SPECTRAL LIFTING IN BANACH ALGEBRAS AND INTERPOLATION IN SEVERAL VARIABLES

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Abstract. Let $A$ be a unital Banach algebra and let $J$ be a closed two-sided ideal of $A$. We prove that if any invertible element of $A/J$ has an invertible lifting in $A$, then the quotient homomorphism $\Phi : A \to A/J$ is a spectral interpolant. This result is used to obtain a noncommutative multivariable analogue of the spectral commutant lifting theorem of Bercovici, Foiaş, and Tannenbaum. This yields spectral versions of Sarason, Nevanlinna–Pick, and Carathéodory type interpolation for $F^\infty_n \mathbin{\hat{\otimes}} B(K)$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra $F^\infty_n$ and $B(K)$, the algebra of bounded operators on a finite dimensional Hilbert space $K$. A spectral tangential commutant lifting theorem in several variables is considered and used to obtain a spectral tangential version of the Nevanlinna-Pick interpolation for $F^\infty_n \mathbin{\hat{\otimes}} B(K)$.

In particular, we obtain interpolation theorems for matrix-valued bounded analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant and not the norm.

1. Introduction and preliminaries

Let $\mathbb{D}$ denote the unit disc in the complex plane, let $z_1, \ldots, z_k \in \mathbb{D}$ be given distinct points, and $F_1, \ldots, F_k$ be complex $m \times m$ matrices. The classical Nevanlinna–Pick problem [N], [P] consists in finding necessary and sufficient conditions for the existence of an analytic $m \times m$ matrix-valued function $F(z)$ with $F(z_j) = F_j$ ($1 \leq j \leq k$) and such that $\|F\|_\infty \leq 1$.

Motivated by problems in control engineering, such as the design of feedback control systems in the presence of parameter uncertainty, Bercovici, Foiaş, and Tannenbaum proved in [BFT] a spectral generalization of the commutant lifting theorem [SzF1], and obtained a spectral version of the Nevanlinna–Pick problem, in which the infinity norm is replaced by

$$\rho(F) := \sup\{\|F(z)\|_{sp} : z \in \mathbb{D}\}$$

($\|A\|_{sp}$ denotes the spectral radius of an operator $A$).

The tangential Nevanlinna–Pick problem considered by Fedcina [F] is to find $F \in H^\infty(\mathbb{D}) \mathbin{\hat{\otimes}} \mathbb{C}^m$ with $F(z_j)u_j = v_j$, $j = 1, \ldots, k$, and $\|F\|_\infty \leq 1$, where $z_j \in \mathbb{D}$ and $u_j, v_j \in \mathbb{C}^m$ are prescribed. The spectral tangential Nevanlinna–Pick interpolation problem, considered by Bercovici and Foiaş [BF], is to find such an $F$ for which $\rho(F) < 1$. This type of interpolation was also motivated by certain control engineering applications.

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In this paper we find noncommutative multivariable analogues of the above-mentioned results obtained by Bercovici, Foiaş, and Tannenbaum (see [BFT] and [BF]) for the noncommutative analytic Toeplitz algebra $F_n^\infty$. In particular, we obtain interpolation results (see Corollary 3.7 and Corollary 4.3) for matrix-valued bounded analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant and not the norm.

We expect these results to play a role in multivariable control and systems theory, as it does in the case $n = 1$. We mention the papers [BV] and [B] for recent results in multivariable linear systems.

We need to recall some facts concerning the noncommutative analytic Toeplitz algebra $F_n^\infty$ and its connection with the function theory on the open unit ball of $\mathbb{C}^n$. Let $F^2(H_n) = C1 \oplus \bigoplus_{m \geq 1} H_n^\otimes_m$ be the full Fock space on $n$ generators, where $H_n$ is an $n$-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ if $n$ is finite, and $\{e_1, e_2, \ldots\}$ if $n = \infty$. For each $i = 1, 2, \ldots$, define the left creation operator by $S_i \xi := e_i \otimes \xi$, $\xi \in F^2(H_n)$.

We shall denote by $P$ the set of all $p \in F^2(H_n)$ which are finite sums of tensor monomials. Define $F_n^\infty$ as the set of all $g \in F^2(H_n)$ such that
\[ \|g\|_\infty := \sup\{|g \otimes p\|_{F^2(H_n)} : p \in P, \|p\|_{F^2(H_n)} \leq 1\} < \infty. \]

We denote by $A_n$ the closure of $P$ in $(F_n^\infty, \|\cdot\|_\infty)$. The Banach algebra $F_n^\infty$ (resp. $A_n$) can be viewed as a noncommutative analogue of the Hardy space $H^\infty(\mathbb{D})$ (resp. disc algebra $A(\mathbb{D})$) when $n = 1$ they coincide.

In [Po7 Theorem 3.1] we proved that $A_n$ is completely isometrically isomorphic to the norm-closed algebra generated by any sequence $V_1, \ldots, V_n$ of isometries with $V_1V_1^* + \cdots + V_nV_n^* \leq I$, and the identity. It follows from [Po5 Theorem 4.3] that the noncommutative analytic Toeplitz algebra $F_n^\infty$ can be identified with the WOT-closed algebra generated by the left creation operators $S_1, \ldots, S_n$, and the identity. The algebras $F_n^\infty$ and $A_n$ were introduced by the author in [Po2] in connection with a noncommutative von Neumann inequality, and have been studied in several papers [Po2, Po3, Po5, Po7, Po9, ArPo1], and recently in [DP1, DP2, ArPo2, DP3, Amc], and [BTV]. In particular, we proved that there is a completely contractive homomorphism
\[ \Phi : F_n^\infty \rightarrow H^\infty(\mathbb{B}_n), \quad f(S_1, \ldots, S_n) \mapsto f(\lambda_1, \ldots, \lambda_n), \]
where $(\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n$. A characterization of the analytic functions in the range of the map $\Phi$ was obtained in [ArPo2] and [DP3]. W. Arveson proved that $\Phi$ is not surjective [Arv] and the functions in its range are the multipliers of a certain function Hilbert space. In [ArPo2, DP3], it was proved that $F_n^\infty / \ker \Phi$ is an operator algebra which can be identified with $W_n^\infty := P_F^2 F_n^\infty |_{F^2}$, the compression to the symmetric Fock space $F^2_n \subseteq F^2(H_n)$. In [Po8, Po9, Arv, ArPo2, DP3, Amc], and [BTV], a good case is made that the appropriate commutative multivariable analogue of $H^\infty(\mathbb{D})$ is the algebra $W_n^\infty$, which is the WOT-closed algebra generated by $B_i := P_{F^2} S_i |_{F^2}$, $i = 1, \ldots, n$, and the identity. In this paper, we provide further evidence that $F_n^\infty$ (resp. $W_n^\infty$) is a noncommutative (resp. commutative) multivariate analogue of $H^\infty(\mathbb{D})$. 
Let \( \mathcal{A} \) be a unital Banach algebra and denote by \( \text{Inv}(\mathcal{A}) \) the group of invertible elements of \( \mathcal{A} \). Given \( a \in \mathcal{A} \), we define the \( \mathcal{A} \)-spectral radius of \( a \) by setting
\[
\rho_{\mathcal{A}}(a) := \inf \{ \| xa^{-1} \| : x \in \text{Inv}(\mathcal{A}) \}.
\]
Since the spectral radius of \( a \in \mathcal{A} \) is \( \| a \|_{sp} = \lim_{n \to \infty} \| a^n \|^{1/n} \), it is clear that \( \| a \|_{sp} = \| xa^{-1} \|_{sp} \) for any \( x \in \text{Inv}(\mathcal{A}) \). Now, it is easy to see that
\[
\| a \|_{sp} \leq \rho_{\mathcal{A}}(a) \leq \| a \|
\]
for any \( a \in \mathcal{A} \). Note that if \( \mathcal{A} = B(\mathcal{H}) \) (or \( \mathcal{A} \) is any \( C^* \)-subalgebra of \( B(\mathcal{H}) \)) then \( \| a \|_{sp} = \rho_{\mathcal{A}}(a) \) (see [R]). There are some other examples of Banach algebras such that \( \| a \|_{sp} = \rho_{\mathcal{A}}(a) \) for any \( a \in \mathcal{A} \). It was proved in [BFT] that this equality holds if \( \mathcal{A} \) is the commutant of an isometry (resp. normal operator) on a Hilbert space.

