\mathcal{A}(V)$ and let $v = X^i \otimes \bar{1} + \cdots$, $(\bar{1} = 1 + I \in V)$ be a nonzero element of N of minimal degree i. The element v is uniquely defined. Then $v \in \mathrm{ann}\,_{\mathcal{A}(V)} I \equiv \mathcal{E}$ and $N \cap \mathcal{A}(V)_{i-1} = 0$. It follows from $\mathcal{A}(V)_{i-1} + \mathcal{A}v = \mathcal{A}(V)$ and $\mathcal{A}v \subseteq N$, that $\mathcal{A}(V) = \mathcal{A}(V)_{i-1} \oplus N = \mathcal{A}(V)_{i-1} \oplus \mathcal{A}v$, hence $N = \mathcal{A}v$, i.e. the map (4.19) is surjective. A monic element v' has minimal degree in the module $\mathcal{A}v'$. By the uniqueness of v', the map (4.19) is injective.

4. Let $\mathcal{A}v \subseteq \mathcal{A}u$ for some nonzero $u, v \in \mathrm{Mon}\,\mathcal{E}$. Then v and u are nonzero elements of the submodules $\mathcal{A}v$ and $\mathcal{A}u$ of minimal degree. By 2, the $\mathcal{A}$-module $\mathcal{A}u$ is isomorphic to $\mathcal{A}(V)$ and $\mathcal{A}v$ is a submodule of $\mathcal{A}u$. By the surjectivity of

the map (4.19), $v = wu$ for some $w \in \mathcal{E}$. If $v = wu$, for some $w, u \in \operatorname{Mon}\mathcal{E}$, then $\mathcal{A}v \subseteq \mathcal{A}u$. If, in addition, $\deg w \geq 1$, then $\mathcal{A}v \subset \mathcal{A}u$. It follows that the map (4.19.1) is correctly defined and bijective, in view of (4.19).

2$\Rightarrow$ 5; 3 $\Rightarrow$ 6; and 4 $\Rightarrow$ 7.

8. Suppose M and N are nonzero Λ-submodules of $\mathcal{A}(V)$ with $M \cap N = 0$. Then $C_*^{-1}M$ and $C_*^{-1}N$ are nonzero $\mathcal{A}$-submodules of $\mathcal{A}(V)$ with $C_*^{-1}M \cap C_*^{-1}N = 0$. The module $C_*^{-1}M$ can be considered as an $\mathcal{A}$-submodule of the factor module $\mathcal{A}(V)/C_*^{-1}N$. By 1, the former has infinite $\mathcal{D}$-length but the latter has finite, a contradiction.

9. If $\operatorname{Soc}_\Lambda \mathcal{A}(V)$ is nonzero, then it is a simple Λ-module (since, by 8, any nonzero Λ-submodule is essential) and the localization $C_*^{-1}(\operatorname{Soc}_\Lambda \mathcal{A}(V))$ is the simple $\mathcal{A}$-submodule of $\mathcal{A}(V)$. By 5, the $\mathcal{A}$-module $\mathcal{A}(V)$ is simple, a contradiction. Thus $\operatorname{Soc}_\Lambda \mathcal{A}(V) = 0$.

By the canonical Λ-monomorphism

$$L := \Lambda/\Lambda \cap \mathcal{A}I \to \mathcal{A}(V) \equiv \mathcal{A}/\mathcal{A}I,\ u + \Lambda \cap \mathcal{A}I \to u + \mathcal{A}I,\ u \in \Lambda,$$

