SPECTRAL LIFTING IN BANACH ALGEBRAS AND INTERPOLATION IN SEVERAL VARIABLES

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Abstract. Let $A$ be a unital Banach algebra and let $J$ be a closed two-sided ideal of $A$. We prove that if any invertible element of $A/J$ has an invertible lifting in $A$, then the quotient homomorphism $\Phi : A \to A/J$ is a spectral interpolant. This result is used to obtain a noncommutative multivariable analogue of the spectral commutant lifting theorem of Bercovici, Foiaş, and Tannenbaum. This yields spectral versions of Sarason, Nevanlinna–Pick, and Carathéodory type interpolation for $F^\infty_n \otimes B(K)$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra $F^\infty_n$ and $B(K)$, the algebra of bounded operators on a finite dimensional Hilbert space $K$. A spectral tangential commutant lifting theorem in several variables is considered and used to obtain a spectral tangential version of the Nevanlinna-Pick interpolation for $F^\infty_n \otimes B(K)$.

In particular, we obtain interpolation theorems for matrix-valued bounded analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant and not the norm.

1. Introduction and preliminaries

Let $\mathbb{D}$ denote the unit disc in the complex plane, let $z_1, \ldots, z_k \in \mathbb{D}$ be given distinct points, and $F_1, \ldots, F_k$ be complex $m \times m$ matrices. The classical Nevanlinna–Pick problem [N], [P] consists in finding necessary and sufficient conditions for the existence of an analytic $m \times m$ matrix-valued function $F(z)$ with $F(z_j) = F_j$ ($1 \leq j \leq k$) and such that $\|F\|_{\infty} \leq 1$.

Motivated by problems in control engineering, such as the design of feedback control systems in the presence of parameter uncertainty, Bercovici, Foiaş, and Tannenbaum proved in [BFT] a spectral generalization of the commutant lifting theorem [SzF1], and obtained a spectral version of the Nevanlinna–Pick problem, in which the infinity norm is replaced by

$$\rho(F) := \sup \{ \|F(z)\|_{sp} : z \in \mathbb{D} \}$$

($\|A\|_{sp}$ denotes the spectral radius of an operator $A$).

The tangential Nevanlinna–Pick problem considered by Fedcina [F] is to find $F \in H^\infty(\mathbb{D}) \otimes \mathbb{C}^m$ with $F(z_j)u_j = v_j$, $j = 1, \ldots, k$, and $\|F\|_\infty \leq 1$, where $z_j \in \mathbb{D}$ and $u_j, v_j \in \mathbb{C}^m$ are prescribed. The spectral tangential Nevanlinna–Pick interpolation problem, considered by Bercovici and Foiaş [BF], is to find such an $F$ for which $\rho(F) < 1$. This type of interpolation was also motivated by certain control engineering applications.
In this paper we find noncommutative multivariable analogues of the abovementioned results obtained by Bercovici, Foiaș, and Tannenbaum (see \cite{BF, BF2} for the noncommutative analytic Toeplitz algebra $F_2^\infty$. In particular, we obtain interpolation results (see Corollary 3.7 and Corollary 4.3) for matrix-valued bounded analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant and not the norm.

We expect these results to play a role in multivariable control and systems theory, as it does in the case $n = 1$. We mention the papers \cite{BV} and \cite{H} for recent results in multivariable linear systems.

We need to recall some facts concerning the noncommutative analytic Toeplitz algebra $F_2^\infty$ and its connection with the function theory on the open unit ball of $\mathbb{C}^n$. Let $F_2^2(H_n) = \mathcal{C}1 \oplus \bigoplus_{m \geq 1} H_n^2$ be the full Fock space on $n$ generators, where $H_n$ is an $n$-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ if $n$ is finite, and $\{e_1, e_2, \ldots\}$ if $n = \infty$. For each $i = 1, 2, \ldots$, define the left creation operator by $S_i \xi := e_i \otimes \xi$, $\xi \in F_2^2(H_n)$.

We shall denote by $\mathcal{P}$ the set of all $p \in F_2^2(H_n)$ which are finite sums of tensor monomials. Define $F_2^\infty$ as the set of all $g \in F_2^2(H_n)$ such that

$$\|g\|_{\infty} := \sup\{\|g \otimes p\|_{F_2^2(H_n)} : p \in \mathcal{P}, \|p\|_{F_2^2(H_n)} \leq 1\} < \infty.$$  

We denote by $\mathcal{A}_n$ the closure of $\mathcal{P}$ in $(F_2^\infty, \cdot, \|\cdot\|_{\infty})$. The Banach algebra $F_2^\infty$ (resp. $\mathcal{A}_n$) can be viewed as a noncommutative analogue of the Hardy space $H_2(H)$ (resp. disc algebra $A(D)$): when $n = 1$ they coincide.

In \cite[Theorem 3.1]{Po1} we proved that $\mathcal{A}_n$ is completely isometrically isomorphic to the norm-closed algebra generated by any sequence $V_1, \ldots, V_n$ of isometries with $V_1 V_1^* + \cdots + V_n V_n^* = I$, and the identity. It follows from \cite[Theorem 4.3]{Po1} that the noncommutative analytic Toeplitz algebra $F_2^\infty$ can be identified with the WOT-closed algebra generated by the left creation operators $S_1, \ldots, S_n$, and the identity.

The algebras $F_2^\infty$ and $\mathcal{A}_n$ were introduced by the author in \cite{Po1} in connection with a noncommutative von Neumann inequality, and have been studied in several papers \cite{Po1}, \cite{Po2}, \cite{Po3}, \cite{Po4}, \cite{Po5}, \cite{Po6}, \cite{Po7}, \cite{ArPo1}, and recently in \cite{DP1}, \cite{DP2}, \cite{ArPo2}, \cite{DP3}, and \cite{Po8}.

We established a strong connection between the algebra $F^\infty_2$ and the function theory on the open unit ball $B_n$ of $\mathbb{C}^n$ through the noncommutative von Neumann inequality \cite{Po1} (see also \cite{Po2}, \cite{Po3}, and \cite{Po4}). In particular, we proved that there is a completely contractive homomorphism

$$\Phi : F^\infty_2 \to H_2^\infty(B_n), \quad f(S_1, \ldots, S_n) \mapsto f(\lambda_1, \ldots, \lambda_n),$$

where $(\lambda_1, \ldots, \lambda_n) \in B_n$. A characterization of the analytic functions in the range of the map $\Phi$ was obtained in \cite{ArPo2} and \cite{DP3}. W. Arveson proved that $\Phi$ is not surjective \cite{Ar} and the functions in its range are the multipliers of a certain function Hilbert space. In \cite{ArPo2}, \cite{DP3}, it was proved that $F^\infty_2 / \ker \Phi$ is an operator algebra which can be identified with $W_2^\infty := P_{F_2^2} F_2^\infty |_{F_2^2}$, the compression to the symmetric Fock space $F_2^2 \subseteq F_2^2(H_n)$. In \cite{Po8}, \cite{Po9}, \cite{Ar}, \cite{ArPo2}, \cite{DP3}, \cite{AMC}, and \cite{BTV}, a good case is made that the appropriate commutative multivariable analogue of $H_2^\infty(D)$ is the algebra $W_2^\infty$, which is the WOT-closed algebra generated by $B_i := P_{F_2^2} S_i |_{F_2^2}$, $i = 1, \ldots, n$, and the identity. In this paper, we provide further evidence that $F^\infty_2$ (resp. $W_2^\infty$) is a noncommutative (resp. commutative) multivariate analogue of $H_2^\infty(D)$. 


Let $A$ be a unital Banach algebra and denote by $\text{Inv}(A)$ the group of invertible elements of $A$. Given $a \in A$, we define the $A$-spectral radius of $a$ by setting $$\rho_A(a) := \inf \{ ||xax^{-1}|| : x \in \text{Inv}(A) \}.$$ Since the spectral radius of $a \in A$ is $\|a\|_{sp} = \lim_{n \to \infty} ||a^n||^{1/n}$, it is clear that $\|a\|_{sp} = ||xax^{-1}||_{sp}$ for any $x \in \text{Inv}(A)$. Now, it is easy to see that $\|a\|_{sp} = \rho_A(a)$.