Let \( \mathcal{A}, \mathcal{B} \) be unital Banach algebras, and \( \Phi : \mathcal{A} \to \mathcal{B} \) be a unital contractive homomorphism. We say that \( \Phi \) is a quotient interpolant if
\[
\| b \| = \inf \{ \| a \| : a \in \mathcal{A}, \Phi(a) = b \}
\]
for any \( b \in \mathcal{B} \). We say that \( b \in \mathcal{B} \) with \( \| b \| \leq 1 \) has a spectral lifting if there exists \( a \in \mathcal{A} \) such that \( \Phi(a) = b \) and \( \rho_{\mathcal{A}}(a) < 1 \). The homomorphism \( \Phi \) is called a spectral interpolant if any \( b \in \mathcal{B} \) has a spectral lifting.

**Problem.** Let \( \Phi : \mathcal{A} \to \mathcal{B} \) be a unital contractive homomorphism which is also a quotient interpolant. When is \( \Phi \) a spectral interpolant?

We show, in Section 2, that this problem has a positive answer if \( \text{Inv}(\mathcal{B}) \subseteq \Phi(\text{Inv}(\mathcal{A})) \). This relation holds, for example, if the group of invertible elements of \( \mathcal{B} \) is connected (in particular, if \( \mathcal{B} \) is finite dimensional or equal to \( B(\mathcal{H}) \)).

The results of Section 2 are used in Section 3 to obtain a noncommutative multivariable analogue (see Theorem 3.1) of the spectral commutant lifting theorem of Bercovici-Foiaş-Tannenbaum. This yields spectral versions of Sarason ([S]), Nevanlinna-Pick, and Carathéodory type interpolation for \( F_{n}^\infty \otimes B(\mathcal{K}) \), the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra \( F_{n}^\infty \) and \( B(\mathcal{K}) \), the algebra of bounded operators on a finite dimensional Hilbert space \( \mathcal{K} \).

In Section 4, we obtain a spectral tangential commutant lifting theorem in several variables (see Theorem 4.1). This leads to a spectral tangential Nevanlinna-Pick interpolation for \( F_{n}^\infty \otimes B(\mathcal{K}) \) (see Theorem 4.2).

Problems concerning the optimal solutions to these spectral interpolation problems in several variables, and explicit algorithm for finding the optimal interpolants will be considered in a future paper.

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## 2. Spectral lifting in Banach algebras

The notation and definitions from Section 1 are used throughout the paper. Let \( \mathcal{A}, \mathcal{B} \) be unital Banach algebras and let \( \Phi : \mathcal{A} \to \mathcal{B} \) be a unital contractive homomorphism. We call \( \Phi \) a norm preserving interpolant if for any \( b \in \mathcal{B} \) there exists \( a \in \mathcal{A} \) such that \( \Phi(a) = b \) and \( \| a \| = \| b \| \). Notice that any norm preserving interpolant is a quotient interpolant. Examples of norm preserving interpolants will be presented in Section 3.
Theorem 2.1. Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital contractive homomorphism with the property that $\text{Inv}(\mathcal{B}) \subseteq \Phi(\text{Inv}(\mathcal{A}))$ and
\[
\|b\| = \inf \{ \|a\| : a \in \mathcal{A}, \Phi(a) = b \}
\]
for any $b \in \mathcal{B}$. Then
\[
\rho_{\mathcal{B}}(b) = \inf \{ \rho_{\mathcal{A}}(a) : a \in \mathcal{A}, \Phi(a) = b \}
\]
for any $b \in \mathcal{B}$. In particular, $\Phi$ is a spectral interpolant.

Proof. Let $b \in \mathcal{B}$ and $a \in \mathcal{A}$ with $\Phi(a) = b$. Since $\Phi$ is a contractive homomorphism and $\Phi(\text{Inv}(\mathcal{A})) \subseteq \text{Inv}(\mathcal{B})$ we have
\[
\rho_{\mathcal{A}}(a) = \inf \{ \|waw^{-1}\| : w \in \text{Inv}(\mathcal{A}) \}
\]
\[
\geq \inf \{ \|\Phi(waw^{-1})\| : w \in \text{Inv}(\mathcal{A}) \}
\]
\[
= \inf \{ \|\Phi(w)b\Phi(w)^{-1}\| : w \in \text{Inv}(\mathcal{A}) \}
\]
\[
\geq \inf \{ \|zbz^{-1}\| : z \in \text{Inv}(\mathcal{B}) \}
\]
\[
= \rho_{\mathcal{B}}(b).
\]

Therefore,
\[
\rho_{\mathcal{B}}(b) \leq \inf \{ \rho_{\mathcal{A}}(a) : a \in \mathcal{A}, \Phi(a) = b \}.
\]

Now, let $\epsilon > 0$ and choose $z \in \text{Inv}(\mathcal{B})$ such that
\[
\|zbz^{-1}\| \leq \rho_{\mathcal{B}}(b) + \frac{\epsilon}{2}.
\]

Since $zbz^{-1} \in \mathcal{B}$, according to the hypothesis, for any $\epsilon > 0$, there exists $d \in \mathcal{A}$ such that
\[
\Phi(d) = zbz^{-1} \quad \text{and} \quad \|d\| \leq \|zbz^{-1}\| + \frac{\epsilon}{2}.
\]

Since $\Phi(\text{Inv}(\mathcal{A})) \supseteq \text{Inv}(\mathcal{B})$, we find $w \in \text{Inv}(\mathcal{A})$ such that $\Phi(w) = z$. Notice that
\[
y := w^{-1}dw \in \mathcal{A}
\]
and
\[
\Phi(y) = \Phi(w)^{-1}\Phi(d)\Phi(w) = z^{-1}(zbz^{-1})z = b.
\]

Now, using (2.2) and (2.3), we infer that
\[
\rho_{\mathcal{A}}(y) \leq \|wyyw^{-1}\| = \|d\| \leq \|zbz^{-1}\| + \frac{\epsilon}{2} \leq \rho_{\mathcal{A}}(b) + \epsilon.
\]

Therefore,
\[
\rho_{\mathcal{B}}(b) \geq \inf \{ \rho_{\mathcal{A}}(a) : a \in \mathcal{A}, \phi(a) = b \}.
\]

Using relation (2.1), it is easy to see that if $b \in \mathcal{B}$, then $\rho_{\mathcal{B}}(b) < 1$ if and only if there exists $a \in \mathcal{A}$ such that $\Phi(a) = b$ and $\rho_{\mathcal{A}}(a) < 1$. This completes the proof.

Corollary 2.2. Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras such that the group $\text{Inv}(\mathcal{B})$ is connected. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital contractive homomorphism which is also a quotient interpolant. Then $\Phi$ is a spectral interpolant.

Proof. Let us prove that
\[
\Phi(\text{Inv}(\mathcal{A})) = \text{Inv}(\mathcal{B}).
\]

The inclusion $\Phi(\text{Inv}(\mathcal{A})) \subseteq \text{Inv}(\mathcal{B})$ is clear. Conversely, let $x \in \text{Inv}(\mathcal{B})$. Since $\text{Inv}(\mathcal{B})$ is connected, it is well known that
\[
x = \exp(z_1) \cdots \exp(z_k)
\]
for some $z_1, \ldots, z_k \in \mathcal{B}$. Due to the hypothesis, there exist $w_1, \ldots, w_k \in \mathcal{A}$ such that $\Phi(w_i) = z_i$, $i = 1, \ldots, k$. Denote $y := \exp(w_1) \cdots \exp(w_k) \in \text{Inv}(\mathcal{A})$ and notice that $\Phi(y) = \exp(\Phi(w_1)) \cdots \exp(\Phi(w_k)) = x$. Hence $\Phi(\text{Inv}(\mathcal{A})) \supset \text{Inv}(\mathcal{B})$ and (2.4) holds.