we identify L with its image in $\mathcal{A}(V)$. The left ideal $\Lambda \cap \mathcal{A}I$ is nonzero, since $C_*^{-1}(\Lambda \cap \mathcal{A}I) = \mathcal{A}I \neq 0$. The Λ-module L is a proper factor module of Λ. It follows from Lemma 4.2.(1) that $\operatorname{GK} L \leq 3$. Let M be a nonzero cyclic Λ-submodule of $\mathcal{A}(V)$. By 8, $N = M \cap L \neq 0$ and $\operatorname{GK} N \leq 3$, hence the Λ-module M is not isomorphic to ${}_\Lambda\Lambda$ (Lemma 4.2(2)). Thus any nonzero finitely generated Λ-submodule P of $\mathcal{A}(V)$ has Gelfand-Kirillov dimension ≤ 3. If $\operatorname{GK} P = 2$ for some P, then P contains a simple Λ-submodule, hence $\operatorname{Soc}_\Lambda \mathcal{A}(V) \neq 0$, a contradiction, i.e. $\operatorname{GK} P = 3$ for any nonzero finitely generated Λ-module P of $\mathcal{A}(V)$.

10. Evident (degree argument). □

Observe that every k-finite dimensional $\mathcal{A}$-module is $\mathcal{D}$-torsion. So,

$$\hat{\mathcal{A}}(\mathcal{D}\text{-torsion}) = \hat{\mathcal{A}}(k\text{-fin.dim}) \cup \hat{\mathcal{A}}(\mathcal{D}\text{-torsion}, k\text{-inf.dim}), \tag{4.21}$$

where a simple $\mathcal{D}$-torsion $\mathcal{A}$-module belongs to the first, respectively the second set, if it is k-finite respectively k-infinite dimensional. By Proposition 4.8, Lemma 4.1.(3) and Lemma 4.6,

$$\hat{\mathcal{A}}(\mathcal{D}\text{-torsion}, k\text{-inf.dim}) = \{[\mathcal{A}(V)],\ \text{where } {}_\mathcal{A}\mathcal{A}(V) \text{ is simple}\}, \tag{4.22}$$

$$\hat{\mathcal{A}}(k\text{-fin.dim}) = \cup\{\operatorname{Sim.fac}\mathcal{A}(V),\ \text{where } {}_\mathcal{A}\mathcal{A}(V) \text{ is not simple}\}, \tag{4.23}$$

where $\operatorname{Sim.fac}\mathcal{A}(V)$ is the set of isoclasses of simple epimorphic images of the $\mathcal{A}$-module $\mathcal{A}(V)$. A module M is called GK-*critical* provided any proper factor module of M has Gelfand-Kirillov dimension less than $\operatorname{GK} M$.

Theorem 4.9. *The map*

$$\hat{\Lambda}(\textit{holonomic})\backslash\hat{C} \otimes \hat{A} \to \hat{\mathcal{A}}(k\textit{-fin.dim}),\ [M] \to [\mathcal{A} \otimes_\Lambda M], \tag{4.24}$$

is bijective with inverse $[N] \to [\operatorname{Soc}_\Lambda N]$.

Proof. For $[M] \in \hat{\Lambda}(\text{holonomic})\backslash\hat{C} \otimes \hat{A}$, $\tilde{M} := \mathcal{A} \otimes_\Lambda M$ is the (nonzero) simple $\mathcal{A}$-module with $\operatorname{Soc}_\Lambda \tilde{M} = M$. By Lemma 4.4, the $\mathcal{A}$-module $\tilde{M}$ is $\mathcal{D}$-torsion, hence, by Lemma 4.5, $\tilde{M}$ is an epimorphic image of $\mathcal{A}(V)$ for some $[V] \in \hat{\mathcal{D}}$. By Lemma 4.6, $\tilde{M}$ is a proper epimorphic image of the non-simple $\mathcal{A}$-module $\mathcal{A}(V)$ and, by (4.23), $[\tilde{M}] \in \hat{\mathcal{A}}(k\text{-fin.dim})$. So, the map (4.24) is well-defined and injective, since $\operatorname{Soc}_\Lambda \tilde{M} = M$.

