CRYSTAL BASES FOR \( U_q(\Gamma(\sigma_1, \sigma_2, \sigma_3)) \)

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ABSTRACT. We construct crystal bases for certain infinite dimensional representations of the \( q \)-deformation of the Lie superalgebra \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \).

1. Introduction

The notion of crystal bases has been generalized to the \( q \)-deformations of certain Lie superalgebras by [1], [11] and [14]. Since the lack of reducibility property in the superalgebra case, in order to generalize crystal base theory to the \( q \)-deformations of Lie superalgebras, one needs to modify the original definition of crystal base. In [1], the notion of pseudo-base was introduced. Roughly speaking, a crystal base for a module \( M \) in the Lie algebra case is a pair \( (L, B) \), where \( L \subset M \) is a lattice over the subring \( A \subset \mathbb{Q}(q) \) consisting of functions regular at \( q = 0 \) and \( B \) is a basis of \( L/qL \) over \( \mathbb{Q} \). In the super case, one needs to allow both a \( \mathbb{Q} \)-basis and its negative, therefore \( B \) is not a basis, and the associated crystal is \( B = \{1\} \).

In this paper, we use an approach similar to that of [1] to construct crystal bases for certain modules of the \( q \)-deformation of the universal enveloping algebra of the Lie superalgebra \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \). The notation \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \) actually stands for a one-parameter family of Lie superalgebras defined over the complex number field \( \mathbb{C} \), though the corresponding \( q \)-deformations can be defined over the field \( \mathbb{C}(q) \), for our purpose, we shall consider the deformations over the field \( \mathbb{Q}(q) \). The construction of crystal bases in [1] depends a crucial fact about the Lie superalgebra \( gl(m, n) \): this Lie superalgebra possesses a natural vector representation \( V \) with a crystal base. This fact enables [1] to construct a crystal base theory for the simple objects in a certain category of \( gl(m, n) \)-modules obtained by taking tensor products of \( V \), although modules in this category are not necessarily completely reducible. Unlike the algebra \( gl(m, n) \), the algebra \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \) does not have a natural vector representation. To construct a crystal base theory for \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \), we first construct a simple module \( V \) with a crystal base, then we show that certain simple modules possess crystal bases by studying tensor products of \( V \). It turns out that finite dimensional nontrivial \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \)-modules do not have bases that behave well with respect to tensor products. The modules that we show to have crystal bases are infinite dimensional \( \Gamma(\sigma_1, \sigma_2, \sigma_3) \)-modules.

2. Preliminary

We use the notation adopted in [12], [13]. Recall that \( G = \Gamma(\sigma_1, \sigma_2, \sigma_3) \) is defined as a contragredient Lie superalgebra with three nonzero complex numbers \( \sigma_1, \sigma_2, \sigma_3 \).
satisfying \( \sigma_1 + \sigma_2 + \sigma_3 = 0 \), with generators \( e_i, f_i, h_i \) \((i = 1, 2, 3)\) and the defining matrix \((a_{ij})_{3 \times 3}\) given by
\[
\begin{pmatrix}
0 & 2\sigma_2 & 2\sigma_3 \\
-1 & 2 & 0 \\
-1 & 0 & 2 \\
\end{pmatrix}.
\]

Let \( G_0 \) be the even part of \( G \) and let \( G_1 \) be the odd part of \( G \), then \( G_0 \cong sl(2) \otimes sl(2) \otimes sl(2) \). We denote the standard generators of \( G_0 \) by \( X_i, Y_i, H_i \), i.e.
\[
X_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, 3.
\]

We use linear functions \( \epsilon_1, \epsilon_2, \epsilon_3 \) to express the roots of \( G \), where \( \epsilon_i(H_j) = \delta_{ij} \). The set of even roots and the set of odd roots are
\[
R_0 = \{ \pm 2\epsilon_i : i = 1, 2, 3 \}, \quad R_1 = \{ \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \}.
\]

We choose \( \alpha_1 = \epsilon_1 - \epsilon_2 - \epsilon_3, \alpha_2 = 2\epsilon_2, \alpha_3 = 3\epsilon_3 \) to be a simple root system and denote the corresponding generators of \( G \) by \( e_i, f_i, h_i \) \((i = 1, 2, 3)\). Note that \( e_i = X_i, f_i = Y_i, h_i = H_i \) \((i = 2, 3)\), and \( h_1 = -\sigma_1 H_1 + \sigma_2 H_2 + \sigma_3 H_3 \). Note also that
\[
\rho = \left( \sum_{\alpha \in \Delta^+_i} \alpha - \sum_{\alpha \in \Delta^-_i} \alpha \right)/2 = -\epsilon_1 + \epsilon_2 + \epsilon_3 = -\alpha_1.
\]

Let \( H = \langle h_1, h_2, h_3 \rangle \). We denote an element \( \lambda = m_1 \epsilon_1 + m_2 \epsilon_2 + m_3 \epsilon_3 \in H^* \) by \((m_1, m_2, m_3)\). We also use the numerical marks to denote the elements in \( H^* \). If \( \lambda(h_i) = a_i \) \((i = 1, 2, 3)\), then we write \( \lambda = [a_1, a_2, a_3] \). Let \( P \) be the set of integral weights in \( H^* \), i.e. \( P \) is the set of \( \lambda = [a_1, a_2, a_3] \) such that \( a_i \in \mathbb{Z} \).

The conditions on the \( \sigma_i \)'s imply that \( G \) depends on only one parameter (compare with [4, 2.5.2]). In order to construct our basic simple highest weight module with a crystal base, we need to assume (see Section 3) that the highest weight \( \lambda = (m_1, m_2, m_3) \) of the module satisfies \( (\lambda + \rho)(h_1) = 0 \), i.e. \(-m_1 \sigma_1 + m_2 \sigma_2 + m_3 \sigma_3 = 0\). Thus the parameter and the highest weight are determined by each other, and if a result holds for one set of \( \sigma_i \), we can adjust the corresponding highest weight for another set of \( \sigma_i \) to obtain a similar result (see the end of Section 3 for an example). In order to simplify our notation, we fix
\[
\sigma_1 = \sigma_2 = \frac{1}{2}, \quad \sigma_3 = -1.
\]

Under our assumption, the defining matrix is now
\[
\begin{pmatrix}
0 & 1 & -2 \\
-1 & 2 & 0 \\
-1 & 0 & 2 \\
\end{pmatrix}.
\]

We use \( \ell_1 = 1, \ell_2 = -1, \ell_3 = 2 \) to symmetrize the matrix.

