

NONRADIAL HÖRMANDER ALGEBRAS OF SEVERAL VARIABLES AND CONVOLUTION OPERATORS

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To our friend Jean Schmets on the occasion of his 60th birthday

ABSTRACT. A characterization of the closed principal ideals in nonradial Hörmander algebras of holomorphic functions of several variables in terms of the behaviour of the generator is obtained. This result is applied to study the range of convolution operators and ultradifferential operators on spaces of quasianalytic functions of Beurling type. Contrary to what is known to happen in the case of non-quasianalytic functions, an ultradistribution on a space of quasianalytic functions is constructed such that the range of the operator does not contain the real analytic functions.

Let $u, v : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, non-negative and even functions which are increasing on the positive real numbers. We assume that v is convex and the quotient $\frac{u(x)}{v(x)}$ tends to zero as $x \rightarrow \infty$. Both functions are extended to \mathbb{R}^N as follows:

$$u(x_1, \dots, x_N) := \sum_{i=1}^N u(x_i), \quad v(x_1, \dots, x_N) := \sum_{i=1}^N v(x_i).$$

Now we put $p(z) := u(\operatorname{Re} z) + v(\operatorname{Im} z)$, $z \in \mathbb{C}^N$, and we suppose that $\log(1 + |z|) = O(p(z))$ and $p(2z) = O(p(z))$ as $|z| \rightarrow \infty$.

Then p is called a non-radial weight. The associated Hörmander algebra A_p is the collection of all entire functions $f \in H(\mathbb{C}^N)$ such that, for some $k \in \mathbb{N}$,

$$|f|_k := \sup_{z \in \mathbb{C}^N} |f(z)| \exp(-kp(z)) < \infty.$$

In the radial case, which occurs if instead of the condition $u = o(v)$ we take $u = v$, it is well-known that each principal ideal in A_p is closed ([2], [13]). However this is not true in general in the non-radial case. A characterization of the closed principal ideals was given by Ehrenpreis [7, 2.2] in the case $p(z) = \log(1 + |z|^2) + |\operatorname{Im} z|$ and by Meise, Taylor and Vogt [18] in the case $v(t) = |t|$ under some particular assumptions on u . For general non-radial Hörmander algebras the closed principal ideals were characterized by Momm [23] in the case $N = 1$. Our proposition 2 is an extension of the results in [23] to the case of functions of several variables. As a consequence,

Received by the editors June 1, 1998.

2000 *Mathematics Subject Classification*. Primary 46E10, 46F05, 46F10, 35R50, 32A15.

Key words and phrases. Hörmander algebras, principal ideal, convolution operators, spaces of quasianalytic functions, real analytic functions.

The research of J. Bonet and A. Galbis was supported in part by DGEIC, Proyecto no. PB97-0333.

some of the results in [4], [5] and [18] concerning the range of convolution operators acting on non-quasianalytic classes of Beurling type are extended to the quasianalytic case in theorem 3 and corollaries 6 and 7. We characterize also in proposition 14 the Roumieu classes which are contained in the range of a given convolution operator, acting on a Beurling class, in terms of the behaviour of the symbol of the operator. Special emphasis is put on the case of ultradifferential operators. See propositions 9 and 15. The limit case which appears when we replace the Roumieu space by the space of the real analytic functions is considered at the end of the paper. Ehrenpreis proved in [7, Theorem II] that the intersection of the ranges of all the convolution operators on $\mathcal{E}(\mathbb{R}^N)$ is the space of real analytic maps $A(\mathbb{R}^N)$. In particular the real analytic functions are contained in the range of every convolution operator on $\mathcal{E}(\mathbb{R}^N)$. This result was extended by Hörmander for convex open sets (cf. [11, 16.4.5]) and for spaces of non-quasianalytic functions of Beurling type in [4]. However our theorem 21 shows the existence of a (non-zero) convolution operator on a quasianalytic class of Beurling type whose range does not contain the space of the real analytic functions. This result is related to the work of Sigurdsson [30] concerning the existence of entire functions f such that the limit set of $\log|f|$ is equal to a prescribed set of subharmonic functions.

Our notation is standard. We refer the reader to the books [9, 10, 11, 15, 20].

Remark. For $|y|$ large enough we have, for some constant $C > 0$ and for all $y \in \mathbb{R}^N$,

$$\begin{aligned} v(|y|) &\leq v\left(\sum_{i=1}^N |y_i|\right) \leq \max v(N|y_i|) \\ &\leq C \max v(|y_i|) \leq Cv(y). \end{aligned}$$

Lemma 1. *Let v be an increasing and convex function on \mathbb{R}_+ such that $v(2t) \leq Kv(t)$ for all $t \geq 1$. Given sequences $(x_j) \subset \mathbb{R}^N$ and $(R_j) \subset \mathbb{R}_+$, there exist a positive constant $D > 0$ and a sequence (f_j) of entire functions with the following properties:*

1. $|f_j(z)| \leq D \exp(Dv(|\operatorname{Im}z|) + C_N \log(1 + |z|^2))$ if $z \in \mathbb{C}^N$, $|z - x_j| \geq R_j + 2$,
2. $|f_j(z)| \leq D \exp(Dv(R_j) + C_N \log(1 + |z|^2))$ if $z \in \mathbb{C}^N$, $|z - x_j| \leq R_j + 2$,
3. $|f_j(x_j)| \geq \exp(v(\frac{2R_j}{\pi\sqrt{N}}) - C_N \log(1 + |x_j|^2))$,

where C_N is a positive constant which only depends on N .

Proof. For each $j \in \mathbb{N}$ and $R > 0$ we consider the function $h_{j,R} : \mathbb{C}^N \rightarrow \mathbb{R}$, which equals $|\operatorname{Im}z|$ if $|z - x_j| \geq R$ and with

$$\begin{aligned} h_{j,R}(z) &:= \sup\{u(z) : u \text{ is plurisubharmonic on } B(x_j, R)\}, \\ \overline{\lim}_{w \rightarrow \xi} u(w) &\leq |\operatorname{Im}\xi| \text{ for every } \xi \text{ with } |\xi - x_j| = R \end{aligned}$$

for $|z - x_j| < R$. By [25], $h_{j,R}$ is plurisubharmonic and continuous on \mathbb{C}^N and satisfies $h_{j,R}(x_j) \geq \frac{2R}{\pi\sqrt{N}}$. Now we put $h_j := h_{j,R_j}$ and define φ_j as the function $\varphi_j(z) := h_{j,R_j+1}(z) + 1$. Since v is an increasing and convex function on \mathbb{R}_+ we have that $\Phi_j := v \circ \varphi_j$ is a plurisubharmonic function which has the following properties (see [4, p.131-132]):

- (i) $\sup_{|w-z| \leq 1} \Phi_j(w) \leq v(|\operatorname{Im}\xi| + 2)$ for every $z \in \mathbb{C}^N$, $|z - x_j| \geq R_j + 2$,
- (ii) $\sup_{|w-z| \leq 1} \Phi_j(w) \leq v(R_j + 4)$ for every $z \in \mathbb{C}^N$, $|z - x_j| < R_j + 2$ and

(iii) $\inf_{|w-x_j|\leq 1} \Phi_j(w) \geq v(\frac{2R_j}{\pi\sqrt{N}})$.

Finally we apply [9, 4.4.4] as in [24, 1.8] to find a sequence (f_j) of entire functions with

$$|f_j(x_j)| \geq \exp(\inf_{|w-x_j|\leq 1} \Phi_j(w) - C_N \log(1 + |x_j|^2))$$

and

$$|f_j(z)| \leq C_N \exp(\sup_{|z-w|\leq 1} \Phi_j(w) + C_N \log(1 + |z|^2)), \quad z \in \mathbb{C}^N,$$

from where the conclusion follows. □

For a non-radial weight p we put $R_k(x) := v^{-1}(ku(x))$. We may assume $v(0) = 0$. Then by the convexity of v and the properties of p , for each $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ and $x_0 > 0$ with $R_k(x) \leq kR_1(x) \leq R_n(x)$, $x \geq x_0$. Hence, in the next proposition we may replace R_k by kR_1 .

