ON THE COMMUTATIVITY OF THE ALGEBRA OF INVARIANT DIFFERENTIAL OPERATORS ON CERTAIN NILPOTENT HOMOGENEOUS SPACES

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Abstract. Let $G$ be a simply connected connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$, $H$ a connected closed subgroup of $G$ with Lie algebra $\mathfrak{h}$ and $\beta \in \mathfrak{h}^*$ satisfying $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$. Let $\chi_{\beta}$ be the unitary character of $H$ with differential $2\sqrt{-1}\pi\beta$ at the origin. Let $\tau = Ind^G_H\chi_{\beta}$ be the unitary representation of $G$ induced from the character $\chi_{\beta}$ of $H$. We consider the algebra $D(G,H,\beta)$ of differential operators invariant under the action of $G$ on the bundle with basis $H \backslash G$ associated to these data. We consider the question of the equivalence between the commutativity of $D(G,H,\beta)$ and the finite multiplicities of $\tau$. Corwin and Greenleaf proved that if $\tau$ is of finite multiplicities, this algebra is commutative. We show that the converse is true in many cases.

1. Notations and formulation of the question

Let $G$ be a simply connected connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $H$ a connected closed subgroup of $G$ with Lie algebra $\mathfrak{h}$. For $l \in \mathfrak{g}^*$, we denote by $\mathfrak{g}(l)$ the Lie algebra of the stabilizer $G(l)$ of $l$ under the co-adjoint action $Ad^*$ of $G$ on $\mathfrak{g}^*$. For $\beta \in \mathfrak{h}^*$ satisfying $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$, the homomorphism $\beta$ induces a character $\chi_{\beta}$ of $H$ with differential $2\sqrt{-1}\pi\beta$ as differential at the origin. We then form the unitary induced representation $\tau = Ind^G_H\chi_{\beta}$ of $G$ in $\mathcal{H}_\tau$ realized, in the usual way, as the completion of a vector subspace of $C^\infty(G,H,\beta)$, namely the vector space of the $C^\infty$ complex functions $f$ on $G$ satisfying the following covariance relation:

$$f(hg) = \chi_{\beta}(h)f(g) \forall h \in H \forall g \in G. \quad (1.1)$$

The action of $G$ is given by right translations:

$$\tau(g')(f)(g) = f(gg') \forall (g,g') \in G \times G \forall f \in C^\infty(G,H,\beta). \quad (1.2)$$

We denote by $\mathcal{K}(G,H,\tau)$ the subspace of $C^\infty(G,H,\beta)$ of elements with compact support modulo $H$. Then the norm $|| \cdot ||_\tau$ on $\mathcal{K}(G,H,\tau)$ is given by

$$|| f ||_\tau^2 = \int_{H \backslash G} |f(g)|^2 \, dg \quad (1.3)$$

where $dg$ denotes a right $G$-invariant measure on $H \backslash G$. The Hilbert space $\mathcal{H}_\tau$ is just the completion of $\mathcal{K}(G,H,\tau)$ relative to this norm. Moreover, the unitary

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representation of $G$ in $\mathcal{H}_r$ decomposes in a continuous sum of unitary irreducible representations of $G$,

\begin{equation}
\tau \simeq \int_G m(\pi)\pi d\mu(\pi)
\end{equation}

where $m(\pi)$ denotes the multiplicity of $\pi$ and $d\mu$ a Plancherel measure on the unitary dual $\hat{G}$ of $G$. Then we know \[3\], \[11\] that for almost all $\pi$ in $\hat{G}$, the multiplicities $m(\pi)$ appearing in (1.4) either are finite and admit a uniform bound, or are all infinite. In the first case, we say that $\tau$ is of finite multiplicities.

Finally, let $D(G, H, \beta)$ be the algebra of linear differential operators leaving $C^\infty(G, H, \beta)$ invariant and commuting with $\tau$, that is

\begin{equation}
D \in D(G, H, \beta) \iff D(\tau(g)f) = \tau(g)(Df) \ \forall g \in G \ \forall f \in C^\infty(G, H, \beta).
\end{equation}

Corwin and Greenleaf established in \[3\] the commutativity of $D(G, H, \beta)$ when $\tau$ is of finite multiplicities and they asked the question:

\begin{equation}
(*) \text{ Is $\tau$ of finite multiplicities if $D(G, H, \beta)$ is commutative?}
\end{equation}

Before we turn to the study of this question, we first recall some facts about parametrization of unipotent actions on vector spaces \[12\].

2. Orbits of $H$ in $\mathfrak{g}^*$

Suppose we are given a $m$-dimensional vector space $V$ admitting a unipotent action of $H$. Let $\{Y_1; \ldots; Y_m\}$ be a basis of $V$ such that the subspaces $V_j = \bigoplus_{i=j+1}^m \mathbb{R}Y_i$ of $V$ are $H$-stable for all $0 \leq j \leq m$ with $V_0 = \{0\}$. We consider the multi-index $e(\psi)$ defined by $(e_0(\psi); \ldots; e_m(\psi))$ for all $\psi \in V$, where $e_j(\psi)$ is the dimension of the $H$-orbit of the projection of $\psi$ on $V/V_j$. We then denote by $\Sigma$ the set of all possible multi-indexes, that is

\begin{equation}
\Sigma = \{ e \in \mathbb{N}^{m+1} \mid \exists \psi \in V, \ e = e(\psi) \}.
\end{equation}

This defines a stratification of $V$ in layers $U_e$ of $H$-orbits, more precisely \[4\]:

\begin{equation}
V = \bigcup_{e \in \Sigma} U_e \ \text{where for $e \in \Sigma$ and $U_e = \{ \psi \in V \mid e(\psi) = e \}.}$
\end{equation}

It happens that among these layers $U_e, e \in \Sigma$, there exists one, and only one, which is a non-empty Zariski open subset of $V$. We shall call it the generic layer (associated to the action of $H$ on $V$), and we will denote it by $V^\text{gene}$. Note that $V^\text{gene}$ is just the subset of $V$ of elements for which the dimensions of $H$-orbits in $V/V_j$ are maximal for $0 \leq j \leq m$. We shall say that a $\psi$ in $V^\text{gene}$ is generic in $V$, and the dimension of the orbit $H \cdot \psi$ will be called the generic dimension of $H$-orbits in $V$.

In the sequel, we will consider a particular $V$. More precisely, fix a sequence $\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ of subalgebras of $\mathfrak{g}$ satisfying the following conditions:

- the subalgebras $\mathfrak{g}_i$ are of dimension $i$; they are normalised by the action of $H$.
- for a certain index $p$, the subalgebra $\mathfrak{g}_p$ coincides with $\mathfrak{h}$.

We choose a weak Malcev basis $\{X_i, 1 \leq i \leq n\}$ of $\mathfrak{g}$ through $\mathfrak{h}$ associated to the sequence $\{\mathfrak{g}_i, 0 \leq i \leq n\}$ such that for all $i \in \{1, \ldots, n\}$, the vectors $\{X_j, 1 \leq j \leq i\}$ form a basis of $\mathfrak{g}_i$. We say that $\{X_i, 1 \leq i \leq n\}$ (resp. $\{\mathfrak{g}_i, 0 \leq i \leq n\}$) is a weak Malcev basis (resp. sequence of subalgebras) of $\mathfrak{g}$ through $\mathfrak{h}$. As usual we denote by $\{X_i^*, 1 \leq i \leq n\}$ the dual basis of $\{X_i, 1 \leq i \leq n\}$. Then we put $V = \mathfrak{g}^*$ and
particular, the layer $\mathfrak{g}$
connected real nilpotent Lie group, exactly the same as the above question (Corwin-Greenleaf).

If the second condition is satisfied, we say that $\mathfrak{g}$
induces the algebra isomorphism generated by $X$
and $Y$.