It remains to be proved that for every non-simple $\mathcal{A}$-module $\mathcal{A}(V)$ the Λ-socle of every simple epimorphic image N of $\mathcal{A}(V)$ has Gelfand-Kirillov dimension 2, i.e. is holonomic (then, evidently, $[N] \in \hat{\Lambda}(\text{holonomic})\backslash \hat{C} \otimes \hat{A}$ and $N = \mathcal{A} \otimes_\Lambda \text{Soc}_\Lambda N$, i.e. the map $[N] \to [\text{Soc}_\Lambda N]$ is inverse to (4.24)). Let $[V = \mathcal{D}/I] \in \hat{\mathcal{D}}$ for some maximal left ideal I of $\mathcal{D}$. Let N be a simple epimorphic image of the $\mathcal{A}$-module $\mathcal{A}(V) \equiv \mathcal{A}/\mathcal{A}I$. By Proposition 4.8.(7), $N = \mathcal{A}(V)/\mathcal{A}v$ for some irreducible monic element $v = v' + \mathcal{A}I$ ($v' \in \mathcal{A}$) of $\mathcal{E} \equiv \bar{\mathcal{I}}(\mathcal{A}I) \equiv \text{ann}_{\mathcal{A}(V)} I \subseteq \mathcal{A}(V) \equiv \mathcal{A}/\mathcal{A}I$. By Proposition 4.8.(5), we have the $\mathcal{A}$-module isomorphism

$$(\cdot)v : \mathcal{A}(V) \to \mathcal{A}v, \quad u \to uv. \tag{4.25}$$

The algebra Λ is somewhat commutative. Fix a finite dimensional filtration of the algebra $\Lambda = \bigoplus_{i\geq 0} \Lambda_i$ such that the associated graded algebra $\text{gr}\,\Lambda = \bigoplus_{i\geq 0} \Lambda_i/\Lambda_{i-1}$ is affine commutative. The $\mathcal{A}$-module $\mathcal{A}(V)$ is not simple, so any nonzero finitely generated Λ-submodule of $\mathcal{A}(V)$ has Gelfand-Kirillov dimension 3 (Proposition 4.8.(9)). Choose a nonzero finitely generated Λ-submodule, say M, of $\mathcal{A}(V)$ which has minimal possible multiplicity $e(M)$ (such a module exists because of Lemma 2.2 and additivity of multiplicity (2.3)). Then M is GK-critical (if not, then there is a nonzero submodule L of M with $\text{GK}(M/L) = 3$. We have the exact sequence of Λ-modules: $0 \to L \to M \to M/L \to 0$ with $\text{GK}\,L = \text{GK}\,M = \text{GK}(M/L) = 3$, hence, by (2.3), $e(L) = e(M) - e(M/L) < e(M)$ which contradicts the choice of M). By Proposition 4.8.(10) we have the descending chain of $\mathcal{A}$-submodules

$$\mathcal{A}(V) \supset \mathcal{A}v \supset \cdots \supset \mathcal{A}v^i \supset \cdots, \ \text{ with } \bigcap_{i=1}^{\infty} \mathcal{A}v^i = 0$$

and each factor $\mathcal{A}v^i/\mathcal{A}v^{i+1}$ is isomorphic to N (since the map $v^i : \mathcal{A}(V) \to \mathcal{A}v$, $u \to uv^i$, is the $\mathcal{A}$-module isomorphism, Proposition 4.8). There exists $i \geq 0$: $M \subseteq \mathcal{A}v^i$ and $M \not\subseteq \mathcal{A}v^{i+1}$. On the one hand, the nonzero finitely generated Λ-submodule $Q = M/M \cap \mathcal{A}v^{i+1}$ is a submodule of $\mathcal{A}v^i/\mathcal{A}v^{i+1} \simeq N$. On the other hand, Q is a proper factor module of the critical Λ-module M, so $\text{GK}\,Q = 2$. The Λ-module Q is holonomic, hence, it contains a simple holonomic submodule, say U. Evidently, $U = \text{Soc}_\Lambda N$. □

Remark. In the proof of the last step of the theorem above we have not used the irreducibility of v. So, in fact, we have proved the following corollary.