Let $A, B$ be unital Banach algebras, and $\Phi : A \to B$ be a unital contractive homomorphism. We say that $\Phi$ is a quotient interpolant if $\|b\| = \inf \{ ||a|| : a \in A, \Phi(a) = b \}$ for any $b \in B$. We say that $b \in B$ with $\|b\| < 1$ has a spectral lifting if there exists $a \in A$ such that $\Phi(a) = b$ and $\rho_A(a) < 1$. The homomorphism $\Phi$ is called a spectral interpolant if any $b \in B$ has a spectral lifting.

**Problem.** Let $\Phi : A \to B$ be a unital contractive homomorphism which is also a quotient interpolant. When is $\Phi$ a spectral interpolant?

We show, in Section 2, that this problem has a positive answer if $\text{Inv}(B) \subseteq \Phi(\text{Inv}(A))$. This relation holds, for example, if the group of invertible elements of $B$ is connected (in particular, if $B$ is finite dimensional or equal to $B(H)$).

The results of Section 2 are used in Section 3 to obtain a noncommutative multivariable analogue (see Theorem 3.1) of the spectral commutant lifting theorem of Bercovici-Foias-Tannenbaum. This yields spectral versions of Sarason ([S]), Nevanlinna-Pick, and Carathéodory type interpolation for $F_n^\infty \otimes B(K)$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra $F_n^\infty$ and $B(K)$, the algebra of bounded operators on a finite dimensional Hilbert space $K$.

In Section 4, we obtain a spectral tangential commutant lifting theorem in several variables (see Theorem 4.1). This leads to a spectral tangential Nevanlinna-Pick interpolation for $F_n^\infty \otimes B(K)$ (see Theorem 4.2).

Problems concerning the optimal solutions to these spectral interpolation problems in several variables, and explicit algorithm for finding the optimal interpolants will be considered in a future paper.

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2. Spectral lifting in Banach algebras

The notation and definitions from Section 1 are used throughout the paper. Let $A, B$ be unital Banach algebras and let $\Phi : A \to B$ be a unital contractive homomorphism. We call $\Phi$ a norm preserving interpolant if for any $b \in B$ there exists $a \in A$ such that $\Phi(a) = b$ and $\|a\| = \|b\|$. Notice that any norm preserving interpolant is a quotient interpolant. Examples of norm preserving interpolants will be presented in Section 3.
Theorem 2.1. Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras and let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a unital contractive homomorphism with the property that $\text{Inv}(\mathcal{B}) \subseteq \Phi(\text{Inv}(\mathcal{A}))$ and
\[
\|b\| = \inf\{\|a\| : a \in \mathcal{A}, \Phi(a) = b\}
\]
for any $b \in \mathcal{B}$. Then
\[
\rho_{\mathcal{B}}(b) = \inf\{\rho_{\mathcal{A}}(a) : a \in \mathcal{A}, \Phi(a) = b\}
\]
for any $b \in \mathcal{B}$. In particular, $\Phi$ is a spectral interpolant.

Proof. Let $b \in \mathcal{B}$ and $a \in \mathcal{A}$ with $\Phi(a) = b$. Since $\Phi$ is a contractive homomorphism and $\Phi(\text{Inv}(\mathcal{A})) \subseteq \text{Inv}(\mathcal{B})$ we have
\[
\rho_{\mathcal{A}}(a) = \inf\{\|waw^{-1}\| : w \in \text{Inv}(\mathcal{A})\}
\]
\[
\geq \inf\{\|\Phi(waw^{-1})\| : w \in \text{Inv}(\mathcal{A})\}
\]
\[
= \inf\{\|\Phi(w)\Phi(w)^{-1}\| : w \in \text{Inv}(\mathcal{A})\}
\]
\[
\geq \inf\{\|zbz^{-1}\| : z \in \text{Inv}(\mathcal{B})\}
\]
\[
= \rho_{\mathcal{B}}(b).
\]
Therefore,
\[
\rho_{\mathcal{B}}(b) \leq \inf\{\rho_{\mathcal{A}}(a) : a \in \mathcal{A}, \Phi(a) = b\}.
\]
Now, let $\epsilon > 0$ and choose $z \in \text{Inv}(\mathcal{B})$ such that
\[
\|zbz^{-1}\| \leq \rho_{\mathcal{B}}(b) + \frac{\epsilon}{2}.
\]
Since $zbz^{-1} \in \mathcal{B}$, according to the hypothesis, for any $\epsilon > 0$, there exists $d \in \mathcal{A}$ such that
\[
\Phi(d) = zbz^{-1} \quad \text{and} \quad \|d\| \leq \|zbz^{-1}\| + \frac{\epsilon}{2}.
\]
Since $\Phi(\text{Inv}(\mathcal{A})) \supseteq \text{Inv}(\mathcal{B})$, we find $w \in \text{Inv}(\mathcal{A})$ such that $\Phi(w) = z$. Notice that $y := w^{-1}dw \in \mathcal{A}$ and
\[
\Phi(y) = \Phi(w)^{-1}\Phi(d)\Phi(w) = z^{-1}(zbz^{-1})z = b.
\]
Now, using (2.2) and (2.3), we infer that
\[
\rho_{\mathcal{A}}(y) \leq \|wzw^{-1}\| = \|d\| \leq \|zbz^{-1}\| + \frac{\epsilon}{2} \leq \rho_{\mathcal{A}}(b) + \epsilon.
\]
Therefore,
\[
\rho_{\mathcal{B}}(b) \geq \inf\{\rho_{\mathcal{A}}(a) : a \in \mathcal{A}, \phi(a) = b\}.
\]
Using relation (2.1), it is easy to see that if $b \in \mathcal{B}$, then $\rho_{\mathcal{B}}(b) < 1$ if and only if there exists $a \in \mathcal{A}$ such that $\Phi(a) = b$ and $\rho_{\mathcal{A}}(a) < 1$. This completes the proof. \hfill \blacksquare

Corollary 2.2. Let $\mathcal{A}, \mathcal{B}$ be unital Banach algebras such that the group $\text{Inv}(\mathcal{B})$ is connected. Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a unital contractive homomorphism which is also a quotient interpolant. Then $\Phi$ is a spectral interpolant.

Proof. Let us prove that
\[
\Phi(\text{Inv}(\mathcal{A})) = \text{Inv}(\mathcal{B}).
\]
The inclusion $\Phi(\text{Inv}(\mathcal{A})) \subseteq \text{Inv}(\mathcal{B})$ is clear. Conversely, let $x \in \text{Inv}(\mathcal{B})$. Since $\text{Inv}(\mathcal{B})$ is connected, it is well known that
\[
x = \exp(z_1) \cdots \exp(z_k)
\]
for some $z_1, \ldots, z_k \in B$. Due to the hypothesis, there exist $w_1, \ldots, w_k \in \mathcal{A}$ such that $\Phi(w_i) = z_i$, $i = 1, \ldots, k$. Denote $y := \exp(w_1) \cdots \exp(w_k) \in \text{Inv}(\mathcal{A})$ and notice that $\Phi(y) = \exp(\Phi(w_1)) \cdots \exp(\Phi(w_k)) = x$. Hence $\Phi(\text{Inv}(\mathcal{A})) \supseteq \text{Inv}(B)$ and (2.4) holds.

Remark 2.3. If $B$ is a finite dimensional algebra, then $\text{Inv}(B) = \exp(B)$, hence $\text{Inv}(B)$ is connected.

Corollary 2.4. Let $\mathcal{A}$ be a unital Banach algebra and let $J$ be a closed two-sided ideal of $\mathcal{A}$. If any invertible element of $\mathcal{A}/J$ has an invertible lifting in $\mathcal{A}$, then the quotient homomorphism $\Phi : \mathcal{A} \to \mathcal{A}/J$ is a spectral interpolant, i.e., $\rho_{\mathcal{A}/J}(a+J) < 1$ if and only if there exists $b \in a + J$ such that $\rho_{\mathcal{A}}(b) < 1$.

Proof. Apply Theorem 2.1 to the quotient homomorphism $\Phi$.

Let us remark that, in general, there are invertible elements in $\mathcal{A}/J$ which can not be lifted to invertible elements in $\mathcal{A}$. For example, if $\pi : B(H^2) \to B(H^2)/K(H^2)$ is the quotient homomorphism into the Calkin algebra, and $S$ is the unilateral shift on the Hardy space $H^2$, then $\pi(S)$ is invertible and there is no invertible operator $T \in B(H^2)$ such that $\pi(T) = \pi(S)$.