Remark 1. It is important to note that the layer $\mathfrak{g}^{*}$
does not necessarily intersect $\Omega_{G,H,\beta}$.
Indeed, a family of simple examples where the condition $\mathfrak{g}^{*} \cap \Omega_{G,H,\beta} \neq \emptyset$
is not satisfied is the Heisenberg group with $\mathfrak{h}$
containing the center and $\beta$
vanishing on the center. More precisely, let $\mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}Z$
with bracket relation $[X,Y] = Z$. If $\mathfrak{h} = \mathbb{R}X \oplus \mathbb{R}Z$ and $\beta = X^{*}$,
then $\Omega_{G,H,\beta} = X^{*} + \mathbb{R}Y^{*}$ and $\mathfrak{g}^{*} \cap \Omega_{G,H,\beta} = \{ zX^{*} + yY^{*} + zZ^{*} \in \mathfrak{g}^{*} | z \neq 0 \}$,
which shows that $\mathfrak{g}^{*} \cap \Omega_{G,H,\beta} = \emptyset$.

Finally, for $l \in \mathfrak{g}^{*}$, we denote by $B_{l}$
the antisymmetric bilinear form on $\mathfrak{g}$
given by $B_{l}(X,Y) = l([X,Y])$. It is well known [3, 11]
that the two following conditions are equivalent:

\begin{enumerate}[(C1)]
\item $\tau$ is of finite multiplicities.
\item $\mathfrak{h} + \mathfrak{g}(l)$ is lagrangian in $\mathfrak{g}$
relative to $B_{l}$ for all generic $l$ in $\Omega_{G,H,\beta}$.
\end{enumerate}

If the second condition is satisfied, we say that $\mathfrak{h} + \mathfrak{g}(l)$ is generically lagrangian in $\mathfrak{g}$.

It turns out that question 6 asked by Duflou in [7]
is, in the case of a simply connected
real nilpotent Lie group, exactly the same as the above question $(\ast)$
of Corwin-Greenleaf.

More precisely, consider the following assertions:

(i) $D(G,H,\beta)$ is a commutative algebra.

(ii) $\mathfrak{h} + \mathfrak{g}(l)$ is generically lagrangian in $\mathfrak{g}$.

(iii) $H \cdot l$ is generically a lagrangian submanifold of $G \cdot l$ relative to $B_{l} : (X,Y) \mapsto l([X,Y])$.

Note that (ii) \iff (iii) is obvious, since if $\mathfrak{h} + \mathfrak{g}(l)$ is lagrangian in $\mathfrak{g}$, then $\dim H \cdot l = \frac{1}{2} \dim G \cdot l$.
But (ii) \implies (i) is a fundamental result proved by Corwin and Greenleaf in [6].
Baklouti and Ludwig have studied in [11] the implication $(ii) \implies (i)$
when $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.
In the sequel, we shall study the implication $(i) \implies (ii)$
in more general cases.

3. A First Result on the Commutativity of $D(G,H,\beta)$

The description of $D(G,H,\beta)$
given in [6] in terms of the enveloping algebra $U(\mathfrak{g})$
of the complexification of $\mathfrak{g}$ will be useful.
Let $a_{\beta}$ be the vector subspace of $U(\mathfrak{g})$
generated by the $X + 2\sqrt{-1}\pi \beta(X)$, $X \in \mathfrak{h}$, and let $U(\mathfrak{g})a_{\beta}$
be the left sided ideal of $U(\mathfrak{g})$ generated by $a_{\beta}$.
If $U(\mathfrak{g},\mathfrak{h},\beta)$
denotes the subalgebra of $U(\mathfrak{g})$
defined by

\begin{equation}
U(\mathfrak{g},\mathfrak{h},\beta) = \{ A \in U(\mathfrak{g}) \mid \forall W \in \mathfrak{h}, \ [A,W] \in U(\mathfrak{g})a_{\beta} \},
\end{equation}

then the left action $L$ of $U(\mathfrak{g})$,
defined for $Y$ in $\mathfrak{g}$ and $f$ in $C^{\infty}(G)$ by

\begin{equation}
L(Y)(f)(g) = \frac{d}{dt} f(e^{-tY}g) \bigg|_{t=0},
\end{equation}
induces the algebra isomorphism $L_{\beta} : U(\mathfrak{g},\mathfrak{h},\beta)/U(\mathfrak{g})a_{\beta} \simeq D(G,H,\beta)$. 

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Next, in the usual way, we shall denote by $S(\mathfrak{g})$ the symmetric algebra of $\mathfrak{g}$ and by $\sigma : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ the symmetrization map. We still denote by $Ad$ (resp. $ad$) the natural continuation of the adjoint action of $G$ (resp. $\mathfrak{g}$) in a $G$-action (resp. $\mathfrak{g}$-action) on $S(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$. Moreover, we shall identify an element of $S(\mathfrak{g})$ with a polynomial function on $\mathfrak{g}^*$. 

**Remark 2.** F. P. Greenleaf has also obtained the following theorem with a different proof [10].

**Theorem 1.** Let $G$ be a simply connected connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$, $H$ a connected closed subgroup of $G$ with Lie algebra $\mathfrak{h}$ and $\beta \in \mathfrak{h}^*$ such that $\beta([\mathfrak{h}, \mathfrak{h}]) = \{0\}$. We assume that the unitary representation $\tau = Ind_H^G \chi_\beta$ of $G$ is of infinite multiplicities. Let $G_0$ be a connected subgroup of codimension one of $G$ with Lie algebra $\mathfrak{g}_0$ containing $\mathfrak{h}$ and such that the unitary representation $\tau_0 = Ind_H^{G_0} \chi_\beta$ of $G_0$ is of finite multiplicities. If we suppose that there is an element $W$ of $\mathcal{U}(\mathfrak{g}, \mathfrak{h}, \beta)$ such that $W \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})a_\beta$, then there exists an element $T$ of $\mathcal{U}(\mathfrak{g}_0, \mathfrak{h}, \beta)$ satisfying $[W, T] \notin \mathcal{U}(\mathfrak{g})a_\beta$.

In other words, if $D(G_0, H, \beta)$ is properly imbedded in $D(G, H, \beta)$, then $D(G, H, \beta)$ is not commutative.

**Proof.** We shall frequently use in the sequel the following property: let $\mathfrak{t}$ be a subalgebra of codimension one of $\mathfrak{g}$, and $l$ be in $\mathfrak{g}^*$, then the dimensions of $\mathfrak{g}(l)$ and of $\mathfrak{t}(l_{\mathfrak{t}})$ differ by 1, and one of those subspaces is imbedded in the other, (see [2], Lemme 1.1.1, p. 49). In the case where $\mathfrak{g}(l) \subset \mathfrak{t}(l_{\mathfrak{t}})$, one has that the dimension of $G.l$ is bigger by 2 than the dimension of $K.(l_{\mathfrak{t}})$ where $K = \exp(\mathfrak{t})$, and the dimensions of polarizations in $\mathfrak{g}$ and $\mathfrak{t}$ are the same. In the other case, the orbits $G.l$ and $K.(l_{\mathfrak{t}})$ have the same dimension, whereas the dimension of polarizations in $\mathfrak{g}$ is bigger by 1 than the dimension of polarizations in $\mathfrak{t}$.

Let us recall that the finite multiplicities situation is characterised by the fact that generically on $\Omega_{G, H, \beta}$, the dimension of an $H$-orbit is half the dimension of a $G$-orbit [5]. Thus, under the assumptions of the theorem, we have $\mathfrak{g}(l) \subset \mathfrak{g}_0$.

Next, we proceed by induction on the dimension of $G$ and the theorem is supposed to be true for all groups of dimension at most $n - 1$, where $n$ is the dimension of $G$. Let $\mathcal{Z}$ be the center of $\mathfrak{g}$. We shall consider two main cases depending on whether $\mathcal{Z}$ is included in $\mathfrak{h}$.

1) **Case:** $\mathcal{Z} \subset \mathfrak{h}$ and $\mathcal{Z} \cap \text{Ker}(\beta) \neq \{0\}$. In this case, we apply the induction hypothesis to the quotient group with Lie algebra $\mathfrak{g}/(\mathcal{Z} \cap \text{Ker}(\beta))$.

In the following cases 2), 3) and 4), we have $\mathcal{Z} \subset \mathfrak{h}$ and $\mathcal{Z} \cap \text{Ker}(\beta) = \{0\}$, so that the center $\mathcal{Z}$ of $\mathfrak{g}$ is necessarily one-dimensional. We put $\mathcal{Z} = \mathbb{R}Z$ with $\beta(Z) = 1$. Moreover, it is easy to check the existence of elements $X$ of $\mathfrak{g}$ and $Y$ of $\mathfrak{g}_0$ such that $[X, Y] = Z$. In the sequel, we shall denote by $\mathfrak{t}$ the centralizer of $Y$ in $\mathfrak{g}$ and by $K$ the connected subgroup of $G$ with Lie algebra $\mathfrak{t}$.