Remark 2.3. If $\mathcal{B}$ is a finite dimensional algebra, then $\text{Inv}(\mathcal{B}) = \exp(\mathcal{B})$, hence $\text{Inv}(\mathcal{B})$ is connected.

Corollary 2.4. Let $\mathcal{A}$ be a unital Banach algebra and let $J$ be a closed two-sided ideal of $\mathcal{A}$. If any invertible element of $\mathcal{A}/J$ has an invertible lifting in $\mathcal{A}$, then the quotient homomorphism $\Phi : \mathcal{A} \to \mathcal{A}/J$ is a spectral interpolant, i.e., $\rho_{\mathcal{A}/J}(a+J) < 1$ if and only if there exists $b \in a + J$ such that $\rho_{\mathcal{A}}(b) < 1$.

Proof. Apply Theorem 2.1 to the quotient homomorphism $\Phi$. $

Let us remark that, in general, there are invertible elements in $\mathcal{A}/J$ which can not be lifted to invertible elements in $\mathcal{A}$. For example, if $\pi : B(H^2) \to B(H^2)/K(H^2)$ is the quotient homomorphism into the Calkin algebra, and $S$ is the unilateral shift on the Hardy space $H^2$, then $\pi(S)$ is invertible and there is no invertible operator $T \in B(H^2)$ such that $\pi(T) = \pi(S)$.

An important particular case, when Corollary 2.4 can be applied, is when the quotient algebra $\mathcal{A}/J$ is finite dimensional. Applications of this result will be considered in the next section.

3. Noncommutative spectral commutant lifting and interpolation

Let $\mathbb{F}_n^+$ be the unital free semigroup on $n$ generators $s_1, \ldots, s_n$, and let $e$ be its neutral element. For any $\sigma := s_{i_1} \cdots s_{i_k} \in \mathbb{F}_n^+$ we define its length $|\sigma| := k$, and $|e| = 0$. On the other hand, if $T_i \in B(\mathcal{H})$, $i = 1, \ldots, n$, we denote $T_\sigma := T_{i_1} \cdots T_{i_k}$ and $T_e := I_\mathcal{H}$.

Let us recall from [Po4], [Po2], and [Po3] some results concerning the noncommutative dilation theory for $n$-tuples of operators. A sequence of operators $T := [T_1, \ldots, T_n], T_i \in B(\mathcal{H}), i = 1, \ldots, n$, is called contractive (or row contraction) if $T_1 T_1^* + \cdots + T_n T_n^* \leq I_\mathcal{H}$. We say that a sequence of isometries $V := [V_1, \ldots, V_n]$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ is a minimal isometric dilation of $T$ if the following properties are satisfied:

(i) $V_1 V_1^* + \cdots + V_n V_n^* \leq I_\mathcal{K}$;
(ii) $V_i^*|_\mathcal{H} = T_i^*, i = 1, \ldots, n$;
(iii) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n} V_\alpha \mathcal{H}$.

The minimal isometric dilation of $T$ is uniquely determined up to an isomorphism. We need to recall the noncommutative commutant lifting theorem [Po4] (see [SzF1], [SzT2], [DMP] for the classical case).

Let $T := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert $\mathcal{K} \supset \mathcal{H}$. If $X \in B(\mathcal{H})$ and $XT_i = T_i X$ for any $i = 1, \ldots, n$, then there exists $X_\infty \in B(\mathcal{K})$ satisfying the following properties:

(i) $X_\infty V_i = V_i X_\infty$, for any $i = 1, \ldots, n$;
(ii) $X_\infty^*|_\mathcal{K} = X^*$;
(iii) $\|X_\infty\| = \|X\|$.
Let $T := [T_1, \ldots, T_n]$ be a row contraction with $T_i \in B(H)$ and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $K \supseteq H$. Let $X \in \{T_1', \ldots, T_n'\}$, and denote
\[ \text{Dil}(X) := \{Y \in \{V_1, \ldots, V_n\}' : P_H Y = X P_H\}, \]
where $P_H$ is the orthogonal projection on $H$. According to the noncommutative commutant lifting theorem, we have $\text{Dil}(X) \neq \emptyset$.

In what follows we obtain a noncommutative multivariable analogue of the spectral commutant lifting theorem of Bercovici-Foias-Tannenbaum [BFT].

**Theorem 3.1.** Let $T := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $H$ and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $K \supseteq H$. If $H$ is finite dimensional and $K \ominus H$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, then
\[ \rho(T_1, \ldots, T_n)^*(X) = \inf \{\rho(V_1, \ldots, V_n)^*(Y) : Y \in \text{Dil}(X)\} \]
for any $X \in \{T_1', \ldots, T_n'\}$.

**Proof.** Let $\Phi : \{V_1, \ldots, V_n\}' \to \{T_1', \ldots, T_n'\}'$ be defined by $\Phi(Y) := P_H Y|_H$.

Since $K \ominus H$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we have $Y^*(H) \subseteq H$ for any $Y \in \{V_1, \ldots, V_n\}'$. Since $V := [V_1, \ldots, V_n]$ is the minimal isometric dilation of $T$, we have $V_i|_H = T_i^*$, $i = 1, \ldots, n$. Now, it is easy to see that
\[ (P_H Y|_H)T_i = T_i(P_H Y|_H) \]
for any $i = 1, 2, \ldots, n$.

Therefore, the mapping $\Phi$ is well-defined. On the other hand, since $K \ominus H$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we infer that $\Phi$ is a unital contractive homomorphism, and $\Phi(Y) = X$ is equivalent to $P_H Y = XP_H$. According to the noncommutative commutant lifting theorem, for any $X \in \{T_1', \ldots, T_n'\}$ there exists $Y \in \{V_1, \ldots, V_n\}'$ such that $P_H Y = XP_H$ and $\|Y\| = \|X\|$. Therefore, $\Phi$ is a norm preserving interpolant. Since $H$ is finite dimensional, the algebra $\{T_1', \ldots, T_n'\}$ is finite dimensional. Applying Theorem 2.1 and Remark 2.3, in the particular case when $A := \{V_1, \ldots, V_n\}'$ and $B := \{T_1', \ldots, T_n'\}'$, the result follows.

**Corollary 3.2.** Let $T := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $H$ and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $K \supseteq H$. If $H$ is finite dimensional and $K \ominus H$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, then, given $X \in \{T_1', \ldots, T_n'\}$, $\rho(T_1, \ldots, T_n)^*(X) < 1$ if and only if there exists $Y \in \text{Dil}(X)$ such that $\rho(V_1, \ldots, V_n)^*(Y) < 1$.

In what follows, we use the noncommutative spectral commutant lifting theorem to obtain spectral versions of Sarason, Nevanlinna–Pick, and Carathéodory type interpolation for $F_n^\infty \odot B(K)$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra $F_n^\infty$ and $B(K)$. In particular, we obtain interpolation results for matrix-valued analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant.

According to Theorem 1.2 from [Po0], the commutant of $F_n^\infty$, which we denote by $R_n^\infty$, is equal to $U^* F_n^\infty U$, where $U$ is the unitary operator on $F^2(H_n)$ defined by $U(e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_k}) = e_{i_{\lambda_1}} \otimes \cdots \otimes e_{i_{\lambda_2}} \otimes e_{i_1}$. Moreover, the commutant of $R_n^\infty$ is equal to $F_n^\infty$.

A complete description of the invariant subspace structure of $F_n^\infty$ was obtained in [Po2] Theorem 2.2 (even in a more general setting). A subspace $N$ of $F^2(H_n)$ is invariant under $S_1, \ldots, S_n$ if and only if $N = \bigoplus_{\lambda \in \Lambda} U^* \varphi_{\lambda} U[F^2(H_n)]$, for some
family \( \{ \varphi_\lambda \in F_n^\infty \colon \lambda \in \Lambda \} \) of isometries with orthogonal ranges (see also [Po8] and [BFT]). Let us remark that \( \mathcal{M} \subseteq F^2(H_n) \) is hyperinvariant for \( \{ S_1, \ldots, S_n \} \), i.e., invariant for \( \{ S_1, \ldots, S_n \}' \), if and only if \( UM \) is invariant for \( \{ S_1, \ldots, S_n \} \).