Corollary 4.10. *Suppose the $\mathcal{A}$-module $\mathcal{A}(V)$ is not simple for some $[V] \in \hat{\mathcal{D}}$. Then any proper factor module of $\mathcal{A}(V)$ contains a simple holonomic Λ-submodule from $\hat{\Lambda}(holonomic)\backslash \hat{C} \otimes \hat{A}$.* □

Corollary 4.11. *Let M be a nonzero simple Λ-module and let $\tilde{M} = \mathcal{A} \otimes_\Lambda M$. Then*

1. $\tilde{M} = 0 \Leftrightarrow [M] \in \hat{C} \otimes \hat{A}$*;*
2. $1 \leq \dim_k \tilde{M} < \infty \Leftrightarrow [M] \in \hat{\Lambda}(holonomic)\backslash \hat{C} \otimes \hat{A}$*;*
3. $\dim_k \tilde{M} = \infty \Leftrightarrow [M] \in \hat{\Lambda}(non\text{-}holonomic)$*.*

Hence, M is holonomic (resp. non-holonomic) iff $\dim_k \tilde{M} < \infty$ (resp. M contains a free $C \otimes K[X]$-module of rank 1).

Proof. 1. It follows from Proposition 4.3.
2 and 3 follow from Theorem 4.9. □

Corollary 4.12. *Let $[V = \mathcal{D}/I] \in \hat{\mathcal{D}}$ for a maximal left ideal I of $\mathcal{D}$ such that the $\mathcal{A}$-module $\mathcal{A}(V) \equiv \mathcal{A}/\mathcal{A}I$ is not simple (i.e. the left ideal $\mathcal{A}I$ of $\mathcal{A}$ is not maximal) and let $v = \tilde{v} + \mathcal{A}I$ ($\tilde{v} \in \mathcal{A}$) be an irreducible element of the ring $\mathcal{E} = \mathcal{E}(\mathcal{A}/\mathcal{A}I)$. Then*

1. *the Λ-module*

$$M(I, \tilde{v}) := \Lambda/\Lambda \cap \mathcal{A}(I, \tilde{v})$$

is a Λ-submodule of $\mathcal{A}(V)$, hence,

$$[\mathrm{Soc}_\Lambda\ M(I, \tilde{v}) = \mathrm{Soc}_\Lambda\ \mathcal{A}(V)/\mathcal{A}v] \in \hat{\Lambda}(holonomic)\backslash \hat{C} \otimes \hat{A};$$

2. *let J be a left ideal of Λ which contains $\Lambda \cap \mathcal{A}(I, \tilde{v})$ and $J/\Lambda \cap \mathcal{A}(I, \tilde{v}) = \mathrm{Soc}_\Lambda\ M(I, \tilde{v})$. Then the left ideal $\mathbf{a} := J \cap C$ of C is nonzero and for any nonzero element $a \in \mathbf{a}$,*

$$[\mathrm{Soc}_\Lambda\ M(I, \tilde{v}) \simeq \Lambda/\Lambda \cap \mathcal{A}(I, \tilde{v})a^{-1}] \in \hat{\Lambda}(holonomic)\backslash \hat{C} \otimes \hat{A}.$$

So, any element of $\hat{\Lambda}(holonomic)\backslash \hat{C} \otimes \hat{A}$ is an isoclass of some Λ-module $\Lambda/\Lambda \cap \mathcal{A}(I, \tilde{v})a^{-1}$ (for some I, $\tilde{v}$ and a as above). Two such simple Λ-modules are isomorphic, $\Lambda/\Lambda \cap \mathcal{A}(I, \tilde{v})a^{-1} \simeq \Lambda/\Lambda \cap \mathcal{A}(I_*, \tilde{v_*})a_*^{-1}$, iff the simple $\mathcal{A}$-modules $\mathcal{A}/\mathcal{A}(I, \tilde{v})$ and $\mathcal{A}/\mathcal{A}(I_*, \tilde{v_*})$ are isomorphic.

Proof. 1. Evident.