2) **Case:** $\mathcal{Z} \subset \mathfrak{h}$, $\mathcal{Z} \cap \text{Ker}(\beta) = \{0\}$, $\mathfrak{h} \subset \mathfrak{t}$ and $Y \subset \mathfrak{h}$. Let $l$ be an element of $\mathfrak{g}^*$ satisfying $l(Z) \neq 0$, (since $\beta(Z) = 1$, this condition is satisfied by any element of $\Omega_{G, H, \beta}$). One has $\mathfrak{t}(l_{\mathfrak{t}}) = \mathfrak{g}(l) \oplus \mathbb{R}Y$. And, as we noticed before, the dimension $[\dim(\mathfrak{g}) + \dim(\mathfrak{g}(l))]/2$ of a polarization of $\mathfrak{g}$ at a point $l \in \mathfrak{g}^*$ is the same as the dimension $[\dim(\mathfrak{t}) + \dim(\mathfrak{t}(l_{\mathfrak{t}}))]/2$ of a polarization of $\mathfrak{t}$ at the point $l_{\mathfrak{t}} \in \mathfrak{t}^*$. Moreover, we also have $\mathfrak{h} + \mathfrak{t}(l_{\mathfrak{t}}) = \mathfrak{h} + \mathfrak{g}(l)$. Next, we choose a weak Malcev basis of
The representation $\tau = \text{Ind}_{H}^{G} \chi_{\beta}$ of $G$ and $\tau' = \text{Ind}_{H}^{K} \chi_{\beta}$ of $K$ are of the same type, that is, both infinite. But $\tau_{0} = \text{Ind}_{H}^{G_{0}} \chi_{\beta}$ is supposed to be of finite multiplicities, so one has $g_{0} \neq \mathfrak{t}$.

On the other hand, for $l \in \Omega_{G,H,\beta}$, we have $(g_{0} \cap \mathfrak{t})(l_{|g_{0} \cap \tau}) = g_{0}(l_{|g_{0}}) \oplus \mathbb{R}Y$. Choosing a weak Malcev basis of $\mathfrak{g}$ passing through $\mathfrak{h}$, $g_{0} \cap \mathfrak{t}$ and $g_{0}$, we see that the unitary representation $\tau_{0}' = \text{Ind}_{H}^{G_{0} \cap K} \chi_{\beta}$ of $G_{0} \cap K$ is, as the representation $\tau_{0} = \text{Ind}_{H}^{G_{0}} \chi_{\beta}$ of $G_{0}$, of finite multiplicities. Let us write the element $W$ of the theorem in the form $W = \sum_{j=0}^{k} X^{j} U_{j}$ with $U_{j}$ in $\mathcal{U}(\mathfrak{t})$. Since the element $Y$ is in $\mathfrak{h}$, the elements $\text{ad}(\mathfrak{Y})'(W) = r! Z' U_{r}$ and $U_{r}$ are in $\mathcal{U}(\mathfrak{g})a_{\beta}$ for all $r \neq 0$. Thus we can suppose $W = U_{0}$ is in $\mathcal{U}(\mathfrak{t}, \mathfrak{h}, \beta)$. Finally, for $W \in \mathcal{U}(\mathfrak{t}, \mathfrak{h}, \beta)$, we apply the induction hypothesis to $K$ and $G_{0} \cap K$ with the representation $\tau' = \text{Ind}_{H}^{K} \chi_{\beta}$ and $\tau_{0}' = \text{Ind}_{H}^{G_{0} \cap K} \chi_{\beta}$ respectively.

3) Case: $\mathcal{Z} \subset \mathfrak{h}$, $\mathcal{Z} \cap \text{Ker}(\beta) = \{0\}$, $\mathfrak{h} \subset \mathfrak{t}$ and $Y \not\subset \mathfrak{h}$. i) $g_{0} = \mathfrak{t}$.

The element $Y$ is in $\mathcal{U}(g_{0}, \mathfrak{h}, \beta)$ and satisfy $[W, Y] \not\in \mathcal{U}(\mathfrak{g})a_{\beta}$. We take $T = Y$.

ii) $g_{0} \neq \mathfrak{t}$ and $W \not\in \mathcal{U}(\mathfrak{t}) + \mathcal{U}(\mathfrak{g})a_{\beta}$.

We still can choose $T$ to be the element $Y$.

iii) $g_{0} \neq \mathfrak{t}$ and $W \in \mathcal{U}(\mathfrak{t}) + \mathcal{U}(\mathfrak{g})a_{\beta}$.

We consider the inclusions: $(g_{0} \cap \mathfrak{t}) \subset g_{0} \subset \mathfrak{g}$ and $(g_{0} \cap \mathfrak{t}) \subset \mathfrak{t} \subset \mathfrak{g}$. Let $l$ be in $\mathfrak{g}^* \cap \mathcal{Z}$ and $l(Z) \neq 0$. We denote by $d$ the dimension of $g(l)$. Under the assumption of the theorem, as we have already remarked at the beginning, we have $g(l) \subset g_{0}$, so that the dimension of $g_{0}(l_{|g_{0}})$ is $d + 1$. The dimension of $\mathfrak{t}(l_{|\mathfrak{t}})$ is also $d + 1$ because the element $Y$ of $[g_{0}(l_{|g_{0}})]$ does not belong to $\mathfrak{g}(l)$. Moreover, $Y$ is in $(g_{0} \cap \mathfrak{t})(l_{|g_{0} \cap \mathfrak{t}})$ but is not in $g_{0}(l_{|g_{0}})$. In other words, the dimension of $(g_{0} \cap \mathfrak{t})(l_{|g_{0} \cap \mathfrak{t}})$ is $d + 2$.

Thus, the representation $\tau' = \text{Ind}_{H}^{K} \chi_{\beta}$ of $K$ is necessarily of infinite multiplicities. Indeed, the representation $\tau_{0}' = \text{Ind}_{H}^{G_{0} \cap K} \chi_{\beta}$ has finite multiplicities and, from the previous calculus of dimensions of stabilizers, we deduce that if the representation $\tau = \text{Ind}_{H}^{K} \chi_{\beta}$ of $K$ has finite multiplicities, the dimensions of $H$–orbits for generic $l \in \Omega_{G,H,\beta}$ would increase when passing from $K \cap G_{0}$ to $K$, which makes impossible the existence of an element $W \in \mathcal{U}(\mathfrak{t}) + \mathcal{U}(\mathfrak{g})a_{\beta}$ that is not in $\mathcal{U}(\mathfrak{t} \cap g_{0}) + \mathcal{U}(\mathfrak{g})a_{\beta}$ as it is shown in [5]. Finally, we apply the induction hypothesis to $K$ and $G_{0} \cap K$ with the representation $\tau'$ and $\tau_{0}'$ respectively.

4) Case: $\mathcal{Z} \subset \mathfrak{h}$, $\mathcal{Z} \cap \text{Ker}(\beta) = \{0\}$ and $\mathfrak{h} \not\subset \mathfrak{t}$. Note that, since $\mathfrak{h} \subset g_{0}$, one has necessarily $g_{0} \neq \mathfrak{t}$. Let us write $g_{0} = (g_{0} \cap \mathfrak{t}) \oplus \mathbb{R}X$, with $X \in \mathfrak{h}$, so that $\mathfrak{g} = \mathfrak{t} \oplus \mathbb{R}X$, and denote by $\beta$ the restriction of $\beta$ to $\mathfrak{h} \cap \mathfrak{t}$. Let $l$ be an element of $\Omega_{G,H,\beta}$. We have $l(Z) \neq 0$. Then $l_{|\mathfrak{t}} = g(l) \oplus \mathbb{R}Y$, so that $(\mathfrak{h} \cap \mathfrak{t}) \cap l_{|\mathfrak{t}} = \mathfrak{h} \cap g(l)$, since $\mathfrak{h} \not\subset \mathfrak{t}$. This forces the vector spaces $\mathfrak{h} + g(l)$ and $(\mathfrak{h} \cap \mathfrak{t}) + t(l_{|\mathfrak{t}})$ to have the same dimension. It follows that if the first subspace is not lagrangian in $\mathfrak{g}$, the second is not lagrangian in $\mathfrak{t}$. Hence, choosing sequences of subalgebras and using equivalence of conditions (C1) and (C2), we obtain that the representation $\tau_{1} = \text{Ind}_{H}^{K \cap G} \chi_{\beta}$ of $K$ is of infinite multiplicities. On the other hand, we have $(g_{0} \cap \mathfrak{t})(l_{|g_{0} \cap \mathfrak{t}}) = g_{0}(l_{|g_{0}}) \oplus \mathbb{R}Y$ and $\text{dim}(\mathfrak{h} + g_{0}(l_{|g_{0}})) = \text{dim}(l \cap \mathfrak{t}) + (g_{0} \cap \mathfrak{t})(l_{|g_{0} \cap \mathfrak{t}})$, which imply that the representation $\tau_{2} = \text{Ind}_{H}^{G_{0} \cap K} \chi_{\beta}$ of $G_{0} \cap K$ is of finite multiplicities.