Let \( q \) be an indeterminate over \( \mathbb{Q} \), let \( q_i = q_i \) \((i = 1, 2, 3)\), and let \( A \) be the subring of the quotient field \( \mathbb{Q}(q) \) consisting of those \( f/g \) with \( g(0) \neq 0 \). We define the algebra \( U' \) to be the \( \mathbb{Z}_2 \)-graded unital associative algebra over \( \mathbb{Q}(q) \) generated by the elements \( E_i, F_i, K_i^{\pm 1} \) \((i = 1, 2, 3)\), with the parities given by
\[
p(E_i) = p(F_i) = 0, \quad i = 2, 3; \quad p(K_i^{\pm 1}) = 0, \quad i = 1, 2, 3; \quad p(E_1) = p(F_1) = 1,
\]
and the following generating relations:

(2.1) \[ K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad 1 \leq i, j \leq 3; \]

(2.2) \[ K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \quad 1 \leq i, j \leq 3; \]

(2.3) \[ E_i F_j - (-1)^{ab} F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, a = p(E_i), b = p(F_j), 1 \leq i, j \leq 3; \]

(2.4) \[ E_2 E_3 = E_3 E_2, \quad F_2 F_3 = F_3 F_2; \]

(2.5) \[ E_i^2 E_1 - (q_i + q_i^{-1}) E_i E_1 + E_1 E_i^2 = 0, \quad i = 2, 3; \]

(2.6) \[ F_i^2 F_1 - (q_i + q_i^{-1}) F_1 F_i + F_1 F_i^2 = 0, \quad i = 2, 3; \]

\[ E_1^2 = F_1^2 = 0. \]

As in [1], to define the Hopf algebra structure, we use the parity operator \( \sigma \) on \( U' \), which is defined by \( \sigma(x) = (-1)^{p(x)} x \) for the generators \( x \) of \( U' \). Let \( U = U' \oplus U' \sigma \) with the algebra structure given by \( \sigma^2 = 1 \) and \( \sigma u \sigma = \sigma(u) \) for \( u \in U' \). Let \( p(i) = p(E_i), \quad i = 1, 2, 3 \). The Hopf algebra structure on \( U \) has the comultiplication \( \Delta \), the antipode \( S \) and the counit \( \varepsilon \) defined by

\[ \Delta(\sigma) = \sigma \otimes \sigma, \quad \Delta(K_i) = K_i \otimes K_i, \]

(2.7) \[ \Delta(E_i) = E_i \otimes K_i^{-1} + \sigma^{p(i)} \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + \sigma^{p(i)} K_i \otimes F_i; \]

(2.8) \[ S(\sigma) = \sigma, S(E_i) = -\sigma^{p(i)} E_i K_i, S(F_i) = -\sigma^{p(i)} K_i^{-1} F_i, S(K_i) = K_i^{-1}; \]

(2.9) \[ \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(\sigma) = \varepsilon(K_i) = 1. \]

There exists an algebra anti-automorphism \( \eta \) of \( U \) defined by

\[ \eta(\sigma) = \sigma, \eta(K_i) = K_i, \eta(E_i) = q_i F_i K_i^{-1}, \eta(F_i) = q_i K_i E_i, \]

(2.10) \[ \eta(uv) = \eta(u) \eta(v), \quad \text{for all } u, v \in U. \] Note that \( \eta^2 = \text{id} \) and

\[ \Delta \circ \eta = (\eta \otimes \eta) \circ \Delta. \]

The adjoint action of \( U' \) on itself is given by

\[ a d_q x(y) = \sum (-1)^{p(b)p(y)} a_i y S(b_i), \]

where \( \Delta x = \sum a_i \otimes b_i. \)

Define a \( \mathbb{Q} \)-algebra anti-automorphism \( \theta \) of \( U' \) by

\[ \theta(E_i) = F_i, \quad \theta(F_i) = E_i, \quad \theta(K_i) = K_i^{-1}, \quad \theta(q) = q^{-1}; \]

(2.11) \[ \theta(uv) = \theta(v) \theta(u), \quad \text{for all } u, v \in U'. \]

Introduce the following elements of \( U' \):

\[ E_{121} = a d_q E_3(E_1) = E_3 E_1 - q_3^{-1} E_1 E_3, \]

(2.12) \[ E_{112} = a d_q E_2(E_1) = E_2 E_1 - q_2^{-1} E_1 E_2, \]

\[ E_{111} = a d_q E_3 a d_q E_2(E_1) = a d_q(E_3 E_2)(E_1), \]

\[ E_0 = (q_2 + q_2^{-1}) E_1 E_{111} + (q_3 + q_3^{-1}) E_{111} E_1 \]

\[ + (q_3 q_2^{-1} - q_3^{-1} q_2) E_{121} E_{112}, \]
and let
\[(2.14)\quad F_{212} = \theta E_{121}, \quad F_{221} = \theta E_{112}, \quad F_{222} = \theta E_{111}, \quad F_0 = 0.\]

Let $\mathcal{U}^+, \mathcal{U}^-, \mathcal{U}^0$ be the subalgebras of $\mathcal{U}$ generated by the $E_i$, the $F_i$, and the $K_i^{\pm 1}$ ($i = 1, 2, 3$) respectively. Then $\mathcal{U}' = \mathcal{U}^- \mathcal{U}^+ \mathcal{U}^0$ and $\mathcal{U}' \cong \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+$ as $\mathbb{Q}(q)$-vector spaces.