Proposition 2. *Let $F \in A_p \setminus (0)$ be given. Then the following assertions are equivalent:*

- (a) *There are $k \in \mathbb{N}$ and $x_0 > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq x_0$ there is $w \in \mathbb{C}^N$ with $|w - x| \leq R_k(x)$ and $|F(w)| \geq \exp(-kp(w))$.*
- (b) *There are $k \in \mathbb{N}$ and $x_0 > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq x_0$ there is $t \in \mathbb{R}^N$ with $|t - x| \leq R_k(x)$ and $|F(t)| \geq \exp(-ku(t))$.*
- (c) *There is $k \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $R > 0$ such that for each $z \in \mathbb{C}^N$, $|z| \geq R$, there exists $w \in \mathbb{C}^N$ with $|w - z| \leq R_k(\operatorname{Re} z) + \frac{|\operatorname{Im} z|}{n}$ and $|F(w)| \geq \exp(-mp(w))$.*
- (d) *For each $k \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and $C > 0$ such that for each $g \in A_p$ with $f := \frac{g}{F} \in A(\mathbb{C}^N)$ we have $f \in A_p$ and $|f|_n \leq C |g|_k$.*

Proof. (a) \Rightarrow (b). Let k, x_0 , be given by condition (a) and fix $x \in \mathbb{R}^N$ with $|x| \geq x_0$. We take $w \in \mathbb{C}^N$ such that $|w - x| \leq R_k(x)$ and $|F(w)| \geq \exp(-kp(w))$ and consider the entire function $f(\xi) := F(\xi, w_2, \dots, w_N)$, $\xi \in \mathbb{C}$. By Levin [15, Theorem 11], applied with $\eta = \frac{3}{16}$ and $H = 2 + \log(\frac{3e}{2\eta})$, there exists $\rho_1 > 0$ with

$$|w_1 - x_1| < \rho_1 < \frac{3}{2} |w_1 - x_1| =: R$$

such that

$$|f(\xi)| \geq |f(w_1)|^{H+1} (\sup_{|\xi-w_1|=2eR} |f(\xi)|^{-H}), \quad |\xi - w_1| = \rho_1.$$

Using that $v(|w_1 - x_1|) \leq v(R_k(x)) \leq ku(x)$ we get the existence of constants C_i depending on p and k such that

$$\begin{aligned} \sup_{|\xi-w_1|=2eR} p(\xi) &\leq u(\operatorname{Re} w_1 + 2eR) + v(\operatorname{Im} w_1 + 2eR) \\ &\leq C_1(p(w) + v(|w_1 - x_1|) + 1) \\ &\leq C_2(p(w) + u(x) + 1). \end{aligned}$$

Now we choose $t_1 \in \mathbb{R}$ with $|t_1 - w_1| = \rho_1$. Then $|t_1 - x_1| \leq \frac{5}{2}R_k(x)$ and $|F(t_1, w_2, \dots, w_N)| \geq \exp(-k_1(p(w) + u(x) + 1))$ for some constant $k_1 > 0$. We again apply Levin [15, Theorem 11] to $f(\xi) := F(t_1, \xi, w_3, \dots, w_N)$ to find $\rho_2 > 0$ with $|w_2 - x_2| < \rho_2 < \frac{3}{2} |w_2 - x_2|$, $|t_2 - w_2| = \rho_2$ and $|F(t_1, t_2, w_3, \dots, w_N)| \geq$

$\exp(-k_2(p(w)+u(x)+u(t_1)+1))$ for some constant $k_2 > 0$. Proceeding by induction in this way we obtain $t \in \mathbb{R}^N$ such that $|t_j - x_j| \leq \frac{5}{2}R_k(x)$ and

$$|F(t)| \geq \exp(-k_N(p(w) + u(x) + u(t) + 1)).$$

To finish we observe that there is $C > 0$ such that $|x - t| \leq CR_k(x)$ and $|\omega - t| \leq CR_k(x)$, from where it easily follows that $|F(t)| \geq \exp(-Au(t))$ for some constant $A > 0$.

(b) \Rightarrow (c). We modify the proof given in [23]. Let k, x_0 satisfy condition (b). We can assume that $|F(z)| \leq C\exp(kp(z))$ for some constant $C > 0$ and every $z \in \mathbb{C}^N$. We choose $k_1 \geq k$ with $\sqrt{2}R_k \leq R_{k_1} + C_1$ for some $C_1 > 0$. Let $n \in \mathbb{N}$ and fix $z \in \mathbb{C}^N$, $z = x + iy$, $x, y \in \mathbb{R}^N$. We may assume that $|x| \geq x_0$. Otherwise we choose w with respect to $z_0 := x_0 + iy$ instead of z . Condition (b) implies that there is $t \in \mathbb{R}^N$ such that $|t - x| \leq R_k(x)$ and $|F(t)| \geq \exp(-ku(t))$. We now consider two cases:

CASE I: $|y| \leq R_k(x)$. We put $w := t$. Then $|w - z| \leq \sqrt{2}R_k(x) \leq R_{k_1}(x) + C_1$.

CASE II: $|y| \geq R_k(x)$. We consider the entire function $f(\xi) := F(\xi, t_2, \dots, t_N)$, $\xi \in \mathbb{C}$. By Levin [15, Theorem 11] with $\eta = \frac{1 - (1 + \frac{1}{\sqrt{2Nn}})^{-1}}{16}$ there exists $\rho_1 > 0$ with

$$(1) \quad |t_1 - z_1| < \rho_1 < (1 + \frac{1}{\sqrt{2Nn}}) |t_1 - z_1| =: R$$

such that

$$(2) \quad |f(\xi)| \geq |f(t_1)|^{H+1} \left(\sup_{|\xi - t_1| = 2eR} |f(\xi)|^{-H} \right), \quad |\xi - t_1| = \rho_1,$$

with $H = 2 + \log(\frac{3e}{2\eta})$. We choose $w_1 \in \mathbb{C}$ with $|\xi - t_1| = \rho_1$ and

$$(3) \quad |w_1 - z_1| < \frac{|t_1 - z_1|}{\sqrt{2Nn}} \leq \frac{|y|}{\sqrt{2Nn}}.$$

For $\xi \in \mathbb{C}$, $|\xi - t_1| \leq 2eR$ we obtain from (1),(3), with suitable constants $C_i > 0$ depending on p and k , that

$$\begin{aligned} p(\xi) &\leq C_2u(t_1) + C_2v(|t_1 - z_1|) + C_2 \\ &\leq C_2u(|x_1| + |t_1 - x_1|) + C_2v(\sqrt{2}|y|) + C_2 \\ &\leq C_3u(x) + C_3u(|t - x|) + C_3v(|y|) + C_3 \\ &= C_4p(z) + C_4. \end{aligned}$$

Hence (since $|w_1 - t_1| = \rho_1$) we get by (b) and (2) that

$$|F(w_1, t_2, \dots, t_N)| \geq \exp(-k'(p(t) + p(z)) - k')$$

for some k' which depends on p and k .

Now we again apply Levin [15, Theorem 11] in the same way to the function $f(\xi) := F(w_1, \xi, t_3, \dots, t_N)$, $\xi \in \mathbb{C}$, and obtain some $\rho_2 > 0$ with

$$|t_1 - z_1| < \rho_1 < (1 + \frac{1}{\sqrt{2Nn}}) |t_1 - z_1| =: R$$

and

$$|f(\xi)| \geq |f(t_2)|^{H+1} \left(\sup_{|\xi - t_2| = 2eR} |f(\xi)|^{-H} \right), \quad |\xi - t_2| = \rho_2.$$

We choose $w_2 \in \mathbb{C}$ with $|w_2 - t_2| = \rho_2$ and

$$|w_2 - z_2| < \frac{|t_2 - z_2|}{\sqrt{2Nn}} \leq \frac{|y|}{\sqrt{2Nn}}.$$

As above we get

$$|F(w_1, w_2, t_3, \dots, t_N)| \geq \exp(-k''(p(t) + p(z)) - k'')$$

for some k'' which depends on p and k .

Hence, by induction, we find $w_j \in \mathbb{C}$ with $|w_j - z_j| \leq \frac{|y|}{\sqrt{2Nn}}$ and $|F(w)| \geq \exp(-k^{(N)}(p(t) + p(z)) - k^{(N)})$, where $k^{(N)}$ depends on p, k and N . In particular $|w - z| \leq \frac{|y|}{n}$. We finally observe that $p(t) + p(z) \leq C_5 p(z) + C_5$ for some $C_5 > 0$. Moreover $|y| \leq |\operatorname{Im} w| + \frac{|y|}{n}$, hence $|y| \leq (1 - \frac{1}{n})^{-1} |\operatorname{Im} w|$ from where it follows that $p(z) = u(x) + v(y) \leq C_6 v(\operatorname{Im} w) + C_6$ for some $C_6 > 0$. Then the desired bounds for $|w - z|$ and $|F(w)|$ are proved. The additive constants which occur can be eliminated applying our result with $n + 1$ instead of n . Then the desired results holds for large $|z| \geq R$.