An important particular case, when Corollary 2.4 can be applied, is when the quotient algebra $\mathcal{A}/J$ is finite dimensional. Applications of this result will be considered in the next section.

3. Noncommutative spectral commutant lifting and interpolation

Let $F_n^+$ be the unital free semigroup on $n$ generators $s_1, \ldots, s_n$, and let $e$ be its neutral element. For any $\sigma := s_{i_1} \cdots s_{i_k} \in F_n^+$ we define its length $|\sigma| := k$, and $|e| = 0$. On the other hand, if $T_i \in B(\mathcal{H})$, $i = 1, \ldots, n$, we denote $T_\sigma := T_{i_1} \cdots T_{i_k}$ and $T_e := I_\mathcal{H}$.

Let us recall from [Po1, Po2, Po4] some results concerning the noncommutative dilation theory for $n$-tuples of operators. A sequence of operators $T := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, $i = 1, \ldots, n$, is called contractive (or row contraction) if $T_1 T_1^* + \cdots + T_n T_n^* \leq I_\mathcal{H}$. We say that a sequence of isometries $V := [V_1, \ldots, V_n]$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ is a minimal isometric dilation of $T$ if the following properties are satisfied:

(i) $V_i V_i^* + \cdots + V_n V_n^* \leq I_\mathcal{K}$;
(ii) $V_i^* |\mathcal{H} = T_i^*$, $i = 1, \ldots, n$;
(iii) $\mathcal{K} = \bigvee_{\alpha \in F_n^+, \alpha \neq e} V_\alpha \mathcal{H}$.

The minimal isometric dilation of $T$ is uniquely determined up to an isomorphism. We need to recall the noncommutative commutant lifting theorem [Po4] (see [SzF1, SzF2, DMP] for the classical case).

Let $T := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$ and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert $\mathcal{K} \supseteq \mathcal{H}$. If $X \in B(\mathcal{H})$ and $X T_i = T_i X$ for any $i = 1, \ldots, n$, then there exists $X_\infty \in B(\mathcal{K})$ satisfying the following properties:

(i) $X_\infty V_i = V_i X_\infty$, for any $i = 1, \ldots, n$;
(ii) $X_\infty^* |\mathcal{H} = X^*$;
(iii) $\|X_\infty\| = \|X\|$.
Let $T := [T_1, \ldots, T_n]$ be a row contraction with $T_i \in B(H)$ and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $K \supseteq H$. Let $X \in \{T_1, \ldots, T_n\}'$, and denote

$$\text{Dil}(X) := \{Y \in \{V_1, \ldots, V_n\}' : P_H Y = X P_H\},$$

where $P_H$ is the orthogonal projection on $H$. According to the noncommutative commutant lifting theorem, we have $\text{Dil}(X) \neq \emptyset$.

In what follows we obtain a noncommutative multivariable analogue of the spectral commutant lifting theorem of Bercovici-Foias-Tannenbaum [BFT].

**Theorem 3.1.** Let $T := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $H$ and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $K \supseteq H$. If $H$ is finite dimensional and $K \supseteq H$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, then

$$\rho(t_1, \ldots, t_n) \cdot (X) = \inf\{\rho(V_1, \ldots, V_n) \cdot (Y) : Y \in \text{Dil}(X)\}$$

for any $X \in \{T_1, \ldots, T_n\}'$.

**Proof.** Let $\Phi : \{V_1, \ldots, V_n\}' \to \{T_1, \ldots, T_n\}'$ be defined by $\Phi(Y) := P_H Y|_H$. Since $K \supseteq H$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we have $Y^*|_H \subseteq H$ for any $Y \in \{V_1, \ldots, V_n\}'$. Since $V := [V_1, \ldots, V_n]$ is the minimal isometric dilation of $T$, we have $V_i^*|_H = T_i^*$, $i = 1, \ldots, n$. Now, it is easy to see that

$$(P_H Y|_H) T_i = T_i (P_H Y|_H) \quad \text{for any } i = 1, 2, \ldots, n.$$ 

Therefore, the mapping $\Phi$ is well-defined. On the other hand, since $K \supseteq H$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, we infer that $\Phi$ is a unital contractive homomorphism, and $\Phi(Y) = X$ is equivalent to $P_H Y = XP_H$. According to the noncommutative commutant lifting theorem, for any $X \in \{T_1, \ldots, T_n\}'$ there exists $Y \in \{V_1, \ldots, V_n\}'$ such that $P_H Y = XP_H$ and $\|Y\| = \|X\|$. Therefore, $\Phi$ is a norm preserving interpolant. Since $H$ is finite dimensional, the algebra $\{T_1, \ldots, T_n\}'$ is finite dimensional. Applying Theorem 2.1 and Remark 2.3, in the particular case when $A := \{V_1, \ldots, V_n\}'$ and $B := \{T_1, \ldots, T_n\}'$, the result follows.  

**Corollary 3.2.** Let $T := [T_1, \ldots, T_n]$ be a contractive sequence of operators on a Hilbert space $H$ and let $V := [V_1, \ldots, V_n]$ be its minimal isometric dilation on a Hilbert space $K \supseteq H$. If $H$ is finite dimensional and $K \supseteq H$ is hyperinvariant for $\{V_1, \ldots, V_n\}$, then, given $X \in \{T_1, \ldots, T_n\}'$, $\rho(t_1, \ldots, t_n) \cdot (X) < 1$ if and only if there exists $Y \in \text{Dil}(X)$ such that $\rho(V_1, \ldots, V_n) \cdot (Y) < 1$.

In what follows, we use the noncommutative spectral commutant lifting theorem to obtain spectral versions of Sarason, Nevanlinna–Pick, and Carathéodory type interpolation for $F_\infty \otimes B(K)$, the WOT-closed algebra generated by the spatial tensor product of the noncommutative analytic Toeplitz algebra $F_\infty$ and $B(K)$. In particular, we obtain interpolation results for matrix-valued analytic functions on the open unit ball of $\mathbb{C}^n$, in which one bounds the spectral radius of the interpolant.

According to Theorem 1.2 from [Po2], the commutant of $F_\infty$, which we denote by $R_\infty$, is equal to $U^* F_\infty U$, where $U$ is the unitary operator on $F^2(H)$ defined by $U(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = e_{i_1} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}$. Moreover, the commutant of $R_\infty$ is equal to $F_\infty$.

A complete description of the invariant subspace structure of $F_\infty$ was obtained in [Po2] Theorem 2.2 (even in a more general setting). A subspace $\mathcal{N}$ of $F^2(H_n)$ is invariant under $S_1, \ldots, S_n$ if and only if $\mathcal{N} = \bigoplus_{\lambda \in \Lambda} U^* \varphi_\lambda U [F^2(H_n)]$, for some
family \( \{ \varphi_\lambda \in F^\infty_n : \lambda \in \Lambda \} \) of isometries with orthogonal ranges (see also \( \text{Po8} \) and \( \text{DFT} \)). Let us remark that \( \mathcal{M} \subseteq F^2(H_n) \) is hyperinvariant for \( \{ S_1, \ldots, S_n \} \), i.e., invariant for \( \{ S_1, \ldots, S_n \}' \), if and only if \( UM \) is invariant for \( \{ S_1, \ldots, S_n \} \).

