

ON THE RELATION BETWEEN UPPER CENTRAL QUOTIENTS AND LOWER CENTRAL SERIES OF A GROUP

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ABSTRACT. Let H be a group with a normal subgroup N contained in the upper central subgroup $Z_c H$. In this article we study the influence of the quotient group $G = H/N$ on the lower central subgroup $\gamma_{c+1} H$. In particular, for any finite group G we give bounds on the order and exponent of $\gamma_{c+1} H$. For G equal to a dihedral group, or quaternion group, or extra-special group we list all possible groups that can arise as $\gamma_{c+1} H$. Our proofs involve: (i) the Baer invariants of G , (ii) the Schur multiplier $\mathcal{M}(L, G)$ of G relative to a normal subgroup L , and (iii) the nonabelian tensor product of groups. Some results on the nonabelian tensor product may be of independent interest.

1. INTRODUCTION

A group H gives rise to an upper central series $1 = Z_0 H \leq Z_1 H \leq \dots$ and a lower central series $H = \gamma_1 H \geq \gamma_2 H \geq \dots$. In this article we consider a normal subgroup $N \triangleleft H$ contained in $Z_c H$, and study the influence of the quotient $G = H/N$ on the lower central group $\gamma_{c+1} H$. An old result of R. Baer [1][18] states that $\gamma_{c+1} H$ is finite whenever the quotient G is finite. We develop Baer's techniques to obtain the following three results.

A. For any finite G we give an upper bound on the order of $\gamma_{c+1} H$. (A previous paper [9] gives a bound on $|\gamma_{c+1} H|$ when G is finite nilpotent; the present result incorporates a small improvement in this case. Several authors have given bounds when $c = 1$. In particular, there are papers by J.A. Green [17], J. Wiegold [29], [30], W. Gaschütz et al. [15], and M.R. Jones [19], [20], [21]. The case $c = 1$ is also studied in [11] where the results are slightly sharper than those obtained by specialising our general bound to $c = 1$.)

B. For any finite G we give an upper bound on the exponent of $\gamma_{c+1} H$. (For $c = 1$ this provides a generalisation of a result of A. Lubotzky and A. Mann [23] on the exponent of the Schur multiplier $\mathcal{M}(G)$ of a powerful p -group G ; it also sharpens a bound of M.R. Jones [21] on the exponent of the Schur multiplier of a prime-power group. Furthermore, for $c \geq 1$ our bound yields a generalisation and sharpening of an estimate, given in [7], on the exponent of the c -nilpotent Baer invariant $M^{(c)}(G)$. This improvement for $c \geq 1$ has the following practical implication. An electronically down-loadable appendix to the paper [12] contains a MAGMA computer program for calculating a number of homotopy-theoretic constructions. In particular, it contains a function for computing $M^{(c)}(G)$ which requires, as input data, a finite presentation of a finite group G together with any positive integer q

Received by the editors February 12, 1999.

2000 *Mathematics Subject Classification*. Primary 20F14, 20F12.

divisible by e^c where e denotes the exponent of $M^{(c)}(G)$. The improved estimate for e helps in choosing a suitable value for q .)

C. For G equal to a dihedral group, or quaternion group, or extra-special group we list all possible groups that can arise as $\gamma_{c+1}H$. (This extends the work of N.D. Gupta and M.R.R. Moghaddam [16] which handles the dihedral 2-groups. It also extends the work of D. MacHale and P.Ó'Murchú [26], and J. Burns et al. [8] which treats all groups G of order at most 30 for $c = 1$, and all groups G of order at most 16 for $c = 2$.)

A precise statement of results A-C is provided in Section 2. Their proofs are given in Sections 4-6 respectively. The proofs involve three techniques with which the reader may not be too familiar. The first is the use of a nonabelian tensor product of groups. The second is the use of a Schur multiplier $\mathcal{M}(N, G)$ of a group G relative to a normal subgroup N . The third is the use of Baer invariants of a group. Relevant details of these techniques are recalled, and developed, in Section 3. Some results in Section 3 (in particular Propositions 5, 8 and 9) may be of independent interest.

2. STATEMENT OF RESULTS

Let a group G be presented as the quotient of a free group F by a normal subgroup R . We state our results in terms of the Baer invariants

$$M^{(c)}(G) = \frac{R \cap \gamma_{c+1}F}{\gamma_{c+1}(R, F)}, \quad c \geq 1,$$

and related invariants

$$\gamma_{c+1}^*(G) = \frac{\gamma_{c+1}F}{\gamma_{c+1}(R, F)}$$

of the group G , where $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$ and $\gamma_{c+1}F = \gamma_{c+1}(F, F)$. It was shown by R. Baer [1] (see also [14] [25]) that these invariants are, up to group isomorphism, independent of the choice of free presentation of G . Note that there are canonical actions of G on $M^{(c)}(G)$ and $\gamma_{c+1}^*(G)$ given by conjugation in $F/\gamma_{c+1}(R, F)$.

If G is of the form $G \cong H/N$ with N a normal subgroup of H contained in $Z_c H$, then it is routine [6] to establish the existence of the canonical short exact sequence

$$A \twoheadrightarrow \gamma_{c+1}^*(G) \twoheadrightarrow \gamma_{c+1}H$$

where A is a submodule of the $\mathbf{Z}G$ -module $M^{(c)}(G)$. We thus have inequalities (in which \leq can be taken to mean 'divides')

$$(1) \quad |\gamma_{c+1}H| \leq |\gamma_{c+1}^*(G)| = |M^{(c)}(G)| |\gamma_{c+1}G|,$$

$$(2) \quad \exp(\gamma_{c+1}H) \leq \exp(\gamma_{c+1}^*(G)),$$

involving the orders and exponents of groups. Since $M^{(c)}(G)$ is a subgroup of $\gamma_{c+1}^*(G)$, we also have an inequality

$$(3) \quad \exp(M^{(c)}(G)) \leq \exp(\gamma_{c+1}^*(G)).$$

Furthermore, a group K arises as $\gamma_{c+1}H$ for some H if and only if

$$(4) \quad K \cong \frac{\gamma_{c+1}^*(G)}{A}$$

with A a submodule of $M^{(c)}(G)$.

Observations (1), (2) and (4) allow us to state results A-C in terms of the invariant $\gamma_{c+1}^*(G)$. For the statement of result A we let $\chi_c(d)$ denote the number of elements in a basis of the free abelian group $\gamma_c F / \gamma_{c+1} F$ with F the free group on d generators. (There is a well-known formula for $\chi_c(d)$ due to E. Witt [27]. Let $\mu(m)$ be the Möbius function, defined for all positive integers by $\mu(1) = 1$, $\mu(p) = -1$ if p is a prime, $\mu(p^k) = 0$ for $k > 1$, and $\mu(ab) = \mu(a)\mu(b)$ if a and b are coprime integers. Witt's formula is

$$\chi_c(d) = (1/c) \sum_{m|c} \mu(m) d^{(c/m)}$$

where m runs through all divisors of c . Thus, for instance, $\chi_2(d) = (d^2 - d)/2$, $\chi_3(d) = (d^3 - d)/3$, $\chi_4(d) = (d^4 - d^2)/4$.