2. It follows from Theorem 4.8 and Lemma 2.7. □

5. The simple holonomic $\mathcal{D}(D_1 \otimes D_2)$-modules

Let K be an algebraically closed field of characteristic zero and let the algebra

$$\Lambda = C \otimes A$$

be the tensor product of rings of differential operators with coefficients from a regular commutative affine domain of Krull dimension 1: $C = \Delta(D_1)$ and $A = \Delta(D_2)$, $D = D_2$. The algebra Λ is isomorphic to the ring of differential operators $\Delta(D_1 \otimes D_2)$ (Lemma 2.5). We keep the notation from Section 4. In this section the simple holonomic Λ-modules will be described.

Repeating the same argument (as in Section 4) for the ring Λ we have the partition (4.6) and Proposition 4.3 is true, hence there is the partition

$$\hat{\Lambda}(\text{holonomic}) = \hat{C} \otimes \hat{A} \cup \hat{\Lambda}(\text{holonomic})\backslash \hat{C} \otimes \hat{A}.$$

The sets $\hat{C}$ and $\hat{A}$ were described in Section 3. So, in order to finish the classification of $\hat{\Lambda}$(holonomic) it remains to describe $\hat{\Lambda}(\text{holonomic})\backslash \hat{C} \otimes \hat{A}$. We keep notation of Section 3 for the algebra $A \equiv \Delta(D)$ (substituting the letter A for Δ). Fix an element $c \in D$ as in Lemma 2.1.(1) applied to the ring $A = \Delta(D)$, i.e.

$$A_c = D_c[X; \delta]$$

is an Ore extension for some X and some derivation δ. The algebras A and A_c are subalgebras of the algebra $B = l[X; \delta]$ which is the localization of A at $D_* = D\backslash\{0\}$ or the localization of A_c at $D_c\backslash\{0\}$. We have the inclusion

$$\Lambda = C \otimes A \subseteq \Lambda_c = C \otimes A_c$$

of algebras. Observe that the algebra Λ_c is of the type considered in Section 4, so the simple holonomic Λ_c-modules are described. Recall that k is the full quotient ring of C. The algebra

$$\mathcal{A} := C_*^{-1}\Lambda_c = k \otimes A_c = k \otimes D_c[X;\delta] = \mathcal{D}_c[X;\delta],$$

is the Ore extension where $\mathcal{D} = k \otimes D$, $\mathcal{D}_c = k \otimes D_c$.

Theorem 5.1. *The map*

$$\hat{\Lambda}(\textit{holonomic})\backslash\hat{C}\otimes\hat{A} \to \hat{\Lambda}_c(\textit{holonomic})\backslash\hat{C}\otimes\hat{A}_c,\ [M]\to[M_c], \tag{5.1}$$

is bijective with inverse $[N] \to [\mathrm{Soc}_\Lambda\ N]$.

Proof. Observe that if $[M] \in \hat{\Lambda}(\text{holonomic})\backslash\hat{C}\otimes\hat{A}$, then $M_c \neq 0$ (otherwise, $[M] \in \hat{C}\otimes\hat{A}$, since $c \in A$) and, by Proposition 2.3, M_c is a simple holonomic Λ_c-module. Since $\mathrm{Soc}_C\ M = 0$ (otherwise, $[M] \in \hat{C}\otimes\hat{A}$) and $\mathrm{Soc}_C\ L \neq 0$ for every $[L] \in \hat{C}\otimes\hat{A}_c$, we conclude that $[M_c] \in \hat{\Lambda}_c(\text{holonomic})\backslash\hat{C}\otimes\hat{A}_c$, i.e. the map (5.1) is well-defined.