For $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)$, where $\delta_i = 0$ or 1, and $m = (m_1, m_2, m_3), m_i \in \mathbb{Z}_{\geq 0}$, let
\[
E^{(\delta, m)} = E_{111}^{\delta_1} E_{121}^{\delta_2} E_{112}^{\delta_3} E_{01}^{m_1} E_{21}^{m_2} E_{31}^{m_3},
\]
\[
F^{(\delta, m)} = F_{222}^{\delta_1} F_{212}^{\delta_2} F_{221}^{\delta_3} F_{01}^{m_1} F_{21}^{m_2} F_{31}^{m_3}.
\]

For $t = (t_1, t_2, t_3), t_i \in \mathbb{Z}$, let $K^t = K_1^{t_1} K_2^{t_2} K_3^{t_3}$. Then by [13], a PBW type theorem holds, i.e., the $K^t$ form a basis of $\mathcal{U}^0$, the elements of the form $E^{(\delta, m)}$ (resp. $F^{(\delta, m)}$) form a basis of $\mathcal{U}^+$ (resp. $\mathcal{U}^-$), and the elements of the form $F^{(\delta, m)} K^t E^{(\delta', m')}$ form a basis of $\mathcal{U}'$.

Following [1], we define the category $\mathcal{O}_{int}$ of $\mathcal{U}$-modules and the modified Kashiwara operators. The category $\mathcal{O}_{int}$ consists of $\mathcal{U}$-modules $M$ such that the following conditions hold:

(i) $M$ is a weight module $M = \sum_{\lambda \in P} M_\lambda$, where
\[M_\lambda = \{u \in M : K_i u = q^{\lambda(h_i)} u, i = 1, 2, 3\}.\]

(ii) $\dim M_\lambda < \infty$ for any $\lambda \in P$.

(iii) For $i = 2, 3$, $M$ is locally $\mathcal{U}_i$-finite, where $\mathcal{U}_i$ is the subalgebra of $\mathcal{U}$ generated by $E_i, F_i, K_i^{\pm 1}$.

(iv) For any $\lambda$ such that $M_\lambda \neq 0, \lambda(h_1) \geq 0$.

(v) For $\lambda \in P$ such that $\lambda(h_1) = 0, E_1 M_\lambda = F_1 M_\lambda = 0$.

The modified Kashiwara operators for the modules $M$ in $\mathcal{O}_{int}$ are defined as follows. For $i = 2, 3, n \geq 0$, let $F_i^{(n)} = F_i^{(n)}/[n]!$ be as usual. For $u \in M$ of weight $\lambda$, let
\[u = \sum_{k \geq 0, -\lambda(h_i)} F_i^{(k)} u_k\]
be the unique expression such that $E_i u_k = 0$ for each $k$.

For $i = 3$, we define
\[(2.16)\quad \tilde{E}_3 u = \sum_k F_3^{(k-1)} u_k, \quad \tilde{F}_3 u = \sum_k F_3^{(k+1)} u_k.\]

For $i = 2$, we let $\ell_k = (\lambda + k \alpha_2)(h_2)$ and define
\[(2.17)\quad \tilde{E}_2 u = \sum_k q_2^{-\ell_k-2k+1} F_2^{(k-1)} u_k, \quad \tilde{F}_2 u = \sum_k q_2^{\ell_k+2k+1} F_2^{(k+1)} u_k.\]

For $i = 1$, we define $\tilde{E}_1 u = q^{-1} K_1 E_1 u, \tilde{F}_1 = F_1 u$.

**Definition 2.1** ([1] Def. 2.3 and Def. 2.4). Let $M \in \mathcal{O}_{int}$. A crystal base of $M$ is a pair $(L, B)$ such that:

(A1) $L$ generates $M$ as a vector space over $\mathbb{Q}(q)$.

(A2) $\sigma L = L$ and $L$ has a weight decomposition $L = \bigoplus_{\lambda \in P} L_\lambda$ with $L_\lambda = L \cap M_\lambda$.

(A3) $\tilde{E}_i L \subset L$ and $\tilde{F}_i L \subset L$ for $i = 1, 2, 3$.

(C1) $B$ is a subset of $L/qL$ such that $\sigma b = \pm b$ for any $b \in B$, and $B$ has a weight decomposition $B = \bigcup_{\lambda \in P} B_\lambda$ with $B_\lambda = B \cap (L_\lambda/qL_\lambda)$. 

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(C2) $B$ is a pseudo-base of $L/qL$, that is, $B = B' \cup (-B')$ for a $\mathbb{Q}$ base $B'$ of $L/qL$.

(C3) $\tilde{E}_i B \subset B \cup \{0\}$ and $\tilde{F}_i B \subset B \cup \{0\}$, for $i = 1, 2, 3$.

(C4) For any $b, b' \in B$ and $i = 1, 2, 3$, $b = \tilde{F}_i b' \iff b = \tilde{E}_i b$.

The crystal associated to the crystal base $(L, B)$ is $B/\{\pm 1\}$.

Definition 2.2. We call a symmetric bilinear form $(\cdot, \cdot)$ on a $U$-module $M$ $\eta$-invariant if it satisfies

\[(2.18) \quad (um, m') = (m, \eta(u)m')\]

for all $m, m' \in M$ and $u \in U$. We say that a crystal base $(L, B)$ for a $U$-module $M$ is polarizable if there exists an $\eta$-invariant form $(\cdot, \cdot)$ on $M$ such that $(L, L) \subset A$, and with respect to the induced $\mathbb{Q}$-valued symmetric bilinear form $(\cdot, \cdot)_0$ on $L/qL$,

\[(2.19) \quad (b, b')_0 = \begin{cases} \pm 1, & \text{if } b' = \pm b, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } b, b' \in B.\]

By [1 Thm. 2.12], a $U$-module $M$ with a polarizable crystal base is completely reducible.