For an arbitrary finite abelian p -group A we define the integer

$$\Lambda_c(A) = e_1 \chi_{c+1}(d_1) + \sum_{j=2}^k e_j \{ \chi_{c+1}(d_1 + \dots + d_j) - \chi_{c+1}(d_1 + \dots + d_{j-1}) \}$$

where the parameters d_j, e_j, k are determined by expressing A uniquely in the form

$$A \cong (C_{p^{e_1}})^{d_1} \times (C_{p^{e_2}})^{d_2} \times \dots \times (C_{p^{e_k}})^{d_k}$$

with $e_1 > e_2 > \dots > e_k \geq 1$.

For an arbitrary finite d -generator p -group P we define the integer

$$\Psi_c(P) = m_{c+1}d + m_c d^2 + \dots + m_2 d^c$$

where the terms of the lower central series of P have orders $|\gamma_j(P)| = p^{m_j}$.

Note that an arbitrary finite group G has a smallest term L in its lower central series, namely the unique group $L = \gamma_r G$ that satisfies $\gamma_r G = \gamma_{r+1} G$. Suppose that P is a d -generator p -Sylow subgroup of G with Frattini subgroup $\Phi(P) = [P, P]P^p$, that $P/(P \cap [G, G])$ is a δ -generator group, that $(L \cap P)/(L \cap \Phi(P))$ has order p^t , that $L \cap P$ has order p^β , and that $[L \cap P, P]$ has order $p^{\beta'}$. We use these various parameters to define the integer

$$\Theta_c(G, L, P) = \beta + (\omega - \beta')(1 + \delta + \delta^2 + \dots + \delta^{c-1}),$$

where

$$\omega = d\beta - (1/2)t(t+1).$$

Theorem A. *Let G be a finite group whose order has prime factors p_1, p_2, \dots, p_n . Let L be the smallest term in the lower central series of G . The quotient G/L is nilpotent and thus a direct product*

$$G/L \cong S_1 \times S_2 \times \dots \times S_n$$

with S_i a (possibly trivial) p_i -group. For each i let P_i be some p_i -Sylow subgroup of G . Then

$$|\gamma_{c+1}^*(G)| \leq \prod_{i=1}^n p_i^{\Lambda_c(S_i^{a_i b_i}) + \Psi_c(S_i) + \Theta_c(G, L, P_i)}.$$

The bound is attained, for instance, when G is abelian.

Note that if G is perfect, then $\Lambda_c(S_i^{ab}) = 0$, $\Psi_c(S_i) = 0$ and $\Theta_c(G, L, P_i) = d_i(2\beta_i - d_i - 1)/2 + \alpha_i$ where $p_i^{\beta_i}$ is the order of a d_i -generator p_i -Sylow subgroup P_i , and $p_i^{\alpha_i}$ is the order of the abelianisation P_i^{ab} . If, at the other extreme, G is nilpotent, then we have $\Theta_c(G, L, P_i) = 0$. If G is abelian, then $\Theta_c(G, L, P_i) = 0$ and $\Psi_c(S_i) = 0$.

The bound in Theorem A can be sharpened by involving the relative Schur multiplier $\mathcal{M}(L, G)$ whose definition is recalled in Section 3. More precisely, in the definition of $\Theta_c(G, L, P)$ we can redefine $\omega = \mu + \beta'$ where the p th primary component of the abelian group $\mathcal{M}(L, G)$ has order $|\mathcal{M}(L, G)_p| = p^\mu$. For example, if $|L|$ is coprime to $|G|/\exp(L)$, then the relative multiplier is trivial (see Proposition 7(ii)) and we can take $\Theta_c(G, L, P_i) = \beta_i$ for $c \geq 1$.

Before stating result B let us recall that A. Lubotzky and A. Mann [23] defined a p -group P to be *powerful* if: $p \geq 3$ and $[P, P] \subset P^p$; or $p = 2$ and $[P, P] \subset P^4$ (where P^i is the subgroup of P generated by all i th powers). In other words, P is powerful if $p \geq 3$ and P/P^p is abelian, or if $p = 2$ and P/P^4 is abelian. They proved a number of results about powerful groups P , one of which states that the exponent $\exp(M^{(1)}(P))$ of the Schur multiplier divides the exponent of P . We shall generalise this. Our generalisation implies, for instance, that $\exp(M^{(c)}(P))$ divides $\exp(P)$ for all $c \geq 1$ and all P in a certain class C_p of p -groups; the class C_p consists of those p -groups P satisfying $[[P^{p^{i-1}}, P], P] \subset P^{p^i}$ for $1 \leq i \leq e$ where $\exp(P) = p^e$. It is shown in [23] that if P is powerful, then $[P^{p^{i-1}}, P] \subset P^{p^i}$. Hence the class C_p contains all powerful p -groups.

Given a normal subgroup $N \trianglelefteq G$ of some group G , we say that the pair (N, G) has *nilpotency class k* if $\gamma_{k+1}(N, G) = 1$ and $\gamma_k(N, G) \neq 1$. For a real number r we let $[r]$ denote the smallest integer n such that $n \geq r$.

Let N be a normal subgroup of a finite p -group P and suppose that N has exponent p^e . We define the integer

$$\Omega(N, P) = [k_1/2] + [k_2/2] + \dots + [k_e/2]$$

where k_j denotes the nilpotency class of the pair $(N^{p^{j-1}}/N^{p^j}, P/N^{p^j})$ for $1 \leq j \leq e$. For N equal to the trivial group we set $\Omega(1, P) = 0$. Note that $\Omega(N, P) \leq [k/2]e$ where k is the nilpotency class of P .

Theorem B. (i) *Let G be a finite group whose order has prime factors p_1, p_2, \dots, p_n . Let L be the smallest term in the lower central series of G . The quotient G/L is thus a direct product*

$$G/L \cong S_1 \times S_2 \times \dots \times S_n$$

with S_i a (possibly trivial) p_i -group. For each i let P_i be a p_i -Sylow subgroup of G . Suppose that $L \cap P_i/[L \cap P_i, P_i]$ has exponent $p_i^{n_i}$. Suppose that the p_i -primary component of G^{ab} has exponent $p_i^{e_i}$ with $e_i \geq 0$, and set $m_i = \min(\Omega(L \cap P_i, P_i), e_i)$. Then, for each $c \geq 1$,

$$\exp(\gamma_{c+1}^*(G)) \text{ divides } \prod_{i=1}^n p_i^{\Omega(L \cap P_i, P_i) + \Omega(S_i, S_i) + n_i + (c-1)m_i}.$$

The bound is attained if G is abelian.