**Theorem 3.3.** Let \( \mathcal{K} \) be a finite dimensional Hilbert space and let \( \mathcal{N} \subseteq F^2(H_n) \) be a finite dimensional subspace with the property that \( \mathcal{N} \) and \( \mathcal{U} \mathcal{N} \) are invariant under \( S_1^* \), \( \ldots \), \( S_n^* \). Then \( X \in B(\mathcal{N} \otimes \mathcal{K}) \) commutes with each \( P_N S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}, i = 1, \ldots, n \), and

\[
\rho_{P_N R_n^\infty|_{\mathcal{N} \otimes B(\mathcal{K})}}(X) < 1
\]

if and only there exists \( \Psi \in R_n^\infty \otimes B(\mathcal{K}) \) such that

\[
P_{\mathcal{N} \otimes \mathcal{K}} \Psi = X P_{\mathcal{N} \otimes \mathcal{K}} \quad \text{and} \quad \rho_{R_n^\infty \otimes B(\mathcal{K})}(\Psi) < 1.
\]

**Proof.** According to [Po8], we have

\[
\mathcal{B} := \{ P_N S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}, i = 1, \ldots, n \}' = P_{\mathcal{N} \otimes \mathcal{K}}(R_n^\infty \otimes B(\mathcal{K}))|_{\mathcal{N} \otimes \mathcal{K}}.
\]

Notice that \( \mathcal{B} \) is a finite dimensional algebra. Let \( \mathcal{A} := R_n^\infty \otimes B(\mathcal{K}) \) and define \( \Phi : \mathcal{A} \rightarrow \mathcal{B} \) by \( \Phi(Y) = P_{\mathcal{N} \otimes \mathcal{K}} Y|_{\mathcal{N} \otimes \mathcal{K}} \). Since \( S_i^* (\mathcal{N} \mathcal{U}) \subseteq \mathcal{U} \mathcal{N} \) for any \( i = 1, \ldots, n \), and \( \{ S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}} \}' = R_n^\infty \otimes B(\mathcal{K}) \), it is easy to see that \( [F^2(H_n) \otimes \mathcal{K}] \) is hyperinvariant for \( \{ S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}} \} \) and the mapping \( \Phi \) is a unital contractive homomorphism. Since \( \mathcal{N} \) is invariant under \( S_1^*, \ldots, S_n^* \), it is clear that the operator matrix \( [P_N S_i|_{\mathcal{N}}, \ldots, P_N S_n|_{\mathcal{N}}] \) is a \( C_0 \)-row contraction and its minimal isometric dilation is \( [S_1, \ldots, S_n] \) (see [Po8]). Therefore, the minimal isometric dilation of \( [P_N S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}, \ldots, P_N S_n|_{\mathcal{N}} \otimes I_{\mathcal{K}}] \) is \( [S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}}] \). According to the noncommutative commutant lifting theorem, for any \( X \in \mathcal{B} \) there exists \( \Psi \in R_n^\infty \otimes B(\mathcal{K}) \), such that \( P_{\mathcal{N} \otimes \mathcal{K}} \Psi = X P_{\mathcal{N} \otimes \mathcal{K}} \) and \( \|X\| = \|\Psi\| \). Therefore, \( \Phi(\Psi) = X \) and \( \Phi \) is a norm preserving interpolant. Applying Corollary 3.2, the result follows. 

Notice that the element \( \Psi \) in Theorem 3.3 satisfies \( \|\Psi\|_{sp} \leq \rho_{R_n^\infty \otimes B(\mathcal{K})}(\Psi) < 1 \). It would be nice to know if \( \rho_{R_n^\infty \otimes B(\mathcal{K})}(\Psi) = \|\Psi\|_{sp} \) for any \( \Psi \in R_n^\infty \otimes B(\mathcal{K}) \). This equality holds if \( n = 1 \) (see [BFT]).

Let us remark that the finite dimensionality hypothesis can be dropped in Theorem 3.3 for those subspaces \( \mathcal{N} \) and \( \mathcal{K} \) for which one can prove that any invertible element \( f \in P_N R_n^\infty|_{\mathcal{N} \otimes B(\mathcal{K})} \) can be lifted to an invertible element \( g \in R_n^\infty \otimes B(\mathcal{K}) \), i.e., \( P_N S_i|_{\mathcal{N} \otimes \mathcal{K}} = f \). We do not have yet any nontrivial example when this lifting property holds and \( \mathcal{N}, \mathcal{K} \) are infinite dimensional.

Let \( J \) be a WOT-closed, two-sided ideal of \( F_n^\infty \) and define \( J(1) := \{ \Psi(1) : \Psi \in J \} \) and \( \mathcal{N}_J := F^2(H_n) \otimes J(1) \). Let us remark that \( \mathcal{N}_J \) and \( \mathcal{U} \mathcal{N}_J \) are invariant subspaces under \( S_i^*, i = 1, \ldots, n \), therefore, Theorem 3.3 works in the case when \( \dim \mathcal{N}_J < \infty \).

**Corollary 3.4.** Let \( \mathcal{K} \) be a finite dimensional Hilbert space and let \( J \) be a WOT-closed two-sided ideal of \( F_n^\infty \) such that \( \dim \mathcal{N}_J < \infty \). Then the quotient homomorphism

\[
\Phi : F_n^\infty \otimes B(\mathcal{K}) \rightarrow F_n^\infty \otimes B(\mathcal{K})/(J \otimes B(\mathcal{K}))
\]

is a spectral interpolant.

**Proof.** According to [ArPo2], the quotient algebra \( F_n^\infty \otimes B(\mathcal{K})/(J \otimes B(\mathcal{K})) \) is completely isometrically isomorphic to \( P_{\mathcal{N}_J} F_n^\infty|_{\mathcal{N}_J} \otimes B(\mathcal{K}) \), which is finite dimensional. Using Theorem 3.3, we infer that \( \Phi \) is a spectral interpolant. The proof is complete.
It will be interesting to see if this result remains true if $N_j$ is infinite dimensional (at least for some particular cases, if not in general). The obstruction in the infinite dimensional case seems to be the lifting of the invertible elements of a quotient algebra $\mathcal{A}/J$ to invertible elements of $\mathcal{A}$ (see Section 2 for an example). In the finite dimensional case, Corollary 3.4 leads to our spectral interpolation results for $F_n^\infty$ (see Theorem 3.6 and Theorem 3.8).

Let $F_s^2(H_n)$ be the symmetric Fock space and $\mathcal{W}_n^\infty$ be the WOT-closed algebra generated by $B_i := F_{F_s^2(H_n)}S_i|F_s^2(H_n)$, $i = 1, \ldots, n$, and the identity. This algebra has been studied in [Po9], [Arv], [ArPo2], [DP3]. The following theorem can be seen as a spectral version of Sarason’s interpolation theorem for $H^\infty(D)$ (see [S]), in a commutative and multivariable setting.