(c) \Rightarrow (d). Let $k \in \mathbb{N}$ and $g \in A_p$ with $f := \frac{g}{F} \in A(\mathbb{C}^N)$ and assume that $|g|_k := \sup_{z \in \mathbb{C}^N} |g(z)| \exp(-kp(z)) < \infty$. We take $L \in \mathbb{N}$ such that $|F|_L < \infty$. Given $z \in \mathbb{C}^N$ we apply [8, 3.2] with $r := R_k(\operatorname{Re} z) + |\operatorname{Im} z|$ to get

$$|f(z)| \leq \frac{(\sup_{|w-z| < 4r} |g(w)|)(\sup_{|w-z| < 4r} |F(w)|)}{(\sup_{|w-z| < r} |F(w)|)^2}.$$

For every $|w - z| \leq 4r$ we have $p(w) \leq u(|\operatorname{Re} z| + |w - z|) + v(|\operatorname{Im} z| + |w - z|)$. Since $u(t) = o(v(t))$, $p(2z) = O(p(z))$ and $v(R_k(\operatorname{Re} z)) = ku(\operatorname{Re} z)$, we get $p(w) \leq A(p(z) + 1)$ for some constant $A > 0$.

On the other hand, condition (c) with $n = 1$ implies that there is $w_0 \in \mathbb{C}^N$ such that $|w_0 - z| \leq r$ and $|F(w_0)| \geq \exp(-mp(w_0))$. Consequently

$$|f(z)| \leq |g|_k |F|_L \exp(np(z))$$

for some constant $n \in \mathbb{N}$. In particular $f \in A_p$.

(d) \Rightarrow (a). Since for every $C > 0$ and $k \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ with $CR_k(x) \leq R_n(x)$ for $n \geq n_0$, we have to show the following condition (*): There are $k \in \mathbb{N}$, $C > 0$ and $x_0 > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq x_0$ there is $w \in \mathbb{C}^N$ with $|w - x| \leq CR_k(x)$ and $|F(w)| \geq \exp(-Ckp(w))$.

Assuming that condition (*) is not satisfied we can find a sequence (x_j) in \mathbb{R}^N with $\lim_{j \rightarrow \infty} |x_j| = \infty$ and such that for each $j \in \mathbb{N}$ and each $\xi \in \mathbb{C}^N$ with $|\xi - x_j| \leq CR_j(x_j)$ we have $|F(\xi)| \leq \exp(-Cjp(\xi))$, where C is a positive constant to be determined later. Since $u(x) = o(v(x))$ as x goes to infinity we can assume $u(x_j) \leq \frac{1}{j}v(\frac{|x_j|}{2})$, hence $R_j(x_j) = v^{-1}(ju(x_j)) \leq \frac{|x_j|}{2}$.

Let (f_j) be a sequence of entire functions on \mathbb{C}^N satisfying the properties (a), (b) and (c) of lemma 1 with $R_j := R_j(x_j)$. It follows from property (b) that $f_j \in A_p$. In order to get a contradiction we show that (Ff_j) is bounded while (f_j) is unbounded in A_p .

To do this we first observe that, since $|x_j| \leq |\operatorname{Re} \xi| + R_j + 2 \leq |\operatorname{Re} \xi| + 2 + \frac{|x_j|}{2}$, we have $\frac{|x_j|}{2} \leq |\operatorname{Re} \xi| + 2$. Hence there are $j_0 \in \mathbb{N}$, $k > 0$ and $D_N > 0$ such that

$$|f_j(\xi)| \leq D_N \exp(kju(\operatorname{Re} \xi) + C_N \log(1 + |\xi|^2))$$

for every $j \geq j_0$ and $\xi \in \mathbb{C}^N$, $|\xi - x_j| < R_j + 2$. Then, $|\xi - x_j| < R_j + 2 (\leq CR_j(x_j))$ implies

$$|F(\xi)f_j(\xi)| \leq D_N \exp(C_N \log(1 + |\xi|^2))$$

if we take C large enough at the beginning of the proof. Moreover

$$|F(\xi)f_j(\xi)| \leq |F(\xi)| D_1 \exp(D_1 p(\xi))$$

for $|\xi - x_j| \geq R_j + 2$. Hence there is $L \in \mathbb{N}$ with

$$\sup_{j \in \mathbb{N}} \sup_{\xi \in \mathbb{C}^N} |F(\xi)f_j(\xi)| \exp(-Lp(\xi)) < \infty,$$

and (Ff_j) is bounded. On the other hand, there is $\epsilon > 0$ with $v(\frac{2R_j(x_j)}{\pi\sqrt{N}}) \geq \epsilon v(R_j(x_j)) = \epsilon ju(x_j)$, which implies

$$|f_j(x_j)| \geq \exp(\epsilon ju(x_j) - C_N \log(1 + |x_j|^2))$$

and (f_j) is not bounded in A_p . □

In [23] $(c) \Rightarrow (d)$ was proved in the case $N = 1$ via another equivalent condition, namely of F being slowly decreasing in the sense of Berenstein and Taylor [3]. We apply Hörmander’s lemma to prove $(c) \Rightarrow (d)$.

The next characterization of the closed principal ideals in a Hörmander algebra follows from proposition 2 using standard arguments from functional analysis as in [23].

Theorem 3. *For each $F \in A_p \setminus (0)$ the following assertions are equivalent:*

- (a) *The principal ideal FA_p is closed.*
- (b) *For every $g \in A_p$ such that $f := \frac{g}{F}$ is an entire function we have $f \in A_p$.*
- (c) *There are $k \in \mathbb{N}$ and $x_0 > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq x_0$ there is $t \in \mathbb{R}^N$ with $|t - x| \leq R_k(x)$ and $|F(t)| \geq \exp(-kp(t))$.*
- (d) *There are $k \in \mathbb{N}$ and $x_0 > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq x_0$ there is $\xi \in \mathbb{C}^N$ with $|\xi - x| \leq R_k(x)$ and $|F(\xi)| \geq \exp(-kp(\xi))$.*

Condition (c) in theorem 3 was introduced by Ehrenpreis [7] in the case $u = \log$ and $v(t) = t$ to characterize the surjective convolution operators on the space of infinitely differentiable functions.

A continuous increasing function $\omega : [0, \infty[\rightarrow [0, \infty[$ is called a *weight* if it satisfies: $\omega(2t) = O(\omega(t))$, $\log(1 + t^2) = O(\omega(t))$, $\omega(t) = o(t)$ as t tends to ∞ and $\varphi : t \rightarrow \omega(e^t)$ is a convex function on \mathbb{R} .

The *Young conjugate* $\varphi^* : [0, \infty[\rightarrow \mathbb{R}$ of φ is defined by

$$\varphi^*(s) := \sup\{st - \varphi(t) : t \geq 0\}.$$

The weight ω is called *non-quasianalytic* if $\int_1^\infty t^{-2}\omega(t)dt$ is finite. See [6].

The space of *ultradifferentiable functions of Beurling type* $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ associated to a weight ω is the Fréchet-Schwartz space of all functions $f \in C^\infty(\mathbb{R}^N)$ such that for each $K \subset \mathbb{R}^N$ compact and each $m \in \mathbb{N}$ we have

$$P_{K,m}(f) := \sup_{x \in K} \sup_{\alpha \in \mathbb{N}^N} |f^{(\alpha)}(x)| \exp(-m\varphi^*(\frac{|\alpha|}{m})) < \infty.$$

The space of *ultradifferentiable functions of Roumieu type* $\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ associated to a weight ω is defined as the space of all functions $f \in C^\infty(\mathbb{R}^N)$ such that for each $K \subset \mathbb{R}^N$ compact there is $m \in \mathbb{N}$ such that

$$\sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m} \varphi^*(|m\alpha|)\right) < \infty.$$

This space is a projective limit of (DFN)-spaces which is ultrabornological. We refer to [6, 22]. Also see [32].

For $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ the convolution operator $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \rightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ is defined by $T_\mu(\varphi)(x) := (\mu * \varphi)(x) = \langle \mu_y, \varphi(x - y) \rangle$, which is a continuous linear operator with transposed map $T_\mu^t : \mathcal{E}'_{(\omega)}(\mathbb{R}^N) \rightarrow \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$, $T_\mu^t(\nu) := \tilde{\mu} * \nu$, where $\tilde{\mu}(\varphi) := \mu(\check{\varphi})$ and $\check{\varphi}(x) := \varphi(-x)$.

If ω is a weight we put $u(t) := \omega(|t|)$, $v(t) := |t|$ and we consider the associated non-radial weight function $p(z) := \omega(|\operatorname{Re} z|) + |\operatorname{Im} z|$, $z \in \mathbb{C}^N$. In this case the corresponding function R_k is given by $R_k(x) = k\omega(x)$.