**Theorem 3.3.** Let \( \mathbb{N} \) be a finite dimensional Hilbert space and let \( \mathcal{N} \subseteq F^2(H_n) \) be a finite dimensional subspace with the property that \( \mathcal{N} \) and \( U \mathcal{N} \) are invariant under \( S_1^*, \ldots, S_n^* \). Then \( X \in B(\mathcal{N} \otimes \mathbb{K}) \) commutes with each \( P_{\mathcal{N}S_1}|_{\mathcal{N} \otimes I_\mathbb{K}}, i = 1, \ldots, n \), and

\[
\rho_{P_{\mathcal{N}S_1}|_{\mathcal{N} \otimes I_\mathbb{K}}}(X) < 1
\]

if and only there exists \( \Psi \in R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K}) \) such that

\[
P_{\mathcal{N} \otimes \mathbb{K}} \Psi = XP_{\mathcal{N} \otimes \mathbb{K}} \quad \text{and} \quad \rho_{R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K})}(\Psi) < 1.
\]

**Proof.** According to \( \text{Po8} \), we have

\[
\mathcal{B} := \{ P_{\mathcal{N}S_1}|_{\mathcal{N} \otimes I_\mathbb{K}}, i = 1, \ldots, n \}' = P_{\mathcal{N} \otimes \mathbb{K}}(R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K}))|_{\mathcal{N} \otimes \mathbb{K}}.
\]

Notice that \( \mathcal{B} \) is a finite dimensional algebra. Let \( \mathcal{A} := R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K}) \) and define \( \Phi : \mathcal{A} \to \mathcal{B} \) by \( \Phi(Y) = P_{\mathcal{N} \otimes \mathbb{K}} Y|_{\mathcal{N} \otimes \mathbb{K}} \). Since \( S_1^* (U \mathcal{N}) \subseteq U \mathcal{N} \) for any \( i = 1, \ldots, n \), and \( \{ S_1 \otimes I_\mathbb{K}, \ldots, S_n \otimes I_\mathbb{K} \}' = R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K}) \), it is easy to see that \( [F^2(H_n) \otimes \mathbb{K}] \subseteq [\mathcal{N} \otimes \mathbb{K}] \) is hyperinvariant for \( \{ S_1 \otimes I_\mathbb{K}, \ldots, S_n \otimes I_\mathbb{K} \} \) and the mapping \( \Phi \) is a unital contractive homomorphism. Since \( \mathcal{N} \) is invariant under \( S_1^*, \ldots, S_n^* \), it is clear that the operator matrix \( [P_{\mathcal{N}S_1}|_{\mathcal{N} \otimes \mathbb{K}}, \ldots, P_{\mathcal{N}S_n}|_{\mathcal{N} \otimes \mathbb{K}}] \) is a \( C_0 \)-row contraction and its minimal isometric dilation is \( [S_1, \ldots, S_n] \) (see \( \text{Po8} \)). Therefore, the minimal isometric dilation of \( [P_{\mathcal{N}S_1}|_{\mathcal{N} \otimes \mathbb{K}}, \ldots, P_{\mathcal{N}S_n}|_{\mathcal{N} \otimes \mathbb{K}}] \) is \( [S_1 \otimes I_\mathbb{K}, \ldots, S_n \otimes I_\mathbb{K}] \).

According to the noncommutative commutant lifting theorem, for any \( X \in \mathcal{B} \) there exists \( \Psi \in R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K}) \), such that \( P_{\mathcal{N} \otimes \mathbb{K}} \Psi = XP_{\mathcal{N} \otimes \mathbb{K}} \) and \( \| X \| = \| \Psi \| \). Therefore, \( \Phi(\Psi) = X \) and \( \Phi \) is a norm preserving interpolant. Applying Corollary 3.2, the result follows.

Notice that the element \( \Psi \) in Theorem 3.3 satisfies \( \| \Psi \|_{sp} \leq \rho_{R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K})}(\Psi) < 1 \). It would be nice to know if \( \rho_{R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K})}(\Psi) = \| \Psi \|_{sp} \) for any \( \Psi \in R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K}) \). This equality holds if \( n = 1 \) (see \( \text{DFT} \)).

Let us remark that the finite dimensionality hypothesis can be dropped in Theorem 3.3 for those subspaces \( \mathcal{N} \) and \( \mathbb{K} \) for which one can prove that any invertible element \( f \in P_{\mathcal{N}R_{\mathcal{N} \otimes \mathbb{K}}^\infty}|_{\mathcal{N} \otimes \mathbb{K}} \) can be lifted to an invertible element \( g \in R_{\mathcal{N} \otimes \mathbb{K}}^\infty \otimes B(\mathbb{K}) \), i.e., \( P_{\mathcal{N} \otimes \mathbb{K}} g|_{\mathcal{N} \otimes \mathbb{K}} = f \). We do not have yet any nontrivial example when this lifting property holds and \( \mathcal{N}, \mathbb{K} \) are infinite dimensional.

Let \( J \) be a WOT-closed, two-sided ideal of \( F_n^\infty \) and define \( J(1) := \{ \Psi(1) : \Psi \in J \} \) and \( \mathcal{N}_J := F^2(H_n) \cap J(1) \). Let us remark that \( \mathcal{N}_J \) and \( U \mathcal{N}_J \) are invariant subspaces under \( S_i^* \), \( i = 1, \ldots, n \), therefore, Theorem 3.3 works in the case when \( \dim \mathcal{N}_J < \infty \).

**Corollary 3.4.** Let \( \mathbb{K} \) be a finite dimensional Hilbert space and let \( J \) be a WOT-closed two-sided ideal of \( F_n^\infty \) such that \( \dim \mathcal{N}_J < \infty \). Then the quotient homomorphism

\[
\Phi : F_n^\infty \otimes B(\mathbb{K}) \to F_n^\infty \otimes B(\mathbb{K})/(J \otimes B(\mathbb{K})){/n}
\]

is a spectral interpolant.

**Proof.** According to \( \text{ArPo2} \), the quotient algebra \( F_n^\infty \otimes B(\mathbb{K})/(J \otimes B(\mathbb{K})) \) is completely isometrically isomorphic to \( P_{\mathcal{N}_J} F_n^\infty|_{\mathcal{N}_J} \otimes B(\mathbb{K}) \), which is finite dimensional. Using Theorem 3.3, we infer that \( \Phi \) is a spectral interpolant. The proof is complete.

\[\Box\]
It will be interesting to see if this result remains true if $\mathcal{N}_f$ is infinite dimensional (at least for some particular cases, if not in general). The obstruction in the infinite dimensional case seems to be the lifting of the invertible elements of a quotient algebra $A/J$ to invertible elements of $A$ (see Section 2 for an example). In the finite dimensional case, Corollary 3.4 leads to our spectral interpolation results for $F_n^\infty$ (see Theorem 3.6 and Theorem 3.8).

Let $F^2(H_n)$ be the symmetric Fock space and $W_n^\infty$ be the WOT-closed algebra generated by $B_i := F^2(H_n)s_i F^2(H_n)$, $i = 1, \ldots, n$, and the identity. This algebra has been studied in [Po9], [Arv], [ArPo2], [DP3]. The following theorem can be seen as a spectral version of Sarason’s interpolation theorem for $H^\infty(\mathbb{B})$ (see [S]), in a commutative and multivariable setting.

**Theorem 3.5.** Let $\mathcal{E} \subseteq F^2(H_n)$ be a finite dimensional invariant subspace under $B_1, \ldots, B_n$ and let $K$ be a finite dimensional Hilbert space. Then $f \in B(\mathcal{E} \otimes K)$ commutes with each $P_{E_i}B_i | \mathcal{E} \otimes I_K$, $i = 1, \ldots, n$, and

$$\rho_{P_{E_i} \otimes K}(W_n^\infty \otimes B(K))_{| \mathcal{E} \otimes K}(f) < 1$$

if and only if there exists $g \in W_n^\infty \otimes B(K)$ such that

$$P_{E_i} \otimes K|g_{| \mathcal{E} \otimes K} = f \quad \text{and} \quad \rho_{W_n^\infty \otimes B(K)}(g) < 1.$$  

**Proof.** Since $F^2_n(H_n)$ is invariant under each $S_i^*$, $i = 1, \ldots, n$, it is easy to see that $\mathcal{E}$ has the same property. Taking into account that $W_n^\infty$ is the compression of $F^\infty$ to the symmetric Fock space, one can see that $f$ commutes with $P_{E \otimes K}(S_i \otimes I_K)_{| \mathcal{E} \otimes K}$. As in the proof of Theorem 3.3, using the noncommutative commutant lifting theorem, we find $\phi \in F^\infty \otimes B(K)$ such that $P_n \otimes B(K)(U^* \otimes I_K)\phi(U \otimes I)_{| \mathcal{E} \otimes K} = f$ and $\|f\| = \|\phi\|$. Hence, $P_{E \otimes K}\phi_{| \mathcal{E} \otimes K} = f$. Setting $g := P_{E_i}B_i | \mathcal{E} \otimes K\phi_{| E_i \otimes K} \in W_n^\infty \otimes B(K)$, we have $P_{E \otimes K}g_{| \mathcal{E} \otimes K} = f$ and $\|f\| \leq \|g\| \leq \|\phi\| = \|f\|$. This shows that $\|f\| = \|g\|$. Define $A := W_n^\infty \otimes B(K)$, $B := P_{E_i \otimes K}(W_n^\infty \otimes B(K))_{| \mathcal{E} \otimes K}$ and let $\Phi : A \rightarrow B$ be defined by $\Phi(g) := P_{E_i \otimes K}(g)_{| \mathcal{E} \otimes K}$. We just proved that $\Phi$ is a unital contractive homomorphism and also a norm preserving interpolant. Now, the result follows by applying the results of Section 2 in our setting.