To finish the proof it suffices to show that each $[N] \in \hat{\Lambda}_c(\text{holonomic})\backslash\hat{C}\otimes\hat{A}_c$ has nonzero $\mathrm{Soc}_\Lambda\ N$. If we take any nonzero cyclic Λ-submodule, say L, of N, then $N = L_c$. By Proposition 2.3, L is a holonomic Λ-module, hence, L contains a simple Λ-submodule. i.e. $\mathrm{Soc}_\Lambda\ N \neq 0$. □

Corollary 5.2. *Let M be a nonzero simple Λ-module and let $\tilde{M} = \mathcal{A}\otimes_\Lambda M$. Then*

1. $\tilde{M} = 0 \Leftrightarrow [M] \in \hat{C}\otimes\hat{A}$*;*
2. $1 \leq \dim_k \tilde{M} < \infty \Leftrightarrow [M] \in \hat{\Lambda}(\textit{holonomic})\backslash\hat{C}\otimes\hat{A}$*;*
3. $\dim_k \tilde{M} = \infty \Leftrightarrow [M] \in \hat{\Lambda}(\textit{non-holonomic})$.

Hence, M is holonomic (resp. non-holonomic) iff $\dim_k \tilde{M} < \infty$ *(resp. M_c contains a free $C\otimes K[X]$-module of rank 1).*

Proof. It follows from Theorem 5.1 and Corollary 4.11. □

There are natural algebra monomorphisms:

$$\Lambda = C\otimes A \to \Lambda_c = C\otimes A_c \to \mathcal{A} = k\otimes A_c = \mathcal{D}_c[X;\delta].$$

Observe that the algebra $\mathcal{A}$ is the localization $\mathcal{A} = S^{-1}\Lambda$ of Λ at the Ore subset $S = \bigcup_{j=0}^{\infty} C_* c^j$ of Λ. The next corollary follows immediately from Theorem 5.1, Corollary 4.12 and Lemma 2.7.

Corollary 5.3. *Let $[V = \mathcal{D}_c/I] \in \hat{\mathcal{D}}_c$ for a maximal left ideal I of $\mathcal{D}_c$ such that the $\mathcal{A}$-module $\mathcal{A}(V) \equiv \mathcal{A}/\mathcal{A}I$ is not simple and let $v = \tilde{v} + \mathcal{A}I$ ($\tilde{v} \in \mathcal{A}$) be an irreducible element of the ring $\mathcal{E} = \mathcal{E}(\mathcal{A}/\mathcal{A}I)$. Then*

1. *the Λ-module*

$$M(I,\tilde{v}) := \Lambda/\Lambda\cap\mathcal{A}(I,\tilde{v})$$

is a Λ-submodule of $\mathcal{A}(V)$, hence,

$$[\mathrm{Soc}_\Lambda\ M(I,\tilde{v}) = \mathrm{Soc}_\Lambda\ \mathcal{A}(V)/\mathcal{A}v] \in \hat{\Lambda}(\textit{holonomic})\backslash\hat{C}\otimes\hat{A};$$

2. *let J be a left ideal of Λ which contains $\Lambda\cap\mathcal{A}(I,\tilde{v})$ and $J/\Lambda\cap\mathcal{A}(I,\tilde{v}) = \mathrm{Soc}_\Lambda\ M(I,\tilde{v})$. Then the set $\mathbf{a} := J\cap S$ of $S = \bigcup_{j=0}^{\infty} C_* c^j$ is non-empty and for any element $a \in \mathbf{a}$,*

$$[\mathrm{Soc}_\Lambda\ M(I,\tilde{v}) \simeq \Lambda/\Lambda\cap\mathcal{A}(I,\tilde{v})a^{-1}] \in \hat{\Lambda}(\textit{holonomic})\backslash\hat{C}\otimes\hat{A}.$$

So, any element of $\hat{\Lambda}(holonomic)\backslash \hat{C} \otimes \hat{A}$ *is an isoclass of some* Λ*-module* $\Lambda/\Lambda \cap \mathcal{A}(I, \tilde{v})a^{-1}$ *for some* I, $\tilde{v}$ *and* a *as above. Two such simple* Λ*-modules are isomorphic,* $\Lambda/\Lambda \cap \mathcal{A}(I, \tilde{v})a^{-1} \simeq \Lambda/\Lambda \cap \mathcal{A}(I_*, \tilde{v_*})a_*^{-1}$, *iff the simple* $\mathcal{A}$*-modules* $\mathcal{A}/\mathcal{A}(I, \tilde{v})$ *and* $\mathcal{A}/\mathcal{A}(I_*, \tilde{v_*})$ *are isomorphic.* □