We now consider the tensor products of $U$-modules. Let $(L, B)$ be a crystal base of a $U$-module $M$. For $b \in B$ and $i = 2, 3$, we define

\[
\varepsilon_i(b) = \max \{ n \in \mathbb{Z}_{\geq 0} : \tilde{E}_i^n b \neq 0 \}, \\
\varphi_i(b) = \max \{ n \in \mathbb{Z}_{\geq 0} : \tilde{F}_i^n b \neq 0 \}.
\]

Note that $\text{wt}(b)(h_i) = \varphi_i(b) - \varepsilon_i(b), i = 2, 3$.

Let $M_1, M_2 \in \mathcal{O}_{\text{int}}$ and suppose that they have crystal bases $(L_1, B_1)$ and $(L_2, B_2)$ respectively. Set $L = L_1 \otimes_A L_2$ and $B = B_1 \otimes B_2$. Then $(L, B)$ is a crystal base for $M_1 \otimes M_2$ and by [1 Prop. 2.8], the actions of $\tilde{E}_i, \tilde{F}_i$ on $b_1 \otimes b_2$ ($b_1 \in B_1$ and $b_2 \in B_2$) are given by

\[
(2.20) \quad \tilde{E}_1(b_1 \otimes b_2) = \begin{cases} \tilde{E}_1(b_1) \otimes b_2, & \text{if } \text{wt}(b_1)(h_1) > 0, \\ \sigma b_1 \otimes \tilde{E}_1(b_2), & \text{if } \text{wt}(b_1)(h_1) = 0, \end{cases}
\]

\[
(2.21) \quad \tilde{F}_1(b_1 \otimes b_2) = \begin{cases} \tilde{F}_1(b_1) \otimes b_2, & \text{if } \text{wt}(b_1)(h_1) > 0, \\ \sigma b_1 \otimes \tilde{F}_1(b_2), & \text{if } \text{wt}(b_1)(h_1) = 0; \end{cases}
\]

\[
(2.22) \quad \tilde{E}_2(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{E}_2(b_2), & \text{if } \varphi_2(b_2) \geq \varepsilon_2(b_1), \\ \tilde{E}_2(b_1) \otimes b_2, & \text{if } \varphi_2(b_2) < \varepsilon_2(b_1), \end{cases}
\]

\[
(2.23) \quad \tilde{F}_2(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{F}_2(b_2), & \text{if } \varphi_2(b_2) \geq \varepsilon_2(b_1), \\ \tilde{F}_2(b_1) \otimes b_2, & \text{if } \varphi_2(b_2) < \varepsilon_2(b_1); \end{cases}
\]

\[
(2.24) \quad \tilde{E}_3(b_1 \otimes b_2) = \begin{cases} \tilde{E}_3(b_1) \otimes b_2, & \text{if } \varphi_3(b_1) \geq \varepsilon_3(b_2), \\ b_1 \otimes \tilde{E}_3(b_2), & \text{if } \varphi_3(b_1) < \varepsilon_3(b_2), \end{cases}
\]

\[
(2.25) \quad \tilde{F}_3(b_1 \otimes b_2) = \begin{cases} \tilde{F}_3(b_1) \otimes b_2, & \text{if } \varphi_3(b_1) \geq \varepsilon_3(b_2), \\ b_1 \otimes \tilde{F}_3(b_2), & \text{if } \varphi_3(b_1) < \varepsilon_3(b_2). \end{cases}
\]
3. The Basic Module and the Main Result

Since a PBW type theorem holds for $\mathcal{U}'$, we can define highest weight modules for $\mathcal{U}$ as usual. We call a $\mathcal{U}$-module $M$ a highest weight module if there is a weight vector $v$ of $M$ such that $M = \mathcal{U}^w(v)$, if this is the case, we call $v$ a highest weight vector. We call a weight vector $v$ maximal if $\mathcal{U}^w(v) = 0$. For a highest weight module $M$ with a highest weight vector $v$, the action of $\sigma$ on $M$ can be specified by the action of $\sigma$ on $v$.

By [3, Thm. 8] (compare with [8, Prop. 2.1]), the simple $\mathcal{U}$-module $L(\lambda)$ with highest weight $\lambda = m_1\epsilon_1 + m_2\epsilon_2 + m_3\epsilon_3$ is finite dimensional only if $m_i \in \mathbb{Z}_{\geq 0}$ ($i = 1, 2, 3$). Consider condition (iv) in the definition of the category $\mathcal{O}_{\text{int}}$. If $L(\lambda)$ is finite dimensional with

$$m_1 > 0 \quad \text{and} \quad \lambda(h_1) = -\frac{1}{2}m_1 + \frac{1}{2}m_2 - m_3 \geq 0,$$

then $m_2 > 0$. By using the action of the Weyl group $W \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ of $G$, we see that there is a weight $\mu$ of the module $L(\lambda)$ such that $\mu(h_1) < 0$. If $m_1 = 0$, then $L(\lambda)$ is finite dimensional if and only if $m_2 = m_3 = 0$, i.e., $L(\lambda)$ is the trivial module. Since condition (iv) (alternatively $\mu(h_1) \leq 0$ for all weights $\mu$ of a $\mathcal{U}$-module in the category) is needed in working with the tensor products of crystal bases, we should consider simple highest weight $\mathcal{U}$-modules with $m_1 < 0$.

Let $\lambda = (-2, 0, 1) = [0, 0, 1]$ (recall that the numbers inside $\langle \rangle$ stand for the coefficients of the $\epsilon_i$'s and the numbers in $[\cdot]$ stand for the numerical marks), denote the $\mathcal{U}$-module $L(\lambda)$ by $V$. Let $v_0$ be a highest weight vector of $V$ and let the action $\sigma$ on $V$ be given by $\sigma v_0 = v_0$. Note that $V$ is infinite dimensional since $\lambda(H_1) = -2$.