Moreover, $W$ can be supposed to belong to $\mathcal{U}(\mathfrak{t}, \mathfrak{h} \cap \mathfrak{t}, \beta)$ since $\mathfrak{g} = \mathfrak{t} \oplus \mathbb{R}X$ with $X \in \mathfrak{h}$, and $X = -2\sqrt{-1} \pi \beta(X)$ modulo $a_{\beta}$. We apply the induction hypothesis to
$K$ and $G_0 \cap K$ with the representation $\tau_1$ and $\tau_2$ respectively to obtain an element $T$ in $\mathcal{U}(g_0 \cap t, h \cap t, \beta)$ such that $[W, T] \not\in \mathcal{U}(t)\alpha_\beta$.

Next, let $X_i = Y$ and $X_i$, $i \in \{2, \ldots, q\}$, be in $g_0 \cap t$, such that if we put $\hat{\gamma}_i = h \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2 \oplus \cdots \oplus \mathbb{R}X_1$ with $g_\alpha = g_0$, then the sequence of subalgebras $(\hat{\gamma}_i, i = 1, \ldots, q)$ of $g_0$ is Jordan-Hölder for the action of $H$ on $g_0$. It is interesting to notice that the sequence $(\hat{\gamma}_i, i = 1, \ldots, q)$ of $g_0$ is Jordan-Hölder for the action of $H \cap K$ on $g_0 \cap t$. We put $\hat{G}_i = \exp(\hat{\gamma}_i)$ and $\hat{G}_i = \exp(\hat{\gamma}_i)$ for $i = 1, \ldots, q$. Then the dimension of generic $H$-orbits in $\Omega_{\hat{G}_i, H, \beta}$ is bigger by one than that of generic $H \cap K$-orbits in $\Omega_{\hat{G}_i, H \cap K, \beta}$.

On the other hand, since the representations $\tau_0 = \text{Ind}_H^K \chi_\beta$ of $G_0$ and $\tau_2 = \text{Ind}_H^K \chi_\beta$ of $G_0 \cap K$ are of finite multiplicities, there are elements $\{\gamma_1, \ldots, \gamma_r\}$ of $\mathcal{U}(g_0, h, \beta)$ and $\{\delta_1 = Y, \delta_2, \ldots, \delta_t\}$ of $\mathcal{U}(g_0 \cap t, h \cap t, \beta)$ given in [5], such that the families $\{\gamma_i = L(\delta_i) | i = 1, \ldots, r\}$ and $\{\delta_i = L(\delta_i) | i = 0, \ldots, r\}$ are rational generators of $D(G_0, H, \beta)$ and $D(G_0 \cap K, H \cap K, \beta)$ respectively. To simplify, we shall call the $\gamma_i$’s and $\delta_i$’s, the Corwin-Greenleaf generators of $D(G_0, H, \beta)$ and $D(G_0 \cap K, H \cap K, \beta)$ respectively. Note that $D(G_0, H, \beta)$ is contained in $D(G_0 \cap K, H \cap K, \beta)$. Moreover, we may suppose that for the element $T$ above, $L(T)$ is one of the Corwin-Greenleaf generators of $D(G_0 \cap K, H \cap K, \beta)$.

Thus, since $[W, Y] = 0$, we denote by $\delta_{j_0}$ the first element of $\{\delta_1, \ldots, \delta_t\}$ satisfying $[\delta_{j_0}, W] \not\in \mathcal{U}(t)\alpha_\beta$. Then, from [5], one can find polynomials $A, B$ and $C$ of $j_0$ variables such that $A(\delta_0, \ldots, \delta_{j_0-1}) = B(\delta_0, \ldots, \delta_{j_0-1})\delta_{j_0} + C(\delta_0, \ldots, \delta_{j_0-1})$, with $A(\delta_0, \ldots, \delta_{j_0-1})$ and $B(\delta_0, \ldots, \delta_{j_0-1})$ non-zero.

It turns out that $[\gamma_{j_0}, W] \not\in \mathcal{U}(g_0)\alpha_\beta$. Indeed, because $D(G, H, \beta)$ has no non-zero divisor of zero, we have $A(\delta_1, \ldots, \delta_{j_0-1})[\gamma_{j_0}, L(W)] = B(\delta_1, \ldots, \delta_{j_0-1})[\delta_{j_0}, L(W)] \not\in 0$, so that $[\gamma_{j_0}, L(W)] \not\in 0$. Hence, we can choose $T = \gamma_{j_0}$, that is $L(T)$ is the Corwin-Greenleaf generator $\gamma_{j_0}$ of $D(G_0, H, \beta)$.

5) Case: $Z \not\subseteq h$. First, remember that an immediate consequence of the assumptions of the theorem is that $Z$ is embeded in $g_0$. Next, let $Z$ be in $Z$ which does not belong to $h$. Denote by $\mathfrak{h}'$ the subalgebra $h \oplus \mathbb{R}Z$ of $g_0'$ and by $H'$ the connected subgroup of $G$ with Lie algebra $\mathfrak{h}'$. Let $\phi$ be a generic element of $\Omega_{G_0, H, \beta}$ and put $\alpha = \phi(Z)$. Define, as usual, the character $\chi_{\phi}$ of $H'$ by $\chi_{\phi}(u) = e^{z^t e u \phi(u)}$ for all $U \in h'$, so that the unitary representation $\tau_0 = \text{Ind}_{H'}^G \chi_{\phi}$ of $G_0$ is of finite multiplicities. Let $\mathfrak{h} \subset h \oplus \mathbb{R}Z \subset h \oplus \mathbb{R}Z \oplus \mathbb{R}X_1 \subset h \oplus \mathbb{R}Z \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \cdots \subset g_0$ be a Jordan-Hölder sequence for the action of $H$ on $g_0$, and consider the sequence $\mathfrak{h}' \subset \mathfrak{h}' \oplus \mathbb{R}X_1 \subset \mathfrak{h}' \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2 \subset \cdots \subset g_0$ which is also a Jordan-Hölder sequence for the action of $H'$ on $g_0$. Actually, since $H$ and $H'$ have the same orbits in $g_0$, if $\{\gamma_1 = L(Z), \gamma_2, \ldots, \gamma_q\}$ is a set of Corwin-Greenleaf generators of $D(G_0, H, \beta)$, then $\{\gamma_2, \ldots, \gamma_q\}$ is a set of Corwin-Greenleaf generators of $D(G_0, H', \phi)$. Any Corwin-Greenleaf generator of $D(G_0, H, \beta)$ can be represented in $\mathcal{U}(g_0, h, \beta)/\mathcal{U}(g_0, h, \beta)$ by an element $C = \sum_{\nu, \mu} a_{\nu, \mu} Z^\nu X_1^\mu \cdots X_{2p}^\mu$ of $\mathcal{U}(g_0, h, \beta)$. And observe that any element of $\mathcal{U}(g_0, h, \beta)$ belongs to $\mathcal{U}(g_0, h, \beta)$. Actually, $C$ acts on $C^\infty(G_0, H, \phi)$ as $C(\alpha) = \sum_{\nu, \mu} a_{\nu, \mu}(2\sqrt{-1} \pi \alpha)^\nu X_1^\mu \cdots X_{2p}^\mu$. On the other hand, the element $W$ of $\mathcal{U}(g_0, h, \beta)$ acts on $C^\infty(G, H', \phi)$ as $W(\alpha)$, so that $[W(\alpha), C(\alpha)] = [W, C](\alpha)$ on $C^\infty(G, H', \phi)$. Moreover, one can choose $\alpha$ in such a way that $W = W(\alpha) + (W - W(\alpha)) = W(\alpha) + \hat{W}[Z + 2\sqrt{-1} \pi \alpha]$, with $\hat{W} \in \mathcal{U}(g)$.
suppose that $g$ be identified to $V$ belong to $S$ above applies to an on-ze o element of $h$ have generically the same dimension as the
Proof.

\[ \text{Proof.} \]
Let \[ h \]
be a simply connected connected real nilpotent Lie group with \[ g \]
be a connected closed subgroup of $G$ with Lie algebra $h$ and $\beta \in h^*$ such that $\beta([h, h]) = \{0\}$. Suppose that $(g; h)$ is a reductive pair. Then the assertions (i), (ii) and (iii) of Section 2 are equivalent.