**Theorem 3.5.** Let $E \subseteq F_s^2(H_n)$ be a finite dimensional invariant subspace under $B_1^*, \ldots, B_n^*$ and let $K$ be a finite dimensional Hilbert space. Then $f \in B(E \otimes K)$ commutes with each $P_E B_i|E \otimes I_K$, $i = 1, \ldots, n$, and

$$\rho_{P_E \otimes K}(\mathcal{W}_n^\infty \bar{\otimes} B(K))_{E \otimes K}(f) < 1$$

if and only if there exists $g \in \mathcal{W}_n^\infty \bar{\otimes} B(K)$ such that

$$P_E \otimes K g_{E \otimes K} = f \quad \text{and} \quad \rho_{\mathcal{W}_n^\infty \bar{\otimes} B(K)}(g) < 1.$$

**Proof.** Since $F_s^2(H_n)$ is invariant under each $S_i^*$, $i = 1, \ldots, n$, it is easy to see that $E$ has the same property. Taking into account that $\mathcal{W}_n^\infty$ is the compression of $F_s^\infty$ to the symmetric Fock space, one can see that $f$ commutes with $P_E \otimes K(S_i \otimes I_K)|_{E \otimes K}$. As in the proof of Theorem 3.3, using the noncommutative commutant lifting theorem, we find $\phi \in \mathcal{W}_n^\infty \bar{\otimes} B(K)$ such that $P_E \otimes K(U^* \otimes I_K)\phi(U \otimes I)|_{E \otimes K} = f$ and $\|f\| = \|\phi\|$. Hence, $P_E \otimes K \phi|_{E \otimes K} = f$. Setting $g := P_{F_s^2(H_n)} \otimes K \phi|_{F_s^2(H_n) \otimes K} \in \mathcal{W}_n^\infty \bar{\otimes} B(K)$, we have $P_E \otimes K g|_{E \otimes K} = f$ and $\|f\| \leq \|g\| \leq \|\phi\| = \|f\|$. This shows that $\|f\| = \|g\|$. Define $\mathcal{A} := \mathcal{W}_n^\infty \bar{\otimes} B(K)$, $\mathcal{B} := P_E \otimes K(\mathcal{W}_n^\infty \bar{\otimes} B(K))_{E \otimes K}$ and let $\Phi : \mathcal{A} \to \mathcal{B}$ be defined by $\Phi(g) := P_E \otimes K(g)_{E \otimes K}$. We just proved that $\Phi$ is a unital contractive homomorphism and also a norm preserving interpolant. Now, the result follows by applying the results of Section 2 in our setting.

Let us remark that a result similar to Corollary 3.4 holds for the algebra $\mathcal{W}_n^\infty \bar{\otimes} B(K)$.

In what follows we obtain a spectral version of Nevanlinna-Pick interpolation for the noncommutative analytic Toeplitz algebra $F_n^\infty$ (see [ArPo2], [DP3], and [Po8]). As mentioned in the first section, there exists a unital contractive homomorphism

$$\Psi : F_n^\infty \bar{\otimes} B(K) \to H^\infty(\mathbb{B}_n) \bar{\otimes} B(K)$$

defined by $[\Psi(f)](\lambda) := f(\lambda)$, $\lambda \in \mathbb{B}_n$.

**Theorem 3.6.** Let $K$ be a finite dimensional Hilbert space, $W_j \in B(K)$, and $\lambda_j$, $j = 1, \ldots, k$, be distinct elements in $\mathbb{B}_n$. Then there exists $\Phi \in F_n^\infty \bar{\otimes} B(K)$ such that

$$\rho_{F_n^\infty \bar{\otimes} B(K)}(\Phi) < 1 \quad \text{and} \quad \Phi(\lambda_j) = W_j, \quad j = 1, \ldots, k,$$

if and only if there exist invertible operators $M_j \in B(K)$, $j = 1, \ldots, k$, such that

$$\left[ I_K - (M_i W_i M_i^{-1})(M_j W_j M_j^{-1})^* \right]_{1 \leq i, j \leq k} > 0.$$
Proof. Let \( \lambda_j := (\lambda_{j1}, \ldots, \lambda_{jn}) \in \mathbb{B}_n, j = 1, \ldots, k. \) For any \( \alpha := s_{j1}s_{j2} \ldots s_{jn} \) in \( F_n^+ \), let \( \lambda_{j\alpha} := \lambda_{j1}\lambda_{j2} \ldots \lambda_{j\alpha} \) and \( \lambda_e := 1. \) Define \( z_{\lambda_j} \in F^2(H_n) \) by setting
\[
z_{\lambda_j} := \sum_{\alpha \in F_n^+} \lambda_{j\alpha} e_\alpha, \quad j = 1, 2, \ldots, k.
\]
Let \( \mathcal{N} := \text{span}(z_{\lambda_j} : j = 1, \ldots, k) \) and \( X \in B(\mathcal{N} \otimes \mathcal{K}) \) be defined by
\[
X^*(z_{\lambda_j} \otimes h) := z_{\lambda_j} \otimes W^*_uh, \quad h \in \mathcal{K}.
\]
Notice that \( S^*_iz_{\lambda_i} = \overline{x}_{ji}z_{\lambda_i} \) for any \( i = 1, \ldots, n; j = 1, \ldots, k. \) Hence, the subspaces \( \mathcal{N} \) and \( U\mathcal{N} \) are invariant under each \( S^*_i; i = 1, \ldots, n. \) Define \( T_i \in B(\mathcal{N} \otimes \mathcal{K}) \) by \( T_i := P_{\mathcal{N}}S_i|_{\mathcal{N}} \otimes I_{\mathcal{K}}. \) Since \( z_{\lambda_1}, \ldots, z_{\lambda_k} \) are linearly independent, the operator \( X \in B(\mathcal{N} \otimes \mathcal{K}) \) given by (3.2) is well defined.

Notice that \( XT_i = T_iX \) for any \( i = 1, \ldots, k. \) Indeed,
\[
T_i^*X^*(z_{\lambda_j} \otimes h) = T_i^*(z_{\lambda_j} \otimes W^*_uh) = S^*_iz_{\lambda_j} \otimes W^*_uh = \overline{x}_{ji}z_{\lambda_j} \otimes W^*_uh
\]
and
\[
X^*T_i^*(z_{\lambda_j} \otimes h) = X^*(\overline{x}_{ji}z_{\lambda_j} \otimes h) = \overline{x}_{ji}z_{\lambda_j} \otimes W^*_uh.
\]
Applying Theorem 3.3, we infer that
\[
\rho(T_1, \ldots, T_n)^*(X) < 1
\]
if and only there exists \( \Phi \in F^\infty_n \otimes B(\mathcal{K}) \) such that
\[
P_{\mathcal{N} \otimes \mathcal{K}}(U^* \otimes I)\Phi(U \otimes I) = X P_{\mathcal{N} \otimes \mathcal{K}} \quad \text{and} \quad \rho_{F^\infty_n \otimes B(\mathcal{K})}(\Phi) < 1.
\]
Since \( [F^2(H_n) \otimes \mathcal{K}] \otimes [\mathcal{N} \otimes \mathcal{K}] \) is hyperinvariant for \( \{S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}}\} \), the first relation in (3.4) is equivalent to
\[
P_{\mathcal{N} \otimes \mathcal{K}}(U^* \otimes I)\Phi(U \otimes I)|_{\mathcal{N} \otimes \mathcal{K}} = X.
\]
Since \( U(z_{\lambda_j}) = z_{\lambda_j}, j = 1, \ldots, k, \) and \( (\phi, z_{\lambda_j}) = \phi(\lambda_i) \) for any \( \phi := \sum_{\alpha \in F_n^+} a_\alpha e_\alpha \) in \( F^2(H_n) \), it is easy to see that
\[
\langle (U^* \otimes I)\Phi(U \otimes I)(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle
\]
\[
= \langle z_{\lambda_j}, z_{\lambda_j} \rangle \langle \Phi(\lambda_j)h, h' \rangle = \langle X(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle
\]
\[
= \langle \Phi(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle = \langle \lambda_{\lambda_j}, z_{\lambda_j} \rangle \langle W^*_uh, h' \rangle.
\]
for any \( j = 1, \ldots, k, \) and \( h, h' \in \mathcal{K} . \) This shows that (3.5) holds if and only if \( \Phi(\lambda_j) = W_j \) for any \( j = 1, \ldots, k. \) Notice that relation (3.3) holds if and only if there exists \( M \in \text{Inv}\{T_1, \ldots, T_n\} \) such that \( \|XM^{-1}\| < 1. \) It is easy to see that
\[
M^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes M^*_uh, \quad h \in \mathcal{K}, \text{ for some invertible operators } M_j \in B(\mathcal{K}), j = 1, \ldots, k.
\]
On the other hand, notice that
\[
M^{-1}X^*M^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes (M_jW_jM_j^{-1})^*h
\]
and \( \|XM^{-1}\| < 1 \) is equivalent to \( I_{\mathcal{N} \otimes \mathcal{K}} - (M^*MX^{-1})(M^*MX^{-1})^* > 0, \) which is equivalent to (3.1). This completes the proof.