(ii) *Suppose that a p -group P satisfies $[[P^{p^{i-1}}, P], P] \subset P^{p^i}$ for all $1 \leq i \leq e$ where p^e is the exponent of P . Then $\Omega(P, P) = e$.*

Note that, by inequality (3), $\exp(M^{(c)}(G))$ divides $\exp(\gamma_{c+1}^*(G))$ for any group G . Thus, for an arbitrary finite p -group P of class k and exponent p^e , Theorem B(i) implies that $\exp(M^{(c)}(P))$ divides $p^{\lfloor k/2 \rfloor e}$; this sharpens the bound $\exp(M^{(c)}(P)) \leq p^{(k-1)e}$ of Corollary 2.6 in [21] (for $c = 1$) and Theorem 6 in [7] (for $c \geq 1$). Theorem B(ii) implies that $\exp(\gamma_{c+1}^*(P))$ divides $\exp(P)$ if, for example, P is a p -group with P/P^p of nilpotency class 2 and P^p contained in the second centre $Z_2(P)$.

The bound in Theorem B(i) can be sharpened by redefining m_i to be $m_i = \min(\epsilon_i, e_i)$ where $p_i^{\epsilon_i}$ and p^{e_i} are the exponents of the p_i -primary components of $\mathcal{M}(L, G)$ and G^{ab} respectively (cf. Proposition 7 in Section 3). The bound is clearly independent of c if G is finite nilpotent, or if G is perfect. We do not know whether the bound can be made independent of c for arbitrary finite groups.

For the statement of result C we let $D_n = \langle a, b \mid a^2 = b^n = (ab)^2 = 1 \rangle$ denote the dihedral group of order $2n$, and $Q_n = \langle a, b \mid a^2 = b^n = (ab)^2 \rangle$ denote the quaternion group of order $4n$. Recall [3] that a p -group E is said to be *extra-special* if its commutator subgroup $[E, E]$, its Frattini subgroup $\Phi(E)$, and its centre $Z_1(E)$ coincide and have order p . The extra-special groups have order p^{2k+1} for $k \geq 1$, with precisely two extra-special p -groups for each k (see [3]). We let $E(p, k)$ denote an arbitrary extra-special p -group of order p^{2k+1} ; we let $E(p, k)^+$ and $E(p, k)^-$ denote the extra-special p -groups of order p^{2k+1} and exponents p and p^2 respectively. For $c \geq 1$ we have the following.

Theorem C. (i) For each $n \geq 2$ we have

$$\gamma_{c+1}^*(D_n) \cong \begin{cases} C_n & \text{odd } n, \\ C_n \times (C_2)^{\chi_{c+1}(2)-1} & \text{even } n. \end{cases}$$

The generator $b \in D_n$ acts trivially on $\gamma_{c+1}^*(D_n)$; the generator $a \in D_n$ acts trivially on elements of order two, and inverts the elements of the cyclic summand C_n .

(ii) For each $n \geq 2$ we have

$$\gamma_{c+1}^*(Q_n) \cong \gamma_{c+1}^*(D_n).$$

The generators $a, b \in Q_n$ act as in (i).

(iii) For each $k \geq 2$ we have

$$\gamma_{c+1}^*(E(p, k)) \cong (C_p)^{\chi_{c+1}(2k)}.$$

The group $E(p, k)$ acts trivially on $\gamma_{c+1}^*(E(p, k))$.

(iii)' For $p \geq 3$ and some $1 \leq r \leq 2^c$ we have

$$\gamma_{c+1}^*(E(p, 1)^+) \cong (C_p)^{\chi_{c+1}(2)+r},$$

$$\gamma_{c+1}^*(E(p, 1)^-) \cong (C_p)^{\chi_{c+1}(2)}.$$

Note that the corresponding Baer invariants $M^{(c)}(G)$ are easily obtained applying the formula $M^{(c)}(G) = \ker(\gamma_{c+1}^*(G) \rightarrow \gamma_{c+1}(G))$ to the precise details of the isomorphisms given in the proof of Theorem C. This extends the computations on dihedral 2-groups given in [16]. (We remark that there is a slip in the statement of the main theorem in [16]; the statement is correct for $\gamma_{c+1}^*(D_{2^n})$ but incorrect for $M^{(c)}(D_{2^n})$.)

The precise value of r in Theorem C(iii)' needs further investigation. The computer program listed in [12] yields the following results for the Burnside group $B(2, 3) = E(3, 1)^+$ of exponent 3 on two generators.

| c | $\gamma_{c+1}^*(B(2, 3))$ | r |
|-----|---------------------------|-----|
| 1 | $(C_3)^3$ | 2 |
| 2 | $(C_3)^5$ | 3 |
| 3 | $(C_3)^9$ | 6 |
| 4 | $(C_3)^{15}$ | 9 |
| 5 | $(C_3)^{27}$ | 18 |

3. PRELIMINARIES

The tensor product of nonabelian groups is a convenient setting for performing commutator calculations. Its functorial properties make it especially suited to the task of relating commutator calculations in a group to those in a homomorphic image of the group. We begin this section by recalling and developing relevant details on this tensor product. We then recall details on a Schur multiplier $\mathcal{M}(N, G)$ defined for pairs of groups. By a *pair* of groups (N, G) we simply mean a group G with normal subgroup N . The advantage of working with pairs is that any finite group G can be expressed as an extension

$$(L, G) \succ \longrightarrow (G, G) \longrightarrow \twoheadrightarrow (G/L, G/L)$$

of a ‘perfect’ pair (L, G) by a ‘nilpotent’ pair $(G/L, G/L)$. Various simplifications apply when dealing with the Schur multiplier of perfect or nilpotent pairs. We end the section with some details on Baer invariants.

Suppose given two groups G and H which act on each other via group actions $G \times H \rightarrow H, (g, h) \mapsto {}^g h$ and $H \times G \rightarrow G, (h, g) \mapsto {}^h g$. Furthermore, suppose that each group acts on itself by conjugation, ${}^x y = xyx^{-1}$. (In keeping with this notation, our convention for commutators is $[x, y] = xyx^{-1}y^{-1}$.) The tensor product $G \otimes H$ is defined [5], [4] to be the group generated by symbols $g \otimes h$ subject to the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h),$$

$$g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h'),$$

for $g, g' \in G, h, h' \in H$. The actions are said to be *compatible* if

$$({}^{g h}) g' = g({}^h (g^{-1} g')), \quad ({}^h g) h' = h(g({}^{h^{-1}} h'))$$

for all $g, g' \in G, h, h' \in H$.