It is well known (see [6, 21, 22, 27]) that the Fourier-Laplace transform $F : \mu \rightarrow \hat{\mu}$, given by $\hat{\mu}(z) := \langle \mu_x, e^{-i\langle x, z \rangle} \rangle$, is an algebra isomorphism between the convolution algebra $(\mathcal{E}'_{(\omega)}(\mathbb{R}^N), *)$ and the Hörmander algebra A_p which is even a topological isomorphism when we endow $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ with the strong topology. We will identify, via Fourier-Laplace transform, the transposed map T_μ^t with the multiplication operator $FT_\mu^t F^{-1} : A_p \rightarrow A_p$, $f \rightarrow \hat{\mu}f$. Sometimes we will identify $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ with A_p . A standard argument gives that $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \rightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ is surjective if, and only if, the principal ideal $\hat{\mu}A_p$ is closed in A_p .

We do not assume in the rest of the article that the weight ω is non-quasianalytic.

Examples 4. (a) $\omega(r) = r^{1/t}$, $t > 1$, is a non-quasianalytic weight. In this case $\mathcal{E}_{\{\omega\}}(\mathbb{R}^N) = \Gamma^{\{t\}}(\mathbb{R}^N)$ is the Gevrey space of order t .

(b) $\omega(r) = \frac{r}{\log(r+e)^q}$, $0 < q \leq 1$, is a quasianalytic weight.

Definition 5. An ultradistribution $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ is called *slowly decreasing* for (ω) if there are $k \in \mathbb{N}$ and $x_0 > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq x_0$ there is $\xi \in \mathbb{C}^N$ with $|\xi - x| \leq k\omega(x)$ and $|\hat{\mu}(\xi)| \geq \exp(-k\omega(\xi) - k|\operatorname{Im} \xi|)$.

It follows from proposition 2 that the ultradistribution $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ is slowly decreasing for (ω) if and only if there exists $k \in \mathbb{N}$ such that for each $j \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $C > 0$, $R > 0$ such that for each $z \in \mathbb{C}^N$, $|z| \geq R$, there exists $w \in \mathbb{C}^N$ satisfying

$$|w - z| \leq k\omega(z) + \frac{1}{j} |\operatorname{Im} z| \quad \text{and} \quad |\hat{\mu}(w)| \geq C \exp(-m(|\operatorname{Im} w| + \omega(w))).$$

The above reformulation of the slowly decreasing condition has been used in [5]. In the case $N = 1$, theorem 3 implies that our definition of (ω) -slowly decreasing coincides with the one used by Meise, Taylor and Vogt in [18].

As an immediate consequence of theorem 3 we get

Corollary 6. *Let ω be a weight and let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$. Then the following conditions are equivalent:*

- (a) $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \rightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ is surjective,
- (b) μ is slowly decreasing for (ω) .

This result was already proved in [18] in the non-quasianalytic case and for $N = 1$. A proof for arbitrary N and non-quasianalytic weights can be found in [4]. The proof in [4], [18] depended on an application of the Phragmén-Lindelöf theorem which was possible because the non-quasianalyticity of the weight ω permits us to extend it to a harmonic function in the upper halfplane. The proof of the corollary for arbitrary weights depends on the equivalence between conditions (a) and (c) of proposition 2.

The next corollary extends [4, Theorem 2] and part of [5, 2.8] to quasianalytic weights.

Corollary 7. *Let $\omega \leq \sigma$ be two weights. The following conditions are equivalent for $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$:*

- (a) $\mathcal{E}_{(\sigma)}(\mathbb{R}^N)$ is contained in $T_\mu(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$.
- (b) If $B \subset \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ satisfies that $T_\mu^t(B)$ is bounded in $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ then B is bounded in $\mathcal{E}'_{(\sigma)}(\mathbb{R}^N)$.
- (c) $T_\mu : \mathcal{E}_{(\sigma)}(\mathbb{R}^N) \rightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ is surjective.

Proof. To prove that (a) implies (b) we may proceed as in [4, Theorem 2], since the proof of (1) \Rightarrow (3) in [4, Proposition 1] remains valid for quasianalytic weights. To prove that (b) implies (c) we first proceed as in [4, Theorem 2] to deduce that μ is slowly decreasing for (σ) and then we apply corollary 6. □

If $G \in \mathcal{H}(\mathbb{C}^N)$ satisfies $\log |G| = \mathcal{O}(\omega)$ then $\mu_G(\varphi) = \sum_{\alpha} (-i)^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} \varphi^{(\alpha)}(0)$ defines an element μ_G of $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ with support $\{0\}$. The operator

$$G(D) : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \rightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N), \quad G(D)f = \mu_G * f,$$

is then called an *ultradifferential operator of class (ω)* . Note that $G(D)f(x) = \sum_{\alpha} i^{|\alpha|} \frac{G^{(\alpha)}(0)}{\alpha!} f^{(\alpha)}(x)$.

For the convenience of the reader we state the following result which will be used in the sequel. It permits us to construct entire functions with certain growth properties and vanishing on a given distribution of points.

Theorem 8 (Rubel and Taylor [28] [29]). *Let $\lambda(r)$, $r \geq 0$, be a positive, increasing and unbounded function such that $\lambda(2r) \leq M\lambda(r)$ for some $M > 0$. Given two sequences $(a_j) \subset \mathbb{C} \setminus \{0\}$ and $(m_j) \subset \mathbb{N}$ we put $n(t) := \sum_{|a_j| \leq t} m_j$ and $N(r) := \int_0^r \frac{n(t)}{t} dt$. If there are positive constants A, B such that $N(r) \leq A\lambda(Br)$, $r > 0$, then there exist an entire function $f \in H(\mathbb{C})$, $f \neq 0$, such that the inequality $|f(z)| \leq \exp(A'\lambda(B'|z|))$ holds for some constants A', B' , and $f(a_j) = 0$ with multiplicity at least m_j ($j \in \mathbb{N}$).*

Our next result exploits ideas and methods of [26].

Proposition 9. *Let $\omega \leq \sigma$ be two weights. Then the following conditions are equivalent:*

- (a) $\mathcal{E}_{(\sigma)}(\mathbb{R}^N)$ is contained in $G(D)(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$ for every ultradifferential operator $G(D)$, of class (ω) .
- (b) There are $C > 0$ and $R_0 > 0$ such that for each $R \geq R_0$ we have

$$\inf_{S > R} \frac{\omega(S)}{\log(\frac{S}{R})} \leq C \frac{\sigma(R)}{\log(\frac{R}{\sigma(R)})}.$$

Proof. (a) \Rightarrow (b). Proceeding by contradiction we assume

$$\forall C > 0 \quad \forall R_0 > 0 \quad \exists R \geq R_0 : \inf_{S > R} \frac{\omega(S)}{\log(\frac{S}{R})} > C \frac{\sigma(R)}{\log(\frac{R}{\sigma(R)})}.$$

Inductively we select an increasing and unbounded sequence (a_j) of positive real numbers and a sequence $(m_j) \subset \mathbb{N}$ such that

(i) $\inf_{S > a_j} \frac{\omega(S)}{\log(\frac{S}{a_j})} > m_j > j \frac{\sigma(a_j)}{\log(\frac{a_j}{\sigma(a_j)})},$

(ii) $n(a_{j-1}) \leq m_j,$

(iii) $n(a_{j-1}) \log(\frac{a_j}{a_1}) \leq \omega(a_j),$ where $n(t) := \sum_{a_j \leq t} m_j.$ To apply the Theorem of

Rubel and Taylor mentioned above, for $\lambda = \omega$ we must estimate $N(r) := \int_0^r \frac{n(t)}{t} dt.$ For $a_j < r < a_{j+1}$ we have

$$\begin{aligned} N(r) &= \int_{a_1}^{a_j} \frac{n(t)}{t} dt + \int_{a_j}^r \frac{n(t)}{t} dt \\ &\leq n(a_{j-1}) \int_{a_1}^{a_j} \frac{dt}{t} + (n(a_{j-1}) + m_j) \int_{a_j}^r \frac{dt}{t} \\ &\leq \omega(a_j) + 2m_j \log\left(\frac{r}{a_j}\right). \end{aligned}$$

Since $\frac{\omega(r)}{\log(\frac{r}{a_j})} > m_j$ we conclude $N(r) \leq \omega(a_j) + 2\omega(r) \leq 3\omega(r).$ Consequently there is an entire function $F \in H(\mathbb{C}), F \neq 0,$ such that $F(a_j) = 0$ with multiplicity at least $m_j, j \in \mathbb{N},$ and $|F(z)| \leq \exp(A\omega(|z|))$ for some $A > 0.$