**Proposition 3.1.** The $\mathcal{U}$-module $V$ has a polarizable crystal base $(L, B)$ with the associated crystal graph (the repeating block is 3,1,2,1):

(3.1) \[ \begin{array}{cccccccccccc}
0 & 3 & 2 & 1 & 2 & 3 & 1 & 4 & 3 & 5 & 1 & 6 & 2 & 7 & 1 & 8 \cdots
\end{array} \]

*Proof.* Define a set of vectors $\{v_i\}_{i \geq 0}$ of $V$ according to the action of the $F_i$'s described in the given crystal graph, i.e.,

$$v_1 = F_3v_0, \quad v_2 = F_1v_1, \quad v_3 = F_2v_2, \quad v_4 = F_1v_3, \ldots$$

We claim that $\{v_i\}_{i \geq 0}$ is a basis of $V$ with the action of $E_i, F_i$ ($i = 1, 2, 3$) given by

(3.2) \[
\begin{align*}
F_1v_i &= \begin{cases} 
0, & i \text{ even,} \\
v_{i+1}, & i \text{ odd,}
\end{cases} \\
F_2v_i &= \begin{cases} 
0, & i \neq 4n + 2, \\
v_{i+1}, & i = 4n + 2,
\end{cases} \\
F_3v_i &= \begin{cases} 
0, & i \neq 4n, \\
v_{i+1}, & i = 4n;
\end{cases}
\]

(3.3) \[
E_1v_i = \begin{cases} 
0, & i = 0 \text{ or odd,} \\
[n]v_{i-1}, & i = 4n, n \geq 1, \\
[n+2]v_{i-1}, & i = 4n + 2, n \geq 0,
\end{cases}
\]
where \([n] = \frac{2^n - q^n}{q - 1}\) (note that \(\ell_1 = 1\) and \(q_1 = q\)).

\[
E_2 v_i = \begin{cases} 
0, & i \neq 4n + 3, \\
v_{i-1}, & i = 4n + 3,
\end{cases} \quad E_3 v_i = \begin{cases} 
0, & i \neq 4n + 1, \\
v_{i-1}, & i = 4n + 1.
\end{cases}
\]

These formulas will imply that the subspace of \(V\) spanned by \(\{v_i\}_{i \geq 0}\) is a submodule of \(V\), thus must be the whole \(V\) since \(V\) is simple. We verify these formulas at the first block:

\[
v_0 \xrightarrow{3} v_1 \xrightarrow{1} v_2 \xrightarrow{2} v_3 \xrightarrow{1}.
\]

The verification at any other block is similar.

Consider \(v_0\). If \(F_1 v_0 \neq 0\), then since \(E_i F_1 v_0 = 0\) for \(i = 1, 2, 3\), \(F_1 v_0\) generates a proper submodule of \(V\), contradicting the simplicity of \(V\). Similarly, \(F_2 v_0 = 0\). Also by \(U_q(\mathfrak{sl}(2))\)-theory, \(v_1 = F_3 v_0 \neq 0\) and \(F_3 v_1 = 0\).

Consider \(v_1\). By (2.4) \(F_2 v_1 = F_2 F_3 v_0 = F_3 F_2 v_0 = 0\). Since \(V\) is infinite dimensional, \(v_2 = F_1 v_1 \neq 0\).

Consider \(v_2\). By (2.5)

\[
F_3 v_2 = F_3 F_1 F_3 v_0 = \frac{1}{q_3 + q_3} (F_3^2 F_1 + F_1 F_3^2) v_0 = 0.
\]

Since \(F_1^2 = 0\), \(F_1 v_2 = 0\). The weight of \(v_2\) is \(wt(v_2) = [2, 1, 0]\). By (2.3), \(E_2 v_2 = 0\), so \(U_q(\mathfrak{sl}(2))\)-theory implies \(v_3 = F_2 v_2 \neq 0\) and \(F_2 v_3 = 0\).

Consider \(v_3\). By (2.4), \(F_3 v_3 = 0\), then \(F_1 v_3 \neq 0\).

Observe that the weights of the vectors in \(\{v_i\}_{i \geq 0}\) are given by

\[
wt(v_{4n}) = [n, 0, 1], \quad wt(v_{4n+1}) = [n + 2, 0, -1],
wt(v_{4n+2}) = [n + 2, 1, 0], \quad wt(v_{4n+3}) = [n + 1, -1, 0];
\]

the actions of \(E_i\) \((i = 1, 2, 3)\) are then clear.

We denote the images of \(\pm v_i\) in \(L/qL\) also by \(\pm v_i\) and let

\[
L = \bigoplus_{i \geq 0} Av_i, \quad B = \{\pm v_i : i \geq 0\}.
\]

To see that \((L, B)\) is a crystal base for \(V\), we only need to verify conditions (A3), (C3) and (C4) in Definition 2.1. For \(i = 2, 3\), by formulas (2.2)–(2.7), we see that \(V\) is a direct sum of one- or two-dimensional modules for the subalgebra \(U_i = \langle E_i, F_i, K_i^{\pm 1} \rangle\), so conditions (A3), (C3) and (C4) clearly hold. For \(i = 1\), \(\tilde{F}_1 = F_1\) and

\[
\tilde{E}_1 v_i = q^{-1} K_1 E_1 v_i = \begin{cases} 
0, & i = 0 \text{ or odd}, \\
\frac{q^{2n+1} - 1}{q^2 - 1} v_{i-1}, & i = 4n + 1, n \geq 1,
\end{cases}
\]

\[
= \begin{cases} 
0, & i = 0 \text{ or odd}, \\
v_{i-1}, & i = 4n, n \geq 1, \quad (\text{mod } qL),
\end{cases}
\]

Thus (A3), (C3) and (C4) also hold for \(\tilde{E}_1\) and \(\tilde{F}_1\).
To show that \((\mathbf{L}, \mathbf{B})\) is polarizable, we define a symmetric bilinear form on \(\mathbf{V}\) by letting
\[
(v_0, v_0) = 1 \quad \text{and} \quad (uv_0, u'v_0) = (v_0, \eta(u)u'v_0),
\]
for all \(u, u' \in \mathcal{U}^- \oplus \mathcal{U}^- \sigma\). Then \((,\) is \(\eta\)-invariant and \((v_i, v_j) = 0\) if \(i \neq j\). To prove that \((\mathbf{L}, \mathbf{L}) \subset A\), we use induction on \(n\) to prove that
\[
(3.7) \quad (v_{4n+i}, v_{4n+i}) \in A, \quad \text{for} \ n \geq 0 \text{ and } 0 \leq i \leq 3.
\]