Proof. (i) \Rightarrow (ii): Under the notation of Section 2, we take $V = \mathfrak{m}^*$, which could be identified to $\Omega_{G, H, \beta}$, so that we have the unipotent action $Ad^*$ of $H$ on $V$. Let $\mathfrak{g}_0$ be an ideal of codimension one of $g$ containing $h$ and $G_0 = \exp(\mathfrak{g}_0)$. One can suppose that $\tau_0 = \text{Ind}_H^G \chi_\beta$ is of finite multiplicities. Then, the $H$-orbits in $V$ have generically the same dimension as the $H$-orbits in $(\mathfrak{m} \cap \mathfrak{g}_0)^*$ (that is the $H$-orbits in $V$ are generically not saturated in the direction $(\mathfrak{m} \cap \mathfrak{g}_0)^*$). This implies the existence of an $H$-invariant homogeneous polynomial $P$ on $V$ which does not belong to $S(\mathfrak{m} \cap \mathfrak{g}_0)$ (Theorem of page 55 in [12]). On the other hand, since $(g; h)$ is a reductive pair then it is easy to check that the symmetrization map $\sigma$ is a vector space isomorphism between $S(\mathfrak{m})^H$ and $\mathcal{U}(\mathfrak{g}, h, \beta)/\mathcal{U}(\mathfrak{g})a_\beta$. In particular, $\sigma(P)$ is a non-zero element of $\mathcal{U}(\mathfrak{g}, h, \beta)$ satisfying $\sigma(P) \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})a_\beta$. So Theorem 1 above applies to $W \equiv \sigma(P)$.

For (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) see the end of Section 2. \qed

Remark 3. An interesting consequence of the Corollary 1 is that if $H$ is one-dimensional then the assertions (i), (ii) and (iii) are equivalent. Indeed, if $h$ is a one-dimensional subalgebra of $g$, then it is easy to see that $(g; h)$ is a reductive pair [5].

Remark 4. Another consequence of the Corollary 1 is the case where $\text{Ker}(\beta)$ is an ideal of $g$. In this case, we just apply the Remark 3 above to the one-dimensional quotient $h/\text{Ker}(\beta)$.

4. A SECOND RESULT ON THE COMMUTATIVITY OF $D(G, H, \beta)$

Here we explain, precisely, how to construct, in some cases, the element $W$ of Theorem 1. We keep the previous notation. In particular, remember that $\sigma : S(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$ denotes the symmetrization map, and $L$ is the left action of $\mathcal{U}(\mathfrak{g})$ on $C^\infty(G)$ defined by (3.2).

4.1. Preliminary results.

Lemma 1. Let $\mathfrak{m}$ be an ideal of codimension one in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathbb{R}X$. If $P \in S(\mathfrak{m})$, then $\sigma(PX) = \sigma(P)X + Q$ where $Q \in \mathcal{U}(\mathfrak{m})$.

Proof. Let $k \geq 1$. If $I_k = (i_1, \cdots, i_k) \in [1; m]^k$, we define $P_{I_k} = X_{i_1} \cdots X_{i_k}$. Let $P \in S(\mathfrak{m})$ be of degree $d$, such that $P = \sum_{k=1}^d \sum_{I_k \in [1; m]^k} a_{I_k} P_{I_k}$, with $a_{I_k} \in \mathbb{C}$.
Thus, we have
\begin{equation}
\sigma(PX) = \sum_{k=1}^{d} \sum_{l \in [1,m]^k} \frac{1}{k!} \sum_{\mu \in S_k} \sum_{j=0}^{k} T_j(X_{\mu(i_1)} \cdots X_{\mu(i_k)})
\end{equation}
where \( S_k \) denotes the symmetric group of \( k \) elements, and, for all \( 0 \leq j \leq k \),
\( T_j(X_{\mu(i_1)} \cdots X_{\mu(i_k)}) = X_{\mu(i_j)} \cdots X_{\mu(i_{j+1})} \cdots X_{\mu(i_k)}. \) Remarking that
\begin{equation}
T_j(X_{\mu(i_1)} \cdots X_{\mu(i_k)}) = T_{j+1}(X_{\mu(i_1)} \cdots X_{\mu(i_k)}) + q, q \in U(m),
\end{equation}
we get that
\begin{equation}
T_j(X_{\mu(i_1)} \cdots X_{\mu(i_k)}) = X_{\mu(i_1)} \cdots X_{\mu(i_k)}X + \tilde{q}, \tilde{q} \in U(m),
\end{equation}
so
\begin{equation}
\sigma(PX) = \sum_{k=1}^{d} \sum_{l \in [1,m]^k} \frac{1}{k!} \sum_{\mu \in S_k} X_{\mu(i_1)} \cdots X_{\mu(i_k)}X + Q
= \sigma(P)X + Q, Q \in U(m).
\end{equation}

\[ \Box \]

Now let \( \{X_1, \cdots, X_n\} \) be a basis of \( \mathfrak{g} \), such that \( \{X_1, \cdots, X_p\} \) is a basis of \( \mathfrak{h} \) and
\begin{equation}
[X_j, X_k] = \sum_{l=1}^{\sup(j,k)-1} a_l(j,k)X_l \quad \text{with} \quad a_l(j,k) \in \mathbb{R}.
\end{equation}

Moreover, for all \( f \in C^\infty(G, H, \beta) \), we define a function \( f^t \) on \( \mathbb{R}^n \) by
\begin{equation}
f^t(x_1, \cdots, x_n) = f(\exp(x_1X_1) \cdots \exp(x_nX_n)).
\end{equation}
If \( X_j \) is in \( \mathfrak{g} \), we put
\begin{equation}
[L(X_j)]^t(f^t) = [L(X_j)(f)]^t \forall f \in C^\infty(G, H, \beta).
\end{equation}
This definition extends naturally to \( U(\mathfrak{g}) \).

**Lemma 2.** We have
\begin{equation}
[L(X_j)]^t = \begin{cases} 
-2\sqrt{-1}\pi\beta(X_j)Id \quad \text{if} \quad 1 \leq j \leq p, \\
-\frac{d}{dx_j} - \sum_{l=p+1}^{p} q_l \frac{d}{dx_l} - 2\sqrt{-1}\pi \sum_{l=1}^{p} q_l \beta(X_l)Id \quad \text{if} \quad p + 1 \leq j \leq n,
\end{cases}
\end{equation}
where the \( q_l \) are polynomials in variables \( x_1, \cdots, x_{j-1} \) such that \( q_l(0) = 0 \).

**Proof.** Let \( f \in C^\infty(G, H, \beta) \). By definition, we have
\begin{equation}
[L(X_j)]f(\exp(x_1X_1) \cdots \exp(x_nX_n))
= \frac{d}{dt} f(\exp(-tX_j) \exp(x_1X_1) \cdots \exp(x_nX_n)) \mid_{t=0}.
\end{equation}
Then we have to consider the two cases \( 1 \leq j \leq p \) and \( p + 1 \leq j \leq n \).