Proposition 1 ([5]). *Suppose that G and H act compatibly on each other.*

(i) *For all $g, g' \in G, h, h' \in H$ the following identities hold in $G \otimes H$:*

$$g' \otimes ({}^g h) h^{-1} = g'(g \otimes h)(g \otimes h)^{-1},$$

$$g({}^h g^{-1}) \otimes h' = (g \otimes h) h' (g \otimes h)^{-1},$$

$$g({}^h g^{-1}) \otimes ({}^{g'} h') h'^{-1} = [g \otimes h, g' \otimes h'].$$

(ii) *There is a homomorphism $\partial_G: G \otimes H \rightarrow G, g \otimes h \mapsto g {}^h g^{-1}$.*

(iii) *There is a ‘diagonal’ action of G on $G \otimes H$ given by $g'(g \otimes h) = (g' g \otimes g' h)$.*

(iv) *There is an isomorphism $G \otimes H \xrightarrow{\cong} H \otimes G, g \otimes h \mapsto h \otimes g$.*

(v) If ${}^g h = h, {}^h g = g$ for all $g \in G, h \in H$, then $G \otimes H \cong G^{ab} \otimes H^{ab}$, where the right-hand side of the isomorphism denotes the usual tensor product of abelian groups.

For each pair of groups (N, G) we can form the tensor product $N \otimes G$ in which all actions are taken to be conjugation in G . Since conjugation yields compatible actions, there is a diagonal action of G on $N \otimes G$. The tensor product $N \otimes G$ acts on G by conjugation in G via the homomorphism $\partial_N: N \otimes G \rightarrow N$. We can thus construct the triple tensor product $(N \otimes G) \otimes G$. One readily checks that the construction preserves ‘compatibility of actions’, and that it can therefore be iterated to form the $(c + 1)$ -fold tensor product

$$\bigotimes^{c+1}(N, G) = (\dots((N \otimes G) \otimes G) \otimes \dots \otimes G), \quad c \geq 1,$$

involving c copies of G and one copy of N .

Proposition 2 ([13]). *Let G be a d -generator p -group with normal subgroup N . Suppose that $|\gamma_i(N, G)| = p^{m_i}$ for $i \geq 1, m_i \geq 0$. Then, for any $c \geq 1$, we have*

$$|\bigotimes^{c+1}(N, G)| \leq p^{m_c d + m_{c-1} d^2 + \dots + m_1 d^c}.$$

Lemma 3. *Let $G_3 \hookrightarrow G_2 \twoheadrightarrow G_1, H_3 \hookrightarrow H_2 \twoheadrightarrow H_1$ be two short exact sequences of groups. Suppose that G_i and H_i act compatibly on one another for $1 \leq i \leq 3$, and that the homomorphisms preserve actions. Then there is an exact sequence of homomorphisms*

$$(G_3 \otimes H_2) \overline{\times} (G_2 \otimes H_3) \longrightarrow G_2 \otimes H_2 \longrightarrow G_1 \otimes H_1 \longrightarrow 1$$

in which $\overline{\times}$ denotes a semi-direct product whose details need not be specified.

Proof. The lemma is a routine adaption of Proposition 9 in [4]. □

Lemma 4. *Let N be a normal subgroup of G for which the commutator $[n, [n, g]]$ is trivial for all $g \in G, n \in N$. In the tensor product $N \otimes G$, with G and H acting by conjugation, the following identity holds for all $g \in G, n \in N$ and all integers $t \geq 2$:*

$$n^t \otimes g = (n \otimes g)^t (n \otimes [n, g])^{t(t-1)/2}.$$

Proof. The case $N = G$ is proved in [2]. The proof of the more general case is analogous; it can also be derived directly using Proposition 1(i). □

Recall that a pair (N, G) is said to be *nilpotent of class k* if $\gamma_{k+1}(N, G) = 1$ and $\gamma_k(N, G) \neq 1$. Also recall that $[k/2]$ denotes the smallest integer n such that $n \geq k/2$.

Proposition 5. *Let G be a group with normal subgroup N . Suppose that N has prime-power exponent p^e and that the pair (N, G) has nilpotency class $\leq k$. Then, for any $c \geq 1$, we have*

$$\exp(\bigotimes^{c+1}(N, G)) \text{ divides } p^{\lfloor k/2 \rfloor e}.$$

Proof. For $t = p^e$ the binomial coefficient $\binom{t}{2}$ is divisible by t when $p \geq 3$, and divisible by $t/2$ when $p = 2$. Thus Lemma 4 proves the proposition for $k = 2, c = 1$ (since for $p = 2, t = p^e$ and $\gamma_3(N, G) = 1$ the identity

$$n \otimes [n, g]^{t/2} = n \otimes [n^{t/2}, g]$$

holds for all $g \in G, n \in N$; but $[n^{t/2}, g] = 1$ because $n^{t/2}$ has order at most 2.)

Let us now consider $k = 2$ and some $c \geq 2$. Then

$$\bigotimes^{c+1}(N, G) = \bigotimes^c(N \otimes G, G)$$

and $N \otimes G$ acts trivially on G . The triviality of this action implies the identity

$$(\cdots((n \otimes g)^t \otimes g_1) \otimes \cdots \otimes g_c) = (\cdots((n \otimes g) \otimes g_1) \otimes \cdots \otimes g_c)^t$$

in $\bigotimes^{c+1}(N, G)$. Hence $\exp(\bigotimes^{c+1}(N, G))$ divides $\exp(N \otimes G)$ and the proposition is proved for $k = 2, c \geq 1$.

Suppose now that the proposition has been proved for some c and all $k < k_0$. Suppose $\gamma_{k_0+1}(N, G) = 1$. Lemma 3 implies an exact sequence

$$(\gamma_{k_0-1}(N, G) \otimes G) \overline{\times} (N \otimes \gamma_{k_0-1}(N, G)) \rightarrow N \otimes G \rightarrow \frac{N}{\gamma_{k_0-1}(N, G)} \otimes \frac{G}{\gamma_{k_0-1}(N, G)}.$$

Working in $N \otimes G$, the image of $\gamma_{k_0-1}(N, G) \otimes G$ contains the image of $N \otimes \gamma_{k_0-1}(N, G)$ by virtue of the identity

$$m \otimes [n, g] = ([n, g] \otimes m)^{-1}$$

which follows from Proposition 1(i) for all $g \in G, m, n \in N$. We thus have an exact sequence

$$\gamma_{k_0-1}(N, G) \otimes G \rightarrow N \otimes G \rightarrow \frac{N}{\gamma_{k_0-1}(N, G)} \otimes \frac{G}{\gamma_{k_0-1}(N, G)}.$$

By applying Lemma 3 to this sequence, and invoking a similar identity, we obtain the exact sequence

$$\begin{aligned} &(\gamma_{k_0-1}(N, G) \otimes G) \otimes G \rightarrow (N \otimes G) \otimes G \\ &\rightarrow \left(\frac{N}{\gamma_{k_0-1}(N, G)} \otimes \frac{G}{\gamma_{k_0-1}(N, G)} \right) \otimes \frac{G}{\gamma_{k_0-1}(N, G)}. \end{aligned}$$

Repetition of the process yields an exact sequence

$$\bigotimes^{c+1}(\gamma_{k_0-1}(N, G), G) \rightarrow \bigotimes^{c+1}(N, G) \rightarrow \bigotimes^{c+1}\left(\frac{N}{\gamma_{k_0-1}(N, G)}, \frac{G}{\gamma_{k_0-1}(N, G)}\right)$$

from which we deduce that $\exp(\bigotimes^{c+1}(N, G)) \leq p^{[(k_0-2)/2]} p^e = p^{[k_0/2]}$. By induction, the proposition is proved for all $c, k \geq 1$. \square

Following J.-L.Loday [22] we say that a pair of groups (N, G) is *perfect* if $N = [N, G]$.