For \(n = 0\), since \(K_3 v_0 = q_3^{\lambda(h_3)} v_0 = q_3 v_0\), we have
\[
(v_1, v_1) = (F_3 v_0, F_3 v_0) = (v_0, \eta(F_3) F_3 v_0) = (v_0, q_3^{-1} K_3 F_3 v_0) = (v_0, v_0) = 1.
\]
Then
\[
(v_2, v_2) = (F_1 v_1, F_1 v_1) = (v_1, q^{-1} K_1 E_1 F_1 v_1) = (v_1, (q^2 + 1)v_1) = q^2 + 1.
\]

Similarly to the case \(n = 0\), we have \((v_2, v_2) = (v_2, v_2)\).

For \(n > 0\), by formulas (3.3) and (3.4) we have
\[
(3.8) \quad \begin{align*}
(v_{4n}, v_{4n}) &= \frac{q^{2n} - 1}{q^2 - 1}(v_{4n-1}, v_{4n-1}), (v_{4n+1}, v_{4n+1}) = (v_{4n}, v_{4n}), \\
(v_{4n+2}, v_{4n+2}) &= \frac{q^{2n+4} - 1}{q^2 - 1}(v_{4n+1}, v_{4n+1}), (v_{4n+3}, v_{4n+3}) = (v_{4n+2}, v_{4n+2}).
\end{align*}
\]
Thus by induction, we see that \((v_i, v_i) \in A\ (i \geq 0)\), hence \((\mathbf{L}, \mathbf{L}) \subset A\). From these computations we also see that the induced \(\mathbb{Q}\)-valued symmetric bilinear form \((,\) on \(\mathbf{L}/q \mathbf{L}\) satisfies (2.19). \(\square\)

By the results in Section 2.4 of [1], we have the following corollary.

Corollary 3.2. For all integers \(n > 0\), the \(\mathcal{U}\)-module \(\mathbf{V}^\otimes_n\) is completely reducible.

Now we can state our main result.

Theorem 3.3. For any \(\mu = [m, 0, n] \ (m, n \in \mathbb{Z}_{\geq 0})\), the simple \(\mathcal{U}\)-module \(L(\mu)\) has a polarizable crystal base.

We will give the proof of Theorem 3.3 in the next section. Although we will see that there are simple \(\mathcal{U}\)-modules \(L(\lambda)\) with \(\lambda(h_2) > 0\) which possess polarizable crystal bases, we cannot expect that all simple \(\mathcal{U}\)-modules \(L(\lambda)\) with \(\lambda(h_i) \geq 0\) \((i = 1, 2, 3)\) possess crystal bases for the same reason as discussed at the beginning of this section. It is clear that if we choose \(\sigma_1 = \sigma_3 = \frac{1}{2}\) and \(\sigma_2 = -1\) instead, then similar results hold, in particular, the statement in Theorem 3.3 can be changed to

Theorem 3.3’. For any \(\mu = [m, n, 0] \ (m, n \in \mathbb{Z}_{\geq 0})\), the simple \(\mathcal{U}\)-module \(L(\mu)\) has a polarizable crystal base.
4. Proof of Theorem 3.3

Consider the decomposition of $V \otimes V$. The actions of $\tilde{E}_i$ and $\tilde{F}_i$ on $B \otimes B$ can be obtained from (2.20)–(2.22) and (3.1). We have

\begin{align*}
\tilde{E}_1(v_i \otimes v_j) &= \begin{cases} 
\tilde{E}_1(v_i) \otimes v_j, & i \neq 0, \\
\sigma v_0 \otimes \tilde{E}_1(v_j), & i = 0,
\end{cases} \\
\tilde{F}_1(v_i \otimes v_j) &= \begin{cases} 
\tilde{F}_1(v_i) \otimes v_j, & i \neq 0, \\
\sigma v_0 \otimes \tilde{F}_1(v_j), & i = 0.
\end{cases}
\end{align*}

(4.1)

\begin{align*}
\tilde{E}_2(v_i \otimes v_j) &= \begin{cases} 
v_i \otimes \tilde{E}_2(v_j), & \text{otherwise}, \\
\tilde{E}_2(v_i) \otimes v_j, & \text{if } i = 4m + 3, j \neq 4n + 2,
\end{cases} \\
\tilde{F}_2(v_i \otimes v_j) &= \begin{cases} 
v_i \otimes \tilde{F}_2(v_j), & \text{if } i \neq 4m + 3, j = 4n + 2, \\
\tilde{F}_2(v_i) \otimes v_j, & \text{otherwise};
\end{cases}
\end{align*}

(4.2)

\begin{align*}
\tilde{E}_3(v_i \otimes v_j) &= \begin{cases} 
\tilde{E}_3(v_i) \otimes v_j, & \text{otherwise}, \\
v_i \otimes \tilde{E}_3(v_j), & \text{if } i \neq 4m, j = 4n + 1,
\end{cases} \\
\tilde{F}_3(v_i \otimes v_j) &= \begin{cases} 
\tilde{F}_3(v_i) \otimes v_j, & \text{if } i = 4m, j \neq 4n + 1, \\
v_i \otimes \tilde{F}_3(v_j), & \text{otherwise}.
\end{cases}
\end{align*}

(4.3)

Let

\[HW(B \otimes B) = \{x \in B \otimes B : \tilde{E}_i x = 0, i = 1, 2, 3\}.\]

The elements of $HW(B \otimes B)$ provide a set of linear independent maximal vectors in $V \otimes V$.