6. A holonomicity criterion of $\mathcal{D}(R)$-modules (R is regular of Krull dimension 2)

Let R be a regular commutative affine domain over the algebraically closed field K of characteristic zero. Let $\Delta(R) = \mathcal{D}(R)$ be the ring of differential operators with coefficients from R. For the first Weyl algebra $A_1 = A_1(K)$ we denote by k its full quotient ring (the *first Weyl skew field*). In this section we give a criterion (Theorem 6.1) when a finitely generated $\Delta(R)$-module is holonomic in terms of finite dimensionality of some left vector spaces over the first Weyl skew field.

Fix

$$\Delta(R) \to \prod_{i=1}^{s} \Delta(R)_{c_i}, \quad c_i \in R,\ i = 1, \dots, s,$$

a faithfully flat extension as in Lemma 2.1. Every ring

$$\Delta(R)_{c_i} = \Delta(R_{c_i}) = R_{c_i}[X_{i,1}, \partial/\partial Y_{i,1}][X_{i,2}, \partial/\partial Y_{i,2}]$$

is an iterated Ore extension (as in Lemma 2.1) and contains the second Weyl algebra $A_2^{(i)}$:

$$\Delta(R)_{c_i} \supseteq A_2^{(i)} := C^{(i)} \otimes A^{(i)},\ C^{(i)} = K[Y_{i,1}][X_{i,1}, \partial/\partial Y_{i,1}],$$

$$A^{(i)} = K[Y_i][X_i, \partial/\partial Y_i],\ X_i = X_{i,2},\ Y_i = Y_{i,2}.$$

The algebras $\{A^{(i)}, C^{(i)}\}$ are isomorphic to the first Weyl algebra. Denote by $k_i = (C_*^{(i)})^{-1} C^{(i)}$ the Weyl skew field associated with $C^{(i)}$. Note the second Weyl algebra $A_2^{(i)} = C^{(i)} \otimes A^{(i)}$ is the example of the algebra $\Lambda = C \otimes A$ from Section 4.

The localization $\mathcal{A}^{(i)}$ of $A_2^{(i)}$ at $C_*^{(i)}$ is the Ore extension

$$\mathcal{A}^{(i)} = k_i \otimes A^{(i)} = k_i \otimes K[Y_i][X_i, \delta_i = \partial/\partial Y_i].$$

Theorem 6.1. *Let* M *be a finitely generated* $\Delta(R)$*-module. Then*

1. *the* $\Delta(R)$*-module* M *is holonomic* $\Leftrightarrow$ $\dim_{k_i} \mathcal{A}^{(i)} \otimes_{A_2^{(i)}} M_{c_i} < \infty$ *for* $i = 1, \dots, s$;
2. *the* $\Delta(R)$*-module* M *is non-holonomic* $\Leftrightarrow$ i *exists such that* $\dim_{k_i} \mathcal{A}^{(i)} \otimes_{A_2^{(i)}} M_{c_i} = \infty$ $\Leftrightarrow$ *there exists a free* $C^{(i)} \otimes K[Y_i]$*-submodule of* M_{c_i} *of rank* 1 *for some* i.

Proof. It follows immediately from Theorem 2.4(2), (3) and Corollary 4.11. □

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Department of Pure Mathematics, University of Sheffield, Hicks Building, Sheffield S3 7RH, UK

E-mail address: vbavula@sheffield.ac.uk

E-mail address: bavula@uia.ua.ac.be

Department of Mathematics and Computer Science, University of Antwerp (U.I.A), Universiteitsplein, 1, B-2610, Wilrijk, Belgium

E-mail address: francin@uia.ua.ac.be