Proposition 6. *Let (N, G) be any perfect pair of groups and set*

$$M = \ker(\partial_N : N \otimes G \rightarrow N).$$

Then M is abelian and, for each $c \geq 1$, there is an exact sequence

$$\bigotimes^{c+1}(M, G^{ab}) \rightarrow \bigotimes^{c+2}(N, G) \rightarrow \bigotimes^{c+1}(N, G) \rightarrow 1$$

where $\otimes^{c+1}(M, G^{ab})$ is the usual iterated tensor product of abelian groups.

Proof. Let $H_3 \hookrightarrow H_2 \twoheadrightarrow H_1$ be a short exact sequence of groups, and let G be a group such that G and H_i act compatibly on each other for $1 \leq i \leq 3$ with the homomorphisms preserving the actions. Then H_3 acts trivially on G via H_2 . Suppose that the action of G on H_2 restricts to a trivial action of G on H_3 . Then Proposition 1(v) and Lemma 3 imply an exact sequence

$$(5) \quad H_3^{ab} \otimes G^{ab} \rightarrow H_2 \otimes G \rightarrow H_1 \otimes G \rightarrow 1.$$

A perfect pair of groups (N, G) gives rise to a short exact sequence $\ker(\partial_N) \hookrightarrow N \otimes G \xrightarrow{\partial} N$. The identity

$$h(n \otimes g) = ([n, g] \otimes h)^{-1} (n \otimes g)$$

which holds in $N \otimes G$ (see Proposition 1) for all $g, h \in G, n \in N$ implies that G acts trivially on $\ker(\partial_N)$. So (5) implies an exact sequence

$$M \otimes G^{ab} \rightarrow (N \otimes G) \otimes G \xrightarrow{\partial \otimes 1} N \otimes G \rightarrow 1.$$

Note that the diagonal action of G on $M \otimes G^{ab}$ is trivial, and hence G acts trivially on $\ker(\partial \otimes 1)$. Thus a second application of (5) yields the exact sequence $\ker(\partial \otimes 1) \otimes G^{ab} \rightarrow \otimes^4(N, G) \rightarrow \otimes^3(N, G) \rightarrow 1$. From this we derive the exact sequence $\otimes^3(M, G^{ab}) \rightarrow \otimes^4(N, G) \rightarrow \otimes^3(N, G) \rightarrow 1$. The proposition follows from a repetition of this argument. \square

Given a pair of groups (N, G) we denote by $\Delta(N, G)$ the subgroup of $N \otimes G$ generated by the elements $n \otimes n$ for $n \in N$. This is a normal subgroup and following [5] we define the *exterior product*

$$N \wedge G = N \otimes G / \Delta(N, G).$$

The homomorphism $\partial_N: N \otimes G \rightarrow N$ clearly induces a homomorphism $\partial_N: N \wedge G \rightarrow N$. The identity

$$[g, n] \otimes [g', n'] = [(g \otimes n), (g' \otimes n')],$$

of Proposition 1(i) implies an isomorphism $N \wedge G \cong N \otimes G$ in the case of perfect pairs.

Definition ([10]). The *Schur multiplier* of a pair of groups (N, G) is the group $\mathcal{M}(N, G)$ defined by

$$\mathcal{M}(N, G) = \ker(\partial_N: N \wedge G \rightarrow N).$$

If the pair is perfect, then, equivalently,

$$\mathcal{M}(N, G) = \ker(\partial_N: N \otimes G \rightarrow N).$$

Proposition 7 ([10]). *Let G be a finite group with normal subgroup $N \trianglelefteq G$.*

(i) *Then $\mathcal{M}(N, G)$ is a finite abelian group with exponent e dividing the order of G .*

(ii) *Let e' denote the exponent of N . Then, in fact, ee' divides the order of G and e divides the order of N .*

(iii) *Let K be any subgroup of G such that each $g \in G$ can be expressed (not necessarily uniquely) as a product $g = nk$ with $n \in N, k \in K$. Then e^2 divides $|N| \times |K|$.*

(iv) Suppose that P is a p -Sylow subgroup of G . Let $\mathcal{M}(P \cap N, P)_p$ denote the p -component of the multiplier, and $\iota: \mathcal{M}(P \cap N, P) \rightarrow \mathcal{M}(N, G)$ the homomorphism induced by inclusion. Then

$$\mathcal{M}(P \cap N, P) \cong \mathcal{M}(N, G)_p \oplus \ker(\iota).$$

Proposition 8. Let G be a δ -generator p -group with normal subgroup $N \trianglelefteq G$ of order $|N| = p^\beta$. Suppose that $|N/(N \cap \Phi(G))| = p^t$ where $\Phi(G) = [G, G]G^p$. Then

$$|N \wedge G| \leq p^{\delta\beta - \frac{t(t+1)}{2}}.$$

Proof. Proposition 2 implies that $|N \otimes G| \leq p^{\delta\beta}$. There is a commutative diagram of group homomorphisms

$$\begin{array}{ccccccc} \Gamma\left(\frac{N}{[N, G]}\right) & \xrightarrow{\Delta'} & N \otimes G & \longrightarrow & N \wedge G & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Gamma\left(\frac{G}{\Phi(G)}\right) & \xrightarrow{\Delta} & \frac{G}{\Phi(G)} \otimes \frac{G}{\Phi(G)} & \longrightarrow & \frac{G}{\Phi(G)} \wedge \frac{G}{\Phi(G)} \longrightarrow 1 \end{array}$$

in which the rows (but not the columns) are exact [5]. The abelian group $\Gamma(A)$, defined for any additive abelian group A , is J.H.C. Whitehead’s universal quadratic construction; it is generated (as an abelian group) by symbols $\gamma(a)$ for $a \in A$ subject to the relations

$$\gamma(-a) = \gamma(a),$$

$$\gamma(a + b + c) + \gamma(a) + \gamma(b) + \gamma(c) = \gamma(a + b) + \gamma(a + c) + \gamma(b + c)$$

for $a, b, c \in A$. The homomorphism Δ is defined on generators by $\Delta(\gamma(x)) = x \otimes x$ for $x \in G/\Phi(G)$. The image of $\Gamma(G/[N, G])$ in $G/\Phi(G) \otimes G/\Phi(G)$ is an elementary abelian group of rank $t(t + 1)/2$. Hence the exactness of the top row implies

$$|N \wedge G| = \frac{|N \otimes G|}{|\Delta'(\Gamma(N/[N, G]))|} \leq \frac{p^{\delta\beta}}{p^{t(t+1)/2}}.$$

This proves the proposition. □

Proposition 9. Let N be a normal subgroup of a group G . If N has exponent p^e , then

$$\exp(N \wedge G) \text{ divides } p^{[k_1/2] + [k_2/2] + \dots + [k_e/2]}$$

where k_i denotes the nilpotency class of the pair $(N^{p^{i-1}}/N^{p^i}, G/N^{p^i})$ for $1 \leq i \leq e$.