By corollary 7, it is enough to show that F is not slowly decreasing for (σ) to conclude. Let $k > 0$ be arbitrary, $w = a_j + re^{i\theta}, 0 \leq r \leq k\sigma(a_j).$ Then we have, by the Jensen-Poisson formula, for $z = a_j, R = |z| = a_j,$

$$\begin{aligned} \log |F(w)| &= \frac{1}{2\pi} \int_0^{2\pi} P_{\frac{r}{R}}(\theta - t) \log |F(z + Re^{it})| dt \\ &\quad - \sum_{|z-a| \leq R, F(a)=0} \log \left| \frac{R^2 - \overline{(a-z)}re^{i\theta}}{R(re^{i\theta} - (a-z))} \right|. \end{aligned}$$

Since each term in the sum is positive

$$\begin{aligned} \log |F(w)| &\leq \frac{1}{2\pi} \int_0^{2\pi} P_{\frac{r}{R}}(\theta - t) \log |F(z + Re^{it})| dt - m_j \log\left(\frac{R}{r}\right) \\ &\leq \sup_{\xi \in \mathbb{C}; |\xi| \leq 2|z|} \log |F(\xi)| - m_j \log\left(\frac{R}{r}\right) \\ &\leq A\omega(2|z|) - m_j \log\left(\frac{a_j}{k\sigma(a_j)}\right). \end{aligned}$$

This means that $\log |F(w)| \leq A\omega(2|z|) - m_j \log\left(\frac{a_j}{k\sigma(a_j)}\right)$ whenever we have $|w - a_j| \leq k\sigma(a_j).$ Now it is enough to note that

$$A\omega(2|z|) - m_j \log\left(\frac{a_j}{k\sigma(a_j)}\right) \leq -k\sigma(a_j)$$

for j large enough in order to conclude that $G(z) := F(z_1), z \in \mathbb{C}^N,$ defines an ultradifferential operator $G(D)$ of class (ω) such that $\mathcal{E}_{(\sigma)}(\mathbb{R}^N)$ is not contained in $G(D)(\mathcal{E}_{(\omega)}(\mathbb{R}^N)).$

(b) \Rightarrow (a). We first assume $G(0) \neq 0$. Without loss of generality we can take $G(0) = 1$. We fix $z \in \mathbb{C}^N$ with $|z| \geq R_0$ (as in (b)) and set $R := |z|$. There is $S > R$ such that

$$\frac{\omega(S)}{\log(\frac{S}{R})} \leq C \frac{\sigma(R)}{\log(\frac{R}{\sigma(R)})}.$$

We take $\eta := \frac{3}{2}e \frac{\sigma(R)}{R} < \frac{3}{2}e$ and $g(\xi) := G(\frac{\xi z}{|z|})$, $\xi \in \mathbb{C}$. By the minimum modulus theorem of Levin [15, Theorem 11], in the more general form given by [26, 4.2], there are discs $(K_j)_{j=1}^n$ the sum of their radii is less than or equal to $4\eta R$ and such that for every $\xi \notin \mathbb{C}$, $|\xi| < R$, $\xi \in \bigcup K_j$ one has

$$\log |g(\xi)| \geq -2 \log M(2R) - \log\left(\frac{3e}{2\eta}\right) \frac{\log M(2S)}{\log(\frac{S}{R})},$$

where $M(t) = \sup_{|z|=t} |g(z)|$. Since $8\eta R = 12e\sigma(R) < A\sigma(R) < R$ for some $A > 0$ and R large enough, we conclude the existence of $\xi \in \mathbb{C}$ with $|\xi - R| < A\sigma(R)$, $|\xi| \leq R$, and

$$|g(\xi)| \geq \exp(-2 \log M(2R) - \log\left(\frac{3e}{2\eta}\right) \frac{\log M(2S)}{\log(\frac{S}{R})}).$$

Then for $w = \frac{\xi z}{|z|}$ we have $|w - z| = |\xi - R| < A\sigma(R)$ and, for some $k > 0$,

$$\begin{aligned} \log |G(w)| &\geq -2 \log M(2R) - \log\left(\frac{3e}{2\eta}\right) \frac{\log M(2S)}{\log(\frac{S}{R})} \\ &\geq -2k\omega(2R) - \log\left(\frac{R}{\sigma(R)}\right)k \frac{\omega(2S)}{\log(\frac{S}{R})} \\ &\geq -2k\omega(2R) - kC\sigma(R) \geq -B\sigma(R) \end{aligned}$$

for some constant $B > 0$. Corollary 7 permits to conclude that $\mathcal{E}_{(\sigma)}(\mathbb{R}^N)$ is contained in $G(D)(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$. In case $G(0) = 0$ we choose $w \in \mathbb{C}^N$ with $G(w) \neq 0$ and proceed as above with the function $G_1(z) := G(z + w)$. □

Examples 10. (a) Let $\omega \leq \sigma$ be two non-quasianalytic weights. If $\mu \in D_{(\sigma)}(\mathbb{R}^N) \subset \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$, then $\mathcal{E}_{(\sigma)}(\mathbb{R}^N)$ is not contained in $T_\mu(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$. In fact, it follows from the Paley-Wiener theorem ([6]) that μ is not slowly decreasing for (σ) . We then apply corollaries 6 and 7.

(b) Let $1 < s < t$ and $\omega(r) = r^{1/t}$, $\sigma(r) = r^{1/s}$. Then $\mathcal{E}_{(\sigma)}(\mathbb{R}^N)$ is contained in $G(D)(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$, for every ultradifferential operator $G(D)$ of class (ω) . In fact, we take $S = 2R$ and observe that

$$\frac{\omega(S)}{\log(\frac{S}{R})} \frac{\log(\frac{R}{\sigma(R)})}{\sigma(R)} \rightarrow 0 \quad \text{for } R \rightarrow \infty.$$

The conclusion follows from proposition 9.

(c) Ehrenpreis [7] proved that $\mathcal{E}_{\{\tau\}}(\mathbb{R}^N)$ is contained in $T_\mu(\mathcal{C}^\infty(\mathbb{R}^N))$ for every $\mu \in \mathcal{E}'(\mathbb{R}^N)$ if $\tau(t) = |t|$.

Lemma 11. Let $\omega \leq \sigma$ be two weights and let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ be an ultradistribution such that the range of $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \rightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ contains the Roumieu class $H :=$

$\mathcal{E}'_{\{\sigma\}}(\mathbb{R}^N)$. If $B \subset \mathcal{E}'_{\{\omega\}}(\mathbb{R}^N)$ satisfies that $T_\mu^t(B)$ is equicontinuous in $\mathcal{E}'_{\{\omega\}}(\mathbb{R}^N)$, then $\{\varphi|_H: \varphi \in B\}$ is equicontinuous in $\mathcal{E}'_{\{\sigma\}}(\mathbb{R}^N)$.

Proof. Let τ denote the quotient topology induced by T_μ on its range. Then $G := (T_\mu(\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)), \tau)$ is a Fréchet space and the inclusion $i : \mathcal{E}_{\{\sigma\}}(\mathbb{R}^N) \rightarrow G$ is continuous, since it is a linear map with closed graph. We now fix a 0-neighbourhood V in $\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ such that $|\langle T_\mu^t \varphi, f \rangle| \leq 1$ for every $\varphi \in B$ and $f \in V$. Then $W := T_\mu(V) \cap \mathcal{E}_{\{\sigma\}}(\mathbb{R}^N)$ is a neighbourhood in $\mathcal{E}_{\{\sigma\}}(\mathbb{R}^N)$ and $|\langle \varphi, g \rangle| \leq 1$ for every $g \in W$ and $\varphi \in B$. \square

Definition 12 ([5]). An ultradistribution $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^N)$ is called *slowly decreasing* for $\{\omega\}$ if for each $m \in \mathbb{N}$ there exists $R > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq R$ there exists $\xi \in \mathbb{C}^N$ satisfying $|x - \xi| \leq \frac{1}{m}\omega(x)$ such that $|\hat{\mu}(\xi)| \geq \exp(-\frac{1}{m}\omega(\xi))$.

Proceeding as in [5, 3.2] it turns out that $\mu \in \mathcal{E}'_{\{\omega\}}(\mathbb{R}^N)$ is slowly decreasing for $\{\omega\}$ if, and only if, there exists a weight function σ with $\sigma = o(\omega)$ such that $\mu \in \mathcal{E}'_{\{\sigma\}}(\mathbb{R}^N)$ and μ is slowly decreasing for (σ) .