Let us remark that the inequality (3.1) can be replaced with
\[
\rho_{F^\infty_n|\mathcal{N} \otimes B(\mathcal{K})}(X) < 1.
\]
In the particular case when \( n = 1 \), we find again Theorem 4 from [BFT]. As mentioned in [BFT], since \( P_{n}F_{n}^{\infty} \otimes B(K) \) is finite dimensional, conditions of type (3.6) can be checked using computer algorithms.

**Corollary 3.7.** Let \( K \) be a finite dimensional Hilbert space, \( W_{j} \in B(K) \), and \( \lambda_{j} \), \( j = 1, \ldots, k \), be distinct elements in \( B_{n} \). If there exist invertible operators \( M_{j} \in B(K) \), \( j = 1, \ldots, k \), such that

\[
\left[ I_{K} - (M_{i}W_{j}M_{j}^{-1})(M_{j}W_{j}M_{j}^{-1})^{*} \right]_{1 \leq i, j \leq k} > 0,
\]

then there exists \( f \in H^{\infty}(B_{n}) \otimes B(K) \) such that

\[
f(\lambda_{j}) = W_{j}, \quad j = 1, \ldots, k,
\]

and \( \sup_{\lambda \in B_{n}} \| f(\lambda) \|_{sp} < 1 \).

**Proof.** Using Theorem 3.6, we find \( f \in F_{n}^{\infty} \otimes B(K) \) such that \( f(\lambda_{j}) = W_{j}, \quad i = 1, \ldots, k \), and \( \rho_{F_{n}^{\infty} \otimes B(K)}(f) < 1 \). As in the proof of Theorem 2.1, we infer that

\[
\| \Psi(f) \|_{sp} \leq \rho_{H^{\infty}(B_{n}) \otimes B(K)}(\Psi(f)) \leq \rho_{F_{n}^{\infty} \otimes B(K)}(f) < 1.
\]

On the other hand, similarly to [BFT] Proposition 3, one can prove that

\[
\| \Psi(f) \|_{sp} = \sup_{\lambda \in B_{n}} \| f(\lambda) \|_{sp}.
\]

This completes the proof. \( \square \)

Let \( P_{m} \) be the set of all polynomials in \( F^{2}(H_{n}) \) of degree \( \leq m \), and let \( P_{m}^{\infty} := \{ p(S_{1}, \ldots, S_{n}) : p \in P_{m} \} \). Let \( J_{>m}^{\infty} \) be the WOT-closed two-sided ideal of \( F_{n}^{\infty} \) generated by \( \{ S_{\alpha} : \alpha \in \mathbb{P}^{n}_{+}, |\alpha| = m + 1 \} \). The following result is a spectral version of the noncommutative Carathéodory interpolation problem for \( F_{n}^{\infty} \) (see [Po6] and [Po8]).

**Theorem 3.8.** Let \( K \) be a finite dimensional Hilbert space and let \( p \in P_{m}^{\infty} \otimes B(K) \). Then there exists \( \Phi \in F_{n}^{\infty} \otimes B(K) \) with

\[
\rho_{F_{n}^{\infty} \otimes B(K)}(\Phi) < 1
\]

such that \( \Phi = p + g \) for some \( g \in J_{>m}^{\infty} \otimes B(K) \) if and only if

\[
(3.7) \quad \rho_{\mathcal{C}}[P_{m} \otimes K(U^{*} \otimes I)p(U \otimes I)|P_{m} \otimes K] < 1
\]

where \( \mathcal{C} := P_{m} \otimes K(R_{n}^{\infty} \otimes B(K))|P_{m} \otimes K \).

**Proof.** Let \( \mathcal{N} := P_{m} \) and \( X := P_{m} \otimes K(U^{*} \otimes I)p(U \otimes I)|P_{m} \otimes K \). Notice that \( X \) commutes with each \( P_{m}S_{i}|P_{m} \otimes K \), \( i = 1, \ldots, n \), and \( P_{m} = U P_{m} \) is invariant under each \( S_{i}^{*}, \ldots, S_{n}^{*} \). According to Theorem 3.3, relation (3.7) holds if and only if there exists \( \Phi \in F_{n}^{\infty} \otimes B(K) \) with \( P_{m} \otimes K(U^{*} \otimes I)\Phi(U \otimes I) = XP_{m} \otimes K \) and \( \rho_{F_{n}^{\infty} \otimes B(K)}(\Phi) < 1 \). Hence, we infer that

\[
(3.8) \quad P_{m} \otimes K(U^{*} \otimes I)(\Phi - p)(U \otimes I)|P_{m} \otimes K = 0.
\]

On the other hand, every element \( f \in F_{n}^{\infty} \otimes B(K) \) has a unique Fourier expansion \( f \sim \sum_{\alpha \in \mathbb{P}^{n}_{+}} S_{\alpha} \otimes W_{(\alpha)} \) determined by

\[
f(1 \otimes h) = \sum_{\alpha \in \mathbb{P}^{n}_{+}} e_{\alpha} \otimes W_{(\alpha)}h \in F^{2}(H_{n}) \otimes K,
\]
where $W_{(\alpha)} \in B(K)$ are given by $\langle W_{(\alpha)} h, k \rangle = \langle f(1 \otimes h), e_\alpha \otimes k \rangle$ for any $h, k \in K$, and $\alpha \in \mathbb{F}_n^+$ (see [Po8]). Using now relation (3.8), one can easily see that $g := \Phi - p \in J_{\infty}^{\infty} B(K)$. This completes the proof. 

Using Theorem 3.5, one can obtain a version of Theorem 3.8 for the algebra $\mathcal{W}_n^{\infty} \otimes B(K)$, in a similar manner. We leave this task to the reader.

4. Spectral tangential commutant lifting in several variables

Let $T := [T_1, \ldots, T_n]$ be a row contraction with $T_i \in B(H)$, and $\mathcal{V} := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $K \supseteq H$. Let $\mathcal{M} \subseteq \mathcal{H}$ be an invariant subspace under each $T_i^*$, $i = 1, \ldots, n$, and $X \in B(H)$ be such that $X \mathcal{H} \subseteq \mathcal{M}$ and

$$\text{(4.1)} \quad (P_{\mathcal{M}}T_i|_{\mathcal{M}})X = XT_i, \quad \text{for any } i = 1, \ldots, n.$$ 

According to the noncommutative commutant lifting theorem, there exists $Y \in \{V_1, \ldots, V_n\}'$ with $P_{\mathcal{M}}Y = XP_{\mathcal{H}}$. Define

$$\text{Dil}_{\mathcal{M}}(X) := \{Y \in \{V_1, \ldots, V_n\}' : P_{\mathcal{M}}Y = XP_{\mathcal{H}}\}$$

and

$$\rho_{\mathcal{M},(T_1,\ldots,T_n)'}(X) := \inf\{\|P_{\mathcal{M}} Z^{-1} XZ\| : Z \in \text{Inv}((\{T_1, \ldots, T_n\}')\}.$$ 

Notice that if $\mathcal{M} = \mathcal{H}$, then $\rho_{\mathcal{M},(T_1,\ldots,T_n)'}(X) = \rho_{(T_1,\ldots,T_n)'}(X)$.

In what follows we extend the spectral tangential commutant lifting theorem of Bercovici and Foias [BF] to our noncommutative multivariable setting.