Proposition 4.1. We have

\[HW(B \otimes B) = \{v_0 \otimes v_j : j = 4n + 1, n \geq 0\} \cup \{v_{4m+3} \otimes v_{4m+2} : m, n \geq 0\}.\]

Proof. We first consider elements of the form $v_0 \otimes v_j$. By (4.1), $\tilde{E}_1(v_0 \otimes v_j) = 0$ if and only if $\tilde{E}_1 v_j = 0$, that is, $j = 0$ or $j$ is odd (see (3.3)). The case $j = 0$ is clear, consider the case $j$ is odd. By (4.2), (4.3), (3.4) we have $\tilde{E}_2(v_0 \otimes v_j) = v_0 \otimes \tilde{E}_2(v_j) = 0 \Leftrightarrow j = 4n + 1$ and $\tilde{E}_3(v_0 \otimes v_j) = \tilde{E}_3(v_0) \otimes v_j = 0$. Therefore $v_0 \otimes v_{4n+1} \in HW(B \otimes B)$.

Then we consider the case $i \neq 0$. We have $\tilde{E}_1(v_i \otimes v_j) = 0 \Leftrightarrow \tilde{E}_1(v_i) = 0 \Leftrightarrow i$ is odd. For $i = 4n + 1$, formulas (4.3) and (3.4) imply that

\[\tilde{E}_3(v_i \otimes v_j) = \begin{cases} 
v_i \otimes \tilde{E}_3(v_j), & j = 4m + 1, \\
\tilde{E}_3(v_i) \otimes v_j, & j \neq 4m + 1,
\end{cases}\]

\[\neq 0.\]

For $i = 4n + 3$, we have

\[\tilde{E}_2(v_i \otimes v_j) = \begin{cases} 
\tilde{E}_2(v_i) \otimes v_j, & j \neq 4m + 2, \\
v_i \otimes \tilde{E}_2(v_j), & j = 4m + 2,
\end{cases} = 0 \Leftrightarrow j = 4m + 2.\]

and $\tilde{E}_3(v_{4m+3} \otimes v_{4m+2}) = \tilde{E}_3(v_{4m+3}) \otimes v_{4m+2} = 0$. Therefore the desired result follows. \qed
Let $v$ be the crystal graph of $V$. Thus (4.5) form a basis of $L$.

**Lemma 4.2.** Let $\lambda_1 = (-2,0,0) = [1,0,0]$, then the $U$-module $L(\lambda_1)$ has a polarizable crystal base.

**Remark.** Since the highest weight of $V$ is $\lambda = [0,0,1]$, we see that Theorem 3.3 follows immediately from Lemma 4.2.

**Proof.** Let $\mu = wt(v_0 \otimes v_1) = (-4,0,0) = [2,0,0]$. Then by Proposition 4.1 and the results in Section 2.4 of [1], the $U$-module $L(\mu)$ has a polarizable crystal base. Let $M(\mu)$ be the Verma module (see [21, p. 72]) with the highest weight $\mu$ and let $M(\lambda_1)$ be the Verma module with the highest weight $\lambda_1$. Let $v_\mu \in M(\mu)$ (resp. $v_{\lambda_1} \in M(\lambda_1)$) be a highest weight vector. Then

$$L(\mu) \cong M(\mu)/(F_{2}v_{\mu}, F_{3}v_{\mu}), \quad L(\lambda_1) \cong M(\lambda_1)/(F_{2}v_{\lambda_1}, F_{3}v_{\lambda_1}).$$

Let $v$ be either $v_\mu$ or $v_{\lambda_1}$. Then

$$F^{(\delta,m)}v = F_{222}^{\delta_1}F_{212}^{\delta_2}F_{221}^{\delta_3}F_{1}^{\delta_4}F_{0}^{m}v, \quad \delta_i = 0, 1, \quad m \in \mathbb{Z}_{\geq 0},$$

form a basis of $L(\mu)$ or $L(\lambda_1)$. The weights of these elements are

$$wt(F^{(\delta,m)}v) = wt(v) - (\delta_1 + \delta_2 + \delta_3 + \delta_4 + 2m)\epsilon_1 + (-\delta_1 - \delta_2 - \delta_3 + \delta_4)\epsilon_2 + (-\delta_1 - \delta_2 + \delta_3 + \delta_4)\epsilon_3.$$  

Thus $wt(F^{(\delta,m)}v_{\lambda_1}) = m + 1 + \delta_1 + 2\delta_2 - \delta_3 \geq 0$ and $L(\lambda_1) \in O_{\text{int}}$.

Formula (4.5) implies that the decomposition of $L(\mu)$ and $L(\lambda_1)$ into simple $U_i$-modules ($i = 2, 3$) are the same, and $L(\mu) \cong L(\lambda_1)$ as $U_i$-modules ($i = 2, 3$).

The corresponding weights of $L(\mu)$ and $L(\lambda_1)$ have different numerical marks on $h_1$, however, the difference is just 1. Therefore, by applying an argument similar to the proof of Proposition 3.1 (in particular, the computations (3.3) and (3.6)), we see that if $(L_\mu, B_\mu)$ is a polarizable crystal base for $L(\mu)$ with

$$L_\mu = \{ F_{i_1} \cdots F_{i_k} v_\mu : (i_1 \cdots i_k) \in J \},$$

where $J$ is a certain set of sequences formed by the numbers in $\{1,2,3\}$, then

$$L_{\lambda_1} = \{ F_{i_1} \cdots F_{i_k} v_{\lambda_1} : (i_1 \cdots i_k) \in J \},$$

with the corresponding $B_{\lambda_1}$ will define a polarizable crystal base for $L(\lambda_1)$.