Let us remark that a result similar to Corollary 3.4 holds for the algebra $W_n^\infty \otimes B(K)$.

In what follows we obtain a spectral version of Nevanlinna-Pick interpolation for the noncommutative analytic Toeplitz algebra $F_n^\infty$ (see [ArPo2], [DP3], and [Po8]). As mentioned in the first section, there exists a unital contractive homomorphism

$$\Psi : F_n^\infty \otimes B(K) \rightarrow H^\infty(\mathbb{B}_n) \otimes B(K)$$

defined by $[\Psi(f)](\lambda) := f(\lambda)$, $\lambda \in \mathbb{B}_n$.

**Theorem 3.6.** Let $K$ be a finite dimensional Hilbert space, $W_j \in B(K)$, and $\lambda_j$, $j = 1, \ldots, k$, be distinct elements in $\mathbb{B}_n$. Then there exists $\Phi \in F_n^\infty \otimes B(K)$ such that

$$\rho_{F_n^\infty \otimes B(K)}(\Phi) < 1 \quad \text{and} \quad \Phi(\lambda_j) = W_j, \quad j = 1, \ldots, k,$$

if and only if there exist invertible operators $M_j \in B(K)$, $j = 1, \ldots, k$, such that

$$(3.1) \quad \left[ I_K - (M_i W_i M_i^{-1})(M_j W_j M_j^{-1})^* \right]_{1 \leq i, j \leq k} > 0.$$
Proof. Let \( \lambda_j := (\lambda_{j1}, \ldots, \lambda_{jn}) \in \mathbb{B}_n, j = 1, \ldots, k \). For any \( \alpha := s_{j1}s_{j2}\ldots s_{jn} \) in \( F_+^n \), let \( \lambda_{j\alpha} := \lambda_{j1}\lambda_{j2}\ldots \lambda_{jn} \) and \( \lambda_c := 1 \). Define \( z_{\lambda_j} \in F^2(H_n) \) by setting
\[
z_{\lambda_j} := \sum_{\alpha \in F_+^n} \lambda_{j\alpha} e_\alpha, \quad j = 1, 2, \ldots, k.
\]

Let \( \mathcal{N} := \text{span}\{z_{\lambda_j} : j = 1, \ldots, k\} \) and \( X \in B(\mathcal{N} \otimes \mathcal{K}) \) be defined by
\[
(3.2) \quad X^*(z_{\lambda_j} \otimes h) := z_{\lambda_j} \otimes W_j^* h, \quad h \in \mathcal{K}.
\]

Notice that \( S_i^* z_{\lambda_j} = \overline{x}_{ij} z_{\lambda_j} \) for any \( i = 1, \ldots, n; j = 1, \ldots, k \). Hence, the subspaces \( \mathcal{N} \) and \( U \mathcal{N} \) are invariant under each \( S_i^* \), \( i = 1, \ldots, n \). Define \( T_i \in B(\mathcal{N} \otimes \mathcal{K}) \) by \( T_i := P_{\mathcal{N}} S_i |_{\mathcal{N} \otimes I_{\mathcal{K}}} \). Since \( z_{\lambda_1}, \ldots, z_{\lambda_k} \) are linearly independent, the operator \( X \in B(\mathcal{N} \otimes \mathcal{K}) \) given by (3.2) is well defined.

Notice that \( X T_i = T_i X \) for any \( i = 1, \ldots, k \). Indeed,
\[
T_i^* X^*(z_{\lambda_j} \otimes h) = T_i^* (z_{\lambda_j} \otimes W_j^* h) = S_i^* z_{\lambda_j} \otimes W_j^* h = \overline{x}_{ij} z_{\lambda_j} \otimes W_j^* h
\]
and
\[
X^* T_i^* (z_{\lambda_j} \otimes h) = X^* (\overline{x}_{ij} z_{\lambda_j} \otimes h) = \overline{x}_{ij} z_{\lambda_j} \otimes W_j^* h.
\]

Applying Theorem 3.3, we infer that
\[
(3.3) \quad \rho(T_1, \ldots, T_n)^N(X) < 1
\]
if and only there exists \( \Phi \in F_+^n \otimes B(\mathcal{K}) \) such that
\[
(3.4) \quad P_{\mathcal{N} \otimes \mathcal{K}}(U^* \otimes I) \Phi(U \otimes I) = X P_{\mathcal{N} \otimes \mathcal{K}} \quad \text{and} \quad \rho_{F_+^n \otimes B(\mathcal{K})}(\Phi) < 1.
\]

Since \( [F^2(H_n) \otimes \mathcal{K}] \otimes [\mathcal{N} \otimes \mathcal{K}] \) is hyperinvariant for \( \{S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}}\} \), the first relation in (3.4) is equivalent to
\[
(3.5) \quad P_{\mathcal{N} \otimes \mathcal{K}}(U^* \otimes I) \Phi(U \otimes I) |_{\mathcal{N} \otimes \mathcal{K}} = X.
\]

Since \( U(z_{\lambda_j}) = z_{\lambda_j}, j = 1, \ldots, k, \) and \( \langle \phi, z_{\lambda_j} \rangle = \phi(\lambda_j) \) for any \( \phi := \sum_{\alpha \in F_+^n} a_\alpha e_\alpha \) in \( F^2(H_n) \), it is easy to see that
\[
\langle (U^* \otimes I) \Phi(U \otimes I)(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle
\]
\[
= \langle z_{\lambda_j}, z_{\lambda_j} \rangle \langle \Phi(\lambda_j) h, h' \rangle = \langle X(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle
\]
\[
= \langle \Phi(z_{\lambda_j} \otimes h), z_{\lambda_j} \otimes h' \rangle = \langle z_{\lambda_j}, z_{\lambda_j} \rangle \langle W_j h, h' \rangle.
\]
for any \( j = 1, \ldots, k, \) and \( h, h' \in \mathcal{K} \). This shows that (3.5) holds if and only if \( \Phi(\lambda_j) = W_j \) for any \( j = 1, \ldots, k \). Notice that relation (3.3) holds if and only if there exists \( M \in \text{Inv}(\{T_1, \ldots, T_n\}^N) \) such that \( \|M X M^{-1}\| < 1 \). It is easy to see that \( M^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes M_j^* h, h \in \mathcal{K}, \) for some invertible operators \( M_j \in B(\mathcal{K}), j = 1, \ldots, k \). On the other hand, notice that
\[
M^{-1} X^* M^*(z_{\lambda_j} \otimes h) = z_{\lambda_j} \otimes (M_j W_j M_j^{-1})^* h
\]
and \( \|M X M^{-1}\| < 1 \) is equivalent to \( I_{\mathcal{N} \otimes \mathcal{K}} - (M X M^{-1})(M X M^{-1})^* > 0 \), which is equivalent to (3.1). This completes the proof.

Let us remark that the inequality (3.1) can be replaced with
\[
(3.6) \quad \rho P_{\mathcal{N} \otimes \mathcal{K}}(X) < 1.
\]
In the particular case when \( n = 1 \), we find again Theorem 4 from [BFT]. As mentioned in [BFT], since \( P_n F_n^\infty \otimes B(K) \) is finite dimensional, conditions of type (3.6) can be checked using computer algorithms.