**Theorem 4.1.** Let $T := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{V} := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $K \supseteq \mathcal{H}$. If $\mathcal{H}$ is finite dimensional, $K \otimes \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, and $\mathcal{M} \subseteq \mathcal{H}$ is an invariant subspace under each $T_i^*$, $i = 1, \ldots, n$, then, for every $X \in B(\mathcal{H})$ such that $X \mathcal{H} \subseteq \mathcal{M}$ and $(P_{\mathcal{M}}T_i|_{\mathcal{M}})X = XT_i, \ i = 1, \ldots, n$, we have

$$\text{(4.2)} \quad \rho_{\mathcal{M},(T_1,\ldots,T_n)'}(X) = \inf\{\rho_{(V_1,\ldots,V_n)'}(Y) : Y \in \text{Dil}_{\mathcal{M}}(X)\}.$$ 

**Proof.** Denote the right hand side of (4.2) by $t$. Let $\epsilon > 0$ and choose $Y \in \text{Dil}_{\mathcal{M}}(X)$ such that $\rho_{(V_1,\ldots,V_n)'}(Y) < t + \epsilon$. Hence, there is $W \in \text{Inv}((\{V_1, \ldots, V_n\}')$ such that $\|W^{-1}YW\| < t + \epsilon$. Since $K \otimes \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we infer that $P_{\mathcal{H}}WP_{\mathcal{H}} = P_{\mathcal{H}}W$. Let $Z := P_{\mathcal{H}}W|_{\mathcal{H}}$ and notice that $Z \in \text{Inv}((\{T_1, \ldots, T_n\}')$ and

$$\text{(4.3)} \quad Z^{-1} = P_{\mathcal{H}}W^{-1}|_{\mathcal{H}}.$$ 

The subspace $\mathcal{M}_* := Z^* \mathcal{M}$ is invariant under each $T_i^*$, $i = 1, \ldots, n$, and satisfies $\mathcal{M}_* = \mathcal{H} \otimes Z^{-1}(\mathcal{H} \otimes \mathcal{M})$. Hence, we deduce the relations

$$\text{(4.4)} \quad P_{\mathcal{M}_*}Z^{-1} = P_{\mathcal{M}}Z^{-1}P_{\mathcal{M}} \quad \text{and} \quad P_{\mathcal{M}}Z = P_{\mathcal{M}}ZP_{\mathcal{M}_*}.$$ 

Since $Y \in \text{Dil}_{\mathcal{M}}(X)$ and $\mathcal{K} \otimes \mathcal{H}$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we can use (4.4) and (4.3) to infer that

$$\|P_{\mathcal{M}_*}Z^{-1}XZ\| = \|P_{\mathcal{M}_*}Z^{-1}(P_{\mathcal{M}}Y|_{\mathcal{H}})Z\| = \|P_{\mathcal{M}}Z^{-1}(P_{\mathcal{H}}Y|_{\mathcal{H}})Z\| 
\quad = \|P_{\mathcal{M}_*}(P_{\mathcal{H}}W^{-1}|_{\mathcal{H}})(P_{\mathcal{H}}Y|_{\mathcal{H}})(P_{\mathcal{H}}W|_{\mathcal{H}})\| \leq \|P_{\mathcal{H}}(W^{-1}YW)|_{\mathcal{H}}\| \leq \|W^{-1}YW\| < t + \epsilon.$$ 

Since $\epsilon > 0$, we deduce that $\rho_{\mathcal{M},(T_1,\ldots,T_n)'}(X) \leq t$. 

Now, let us prove the converse. Let \( \epsilon > 0 \) and choose \( Z \in \text{Inv}(\{T_1, \ldots, T_n\}') \) such that
\[
\|P_M Z^{-1}XZ\| \leq \rho_{M,\{T_1, \ldots, T_n\}'}(X) + \epsilon.
\]
Since \( \{T_1, \ldots, T_n\}' \) is finite dimensional, we use Theorem 2.1 and Remark 2.3 when \( \Phi : \{V_1, \ldots, V_n\} \to \{T_1, \ldots, T_n\}' \) and \( \Phi(W) = P_W|_H \), to find \( W \in \text{Inv}(\{V_1, \ldots, V_n\}') \) such that \( Z = P_W|_H \). Denote \( X_* := P_M Z^{-1}XZ \) and notice that
\[
(P_M T_i|_{M_*) X_* = X_i T_i, \quad i = 1, \ldots, n.
\]
Indeed, since \( M_* \) is invariant under each \( T_i^* \), \( i = 1, \ldots, n \), we have \( P_M T_i P_M = P_M T_i \), \( i = 1, \ldots, n \). Using this relation together with (4.1) and (4.4), we infer that, for any \( i = 1, \ldots, n \),
\[
X_i T_i = P_M Z^{-1}XZ T_i = P_M Z^{-1}XT_i Z
= P_M Z^{-1}(P_M T_i|_{M_*)} XZ = P_M Z^{-1}T_i XZ
= P_M X_i Z^{-1}XZ = P_M P_M Z^{-1}XZ
= P_M T_i X_i.
\]
According to (4.6), the noncommutative commutant lifting theorem, and relation (4.5), we find \( Y_* \in \text{Dil}_{M_*}(X_*) \) satisfying
\[
\|Y_*\| = \|X_*\| \leq \rho_{M,\{T_1, \ldots, T_n\}'}(X) + \epsilon.
\]
Set \( Y := W Y_* W^{-1} \) and let us show that \( Y \in \text{Dil}_M(X) \). Notice that
\[
X = P_M ZX_* Z^{-1}.
\]
Indeed, using (4.4), we have
\[
P_M ZX_* Z^{-1} = P_M Z(P_M Z^{-1}XZ)Z^{-1} = P_M ZP_M Z^{-1}X
= P_M ZX_* Z^{-1} = P_M X = X.
\]
Since \( P_M Y_* = X_* P_H, Z^{-1} = P_H W^{-1}|_H \), and \( Y(K \oplus H) \subseteq K \oplus H \), we can use relation (4.8) to obtain
\[
XP_H = P_M ZX_* Z^{-1}P_H = P_M ZP_M Y_* Z^{-1}P_H
= P_M ZP_H Y_* Z^{-1}P_H = P_M (P_H Z|_H)(P_H Z^{-1}|_H)(P_H W^{-1}|_H) P_H
= P_M (P_H W Y_* W^{-1}|_H) P_H = P_M Y.P_H = P_M Y.
\]
According to (4.7), we have \( \|W^{-1}Y\| = \|Y_*\| \leq \rho_{M,\{T_1, \ldots, T_n\}'}(X) + \epsilon \). Hence \( \rho_{\Phi,\{\Phi(T_1, \ldots, T_n)\}'}(Y) \leq \rho_{M,\{T_1, \ldots, T_n\}'}(X) + \epsilon \) and \( t \leq \rho_{M,\{T_1, \ldots, T_n\}'}(X) + \epsilon \). This completes the proof.

The following result is a spectral version of the tangential Nevanlinna-Pick interpolation problem for \( F_n^\infty \) (see [Po8]).

**Theorem 4.2.** Let \( \lambda_j, j = 1, \ldots, k \), be distinct elements in \( \mathbb{B}_n \) and let \( K \) be a finite dimensional Hilbert space. If \( u_1, \ldots, u_k, v_1, \ldots, v_k \in K \) with \( u_i \neq 0, j = 1, \ldots, k \), and \( \delta > 0 \), then there exists \( \Phi \in F_n^\infty \otimes B(K) \) such that
\[
\Phi(\lambda_j) u_j = v_j, \quad j = 1, \ldots, k, \quad \text{and} \quad \rho_{F_n^\infty \otimes B(K)}(\Phi) < \delta.
\]
if and only if there exist invertible operators $Z_j \in B(K)$, $j = 1, \ldots, k$, such that

$$\left(\delta Z_j u_j, \delta Z_i u_i - \langle Z_j v_j, Z_i v_i \rangle\right)_{1 \leq i, j \leq k} > 0. \tag{4.9}$$

Proof. Let $\mathcal{N} := \text{span}\{z_{\lambda_j} : j = 1, \ldots, k\}$ and $\mathcal{M} := \mathbb{C} z_{\lambda_1} \otimes u_1 + \cdots + \mathbb{C} z_{\lambda_k} \otimes u_k$ be a subspace of $\mathcal{N} \otimes K$. Define $X(\{\lambda_j\}, \{u_j\}, \{v_j\}) \in B(\mathcal{N} \otimes K, \mathcal{M})$ by setting $X(\{\lambda_j\}, \{u_j\}, \{v_j\}) := z_{\lambda_j} \otimes v_j$, $j = 1, \ldots, k$. For each $i = 1, \ldots, n$, define $T_i := P_N S_i |_{\mathcal{N} \otimes K}$ and notice that $T_i^* X^* = X T_i^* |_{\mathcal{M}}$, where $X := X(\{\lambda_j\}, \{u_j\}, \{v_j\})$. Hence, $XT_i = PM T_i X$ for any $i = 1, \ldots, n$.