By [6, 22, 27], the Fourier-Laplace transform $F : \mu \rightarrow \hat{\mu}$ defines a topological isomorphism between the (LF)-spaces $\mathcal{E}'_{\{\omega\}}(\mathbb{R}^N)$ and $\text{ind}_j \text{proj}_k A(\omega_{jk}, \mathbb{C}^N)$, where

$$A(\omega_{jk}, \mathbb{C}^N) := \{f \in H(\mathbb{C}^N) : \sup_{z \in \mathbb{C}^N} \omega_{jk}(z) |f(z)| < \infty\},$$

and

$$\omega_{jk}(z) := \exp(-j|\text{Im}z| - \frac{1}{k}\omega(z)).$$

The following lemma extends [19, 3.2,3.3] to quasianalytic classes. We indicate the similar proof. We thank A. Albanese and R. Meise for conversations on the lemma.

Lemma 13. *Let σ be a weight. For every equicontinuous set $A \subset \mathcal{E}'_{\{\sigma\}}(\mathbb{R}^N)$, there is a weight $\tau = o(\sigma)$ and there is a seminorm P on $\mathcal{E}_{\{\tau\}}(\mathbb{R}^N)$ such that $|\varphi(f)| \leq P(f)$ for every $\varphi \in B$ and every $f \in \mathcal{E}'_{\{\sigma\}}(\mathbb{R}^N)$.*

Proof. We identify spaces via the Fourier-Laplace transform. Since A is equicontinuous, it is contained in a Banach disc in $\text{ind}_j \text{proj}_k A(\omega_{jk}, \mathbb{C}^N)$. By Grothendieck’s factorization theorem, it is bounded in a step, i.e.

$$\exists j \forall k \exists M_k \forall \nu \in A \forall z \in \mathbb{C}^N :$$

$$|\hat{\nu}(z)| \leq M_k \exp(j|\text{Im}z| + \frac{1}{k}\sigma(z)).$$

We get, for each k and $r > 0$

$$g(r) := \sup_{|z|=r} \log^+ (|\hat{\nu}(z)| \exp(-j|\text{Im}z|)) \leq \frac{1}{k}\sigma(r) + \log M_k,$$

i.e., $g = o(\sigma)$. By [6, 1.7 and 1.8], there is a weight $\tau = o(\sigma)$ such that $g = o(\sigma)$. This yields

$$\forall \delta > 0 \exists C_\delta > 0 \forall \nu \in B \forall z \in \mathbb{C}^N :$$

$$|\hat{\nu}(z)| \leq C_\delta \exp(j|\text{Im}z| + \delta\tau(z)).$$

This implies that A is bounded, hence equicontinuous in $\mathcal{E}'_{\{\tau\}}(\mathbb{R}^N)$ and the proof is complete. \square

Proposition 14. *Let $\omega \leq \sigma$ be two weights, $\omega = o(\sigma)$ and let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$. Then, the following conditions are equivalent:*

- (a) $\mathcal{E}_{\{\sigma\}}(\mathbb{R}^N)$ is contained in $T_\mu(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$.
- (b) There is a weight τ satisfying $\omega = o(\tau)$ and $\tau = o(\sigma)$ such that $\mathcal{E}_{(\tau)}(\mathbb{R}^N)$ is contained in $T_\mu(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$.
- (c) μ is slowly decreasing for $\{\sigma\}$.

Proof. (a) \Rightarrow (b). Let (C_n) be a fundamental sequence of bounded sets in $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$. We put $G := \mathcal{E}_{\{\sigma\}}(\mathbb{R}^N)$. By lemma 11, for each $n \in \mathbb{N}$, the set $B_n := \{\varphi \mid_G : T_\mu^t \varphi \in C_n\}$ is equicontinuous in $\mathcal{E}'_{\{\sigma\}}(\mathbb{R}^N)$. We can apply lemma 13 to find a weight $\sigma_n = o(\sigma)$ and a seminorm P_n on $\mathcal{E}_{(\sigma_n)}(\mathbb{R}^N)$ such that $|\varphi(f)| \leq P_n(f)$ for every $\varphi \in B_n$ and $f \in G$. By [6, 1.9] we find a weight τ with $\sigma_n = o(\tau)$ for every $n \in \mathbb{N}$, $\omega = o(\tau)$ and $\tau = o(\sigma)$. Clearly $\mathcal{E}_{\{\sigma\}}(\mathbb{R}^N) \subset \mathcal{E}_{(\tau)}(\mathbb{R}^N) \subset \mathcal{E}_{(\omega)}(\mathbb{R}^N)$. Fix $B \subset \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ such that $T_\mu^t(B)$ is a bounded set in $\mathcal{E}'_{(\omega)}(\mathbb{R}^N)$. There is $m \in \mathbb{N}$ with $T_\mu^t(B) \subset C_m$. Accordingly B is bounded in $\mathcal{E}'_{(\sigma_m)}(\mathbb{R}^N)$, hence in $\mathcal{E}'_{(\tau)}(\mathbb{R}^N)$. We apply corollary 7 to conclude that $\mathcal{E}_{(\tau)}(\mathbb{R}^N)$ is contained in $T_\mu(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$.

(b) \Rightarrow (c). Since $\tau = o(\sigma)$, we can apply corollaries 6 and 7 to conclude that μ is slowly decreasing for (τ) .

(c) \Rightarrow (a) follows from the inclusion $\mathcal{E}_{\{\sigma\}}(\mathbb{R}^N) \subset \mathcal{E}_{(\tau)}(\mathbb{R}^N)$. □

Proposition 15. *Let $\omega \leq \sigma$ be two weights such that $\omega = o(\sigma)$. The following conditions are equivalent:*

- (a) $\mathcal{E}_{\{\sigma\}}(\mathbb{R}^N)$ is contained in $G(D)(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$ for every ultradifferential operator $G(D)$ of class (ω) .
- (b) For every $\epsilon > 0$ there is $R_0 > 0$ such that for each $R \geq R_0$ we have

$$\inf_{S > R} \frac{\omega(S)}{\log(\frac{S}{R})} \leq \epsilon \frac{\sigma(R)}{\log(\frac{R}{\epsilon\sigma(R)})}.$$

- (c) There is a weight function τ , $\omega \leq \tau$, $\tau = o(\sigma)$, such that $\mathcal{E}_{(\tau)}(\mathbb{R}^N)$ is contained in $G(D)(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$ for every ultradifferential operator $G(D)$ of class (ω) .

Proof. (a) \Rightarrow (b). Proceeding by contradiction we assume

$$\exists 0 < \epsilon < \frac{1}{3} \quad \forall R_0 > 0 \quad \exists R \geq R_0 : \inf_{S > R} \frac{\omega(S)}{\log(\frac{S}{R})} > \epsilon \frac{\sigma(R)}{\log(\frac{R}{\epsilon\sigma(R)})}.$$

Inductively we select an increasing and unbounded sequence (a_j) of positive real numbers and a sequence $(m_j) \subset \mathbb{N}$ such that

- (i) $\inf_{S > a_j} \frac{\omega(S)}{\log(\frac{S}{a_j})} > m_j > 3\epsilon \frac{\sigma(a_j)}{\log(\frac{a_j}{\epsilon\sigma(a_j)})}$,
- (ii) $n(a_{j-1}) \leq m_j$,
- (iii) $n(a_{j-1}) \log(\frac{a_j}{a_1}) \leq \omega(a_j)$, where $n(t) := \sum_{a_j \leq t} m_j$. To apply the Theorem of

Rubel and Taylor for $\lambda = \omega$ we must estimate $N(r) := \int_0^r \frac{n(t)}{t} dt$. For $a_j < r < a_{j+1}$ we have (as in the proof of proposition 9)

$$N(r) \leq \omega(a_j) + 2m_j \log(\frac{r}{a_j}).$$

Since $\frac{\omega(r)}{\log(\frac{r}{a_j})} > \epsilon m_j$ we conclude $N(r) \leq (1 + \frac{2}{\epsilon})\omega(r)$. Consequently there is an entire function $F \in H(\mathbb{C})$, $F \neq 0$, such that $F(a_j) = 0$ with multiplicity at least m_j , $j \in \mathbb{N}$, and $|F(z)| \leq \exp(A\omega(|z|))$ for some $A > 0$. To conclude, it is enough

to show that F is not slowly decreasing for $\{\sigma\}$. Let $w = a_j + re^{i\theta}$, $0 \leq r \leq \epsilon\sigma(a_j)$. Then we have, by the Jensen-Poisson formula, for $z = a_j$, $R = |z| = a_j$,

$$\log |F(w)| \leq A\omega(2|z|) - m_j \log\left(\frac{a_j}{\epsilon\sigma(a_j)}\right).$$

This means that $\log |F(w)| \leq A\omega(2a_j) - m_j \log\left(\frac{a_j}{\epsilon\sigma(a_j)}\right)$ whenever $|w - a_j| \leq \epsilon\sigma(a_j)$. Now it is enough to note that

$$A\omega(2a_j) - m_j \log\left(\frac{a_j}{\epsilon\sigma(a_j)}\right) \leq -\epsilon\sigma(a_j)$$

for j large enough to conclude that $G(z) := F(z_1)$, $z \in \mathbb{C}^N$, defines an ultradifferential operator $G(D)$ of class (ω) such that $\mathcal{E}_{\{\sigma\}}(\mathbb{R}^N)$ is not contained in $G(D)(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$.