Proof. Let K, M be normal subgroups of G with $K \leq M$. Using the identity $m \otimes k = (k \otimes m)^{-1}$ which holds in $M \wedge G$ for all $k \in K, m \in M$, one readily develops the short exact sequence

$$(6) \quad K \wedge G \rightarrow M \wedge G \rightarrow M/K \wedge G/K \rightarrow 1$$

from Lemma 3. Now (6) yields the exact sequences

$$N^{p^i} \wedge G \rightarrow N^{p^{i-1}} \wedge G \rightarrow N^{p^{i-1}}/N^{p^i} \wedge G/N^{p^i} \rightarrow 1$$

for $i \geq 1$. Hence

$$\exp(N \wedge G) \leq \prod_{i=1}^e \exp(N^{p^{i-1}}/N^{p^i} \wedge G/N^{p^i}).$$

Proposition 5 implies $\exp(N^{p^{i-1}}/N^{p^i} \wedge G/N^{p^i}) \leq p^{\lfloor k_i/2 \rfloor}$. □

Tensor products are related to Baer invariants by the following slight generalisation of a result of A.-S.T. Lue [24] (cf. [6]).

Proposition 10 ([24]). *For any group G with normal subgroup $N \trianglelefteq G$, and for $c \geq 1$, there is an exact sequence*

$$\bigotimes^{c+1}(N, G) \rightarrow \gamma_{c+1}^*(G) \rightarrow \gamma_{c+1}^*(G/N) \rightarrow 1.$$

It is convenient to set

$$\overline{\gamma}_{c+1}^*(N, G) = \ker(\gamma_{c+1}^*(G) \rightarrow \gamma_{c+1}^*(G/N)).$$

Note that $\gamma_{c+1}^*(G) = \overline{\gamma}_{c+1}^*(G, G)$. (The bar is intended to suggest that $\overline{\gamma}_{c+1}^*(N, G)$ is a quotient of some functor $\gamma_{c+1}^*(N, G)$. For example, we can take $\gamma_2^*(N, G) = N \wedge G$ [6]. Proposition 9 and Proposition 11 could be subsumed under a single result concerning $\gamma_{c+1}^*(N, G)$.)

Proposition 11. *Suppose that N is a normal subgroup of a group G . If N has exponent p^e , then*

$$\exp(\overline{\gamma}_{c+1}^*(N, G)) \text{ divides } p^{\lfloor k_1/2 \rfloor + \lfloor k_2/2 \rfloor + \dots + \lfloor k_e/2 \rfloor}$$

where k_i denotes the nilpotency class of the pair $(N^{p^{i-1}}/N^{p^i}, G/N^{p^i})$ for $1 \leq i \leq e$.

Proof. The proof is analogous to that of Proposition 9, but with (6) replaced by the exact sequence

$$\overline{\gamma}_{c+1}^*(K, G) \rightarrow \overline{\gamma}_{c+1}^*(M, G) \rightarrow \overline{\gamma}_{c+1}^*(M/K, G/K) \rightarrow 1.$$

We leave the verification of the exactness of this (canonical) sequence to the reader. □

The *upper epicentral series* of an arbitrary group G was introduced in [6]. This is a family of characteristic subgroups $1 = Z_0^*(G) \leq Z_1^*(G) \leq Z_2^*(G) \leq \dots$ with various useful properties such as those listed in the next proposition. Part (i) of the following proposition can be taken as the definition of $Z_c^*(G)$.

Proposition 12 ([6]). *Let $c \geq 1$.*

(i) $Z_c^*(G)$ is the smallest normal subgroup of G , contained in $Z_c(G)$, such that the quotient $G/Z_c^*(G)$ is isomorphic to $H/Z_c H$ for some group H .

(ii) $Z_{c+1}^*(G)$ contains $Z_c^*(G)$.

(iii) $Z_c^*(G) = 1$ if and only if there exists an isomorphism $G \cong H/Z_c H$ for some group H .

(iv) Let N be a normal subgroup of G . Then $N \leq Z_c^*(G)$ if and only if the quotient homomorphism $G \twoheadrightarrow G/N$ induces an isomorphism $\gamma_{c+1}^*(G) \xrightarrow{\cong} \gamma_{c+1}^*(G/N)$.

Let A be a d -generator abelian group with generators a_1, \dots, a_d . Let A_i denote the cyclic subgroup of A generated by a_i . Let $\mathcal{L}(d)$ denote the set of basic commutators on the d symbols a_i . To each basic commutator $\lambda = [a_{i_1}, \dots, a_{i_k}]$ of weight k we associate the k -fold tensor product of abelian groups $T(\lambda) = A_{i_1} \otimes \dots \otimes A_{i_k}$.

Thus T is a cyclic group of order equal to the highest common factor of the orders of the A_{i_j} . It is explained in [9] that the invariant $\gamma_{c+1}^*(A)$ is isomorphic to a direct sum of cyclic groups

$$\gamma_{c+1}^*(A) \cong \bigoplus_{\lambda \in \mathcal{L}(d)} T(\lambda).$$

The following proposition is an immediate corollary to this isomorphism. An alternative derivation of the proposition can be found in [28].

Proposition 13. *Let A be a direct product of cyclic groups*

$$A = (C_{n_1})^{d_1} \times (C_{n_2})^{d_2} \times \dots \times (C_{n_k})^{d_k}$$

with each n_i divisible by n_{i+1} . Then

$$\gamma_{c+1}^*(A) \cong (C_{n_1})^{\chi_{c+1}(d_1)} \times \prod_{j=2}^k (C_{n_j})^{\{\chi_{c+1}(d_1+\dots+d_j) - \chi_{c+1}(d_1+\dots+d_{j-1})\}}.$$

Proposition 14 ([9]). *Let $G = S_1 \times S_2 \times \dots \times S_k$ be a direct product of groups whose abelianisations S_i^{ab} have finite, and mutually coprime, orders. Then, for each $c \geq 1$, there is an isomorphism*

$$\gamma_{c+1}^*(G) \cong \gamma_{c+1}^*(S_1) \times \dots \times \gamma_{c+1}^*(S_k).$$

Proposition 15. *Let N be a nontrivial normal subgroup of a p -group G . Let K denote the kernel of the canonical surjection $\gamma_{c+1}^*(G) \twoheadrightarrow \gamma_{c+1}^*(G/N)$.*

(i) K is nontrivial if and only if there exists some group H for which $H/Z_c H \cong G$.