5. **Crystal graphs**

From the proof of Lemma 4.2 we see that we can identify the crystal graph of $L([m,0,0])$ ($m > 0$) with the crystal graph of $L([2,0,0])$. We also identify the crystal graph of $L([2,0,0])$ with the connected component generated by $\square \otimes \square$ in the crystal graph of $V \otimes V$. Write

$$\square \otimes \square = \begin{array}{c} i \\ j \end{array}, \quad i, j \geq 0.$$
The crystal graph of $L([2,0,0])$ is given by the diagram

\[
\begin{array}{c}
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
\end{array}
\end{array}
\xrightarrow{1}
\begin{array}{c}
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\end{array}
\xrightarrow{2}
\begin{array}{c}
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\end{array}
\xrightarrow{1}
\begin{array}{c}
\begin{array}{c}
B_0 \\
B_1 \\
\end{array}
\end{array}
\xrightarrow{1}
\ldots
\end{array}
\]

(5.1)

where the blocks $B_i$ ($i = 0,1,2,...$) are given by

\[
\begin{array}{c}
\begin{array}{c}
\frac{1}{4} \\
\frac{1}{4} \\
\frac{3}{4} \\
\frac{3}{4} \\
\end{array}
\end{array}
\xrightarrow{1}
\begin{array}{c}
\begin{array}{c}
\frac{4i}{4} \\
\frac{4i+1}{4} \\
\frac{4i+2}{4} \\
\frac{4i+3}{4} \\
\end{array}
\end{array}
\xrightarrow{2}
\begin{array}{c}
\begin{array}{c}
\frac{4i+4}{5} \\
\frac{4i+4}{5} \\
\frac{4i+4}{5} \\
\frac{4i+4}{5} \\
\end{array}
\end{array}
\xrightarrow{1}
\ldots
\end{array}
\]

(5.2)

and the blocks $B'_j$ ($j = 1,2,...$) are given by

\[
\begin{array}{c}
\begin{array}{c}
\frac{4j}{3} \\
\frac{4j+1}{3} \\
\frac{4j+2}{3} \\
\frac{4j+3}{3} \\
\end{array}
\end{array}
\xrightarrow{1}
\begin{array}{c}
\begin{array}{c}
\frac{4j}{3} \\
\frac{4j+1}{3} \\
\frac{4j+2}{3} \\
\frac{4j+3}{3} \\
\end{array}
\end{array}
\xrightarrow{2}
\ldots
\end{array}
\]

(5.3)

These graphs can be verified by using (4.1)-(4.3) and (3.1).

The crystal graph of $L([0,0,n])$ can be identified with the connected component generated by $\begin{array}{c}
\begin{array}{c}
\otimes \\
\otimes \\
\otimes \\
\otimes \\
\end{array}
\end{array}$ ($n$ copies) in the crystal graph of $V^\otimes n$, and is given by

\[
\begin{array}{c}
\begin{array}{c}
0 \\
0 \\
\cdots \\
0 \\
\end{array}
\end{array}
\xrightarrow{3}
\begin{array}{c}
\begin{array}{c}
B_1 \\
B_2 \\
\end{array}
\end{array}
\xrightarrow{3}
\cdots
\]

(5.4)
where the blocks $B_i (i = 0, 1, 2, \ldots)$ are given by

The crystal graph of $L([m, 0, n])$ ($m, n > 0$) can be identified with the connected component generated by

in which

The crystal graph of $L([m, 0, n])$ ($m, n > 0$) can be identified with the connected component generated by

in the crystal graph of $L([0, 0, n]) \otimes L([m, 0, 0])$ (or $L([0, 0, n]) \otimes L([2, 0, 0])$). We should just give the crystal graph in the case $n = 1$ as an example.

**Example.** The crystal graph of $L([m, 0, 1])$ is given by

where the blocks $C_i (i = 0, 1, 2, \ldots)$ are given by

\[ 3 \rightarrow \begin{array}{c} 4i + 1 \otimes 0 \\ 4i + 2 \otimes 0 \\ 4i + 3 \otimes 0 \\ 4i + 4 \otimes 0 \end{array} \]
the block $\{D\}$ is given by
\[
1 \xrightarrow{0 \otimes \frac{0}{2}} 2 \xrightarrow{0 \otimes \frac{1}{3}} 3 \xrightarrow{0 \otimes \frac{1}{3}} 1
\]
\[
1 \otimes \frac{0}{2} \xrightarrow{2} 1 \otimes \frac{0}{3} \xrightarrow{3} 1 \otimes \frac{1}{3}
\]
\[
1 \otimes \frac{1}{2} \xrightarrow{2} 1 \otimes \frac{1}{2} \xrightarrow{3} 1 \otimes \frac{1}{2}
\]
\[
2 \otimes \frac{1}{2} \xrightarrow{2} 2 \otimes \frac{1}{2} \xrightarrow{2} 2 \otimes \frac{1}{2} \xrightarrow{1} 4 \otimes \frac{1}{2}
\]
and the blocks $\{D_i\} (i = 0, 1, 2, \ldots)$ are given by
\[
1 \xrightarrow{4i \otimes \frac{0}{4}} 3 \xrightarrow{1,1} 1 \xrightarrow{0,2} 2 \xrightarrow{0,3} 1
\]
\[
1,1 \xrightarrow{1,2} 2 \xrightarrow{1,3} 1 \xrightarrow{1,4}
\]
\[
2,1 \xrightarrow{2,2} 2 \xrightarrow{2,3} 1 \xrightarrow{2,4}
\]
in which
\[
0, t = 4i + t \otimes \frac{0}{4}, \quad 1 \leq t \leq 3,
\]
\[
1, t = 4i + t \otimes \frac{1}{4}, \quad 1 \leq t \leq 4,
\]
\[
2, t = 4i + t \otimes \frac{1}{5}, \quad 1 \leq t \leq 4.
\]

**Remark.** From graph (5.1) we see that
\[
\tilde{E}_i \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \begin{array}{c} 0 \\ 2 \end{array} = 0, \quad i = 1, 2, 3.
\]

So \( \begin{array}{c} 0 \\ 1 \end{array} \otimes \begin{array}{c} 0 \\ 2 \end{array} \) generates a connected component in the crystal graph of \( L([2, 0, 0]) \otimes L([2, 0, 0]) \). Since
\[
\text{wt} \left( \begin{array}{c} 0 \\ 1 \end{array} \otimes \begin{array}{c} 0 \\ 2 \end{array} \right) = [4, 1, 1],
\]
we see that there are \( L(\nu) \) with \( \nu(h_2) > 0 \) which possess polarizable crystal bases.
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