(b) \Rightarrow (c). We find an increasing sequence (R_m) of positive real numbers such that for every $r \geq R_m$:

$$\inf_{S>R} \frac{\omega(S)}{\log\left(\frac{S}{R}\right)} \leq \frac{1}{m} \frac{\sigma(R)}{\log\left(\frac{mR}{\sigma(R)}\right)}.$$

We define $g(t) = 0$ for $t \in [0, R_1[$ and $g(t) = \frac{\sigma(x)}{m}$ for $t \in [R_m, R_{m+1}[$ and apply [6, 1.7;1.8] to find a weight function τ such that $\omega \leq \tau$, $g = o(\tau)$, $\tau = o(\sigma)$. We apply proposition 9 to get that $\mathcal{E}_{(\tau)}(\mathbb{R}^N)$ is contained in $G(D)(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$ for every ultradifferential operator $G(D)$ of class (ω) . □

Example 16. Let $\omega(r) = o\left(\frac{r^{1/t}}{\log r}\right)$, $t > 1$, and $\sigma(r) = r^{1/t}$. Then $\mathcal{E}_{\{\sigma\}}(\mathbb{R}^N)$ is contained in $G(D)(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$, for every ultradifferential operator $G(D)$ of class (ω) .

In fact, we put $\Phi(R) := \inf_{S>R} \frac{\omega(S)}{\log\left(\frac{S}{R}\right)}$. Using that $\Phi(R) \leq \frac{\omega(2R)}{\log 2}$ we deduce that $\Phi(R) = o\left(\frac{\sigma(R)}{\log R}\right)$. Consequently, for every $0 < \epsilon < 1$ there is $R_1 > 0$ such that $R \geq R_1$ implies $\Phi(R) \leq \epsilon \frac{\sigma(R)}{\log R}$. Hence $\Phi(r) \frac{\log\left(\frac{R}{\epsilon\sigma(R)}\right)}{\sigma(R)} \leq \epsilon \frac{\log\left(\frac{R}{\epsilon\sigma(R)}\right)}{\log R}$. Since the term in the right-hand side converges to $(1 - \frac{1}{t})\epsilon$ as $R \rightarrow \infty$ it follows that condition (b) in the proposition 15 is satisfied.

The example above is “sharp” since for a weight ω with $2\omega(r) \leq \omega(Ar)$, $r \geq r_0$, for some $A > 1$ we have $\inf_{S>R} \frac{\omega(S)}{\log\left(\frac{S}{R}\right)} \sim \omega(R)$ by [24, ex.4.6(b)].

Remark. The condition (b) in proposition 15 is equivalent to the following:

For every $\epsilon > 0$ there is $R_0 > 0$ such that for each $R \geq R_0$ we have

$$\inf_{S>R} \frac{\omega(S)}{\log\left(\frac{S}{R}\right)} \leq \epsilon \frac{\sigma(R)}{\log\left(\frac{R}{\sigma(R)}\right)}.$$

The limit case $t = 1$, which occurs when we replace the Roumieu class $\mathcal{E}_{\{\sigma\}}(\mathbb{R}^N)$ by the space of real-analytic functions, is the content of corollary 18.

Ehrenpreis [7, Theorem II, p. 556] proved that, in case $\omega = \log$, the intersection of the ranges of all the convolution operators on $\mathcal{E}_{(\omega)}(\mathbb{R}^N)$ coincides with the space of the real-analytic functions, showing that the phenomenon in the example cannot happen in the larger class of arbitrary convolution operators.

Proposition 17. Let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$. The following conditions are equivalent:

(a) The space $A(\mathbb{R}^N)$ of real analytic functions is contained in $T_\mu(\mathcal{E}_{(\omega)}(\mathbb{R}^N))$.

(b) μ satisfies the following condition (E):

For each $m \in \mathbb{N}$ there exists $R > 0$ such that for each $x \in \mathbb{R}^N$ with $|x| \geq R$ there exists $\xi \in \mathbb{C}^N$ satisfying $|x - \xi| \leq \frac{1}{m}|x|$ such that $|\hat{\mu}(\xi)| \geq \exp(-\frac{|\xi|}{m})$.

(c) There exists a weight function $\sigma, \sigma \geq \omega$, such that $T_\mu : \mathcal{E}_\sigma(\mathbb{R}^N) \rightarrow \mathcal{E}_\sigma(\mathbb{R}^N)$ is surjective.

Proof. (a) \Rightarrow (b). We proceed by contradiction. We assume that there are $k \in \mathbb{N}$ and an unbounded sequence (x_j) in \mathbb{R}^N such that for each $j \in \mathbb{N}$ and each $\xi \in \mathbb{C}^N$ with $|\xi - x_j| \leq \frac{|x_j|}{k}$ we have $|\hat{\mu}(\xi)| \leq \exp(-\frac{|\xi|}{k})$. Let (f_j) the sequence of entire functions given by lemma 1 with $v(t) := t$ and $R_j := \frac{|x_j|}{3k}$.

For $\xi \in \mathbb{C}^N$ with $|\xi - x_j| \geq R_j + 2$ we have

$$|f_j(\xi)| \leq C_N \exp(|\operatorname{Im} \xi| + 2 + C_N \log(1 + |\xi|^2)),$$

which implies f_j is the Fourier-Laplace transform of some distribution with compact support. Let $\nu_j \in \mathcal{E}'_\omega(\mathbb{R}^N)$ with $\hat{\nu}_j = f_j$. We show that $\{T_\mu^t(\nu_j) : j \in \mathbb{N}\}$ is a bounded set in $\mathcal{E}'_\omega(\mathbb{R}^N)$. If $|\xi - x_j| < R_j + 2$, then

$$|f_j(\xi)| \leq C_N \exp(R_j + 4 + C_N \log(1 + |\xi|^2)).$$

and

$$|\widehat{\mu}(\xi)f_j(\xi)| \leq C_N \exp(\frac{-3|\xi| + |x_j|}{3k} + 4 + C_N \log(1 + |\xi|^2)).$$

Since $|x_j| < |\xi| + R_j + 2 = |\xi| + \frac{|x_j|}{3k} + 2$, we have $|x_j| \leq 3|\xi|$ for j large enough and

$$|\widehat{\mu}(\xi)f_j(\xi)| \leq C_N \exp(4 + C_N \log(1 + |\xi|^2))$$

for every $\xi \in \mathbb{C}^N$ with $|\xi - x_j| < R_j + 2, j \geq j_0$. Moreover, $|\xi - x_j| \geq R_j + 2$ implies

$$|\widehat{\mu}(\xi)f_j(\xi)| \leq |\widehat{\mu}(\xi)| C_N \exp(|\operatorname{Im} \xi| + 2 + C_N \log(1 + |\xi|^2)),$$

whence it follows that $\{T_\mu^t(\nu_j) : j \in \mathbb{N}\}$ is a bounded set in $\mathcal{E}'_\omega(\mathbb{R}^N)$. We can proceed as in the lemma 11 to conclude that $\{\nu_j | A(\mathbb{R}^N) : j \in \mathbb{N}\}$ is equicontinuous in $A'(\mathbb{R}^N)$, i.e. ([11, 15.1.5])

$$\exists A > 0 \quad \forall \epsilon > 0 : \sup_j \sup_z |f_j(z)| \exp(-A|\operatorname{Im} z| - \epsilon|z|) < \infty.$$

On the other hand,

$$|f_j(x_j)| \exp(-\epsilon|x_j|) \geq \exp((\frac{2}{3k\pi\sqrt{N}} - \epsilon)|x_j| - C_N \log(1 + |x_j|^2)).$$

Now, for $0 < \epsilon < \frac{2}{3k\pi\sqrt{N}}$, this implies $\lim_{j \rightarrow \infty} |f_j(x_j)| \exp(-\epsilon|x_j|) = \infty$, which is a contradiction.