(ii) If $p \geq 3$ and $N \subset G^p \cap Z_2 G$, or if $p = 2$ and $N \subset G^{p^2} \cap Z_2 G$, then K is contained in the Frattini subgroup of $\gamma_{c+1}^(G)$.*

(iii) If N is a proper subgroup of a cyclic normal subgroup in G , and if $N \subset Z_2 G$, then K is contained in the Frattini subgroup of $\gamma_{c+1}^(G)$.*

Proof. Proposition 12 implies (i).

Proposition 10 implies that K is generated by the image of tensors of the form $(\dots((n \otimes g_1) \otimes g_2) \otimes \dots \otimes g_c)$. The hypothesis of (ii) with Lemma 4 implies that the canonical image in $\bigotimes^{c+1}(G, G)$ of each such tensor lies in the subgroup $\bigotimes^{c+1}(G, G)^p$ generated by p th powers of tensors. The hypothesis of (iii) implies that the image lies in the subgroup generated by p th powers of tensors together with tensors of the form $(\dots((g \otimes g^t) \otimes g_2) \otimes \dots \otimes g_c)$. In both cases K lies in the Frattini subgroup of the p -group $\gamma_{c+1}^*(G)$. \square

4. PROOF OF THEOREM A

Let G, S_i, P_i, L be as in the statement of Theorem A. For each prime p_i let $\bigotimes^{c+1}(L, G)_{p_i}$ denote some p_i -Sylow subgroup of $\bigotimes^{c+1}(L, G)$, and set

$$\Lambda_c^i = \log_{p_i} |\gamma_{c+1}^*(S_i^{ab})|,$$

$$\Psi_c^i = \log_{p_i} \left| \bigotimes^{c+1}([S_i, S_i], S_i) \right|,$$

$$\Theta_c^i = \log_{p_i} \left| \bigotimes^{c+1}(L, G)_{p_i} \right|.$$

Propositions 10 and 14 imply exact sequences

$$\bigotimes^{c+1}(L, G) \rightarrow \gamma_{c+1}^*(G) \rightarrow \prod_{i=1}^n \gamma_{c+1}^*(S_i) \rightarrow 1,$$

$$\bigotimes^{c+1}([S_i, S_i], S_i) \rightarrow \gamma_{c+1}^*(S_i) \rightarrow \gamma_{c+1}^*(S_i^{ab}) \rightarrow 1.$$

Hence

$$|\gamma_{c+1}^*(G)| \leq \prod_{i=1}^n p_i^{\Lambda_c^i + \Psi_c^i + \Theta_c^i}.$$

To complete the proof we must find appropriate upper bounds $\Lambda_c(S_i^{ab})$, $\Psi_c(S_i)$, $\Theta_c(G, L, P_i)$ for Λ_c^i , Ψ_c^i , Θ_c^i .

Proposition 13 furnishes the appropriate formula for $\Lambda_c(S_i^{ab})$. Proposition 2 provides the appropriate formula for $\Psi_c(S_i)$. Suppose that P_i is a d_i -generator group, that $P_i/(P_i \cap [G, G])$ is a δ_i -generator group, that $(L \cap P_i)/(L \cap \Phi(P_i))$ has order p^{t_i} , that $L \cap P_i$ has order $p_i^{\beta_i}$, and that $[L \cap P_i, P_i]$ has order $p_i^{\beta'_i}$. Set $M = \mathcal{M}(L, G) = \ker(L \otimes G \twoheadrightarrow L)$ and let M_{p_i} denote the p_i -primary component of M . Since the pair (L, G) is perfect, Proposition 6 implies

$$\begin{aligned} |\bigotimes^{c+1}(L, G)| &\leq |\bigotimes^c(M, G^{ab})| \times |\bigotimes^{c-1}(M, G^{ab})| \times \dots \\ &\quad \times |\bigotimes^2(M, G^{ab})| \times |M| \times |L|, \end{aligned}$$

and thus

$$\begin{aligned} |\bigotimes^{c+1}(L, G)_{p_i}| &\leq |\bigotimes^c(M_{p_i}, G^{ab})| \times |\bigotimes^{c-1}(M_{p_i}, G^{ab})| \times \dots \\ &\quad \times |\bigotimes^2(M_{p_i}, G^{ab})| \times |M_{p_i}| \times |L \cap P_i|. \end{aligned}$$

Proposition 7(iv) implies that $M_{p_i} \subset \mathcal{M}(L \cap P_i, P_i)$. Since $|L \cap P_i| = p_i^{\beta_i}$, Proposition 8 implies that

$$|\mathcal{M}(L \cap P_i, P_i)| \times |[L \cap P_i, P_i]| \leq p_i^{d_i \beta_i - \frac{t_i(t_i+1)}{2}}.$$

Setting $\omega_i = d_i \beta_i - (1/2)t_i(t_i + 1)$, we have

$$|M_{p_i}| \leq |\mathcal{M}(L \cap P_i, P_i)| \leq p_i^{\omega_i - \beta'_i}$$

and hence

$$|\bigotimes^{c+1}(L, G)_{p_i}| \leq p_i^{(\omega_i - \beta'_i)\delta_i^c} p_i^{(\omega_i - \beta'_i)\delta_i^{(c-1)}} \dots p_i^{(\omega_i - \beta'_i)\delta_i} p_i^{(\omega_i - \beta'_i)} p_i^{\beta_i}.$$

This yields the appropriate formula for $\Theta_c(G, L, P_i)$.

5. PROOF OF THEOREM B

Let G, S_i, P_i, L be as in the statement of Theorem B. Let $\overline{\gamma}_{c+1}^*(L, G)_{P_i}$ denote a P_i -Sylow subgroup of the group $\overline{\gamma}_{c+1}^*(L, G)$. The short exact sequence $\overline{\gamma}_{c+1}^*(L, G) \hookrightarrow \gamma_{c+1}^*(G) \twoheadrightarrow \gamma_{c+1}^*(G/L)$ with Proposition 14 implies

$$\exp(\gamma_{c+1}^*(G)) \leq \prod_{i=1}^n \exp(\overline{\gamma}_{c+1}^*(L, G)_{P_i}) \exp(\gamma_{c+1}^*(S_i)).$$

We apply Propositions 6 and 10 and the surjection $\otimes^{c+1}(L, G) \twoheadrightarrow \overline{\gamma}_{c+1}^*(L, G)$ to obtain

$$\exp(\gamma_{c+1}^*(G)) \leq \prod_{i=1}^n \exp((\mathcal{M}(L, G) \otimes G^{ab})_{P_i})^{c-1} \exp((L \otimes G)_{P_i}) \exp(\gamma_{c+1}^*(S_i)).$$