**Corollary 3.7.** Let \( K \) be a finite dimensional Hilbert space, \( W_j \in B(K) \), and \( \lambda_j \), \( j = 1, \ldots, k \), be distinct elements in \( \mathbb{B}_n \). If there exist invertible operators \( M_j \in B(K) \), \( j = 1, \ldots, k \), such that

\[
\left[ I_K - (M_i W_j M_i^{-1})(M_j W_j M_j^{-1})^* \right]_{1 \leq i,j \leq k} > 0,
\]

then there exists \( f \in H^\infty(\mathbb{B}_n) \otimes B(K) \) such that

\[
f(\lambda_j) = W_j, \quad j = 1, \ldots, k, \quad \text{and} \quad \sup_{\lambda \in \mathbb{B}_n} \|f(\lambda)\|_{sp} < 1.
\]

**Proof.** Using Theorem 3.6, we find \( f \in F_n^\infty \otimes B(K) \) such that \( f(\lambda_j) = W_j, \quad i = 1, \ldots, k \), and \( \rho_{F_n^\infty \otimes B(K)}(f) < 1 \). As in the proof of Theorem 2.1, we infer that

\[
\|\Psi(f)\|_{sp} \leq \rho_{H^\infty(\mathbb{B}_n) \otimes B(K)}(\Psi(f)) \leq \rho_{F_n^\infty \otimes B(K)}(f) < 1.
\]

On the other hand, similarly to [BFT] Proposition 3, one can prove that

\[
\|\Psi(f)\|_{sp} = \sup_{\lambda \in \mathbb{B}_n} \|f(\lambda)\|_{sp}.
\]

This completes the proof. \( \blacksquare \)

Let \( \mathcal{P}_m \) be the set of all polynomials in \( F^2(H_n) \) of degree \( \leq m \), and let \( \mathcal{P}_m^\infty := \{ p(S_1, \ldots, S_n) : \ p \in \mathcal{P}_m \} \). Let \( J_{> m}^\infty \) be the WOT-closed two-sided ideal of \( F_n^\infty \) generated by \( \{ S_\alpha : \alpha \in \mathbb{F}_m^n, |\alpha| = m + 1 \} \). The following result is a spectral version of the noncommutative Carathéodory interpolation problem for \( F_n^\infty \) (see [Po8] and [Po8]).

**Theorem 3.8.** Let \( K \) be a finite dimensional Hilbert space and let \( p \in \mathcal{P}_m^\infty \otimes B(K) \). Then there exists \( \Phi \in F_n^\infty \otimes B(K) \) with

\[
\rho_{F_n^\infty \otimes B(K)}(\Phi) < 1
\]

such that \( \Phi = p + g \) for some \( g \in J_{> m}^\infty \otimes B(K) \) if and only if

\[
\rho_C[P_{\mathcal{P}_m \otimes K}(U^* \otimes I)p(U \otimes I)|_{\mathcal{P}_m \otimes K}] < 1
\]

where \( C := P_{\mathcal{P}_m \otimes K}(R_n^\infty \otimes B(K))|_{\mathcal{P}_m \otimes K} \).

**Proof.** Let \( \mathcal{N} := \mathcal{P}_m \) and \( X := P_{\mathcal{P}_m \otimes K}(U^* \otimes I)p(U \otimes I)|_{\mathcal{P}_m \otimes K} \). Notice that \( X \) commutes with each \( P_{\mathcal{P}_m S_i}|_{\mathcal{P}_m \otimes I_K} \), \( i = 1, \ldots, n \), and \( P_{\mathcal{P}_m} = UP_{\mathcal{P}_m} \) is invariant under each \( S_1^*, \ldots, S_n^* \). According to Theorem 3.3, relation (3.7) holds if and only if there exists \( \Phi \in F_n^\infty \otimes B(K) \) with \( P_{\mathcal{P}_m \otimes K}(U^* \otimes I)\Phi(U \otimes I) = XP_{\mathcal{P}_m \otimes K} \) and \( \rho_{F_n^\infty \otimes B(K)}(\Phi) < 1 \). Hence, we infer that

\[
P_{\mathcal{P}_m \otimes K}(U^* \otimes I)(\Phi - p)(U \otimes I)|_{\mathcal{P}_m \otimes K} = 0.
\]

On the other hand, every element \( f \in F_n^\infty \otimes B(K) \) has a unique Fourier expansion \( f \sim \sum_{\alpha \in \mathbb{F}_n^+} S_\alpha \otimes W(\alpha) \) determined by

\[
f(1 \otimes h) = \sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes W(\alpha) h \in F^2(H_n) \otimes K,
\]
where \( W_\alpha \in B(\mathcal{K}) \) are given by \( \langle W_\alpha h, k \rangle = \langle f(1 \otimes h), e_\alpha \otimes k \rangle \) for any \( h, k \in \mathcal{K} \), and \( \alpha \in \mathbb{F}_n^+ \) (see [Po8]). Using now relation (3.8), one can easily see that \( g := \Phi - p \in J_{\infty}^\omega \otimes B(\mathcal{K}) \). This completes the proof.

Using Theorem 3.5, one can obtain a version of Theorem 3.8 for the algebra \( W_n^\omega \otimes B(\mathcal{K}) \), in a similar manner. We leave this task to the reader.

4. Spectral tangential commutant lifting in several variables

Let \( T := [T_1, \ldots, T_n] \) be a row contraction with \( T_i \in B(\mathcal{H}) \), and \( V := [V_1, \ldots, V_n] \) be its minimal isometric dilation on a Hilbert space \( \mathcal{K} \supseteq \mathcal{H} \). Let \( \mathcal{M} \subseteq \mathcal{H} \) be an invariant subspace under each \( T_i^* \), \( i = 1, \ldots, n \), and \( X \in B(\mathcal{H}) \) be such that \( X\mathcal{H} \subseteq \mathcal{M} \) and

\[
(P_{\mathcal{M}}T_i|_{\mathcal{M}})X = XT_i, \quad \text{for any } i = 1, \ldots, n.
\]

According to the noncommutative commutant lifting theorem, there exists \( Y \in \{V_1, \ldots, V_n\}' \) with \( P_{\mathcal{M}}Y = XP_\mathcal{H} \). Define

\[
Dil_\mathcal{M}(X) := \{Y \in \{V_1, \ldots, V_n\}' : P_{\mathcal{M}}Y = XP_\mathcal{H}\}
\]

and

\[
\rho_{\mathcal{M},\{T_1, \ldots, T_n\}'}(X) := \inf\{\|P_\mathcal{M}Z^{-1}XZ\| : Z \in \text{Inv}(\{T_1, \ldots, T_n\}')\}.
\]

Notice that if \( \mathcal{M} = \mathcal{H} \), then \( \rho_{\mathcal{M},\{T_1, \ldots, T_n\}'}(X) = \rho_{\{T_1, \ldots, T_n\}'}(X) \).

In what follows we extend the spectral tangential commutant lifting theorem of Bercovici and Foias [BF] to our noncommutative multivariable setting.

**Theorem 4.1.** Let \( T := [T_1, \ldots, T_n] \) be a contractive sequence of operators on a Hilbert space \( \mathcal{H} \) and let \( V := [V_1, \ldots, V_n] \) be its minimal isometric dilation on a Hilbert space \( \mathcal{K} \supseteq \mathcal{H} \). If \( \mathcal{H} \) is finite dimensional, \( \mathcal{K} \otimes \mathcal{H} \) is hyperinvariant for \( \{V_1, \ldots, V_n\} \) and \( \mathcal{M} \subseteq \mathcal{H} \) is an invariant subspace under each \( T_i^* \), \( i = 1, \ldots, n \), then, for every \( X \in B(\mathcal{H}) \) such that \( X\mathcal{H} \subseteq \mathcal{M} \) and \( (P_{\mathcal{M}}T_i|_{\mathcal{M}})X = XT_i, \quad i = 1, \ldots, n \), we have

\[
\rho_{\mathcal{M},\{T_1, \ldots, T_n\}'}(X) = \inf\{\rho_{\{V_1, \ldots, V_n\}'}(Y) : Y \in Dil_\mathcal{M}(X)\}.
\]

**Proof.** Denote the right hand side of (4.2) by \( t \). Let \( \epsilon > 0 \) and choose \( Y \in \text{Dil}_\mathcal{M}(X) \) such that \( \rho_{\{V_1, \ldots, V_n\}'}(Y) \leq t + \epsilon \). Hence, there is \( W \in \text{Inv}(\{V_1, \ldots, V_n\}') \) such that \( \|W^{-1}YW\| < t + \epsilon \). Since \( \mathcal{K} \otimes \mathcal{H} \) is hyperinvariant for \( \{V_1, \ldots, V_n\} \), we infer that \( P_\mathcal{H}WP_\mathcal{H} = P_\mathcal{H}W \). Let \( Z := P_\mathcal{H}W|_{\mathcal{H}} \) and notice that \( Z \in \text{Inv}(\{T_1, \ldots, T_n\}') \) and

\[
Z^{-1} = P_\mathcal{H}W^{-1}|_{\mathcal{H}}.
\]