(b) \Rightarrow (c). Applying (b) inductively, we find a strictly increasing sequence (R_m) tending to infinity so that the conclusion of (b) holds with m replaced by $2m$ for all $x \in \mathbb{R}^N$ satisfying $|x| \geq R_m$. Then we define $g(t) := 0$ for $t \in [0, R_1[$ and $g(t) := \frac{t}{m}$ for $t \in [R_m, R_{m+1}[$. Since $g(t) = o(t)$ as t goes to infinity we can apply [6, 1.7;1.8] to get a weight function $\sigma \geq \omega$ such that $g(t) = o(\sigma(t))$ and $\sigma(t) = o(t)$ as t goes to infinity. Proceeding as in [5, 3.2] we conclude that μ is slowly decreasing for (σ) . It follows from corollary 6 that $T_\mu : \mathcal{E}_\sigma(\mathbb{R}^N) \rightarrow \mathcal{E}_\sigma(\mathbb{R}^N)$ is surjective. \square

We now include some result on the limit case of the space of real analytic functions. Compare with example 13.

Corollary 18. *Let ω be an arbitrary weight and let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$ satisfy $\text{supp}\mu = (0)$. Then the range of $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \rightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N)$ contains the space $A(\mathbb{R}^N)$ of real analytic functions.*

Proof. In fact, μ satisfies the condition (E) by [17, Th.7; Lemme 15] (see also [14, 1.8]). □

The next result was already proved in [4] with a different approach. It should be compared with corollary 18 and example 16 where a restriction on the support of μ was needed.

Corollary 19. *Let ω be a non-quasianalytic weight and let $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R}^N)$. Then the space $A(\mathbb{R}^N)$ of real analytic functions is contained in the range of the convolution operator $T_\mu : \mathcal{E}_{(\omega)}(\mathbb{R}^N) \rightarrow \mathcal{E}_{(\omega)}(\mathbb{R}^N)$.*

Proof. It follows from [7, 4.5], after taking convolution with a fixed test function, that μ satisfies condition (E). □

Lemma 20. *Let $\mu \in A'(\mathbb{R}^N)$ be an analytic functional. Then there are a (quasianalytic) weight σ and an ultradistribution $\nu \in \mathcal{E}'_{(\sigma)}(\mathbb{R}^N)$ such that μ is the restriction of ν to $A(\mathbb{R}^N)$.*

Proof. In fact, for every $\epsilon > 0$ we put

$$C_\epsilon := \sup_{z \in \mathbb{C}} | \hat{\mu}(z) | \exp(- | \text{Im}z | - \epsilon | z |) < \infty$$

and define $g : [0, +\infty[\rightarrow [0, +\infty[$ by $\exp(g(t)) := \sup_{|z|=t} | \hat{\mu}(z) | \exp(- | \text{Im}z |)$. Then $\exp(g(t)) \leq C_\epsilon \exp(\epsilon t)$, from where it follows that $g(t) = o(t)$ as t goes to ∞ . Now we apply [6, 1.7,1.8] to find a weight σ such that $g(t) = o(\sigma(t))$ and $\sigma(t) = o(t)$. Then, $g(| z |) \leq K\sigma(| z |)$ for some $K > 0$ and every $z \in \mathbb{C}$ and

$$\sup_{z \in \mathbb{C}} | \hat{\mu}(z) | \exp(- | \text{Im}z | - K\sigma(| z |)) < \infty. \quad \square$$

If p is a subharmonic function on \mathbb{C} satisfying a growth condition of the form $p(z) \leq A + B | z |$, $z \in \mathbb{C}$, then the *limit set* $L(p)$ of p is the set of all subharmonic functions in \mathbb{C} which are limits in $L^1_{\text{loc}}(\mathbb{R}^2)$ (or equivalently in $\mathcal{D}'(\mathbb{R}^2)$) of sequences of the form $t_j^{-1}p(t_j)$, where t_j goes to infinity. The indicator function j_p of p is the least upper semicontinuous majorant of $\overline{\lim}_{t \rightarrow \infty} t^{-1}p(t)$. The *indicator function* i_f of an entire function f is the indicator function of the subharmonic function $p = \log | f |$. The next theorem depends heavily on a result of Sigurdsson [30] on the existence of analytic functions f such that $\log | f |$ has the largest limit set among all subharmonic functions having a fixed subharmonic and positively homogeneous function r as indicator function.

Theorem 21. *There are a quasianalytic weight σ and an ultradistribution $\mu \in \mathcal{E}'_{(\sigma)}(\mathbb{R})$ such that the range of $T_\mu : \mathcal{E}_{(\sigma)}(\mathbb{R}) \rightarrow \mathcal{E}_{(\sigma)}(\mathbb{R})$ does not contain the space $A(\mathbb{R})$ of real analytic functions.*

Proof. We take $r(z) := | \text{Im}z |$ and apply [30, 1.3.2] to get an entire function $f \in H(\mathbb{C})$ such that the limit set of $p := \log | f |$ coincides with

$$L(p) = \{u \text{ subharmonic} : u(0) = 0, u(z) \leq | \text{Im}z |\}.$$

By [30, 1.1.2] (see also [31, 2.2(ii)]), the indicator function of f is given by $i_f(z) := \overline{\lim}_{t \rightarrow \infty} \frac{p(tz)}{t} = |\operatorname{Im} z|$, $z \in \mathbb{C}$. From the Hartogs lemma [16] it follows that for all $\epsilon > 0$ there is t_ϵ such that $\frac{p(tz)}{t} \leq |\operatorname{Im} z| + \epsilon$ for every $t \geq t_\epsilon$, $|z| = 1$. Consequently f is the Fourier-Laplace transform of an analytic functional. We apply the lemma 20 to find a quasianalytic weight σ and an ultradistribution $\mu \in \mathcal{E}'_{(\sigma)}(\mathbb{R})$ such that $f = \hat{\mu}$. To finish we show that f does not satisfy the condition (E). To do this, we put $u(z) := \log |\cos z|$, which is an element of $L(p)$. There is an increasing and unbounded sequence (t_j) such that $\lim_{j \rightarrow \infty} t_j^{-1} \log |f(t_j \cdot)| = u(\cdot)$ in $L^1_{\text{loc}}(\mathbb{R}^2)$. We take $x_0 = \frac{\pi}{4}$ and $0 < \epsilon < 1$ satisfying $u(z) < -2\epsilon$ for every $z \in \mathbb{C}$ with $|z - x_0| \leq \epsilon$. Since

$$\overline{\lim}_{j \rightarrow \infty} \left(\sup_{|z - x_0| \leq \epsilon} t_j^{-1} \log |f(t_j z)| \right) \leq \sup_{|z - x_0| \leq \epsilon} u(z) \leq -2\epsilon$$

by [10, 4.1.9], there is $j_0 \in \mathbb{N}$ such that $t_j^{-1} \log |f(t_j z)| \leq -2\epsilon$ for every $j \geq j_0$ and $|z - x_0| \leq \epsilon$. Then, as $|z| \leq |x_0| + 1$, we obtain $\log |f(t_j z)| \leq -2\epsilon' |t_j z|$ for every $j \geq j_0$ and $|z - x_0| \leq \epsilon$, with $\epsilon' = \frac{\epsilon}{|x_0| + 1}$. Finally we put $a_j := t_j x_0$, $j \geq j_0$. Then $|f(\xi)| \leq \exp(-2\epsilon' |\xi|)$ for every $\xi \in \mathbb{C}$ with $|\xi - a_j| \leq \epsilon' |a_j|$ and f does not satisfy condition (E). \square

By Sigurdsson [31, Thm. and Def. 2.5], for each $\mu \in \mathcal{E}'_{(\sigma)}(\mathbb{R}^N)$, $\hat{\mu}$ satisfies condition (E) if and only if $u(x) = 0$ for all $x \in \mathbb{R}^N$ and $u \in L(\log |\hat{\mu}|)$. (The uniformity of R —which a priori depends on the direction $x \in \mathbb{R}^N \setminus (0)$ —follows via [31, 2.5 (v)] from Hartogs' lemma [16, Thm. 1.31]. See also [16, Thm. 4.9].) These conditions are equivalent to several notions of regular growth of $|\hat{\mu}|$ on the cone \mathbb{R}^N ([31, 2.5]). By Hörmander and Sigurdsson [12], in the case $N = 1$, the equivalent conditions on $\hat{\mu}$ are fulfilled if ω is a non-quasianalytic weight. Of course, the function f in Theorem 21 is not of regular growth on \mathbb{R} .

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