Proposition 7(iv) yields

$$\exp(\gamma_{c+1}^*(G)) \leq \prod_{i=1}^n \exp(\mathcal{M}(L \cap P_i, P_i) \otimes G^{ab})^{c-1} \exp((L \otimes G)_{P_i}) \exp(\gamma_{c+1}^*(S_i)).$$

The exact sequence

$$(L \cap P_i) \wedge P_i \rightarrow (L \otimes G)_{P_i} \rightarrow L \cap P_i/[L \cap P_i, P_i] \rightarrow 1$$

is readily derived, and yields

$$\begin{aligned} \exp(\gamma_{c+1}^*(G)) &\leq \prod_{i=1}^n \exp(\mathcal{M}(L \cap P_i, P_i) \otimes G^{ab})^{c-1} \exp((L \cap P_i) \wedge P_i) \\ &\quad \times \exp\left(\frac{L \cap P_i}{[L \cap P_i, P_i]}\right) \exp(\gamma_{c+1}^*(S_i)). \end{aligned}$$

The bound of Theorem B(i) now follows from Propositions 9 and 11. (We take $N = G = S_i$ in Proposition 11.)

To prove Theorem B(ii) it suffices to note that the condition $[[P^{p^{i-1}}, P], P] \subset P^{p^i}$ is equivalent to saying that the pair $(P^{p^{i-1}}/P^{p^i}, P/P^{p^i})$ has nilpotency class at most 2.

6. PROOF OF THEOREM C

Consider the dihedral group $D_n = \langle a, b \mid a^2 = b^n = (ab) \rangle$ with $n = 2^r m$ where $m \geq 1$ is odd. The smallest term of the lower central series of D_n is $L = \gamma_r(D_n) \cong C_m$. Proposition 7(ii) implies that the relative multiplier $\mathcal{M}(L, G)$ is trivial. Proposition 6 therefore implies an isomorphism $\otimes^{c+1}(L, G) \cong L \otimes G$ for all $c \geq 1$. So Proposition 10 yields a commutative diagram

$$\begin{array}{ccccccc} \mathcal{M}(L, D_n) = 1 & \longrightarrow & M^{(c)}(D_n) & \xrightarrow{\cong} & M^{(c)}(D_{2^r}) & \longrightarrow & L/\gamma_{c+1}(L, D_n) = 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ L \otimes D_n & \longrightarrow & \gamma_{c+1}^*(D_n) & \longrightarrow & \gamma_{c+1}^*(D_{2^r}) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & C_m & \longrightarrow & \gamma_{c+1} D_n & \longrightarrow & \gamma_{c+1} D_{2^r} \longrightarrow 1 \end{array}$$

in which the rows are exact and the columns are short exact. From this, and the isomorphism $\gamma_{c+1}D_n \cong C_m \times \gamma_{c+1}D_{2^r}$, we derive the isomorphism

$$\gamma_{c+1}^*(D_n) \cong C_m \times \gamma_{c+1}^*(D_{2^r}).$$

A description of $\gamma_{c+1}^*(D_{2^r})$ is given in [16]. Alternatively, the description can be re-obtained as follows. The central extensions $C_2 \hookrightarrow D_{2^r} \twoheadrightarrow D_{2^{r-1}}$ and repeated applications of Proposition 15(i) imply that $|\gamma_{c+1}^*(D_{2^r})| \geq |\gamma_{c+1}^*(C_2 \times C_2)| \times 2^{r-1}$. These central extensions together with repeated applications of Proposition 15(iii) imply that $\gamma_{c+1}^*(D_{2^r})$ has the same number of generators as $\gamma_{c+1}^*(C_2 \times C_2)$, namely one generator for each basic commutator on two generators a, b . Theorem B implies that $\exp(\gamma_{c+1}^*(D_{2^r})) \leq 2^r$. So, to obtain the isomorphism $\gamma_{c+1}^*(D_{2^r}) \cong C_{2^r} \oplus (C_2)^{\chi_{c+1}(2)-1}$ it suffices to verify that at least all but one of the generators are of order 2, and that the generators a, b act as stated in the theorem. (Note that the invariant $\gamma_{c+1}^*(G)$ is abelian precisely when $\gamma_{c+1}(G)$ acts trivially on it.) This verification, which we leave to the reader, yields the desired description of $\gamma_{c+1}^*(D_{2^r})$ and completes the proof of part (i) of the theorem.

To prove part (ii) we note that the quaternion group Q_n is not of the form $H/Z(H)$ for any group H with centre $Z(H)$ [3]. Proposition 12(ii)(iii) thus implies that the c th term $Z_c^*(Q_n)$ of the upper epicentral series of Q_n contains the centre $Z(Q_n) \cong C_2$. Since $Q_n/Z(Q_n) \cong D_n$, the isomorphism $\gamma_{c+1}^*(Q_n) \cong \gamma_{c+1}^*(D_n)$ follows from Proposition 12(iv).

To prove part (iii) we note that an extraspecial group $E(p, k)$, $k \geq 2$, is not of the form $H/Z(H)$ for any group H with centre $Z(H)$ [3]. Arguing as in the previous paragraph, we see that $\gamma_{c+1}^*(E(p, k)) \cong \gamma_{c+1}^*(E(p, k)/Z(E(p, k))) \cong \gamma_{c+1}^*(C_p \times C_p)$. Proposition 13 completes the proof of part (iii). This argument also holds for $E(p, 1)^-$, $p \geq 3$.

To obtain our partial description of $\gamma_{c+1}^*(E(p, 1)^+)$, $p \geq 3$, we first remark that $Z_1^*(G)$ is trivial for the group $G = E(p, 1)^+$ [3]. Letting $Z = Z(G)$ denote the centre of this group, Proposition 10 and Proposition 1(v) yield an exact sequence $\bigotimes^{c+1}(Z, G) = Z \otimes G^{ab} \otimes \dots \otimes G^{ab} \rightarrow \gamma_{c+1}^*(G) \xrightarrow{\phi} \gamma_{c+1}^*(C_p \times C_p) \rightarrow 1$. The group $\bigotimes^{c+1}(Z, G)$ is elementary abelian of rank 2^c , and Proposition 12(iv) implies that ϕ has non-trivial kernel. Theorem B implies that $\exp(\gamma_{c+1}^*(G)) = p$. The commutator subgroup $[G, G] = Z$ acts trivially on $\gamma_{c+1}^*(G)$, and so $\gamma_{c+1}^*(G)$ is abelian. Hence $\gamma_{c+1}^*(G)$ is elementary abelian of rank at most $\chi_{c+1}(2) + 2^c$, and at least $\chi_{c+1}(2) + 1$.

ACKNOWLEDGEMENTS

This research was carried out at the Max Planck Institut für Mathematik, Bonn. I would like to thank the Institute for its very generous hospitality.

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