The subspace \( \mathcal{M}_i := Z^*\mathcal{M} \) is invariant under each \( T_i^* \), \( i = 1, \ldots, n \), and satisfies \( \mathcal{M}_i = \mathcal{H} \otimes Z^{-1}(\mathcal{H} \otimes \mathcal{M}) \). Hence, we deduce the relations

\[
P_{\mathcal{M}_i}Z^{-1} = P_{\mathcal{M}_i}Z^{-1}P_{\mathcal{M}_i} \quad \text{and} \quad P_{\mathcal{M}_i}Z = P_{\mathcal{M}_i}ZP_{\mathcal{M}_i}.
\]

Since \( Y \in \text{Dil}_\mathcal{M}(X) \) and \( \mathcal{K} \otimes \mathcal{H} \) is hyperinvariant for \( \{V_1, \ldots, V_n\} \), we can use (4.4) and (4.3) to infer that

\[
\|P_{\mathcal{M}_i}Z^{-1}XZ\| = \|P_{\mathcal{M}_i}Z^{-1}(P_{\mathcal{M}_i}Y|_{\mathcal{H}})Z\| = \|P_{\mathcal{M}_i}Z^{-1}(P_\mathcal{H}Y|_{\mathcal{H}})Z\| = \|P_{\mathcal{M}_i}(P_\mathcal{H}W^{-1}|_{\mathcal{H}})(P_\mathcal{H}Y|_{\mathcal{H}})(P_\mathcal{H}W|_{\mathcal{H}})\| \leq \|P_\mathcal{H}(W^{-1}YW)|_{\mathcal{H}}\| \leq \|W^{-1}YW\| < t + \epsilon.
\]

Since \( \epsilon > 0 \), we deduce that \( \rho_{\mathcal{M},\{T_1, \ldots, T_n\}'}(X) \leq t \).
Now, let us prove the converse. Let \( \epsilon > 0 \) and choose \( Z \in \text{Inv}(\{T_1, \ldots, T_n\}') \) such that
\[
\|P_{M_i}Z^{-1}XZ\| \leq \rho_{M_i,\{T_1, \ldots, T_n\}'}(X) + \epsilon.
\]
Since \( \{T_1, \ldots, T_n\}' \) is finite dimensional, we use Theorem 2.1 and Remark 2.3 when \( \Phi : \{V_1, \ldots, V_n\} \to \{T_1, \ldots, T_n\}' \) and \( \Phi(W) = P_{\mathcal{H}}W|_{\mathcal{H}} \), to find \( W \in \text{Inv}(\{V_1, \ldots, V_n\}') \) such that \( Z = P_{\mathcal{H}}W|_{\mathcal{H}} \). Denote \( X_* := P_{M_i}Z^{-1}XZ \) and notice that
\[
(P_{M_i}T_i|M_i)X_* = X_iT_i, \quad i = 1, \ldots, n.
\]
Indeed, since \( \mathcal{M}_* \) is invariant under each \( T_i^* \), \( i = 1, \ldots, n \), we have \( P_{M_i}T_iP_{M_i} = P_{M_i}T_i \), \( i = 1, \ldots, n \). Using this relation together with (4.1) and (4.4), we infer that, for any \( i = 1, \ldots, n \),
\[
X_iT_i = P_{M_i}Z^{-1}XZ = P_{M_i}Z^{-1}(P_{M_i}T_i|M_i)XZ = P_{M_i}Z^{-1}T_iXZ
= P_{M_i}T_iZ^{-1}XZ = P_{M_i}T_iP_{M_i}Z^{-1}XZ
= P_{M_i}T_iX_*.
\]
According to (4.6), the noncommutative commutant lifting theorem, and relation (4.5), we find \( Y_* \in \text{Dil}_{M_i}(X_*) \) satisfying
\[
\|Y_*\| = \|X_*\| \leq \rho_{M_i,\{T_1, \ldots, T_n\}'}(X) + \epsilon.
\]
Set \( Y := WY_*W^{-1} \) and let us show that \( Y \in \text{Dil}_M(X) \). Notice that
\[
X = P_{M}ZX_*Z^{-1}.
\]
Indeed, using (4.4), we have
\[
P_{M}ZX_*Z^{-1} = P_{M}Z(P_{M_i}Z^{-1}XZ)Z^{-1} = P_{M}ZP_{M_i}Z^{-1}X
= P_{M}ZZ^{-1}X = P_{M}X = X.
\]
Since \( P_{M_i}Y_* = X_iP_{\mathcal{H}}, \) \( Z^{-1} = P_{\mathcal{H}}W^{-1}|_{\mathcal{H}}, \) and \( Y(\mathcal{K} \ominus \mathcal{H}) \subseteq \mathcal{K} \ominus \mathcal{H} \), we can use relation (4.8) to obtain
\[
XP_{\mathcal{H}} = P_{M}ZX_*Z^{-1}P_{\mathcal{H}} = P_{M}ZP_{M_i}Y_*Z^{-1}P_{\mathcal{H}}
= P_{M}ZP_{\mathcal{H}}Y_*Z^{-1}P_{\mathcal{H}} = P_{M}(P_{\mathcal{H}}Z|_{\mathcal{H}})P_{\mathcal{H}}(P_{\mathcal{H}}Y_*|_{\mathcal{H}})(P_{\mathcal{H}}W^{-1}|_{\mathcal{H}})P_{\mathcal{H}}P_{\mathcal{H}} = P_{M}Y_{\mathcal{H}}Y_{\mathcal{H}}W^{-1}|_{\mathcal{H}}P_{\mathcal{H}} = P_{M}Y_{\mathcal{H}}Y_{\mathcal{H}}W^{-1}|_{\mathcal{H}}.
\]
According to (4.7), we have \( \|W^{-1}YW\| = \|Y_*\| \leq \rho_{M_i,\{T_1, \ldots, T_n\}'}(X) + \epsilon \). Hence \( \rho_{\{V_1, \ldots, V_n\}'}(Y) \leq \rho_{M_i,\{T_1, \ldots, T_n\}'}(X) + \epsilon \) and \( t \leq \rho_{M_i,\{T_1, \ldots, T_n\}'}(X) + \epsilon \). This completes the proof.

The following result is a spectral version of the tangential Nevanlinna-Pick interpolation problem for \( F_{\infty}^{\infty} \) (see [Po8]).

**Theorem 4.2.** Let \( \lambda_j, \quad j = 1, \ldots, k, \) be distinct elements in \( B_\infty \) and let \( \mathcal{K} \) be a finite dimensional Hilbert space. If \( u_1, \ldots, u_k, v_1, \ldots, v_k \in \mathcal{K} \) with \( u_i \neq 0, j = 1, \ldots, k, \) and \( \delta > 0 \), then there exists \( \Phi \in F_{\infty}^{\infty} \otimes B(\mathcal{K}) \) such that
\[
\Phi(\lambda_j)^*u_j = v_j, \quad j = 1, \ldots, k, \quad \text{and} \quad \rho_{F_{\infty}^{\infty} \otimes B(\mathcal{K})}(\Phi) < \delta.
\]
if and only if there exist invertible operators $Z_j \in B(K)$, $j = 1, \ldots, k$, such that

$$
(4.9) \quad \left| \frac{\delta Z_j u_j, \delta Z_i u_i - \langle Z_j v_j, Z_i v_i \rangle}{1 - \lambda_j \lambda_i} \right| > 0.
$$

Proof. Let $\mathcal{N} := \text{span}\{z_\lambda : j = 1, \ldots, k\}$ and $\mathcal{M} := \mathbb{C}z_\lambda \otimes u_1 + \cdots + \mathbb{C}z_\lambda \otimes u_k$ be a subspace of $\mathcal{N} \otimes K$. Define $X((\lambda_j), \{u_j\}, \{v_j\}) \in B(\mathcal{N} \otimes K, \mathcal{M})$ by setting $X((\lambda_j), \{u_j\}, \{v_j\}) := z_\lambda \otimes v_j$, $j = 1, \ldots, k$. For each $i = 1, \ldots, n$, define $T_i := P_N S_i |_{\mathcal{N} \otimes I_K}$ and notice that $T_i^* X^* = X^* T_i^* |_{\mathcal{M}}$, where $X := X((\lambda_j), \{u_j\}, \{v_j\})$. Hence, $X T_i = P_M T_i X$ for any $i = 1, \ldots, n$.