**Case:** \( 1 \leq j \leq p \). Since \( f \) is \( H \)-covariant, it follows that
\begin{equation}
[L(X_j)]f(\exp(x_1X_1) \cdots \exp(x_nX_n))
= \frac{d}{dt} \exp(-2\sqrt{-1}\pi t\beta(X_j)) f(\exp(x_1X_1) \cdots \exp(x_nX_n)) \mid_{t=0}
\end{equation}
which means that

\[(4.11) \quad [L(X_j)]^t = -2\sqrt{-1} \pi \beta(X_j) Id \quad \forall 1 \leq j \leq p.\]

Case: \(p + 1 \leq j \leq n\). First note that

\[
\exp(-tX_j) \exp(x_1X_1) \cdots \exp(x_nX_n)
\]

\[= \prod_{k=1}^{j-1} \exp(Ad(\exp(-tX_j)(x_kX_k)))] \times \exp((x_j - t)X_j) \exp(x_{j+1}X_{j+1}) \cdots \exp(x_nX_n),
\]

where

\[
Ad(\exp(-tX_j))(x_kX_k) = \exp(-tad(X_j))(x_kX_k)
\]

\[= x_kX_k - tx_k \sum_{l=1}^{j-1} P_l(k,t)X_l \quad \forall 1 \leq k \leq j - 1,
\]

where \(P_l(k,t)\) is a polynomial in the variable \(t\). Moreover, the Campbell-Hausdorff formula in [4] allows us to write that

\[
\prod_{k=1}^{j-1} \exp(x_kX_k - tx_k \sum_{l=1}^{j-1} P_l(k,t)X_l) = \exp(\sum_{k=1}^{j-1} x_kX_k - t \sum_{l=1}^{j-1} q_l(j - 1; t, x)X_l)
\]

where \(q_l(j - 1; t, x)\) is a polynomial in the variables \(t\) and \(x = (x_1, \cdots, x_{j-1})\) such that \(q_l(j - 1; t, 0) = 0\) for all \(1 \leq l \leq j - 1\). The idea is then to rewrite the right side of (4.14) as follows:

\[
\exp(\sum_{k=1}^{j-1} x_kX_k - t \sum_{l=1}^{j-1} q_l(j - 1; t, x)X_l)
\]

\[= \exp(\sum_{k=1}^{j-2} x_kX_k - t \sum_{l=1}^{j-2} q_l(j - 1; t, x)X_l + (x_{j-1} - t q_{j-1}(j - 1; t, x))X_{j-1})
\]

\[= \exp(\sum_{k=1}^{j-2} x_kX_k - t \sum_{l=1}^{j-2} q_l(j - 1; t, x)X_l + (x_{j-1} - t q_{j-1}(j - 1; t, x))X_{j-1}) \times \exp(-(x_{j-1} - t q_{j-1}(j - 1; t, x))X_{j-1}) \exp((x_{j-1} - t q_{j-1}(j - 1; t, x))X_{j-1}).
\]

Again the Campbell-Hausdorff formula implies that

\[
\exp(\sum_{k=1}^{j-1} x_kX_k - t \sum_{l=1}^{j-1} q_l(j - 1; t, x)X_l)
\]

\[= \exp(\sum_{k=1}^{j-2} x_kX_k - t \sum_{l=1}^{j-2} q_l(j - 2; t, x)X_l) \exp((x_{j-1} - t q_{j-1}(j - 1; t, x))X_{j-1})
\]
where \( q(j - 2; t, x) \) is a polynomial in the variables \( t \) and \( x = (x_1, \cdots, x_{j-1}) \), such that \( q_i(j - 2; t, 0) = 0 \). We apply the same process to

\[
\exp\left(\sum_{k=1}^{j-2} x_k X_k - t \sum_{j=1}^{j-2} q(j - 2; t, x)X_i\right).
\]

After \( j - 2 \) steps, we obtain that

\[
(4.17) \quad \exp\left(\sum_{k=1}^{j-1} x_k X_k - t \sum_{l=1}^{j-1} q(l - 1; t, x)X_l\right) = \prod_{k=1}^{j-1} \exp((x_k - t q_k(k; t, x))X_k)
\]

where, for all \( 1 \leq k \leq j - 1 \), \( q_k(k; t, x) \) is a polynomial in the variables \( t \) and \( x = (x_1, \cdots, x_{j-1}) \) such that \( q_k(k; t, 0) = 0 \). Thus, we have

\[
(4.18) \quad [L(X_j)]^2 f^t(x_1, \cdots, x_n) = \frac{d}{dt} f^t(x_1 - t q_1(1; t, x), \cdots, x_{j-1} - t q_{j-1}(j - 1; t, x), x_j - t, x_{j+1}, \cdots, x_n) \Big|_{t=0}.
\]

If we put \( q_k(x) = q_k(k; 0, x), 1 \leq k \leq j - 1 \), we obtain the result using the \( H \)-covariance of \( f \) in \( C^\infty(G, H, \beta) \).

As we said before (Section 3), we view the symmetric algebra \( S(\mathfrak{g}) \) (resp. \( S^m(\mathfrak{g}) \)) of \( \mathfrak{g} \) as the algebra \( C[\mathfrak{g}^*] \) of polynomials (resp. polynomials of degree \( m \)) on \( \mathfrak{g}^* \). Denote by \( S(\mathfrak{g})^H \) (resp. \( S^m(\mathfrak{g})^H \)) its subalgebra of \( H \)-invariant polynomials on \( \mathfrak{g}^* \) defined by

\[
(4.19) \quad C[\mathfrak{g}^*]^H = \{ P \in C[\mathfrak{g}^*] \mid Ad(h)(P)(l) = P(l) \forall h \in H \forall l \in \mathfrak{g}^* \}.
\]

It is clear that any polynomial on \( \mathfrak{g}^* \) can be written as a finite sum of homogeneous polynomials. Then we have

\[
(4.20) \quad \forall m \in \mathbb{N} \quad \forall Y \in H \quad \forall P \in S^m(\mathfrak{g}), \quad ad(Y)(P) \in S^m(\mathfrak{g})
\]

so that

\[
(4.21) \quad S(\mathfrak{g})^H = \bigoplus_{m \geq 0} S^m(\mathfrak{g})^H.
\]

On the other hand, using the basis \( \{X_1, \cdots, X_n\} \) defined by (4.5), we write for multi-indexes in \( \mathbb{N}^n \), \( (\nu, \alpha) = (\nu_1, \cdots, \nu_p, \alpha_{p+1}, \cdots, \alpha_n) \). Then, following the Poincaré-Birkhoff-Witt Theorem, any element of \( \mathcal{U}(\mathfrak{g}) \) can be written as

\[
(4.22) \quad \sum_{(\alpha, \nu) \in \mathbb{N}^{n-p} \times \mathbb{N}^p} a_{\nu, \alpha} X_1^{\alpha_1} \cdots X_{p+1}^{\nu_{p+1}} X_p^{\nu_p} \cdots X_1^{\nu_1} \equiv \sum_{(\alpha, \nu) \in \mathbb{N}^{n-p} \times \mathbb{N}^p} a_{\nu, \alpha} X^\alpha X^\nu \text{ with } a_{\nu, \alpha} \in \mathbb{C}.
\]

However, to avoid confusion between \( \mathcal{U}(\mathfrak{g}) \) and \( S(\mathfrak{g}) \), we shall use small letters for the basis of \( \mathfrak{g} \) defined by (4.5) to write any polynomial on \( \mathfrak{g}^* \) as

\[
(4.23) \quad \sum_{(\alpha, \nu) \in \mathbb{N}^{n-p} \times \mathbb{N}^p} a_{\nu, \alpha} x_1^{\alpha_1} \cdots x_{p+1}^{\nu_{p+1}} x_p^{\nu_p} \cdots x_1^{\nu_1} \equiv \sum_{(\alpha, \nu) \in \mathbb{N}^{n-p} \times \mathbb{N}^p} a_{\nu, \alpha} x^\alpha x^\nu \text{ with } a_{\nu, \alpha} \in \mathbb{C}.
\]
As usual, if \( \lambda \in \mathbb{N}^n \) is a multi-index, we shall denote its length as the number 
\[ |\lambda| = \sum_{k=1}^{n} \lambda_k; \]
so that the degree of the element \( \sum_{(\alpha, \nu) \in \mathbb{N}^n} a_{\nu, \alpha} X^\alpha X^\nu \) of \( \mathcal{U}(\mathfrak{g}) \) is the number \( |\alpha| + |\nu| \). In the sequel, we shall denote by \( \mathcal{U}_m(\mathfrak{g}) \) the vector subspace of \( \mathcal{U}(\mathfrak{g}) \) of the elements with degree at most \( m \).