As in the proof of Theorem 3.3, the minimal isometric dilation of the sequence $[T_1, \ldots, T_n]$ is $[S_1 \otimes I_K, \ldots, S_n \otimes I_K]$ and $[F^2(H_n) \otimes K] \otimes [\mathcal{N} \otimes K]$ is hyperinvariant for $\{S_1 \otimes I_K, \ldots, S_n \otimes I_K\}$. Since $\mathcal{M} \subseteq \mathcal{N} \otimes K$ is invariant under each $T_i^*$, $i = 1, \ldots, n$, we can apply Theorem 4.1 and infer that

$$\rho_{\mathcal{M}, \{T_1, \ldots, T_n\}^\prime}(X) = \inf \{\rho_{\{S_1 \otimes I_K, \ldots, S_n \otimes I_K\}^\prime}(Y) : Y \in \text{Dil}_K(X)\}. \tag{4.10}$$

Since $\{S_1 \otimes I_K, \ldots, S_n \otimes I_K\}^\prime = U\star F_n^\prime U \overset{\otimes}{\otimes} B(K)$, we can see that

$$\rho_{\mathcal{M}, \{T_1, \ldots, T_n\}^\prime}(X) < \delta$$

if and only if there exists $\Phi \in F_n^\prime \otimes B(K)$ such that $\rho_{F_n^\prime \otimes B(K)}(\Phi) < \delta$ and

$$P_{\mathcal{M}}(U^* \otimes I)\Phi(U \otimes I) = X P_{\mathcal{N} \otimes K}. \tag{4.11}$$

Notice that

$$\langle P_{\mathcal{M}}(U^* \otimes I)\Phi(U \otimes I)(z_{\lambda_i} \otimes k), z_{\lambda_j} \otimes u_j \rangle = \langle \Phi(z_{\lambda_i} \otimes k), z_{\lambda_j} \otimes u_j \rangle$$

$$= \langle z_{\lambda_i}, z_{\lambda_j} \rangle \langle \Phi(\lambda_j) k, u_j \rangle$$

$$= \langle z_{\lambda_i}, z_{\lambda_j} \rangle \langle k, \Phi(\lambda_j)^* u_j \rangle$$

and

$$\langle X(z_{\lambda_i} \otimes k), z_{\lambda_j} \otimes u_j \rangle = \langle z_{\lambda_i}, z_{\lambda_j} \rangle \langle k, v_j \rangle$$

for any $k \in K$ and $i, j = 1, \ldots, k$. Therefore, the relation (4.11) holds if and only if $\Phi(\lambda_j)^* u_j = v_j$, $j = 1, \ldots, k$. On the other hand, if $Z \in \{T_1, \ldots, T_n\}^\prime$ then

$$Z^*(z_{\lambda_i} \otimes k) = z_{\lambda_i} \otimes Z_k, \quad k \in K, \tag{4.12}$$

for some $Z_j \in B(K), j = 1, \ldots, k$. Notice that $Z$ is invertible if and only if $Z_j$ is invertible for any $j = 1, \ldots, k$. Moreover, using the definition of $X = X(\{\lambda_j\}, \{u_j\}, \{v_j\})$ and (4.12), we have

$$Z^* X^*(\{\lambda_j\}, \{u_j\}, \{v_j\}) Z^* X^*(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\}).$$

Therefore,

$$\rho_{\mathcal{M}, \{T_1, \ldots, T_n\}^\prime}(X) = \inf \{\|X(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\})\| : Z_j \in B(K) \text{ are invertible}\}$$

and relation (4.10) holds if and only if there exist invertible operators $Z_j \in B(K)$ such that $\|X(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\})\| < \delta$. This inequality is equivalent to

$$\delta^2 I - X(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\}) X^*(\{\lambda_j\}, \{Z_j u_j\}, \{Z_j v_j\}) > 0,$$

which is equivalent to (4.9). This completes the proof. \hfill \blacksquare

We remark that (4.9) can be replaced by relation (4.10). As a consequence of Theorem 4.2, when the distinct elements in $\mathbb{F}_n$ are $\lambda_j, j = 1, \ldots, k$, we infer the following spectral tangential interpolation result for matrix-valued bounded analytic functions in the unit ball of $\mathbb{C}^n$. 


Corollary 4.3. Let \( \lambda_j, j = 1, \ldots, k \), be distinct elements in \( \mathbb{B}_n \) and let \( K \) be a finite dimensional Hilbert space. If \( u_1, \ldots, u_k, v_1, \ldots, v_k \in K \) with \( u_i \neq 0 \), \( j = 1, \ldots, k \), \( \delta > 0 \), and there exist invertible operators \( Z_j \in B(K), \ j = 1, \ldots, k \), such that
\[
\left[ \frac{\langle \delta Z_j u_j, \delta Z_i u_i \rangle - \langle Z_j v_j, Z_i v_i \rangle}{1 - \langle \lambda_i, \lambda_j \rangle} \right]_{1 \leq i, j \leq k} > 0,
\]
then there exists \( F \in H^\infty(\mathbb{B}_n) \otimes B(K) \) such that
\[
\sup_{\lambda \in \mathbb{B}_n} \| F(\lambda) \|_p < \delta \quad \text{and} \quad F(\lambda_j) u_j = v_j, \ j = 1, \ldots, k.
\]

Let us make some remarks on the dependence of \( \rho_{M, \{T_1, \ldots, T_n\}}(X) \) on the given interpolation data. For each \( m = 1, \ldots, k \), we define
\[
\rho_m := \inf \{ \| X(\{\lambda_j\}^m_{j=1}, \{Z_j u_j\}^m_{j=1}, \{Z_j v_j\}^m_{j=1}) \| : Z_j \in B(K) \text{ are invertible} \}.
\]

A multivariable analogue of [BF, Proposition 4] holds. More precisely, one can prove that if \( u_k \) and \( v_k \) are linearly independent, then \( \rho_{k-1} = \rho_k \). Indeed, suppose that \( \rho_{k-1} < \rho_k \). Using Theorem 4.2, we find \( \Phi \in F^\infty_n \otimes B(K) \) such that \( \rho_{F^\infty_n \otimes B(K)}(\Phi) < \rho_k \) and \( \Phi(\lambda_j) u_j = v_j, \ j = 1, \ldots, k - 1 \). We may suppose that \( \Phi(\lambda_k)^* \notin CI_K \) because, otherwise, we can replace \( \Phi \) by \( \Phi + \Psi \) for some \( \Psi \in F^\infty_n \otimes B(K) \) satisfying \( \Phi(\lambda_j) = 0, \ j = 1, \ldots, k - 1 \), and \( \Psi(\lambda_k) \notin CI_K \). Since we can choose \( \Psi \) with very small norm we have \( \rho_{F^\infty_n \otimes B(K)}(\Phi + \Psi) < \rho_k \).

Therefore, since \( \Phi(\lambda_k)^* \notin CI_K \), there exist linearly independent vectors \( u \) and \( v \) such that \( \Phi(\lambda_k)^* u = v \). Since \( u_k, v_k \) are linearly independent, we can find \( Z_k \in B(K) \) invertible with \( Z_k u_k = u \) and \( Z_k v_k = v \). Hence, we infer that \( \rho_k \leq \rho_{F^\infty_n \otimes B(K)}(\Phi) < \rho_k \), which is a contradiction. Since \( \rho_{k-1} \leq \rho_k \), we must have \( \rho_k = \rho_{k-1} = \rho_k \). This shows that in Theorem 4.2 we can assume, without loss of generality, that \( v_j = \mu_j u_j \), for some \( \mu_j \in \mathbb{C}, \mu_j \neq 0, \ j = 1, \ldots, k \). Similarly to [BF, Proposition 5], one can show that if \( k \leq \dim K \), then
\[
\rho_k = \max \{ |\mu_1|, \ldots, |\mu_k| \}.
\]

The case when the number of dependent vector pairs \( (u_j, v_j) \) exceeds the dimension of \( K \), and the problem of optimal solutions will be considered in a future paper.

References


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