As in the proof of Theorem 3.3, the minimal isometric dilation of the sequence $[T_1, \ldots, T_n]$ is $[S_1 \otimes I_K, \ldots, S_n \otimes I_K]$ and $[F^2(H_n) \otimes K] \ominus [N \otimes K]$ is hyperinvariant for the sets $\{S_1 \otimes I_K, \ldots, S_n \otimes I_K\}$. Since $\mathcal{M} \subseteq N \otimes K$ is invariant under each $T_i^*$, $i = 1, \ldots, n$, we can apply Theorem 4.1 and infer that

$$
\rho_{\mathcal{M}, \{T_1, \ldots, T_n\}'}(X) = \inf \{\rho_{\mathcal{S}_1 \otimes I_K, \ldots, \mathcal{S}_n \otimes I_K}': Y \in \text{Dil}_K(X)\}.
$$

Since $\{S_1 \otimes I_K, \ldots, S_n \otimes I_K\}' = U^* F_n^\otimes U \otimes B(K)$, we can see that

$$
(4.10) \quad \rho_{\mathcal{M}, \{T_1, \ldots, T_n\}'}(X) < \delta
$$

if and only if there exists $\Phi \in F_n^\otimes B(K)$ such that $\rho_{F_n^\otimes B(K)}(\Phi) < \delta$ and

$$
(4.11) \quad P_M (U^* \otimes I) \Phi (U \otimes I) = X P_N \otimes K.
$$

Notice that

$$
\langle P_M (U^* \otimes I) \Phi (U \otimes I) (z_\lambda \otimes k), z_\lambda \otimes u_j \rangle = \langle \Phi(z_\lambda \otimes k), z_\lambda \otimes u_j \rangle = \langle z_\lambda, z_\lambda \rangle \langle \Phi(\lambda_j) k, u_j \rangle = \langle z_\lambda, z_\lambda \rangle \langle k, \Phi(\lambda_j) u_j \rangle
$$

and $\langle X(z_\lambda \otimes k), z_\lambda \otimes u_j \rangle = \langle z_\lambda, z_\lambda \rangle \langle k, v_j \rangle$ for any $k \in K$ and $i, j = 1, \ldots, k$. Therefore, the relation (4.11) holds if and only if $\Phi(\lambda_j) u_j = v_j$, $j = 1, \ldots, k$. On the other hand, if $Z \in \{T_1, \ldots, T_n\}'$ then

$$
(4.12) \quad Z^* (z_\lambda \otimes k) = z_\lambda \otimes Z_j k, \quad k \in K,
$$

for some $Z_j \in B(K), j = 1, \ldots, k$. Notice that $Z$ is invertible if and only if $Z_j$ is invertible for any $j = 1, \ldots, k$. Moreover, using the definition of $X = X((\lambda_j), \{u_j\}, \{v_j\})$ and (4.12), we have

$$
Z^* X^* ((\lambda_j), \{u_j\}, \{v_j\}) Z^{-1} |_{Z^* \mathcal{M}} = X^* ((\lambda_j), \{Z_j u_j\}, \{Z_j v_j\}).
$$

Therefore,

$$
\rho_{\mathcal{M}, \{T_1, \ldots, T_n\}'}(X) = \inf \{\|X((\lambda_j), \{Z_j u_j\}, \{Z_j v_j\})\| : Z_j \in B(K) \text{ are invertible}\}
$$

and relation (4.10) holds if and only if there exist invertible operators $Z_j \in B(K)$ such that $\|X((\lambda_j), \{Z_j u_j\}, \{Z_j v_j\})\| < \delta$. This inequality is equivalent to

$$
\delta^2 I - X((\lambda_j), \{Z_j u_j\}, \{Z_j v_j\}) X^* ((\lambda_j), \{Z_j u_j\}, \{Z_j v_j\}) > 0,
$$

which is equivalent to (4.9). This completes the proof. \qed

We remark that (4.9) can be replaced by relation (4.10). As a consequence of Theorem 4.2, when the distinct elements in $\mathbb{K}$ are $\lambda_j$, $j = 1, \ldots, k$, we infer the following spectral tangential interpolation result for matrix-valued bounded analytic functions in the unit ball of $\mathbb{C}^n$. 

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Corollary 4.3. Let $\lambda_j, j = 1, \ldots, k$, be distinct elements in $\mathbb{B}_n$ and let $K$ be a finite dimensional Hilbert space. If $u_1, \ldots, u_k, v_1, \ldots, v_k \in K$ with $u_i \neq 0, j = 1, \ldots, k$, $\delta > 0$, and there exist invertible operators $Z_j \in B(K)$, $j = 1, \ldots, k$, such that
\[
\frac{\langle \delta Z_j u_j, Z_i v_i \rangle - \langle Z_j v_j, Z_i u_i \rangle}{1 - \langle \lambda_i, \lambda_j \rangle} > 0,
\]
then there exists $F \in H^\infty(\mathbb{B}_n) \otimes B(K)$ such that
\[
\sup_{\lambda \in \mathbb{B}_n} |F(\lambda)|_{\infty} < 1 \quad \text{and} \quad F(\lambda_j) u_j = v_j, \ j = 1, \ldots, k.
\]

Let us make some remarks on the dependence of $\rho_m, \{T_1, \ldots, T_n\} (X)$ on the given interpolation data. For each $m = 1, \ldots, k$, we define
\[
\rho_m := \inf \{ \| X(\{\lambda_j\}_{j=1}^m, \{Z_j u_j\}_{j=1}^m, \{Z_j v_j\}_{j=1}^m) \| : \ Z_j \in B(K) \text{ are invertible} \}.
\]
A multivariable analogue of [BF, Proposition 4] holds. More precisely, one can prove that if $u_k$ and $v_k$ are linearly independent, then $\rho_{k-1} = \rho_k$. Indeed, suppose that $\rho_{k-1} < \rho_k$. Using Theorem 4.2, we find $\Phi \in F_n^\infty \otimes B(K)$ such that $\rho_{F_n^\infty \otimes B(K)}(\Phi) < \rho_k$ and $\Phi(\lambda_j)^* u_j = v_j, \ j = 1, \ldots, k - 1$. We may suppose that $\Phi(\lambda_k)^* \notin CI_K$ because, otherwise, we can replace $\Phi$ by $\Phi + \Psi$ for some $\Psi \in F_n^\infty \otimes B(K)$ satisfying $\Phi(\lambda_j) = 0, \ j = 1, \ldots, k - 1$, and $\Psi(\lambda_k) \notin CI_K$. Since we can choose $\Psi$ with very small norm we have $\rho_{F_n^\infty \otimes B(K)}(\Phi + \Psi) = \rho_k$.

Therefore, since $\Phi(\lambda_k)^* \notin CI_K$, there exist linearly independent vectors $u$ and $v$ such that $\Phi(\lambda_k)^* u = v$. Since $u_k$ and $v_k$ are linearly independent, we can find $Z_k \in B(K)$ invertible with $Z_k u_k = u$ and $Z_k v_k = v$. Hence, we infer that $\rho_k \leq \rho_{F_n^\infty \otimes B(K)}(\Phi) < \rho_k$, which is a contradiction. Since $\rho_{k-1} \leq \rho_k$, we must have $\rho_{k-1} = \rho_k$. This shows that in Theorem 4.2 we can assume, without loss of generality, that $v_j = \mu_j u_j$, for some $\mu_j \in \mathbb{C}$, $\mu_j \neq 0, \ j = 1, \ldots, k$. Similarly to [BF, Proposition 5], one can show that if $k \leq \dim K$, then
\[
\rho_k = \max \{ |\mu_1|, \ldots, |\mu_k| \}.
\]

The case when the number of dependent vector pairs $(u_j, v_j)$ exceeds the dimension of $K$, and the problem of optimal solutions will be considered in a future paper.

References

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