**Lemma 3.** Let \( G \) be a simply connected connected nilpotent Lie group. Assume \( H \) is a commutative subgroup of \( G \). Let \( \mathcal{P} \) be an \( H \)-invariant homogeneous polynomial on \( \mathfrak{g}^* \) such that \( \mathcal{P} \) does not vanish identically on \( \mathfrak{g}^* \). Then there exists a non-empty Zariski open subset \( \mathcal{O} \) of \( \mathfrak{h}^* \), such that for all \( \beta \) in \( \mathcal{O} \), we have \( (L_\beta \circ \sigma)(\mathcal{P}) \neq 0 \) in \( \mathcal{D}(G, H, \beta) \), where \( L_\beta \) is the isomorphism induced by (3.2). 

**Proof.** Suppose that \( \mathcal{P} \) is a homogeneous polynomial of degree \( d \) which does not vanish identically on \( \mathfrak{g}^* \). We can write as in (4.23):
\[
(4.24) \quad \mathcal{P} = \sum_{(\alpha, \nu) \in \mathbb{N}^n \times \mathbb{N}^n, |\alpha| + |\nu| = d} a_{\nu, \alpha} x^\alpha x^\nu
\]
with \( a_{\nu, \alpha} \) in \( \mathbb{C} \). Then applying the symmetrization map to \( \mathcal{P} \), we get
\[
(4.25) \quad \sigma(\mathcal{P}) = \sum_{(\alpha, \nu) \in \mathbb{N}^n \times \mathbb{N}^n, |\alpha| + |\nu| = d} (a_{\nu, \alpha} X^\alpha X^\nu + W_{\alpha, \nu})
\]
where
\[
(4.26) \quad W_{\alpha, \nu} = \sum_{(\alpha', \nu') \in \mathbb{N}^n \times \mathbb{N}^n, |\alpha'| + |\nu'| < |\alpha| + |\nu|} b_{\nu', \alpha'} X^{\alpha'} X^{\nu'}.
\]
Actually, we can rewrite (4.25) as follows:
\[
(4.27) \quad \sigma(\mathcal{P}) = \sum_{\alpha \in \mathbb{N}^n} X^\alpha \left( \sum_{\nu \in \mathbb{N}^n, |\nu| = d - |\alpha|} a_{\nu, \alpha} X^\nu + \sum_{\nu' \in \mathbb{N}^n, |\nu'| < d - |\alpha|} b_{\alpha, \nu'} X^{\nu'} \right).
\]
Let us define the polynomial \( \mathcal{P}_\alpha \) on \( \mathfrak{h}^* \) as
\[
(4.28) \quad \mathcal{P}_\alpha = \sum_{\nu \in \mathbb{N}^n, |\nu| = d - |\alpha|} a_{\alpha, \nu} x^\nu + \sum_{\nu' \in \mathbb{N}^n, |\nu'| < d - |\alpha|} b_{\alpha, \nu'} x^{\nu'},
\]
such that
\[
(4.29) \quad (L_\beta \circ \sigma)(\mathcal{P}) = \sum_{\alpha \in \mathbb{N}^n} \mathcal{P}_\alpha (-2\sqrt{-1} \pi \beta) X^\alpha.
\]
Define the subset \( \mathcal{A}_\mathcal{P} \) of multi-indexes in \( \mathbb{N}^n \) by
\[
(4.30) \quad \mathcal{A}_\mathcal{P} = \{ \alpha \in \mathbb{N}^n \mid \mathcal{P}_\alpha \neq 0 \}.
\]
Since \( \mathcal{P} \) does not vanish identically on \( \mathfrak{g}^* \), there exists a multi-index \( \alpha \) in \( \mathbb{N}^n \) such that \( \mathcal{P}_\alpha \) does not vanish identically on \( \mathfrak{h}^* \), so that the subset \( \mathcal{A}_\mathcal{P} \) is not empty. Next define the variety \( \mathcal{M}_\mathcal{P} \) of \( \mathfrak{h}^* \) by
\[
(4.31) \quad \mathcal{M}_\mathcal{P} = \bigcap_{\alpha \in \mathcal{A}_\mathcal{P}} \{ \mathcal{P}_\alpha = 0 \}.
\]
It is clear that \( \mathcal{M}_\mathcal{P} \) is a non-empty Zariski closed subset of \( \mathfrak{h}^* \) which differs from \( \mathfrak{h}^* \). Then we define \( \mathcal{O}_\mathcal{P} \) as the non-empty Zariski open subset of \( \mathfrak{h}^* \):
\[
(4.32) \quad \mathcal{O}_\mathcal{P} = \mathfrak{h}^* \setminus \mathcal{M}_\mathcal{P}.
\]
On the other hand, for all linear forms \( l \) in \( \mathfrak{h}^* \), define the subset \( \mathcal{A}_{\mathcal{P}, l} \) of \( \mathcal{A}_{\mathcal{P}} \) by
\[
(4.33) \quad \mathcal{A}_{\mathcal{P}, l} = \{ \alpha \in \mathcal{A}_{\mathcal{P}} \mid \mathcal{P}_l(l) \neq 0 \}.
\]
Note that if \( l \) is in \( \mathcal{O}_{\mathcal{P}} \), then \( \mathcal{A}_{\mathcal{P}, l} \) is not empty.

Finally, fix \( \beta \) in \( \mathcal{O}_{\mathcal{P}} \). Let \( \xi \) be an element of maximal length in \( \mathcal{A}_{\mathcal{P}, \beta} \). We define a function \( \phi_\xi \) in \( C^\infty(G, H, \beta) \) as follows:
\[
(4.34) \quad \phi_\xi(t) = \chi_\beta(t) \quad \xi = (t_1, \ldots, t_n).
\]
\( \phi_\xi \) is a homogeneous function of degree \( \xi \) in the variables \( t_1, \ldots, t_n \). Using Lemma 2 together with (4.29), we obtain that
\[
(4.35) \quad [L_\beta \circ \sigma(\mathcal{P})]^{\xi}(\phi_\xi) = \mathcal{P}_\xi(-2\sqrt{-1}\pi \beta)t_{p+1}! \cdots t_n!.
\]
We have \( [L_\beta \circ \sigma(\mathcal{P})]^{\xi}(\phi_\xi) \neq 0 \). Hence \( (L_\beta \circ \sigma(\mathcal{P}))\phi_\xi \neq 0 \). This shows that \( (L_\beta \circ \sigma(\mathcal{P})) \neq 0 \).

4.2. A second theorem.

**Theorem 2.** Let \( G \) be a simply connected connected real nilpotent Lie group with Lie algebra \( \mathfrak{g} \), \( H \) a connected closed commutative subgroup of \( G \) with Lie algebra \( \mathfrak{h} \). Consider a weak Malcev basis passing through \( \mathfrak{h} \). Then, if \( \mathfrak{h} + \mathfrak{g}(l) \) is not lagrangian in \( \mathfrak{g} \) for generic \( l \) in \( \Omega_{G, H, \beta} \), the algebra \( \mathcal{D}(G, H, \beta) \) is not commutative, for all \( \beta \) in a non-empty Zariski open subset of \( \mathfrak{h}^* \).

**Proof.** Let \( \mathfrak{g}_0 \) be an ideal of codimension one of \( \mathfrak{g} \) containing \( \mathfrak{h} \) and \( G_0 = \exp(\mathfrak{g}_0) \). One can suppose that \( \tau_0 = Ind_H^G \chi_{\beta} \) is of finite multiplicities. Under the notation of Section 2, we take \( V = \mathfrak{g}^* \). So it is clear that \( V^{\text{gene}} \cap \Omega_{G, H, \beta} \neq \emptyset \) for almost all \( \beta \) in \( \mathfrak{h}^* \). Under the assumptions of the Theorem 2, the Pukanszky parametrization of the \( H \)-orbits in \( \mathfrak{g}^* \), outlined in Section 2, gives a non-zero \( H \)-invariant polynomial \( \mathcal{P} \) on \( \mathfrak{g}^* \) such that \( \mathcal{P} \notin \mathcal{S}(\mathfrak{g}_0) \). Moreover, using (4.20) – (4.21), one can suppose that \( \mathcal{P} \) is homogeneous. Then, from Lemma 3, \( \sigma(\mathcal{P}) \notin \mathcal{U}(\mathfrak{g}_0) + \mathcal{U}(\mathfrak{g})_{\alpha_{\beta}} \) and \( (L_\beta \circ \sigma)(\mathcal{P}) \) is a non-zero element of \( \mathcal{D}(G, H, \beta) \), for all \( \beta \) in \( \mathcal{O}_{\mathcal{P}} \), as defined by (4.32). Thus, we apply Theorem 1 to get an element \( T \) of \( \mathcal{U}(\mathfrak{g}_0, \mathfrak{h}, \beta) \) such that \( [L_\beta \circ \sigma(\mathcal{P}), L_\beta(T)] \neq 0 \) in \( \mathcal{D}(G, H, \beta) \). \( \Box \)

4.3. The case where \( \mathfrak{h} \) is an ideal of \( \mathfrak{g} \).

**Corollary 2.** Let \( G \) be a simply connected connected real nilpotent Lie group with Lie algebra \( \mathfrak{g} \) and \( H \) a connected closed normal subgroup of \( G \) with Lie algebra \( \mathfrak{h} \). Then, for almost all \( \beta \) in \( \mathfrak{h}^* \) satisfying \( \beta([\mathfrak{h}, \mathfrak{h}]) = \{ 0 \} \), the assertions (i), (ii) and (iii) of Section 2 are equivalent.

**Proof.** (i) \( \Rightarrow \) (ii): Under the notation of Section 2, we take \( V = [\mathfrak{h}, \mathfrak{h}]^\perp \) and we choose \( \beta \) in the fundamental layer of \( V \) to apply Theorem 2.

For (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (i) see the end of Section 2. \( \Box \)

5. Characterization of \( \mathcal{D}(G, H, \beta) \) in Terms of the Algebra of \( \mathcal{A}^\infty(H) \)-invariant Rational Functions on \( \Omega_{G, H, \beta} \)

We shall denote by \( \tau_\beta \) the representation associated to \( l \in \Omega_{G, H, \beta}/H \) by the Kirillov map and by \( d\mu \) the measure on \( \Omega_{G, H, \beta}/H \) the Lebesgue measure on \( \Omega_{G, H, \beta} \).
If $\phi = \int_{\Omega_{G,H,\beta}}^\oplus \phi_\pi d\mu(l)$, then

$$D\phi = \int_{\Omega_{G,H,\beta}}^\oplus \Theta^*(D,l)\phi_\pi d\mu(l) \quad \forall D \in \mathcal{D}(G,H,\beta)$$

where $\Theta^*(D,\cdot)$ belongs to $\mathbb{C}(\Omega_{G,H,\beta})^H$, the algebra of $Ad^*(H)$-invariant rational functions on $\Omega_{G,H,\beta}$. The application $\Theta^*: \mathcal{D}(G,H,\beta) \rightarrow \mathbb{C}(\Omega_{G,H,\beta})^H$ is an isomorphism between $\mathcal{D}(G,H,\beta)$ and a subalgebra of $\mathbb{C}(\Omega_{G,H,\beta})^H$. Actually Fujiwara proved that if there exists a common polarization for almost all linear forms on $\mathfrak{g}$ whose restriction to $\mathfrak{h}$ is $\beta$ or if $\mathfrak{h}$ is 1-dimensional, then $\Theta^*$ is an isomorphism between $\mathcal{D}(G,H,\beta)$ and $\mathbb{C}[\Omega_{G,H,\beta}]^H$, the algebra of $Ad^*(H)$-invariant polynomials on $\Omega_{G,H,\beta}$ [3]. This gives a partial answer to a question of Corwin and Greenleaf [5], also asked by Duflo (Problème 3 of [7]) in a more general context.

In the particular cases studied above, we have

**Corollary 3.** Let $G$ be a connected simply connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $H$ a connected closed subgroup of $G$ with Lie algebra $\mathfrak{h}$. The following assertions (a) and (b) are equivalent:
- for all $\beta$ in $\mathfrak{h}^*$ satisfying $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$ when $(\mathfrak{g},\mathfrak{h})$ is a reductive pair,
- for almost all $\beta$ in $\mathfrak{h}^*$ satisfying $\beta([\mathfrak{h},\mathfrak{h}]) = \{0\}$ when $\mathfrak{h}$ is commutative or $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

(a) $\mathcal{D}(G,H,\beta)$ is a commutative algebra.
(b) $\mathcal{D}(G,H,\beta)$ is isomorphic, via $\Theta^*$, to a subalgebra of $\mathbb{C}(\Omega_{G,H,\beta})^H$.

**Proof.** (a) $\Rightarrow$ (b): From Theorem 2 and Corollaries 1 and 2 if $\mathcal{D}(G,H,\beta)$ is commutative, then $\tau$ is of finite multiplicities, so that the results of [3] apply.

(b) $\Rightarrow$ (a) is obvious.

**Remark 5.** Note that in the particular reductive case where $H$ is one-dimensional (Remark 3) or if $H$ is a normal subgroup of $G$ (Corollary 2), then from [8], the image of $\mathcal{D}(G,H,\beta)$ under $\Theta^*$ is, actually, the algebra $\mathbb{C}[\Omega_{G,H,\beta}]^H$ of $Ad^*(H)$-invariant polynomials on $\Omega_{G,H,\beta}$.

### 6. Examples

In the following examples $\mathfrak{g}$ will be the real nilpotent Lie algebra of dimension 7 generated by the vectors $\{X_i, 1 \leq i \leq 7\}$ with the following non-zero brackets:

- $[X_1, X_3] = X_2$, $[X_1, X_4] = X_3$, $[X_1, X_5] = X_4$, $[X_1, X_7] = X_6$, $[X_4, X_5] = X_6$,
- $[X_5, X_6] = X_2$ and $[X_4, X_7] = -X_2$.

Moreover, in the following examples, $\tau = Ind_{\mathfrak{h}}^{\mathfrak{g}} X_\beta$ is of infinite multiplicities.

**Example 1.** Take $\mathfrak{h} = \mathbb{R} X_1$ and $\beta = \xi_1 X_1^*$. Put $l = \sum_{i=1}^7 \xi_i X_i^*$ with $\xi_2 \neq 0$. Corollary 1 and Remark 3 apply in this situation. We take the Malcev basis ordered in the following way: $X_1, X_2, X_3, X_4, X_5, X_6$ and $X_7$. This defines a Jordan-Hölder sequence of subalgebras of $\mathfrak{g}$. It happens that

$$Ad^*(\exp(-tX_1))(\sum_{i=1}^7 \xi_i X_i^*) = \sum_{i=1}^7 \xi_i(t) X_i^*$$

with

- $\xi_1(t) = \xi_1$, $\xi_2(t) = \xi_2$, $\xi_3(t) = \xi_3 + t\xi_2$,
- $\xi_4(t) = \xi_4 + t\xi_3 + \frac{t^2}{2}\xi_2$, $\xi_5(t) = \xi_5$,
- $\xi_5(t) = \xi_5 + t\xi_4 + \frac{t^2}{2}\xi_3 + \frac{t^3}{3}\xi_2$, $\xi_7(t) = \xi_7 + t\xi_6$. 

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We parametrize the $H$-orbits by $u = \xi_3 + t\xi_2$. The orbit of $\ell$ is of dimension 1 and is exactly the set \( \{ \ell(u) = \sum r_i(\ell, u)X_i^i, \, u \in \mathbb{R} \} \) where

\[
\begin{align*}
\ell_1(\ell, u) &= \xi_1, \\
\ell_2(\ell, u) &= \xi_2, \\
\ell_3(\ell, u) &= u, \text{ at this step, dimension of orbits passes from 0 to 1} \\
\ell_4(\ell, u) &= \frac{2\xi_1 - \xi_2^2 - \xi_3^2}{2\xi_2} + \frac{1}{2\xi_2}u^2, \\
\ell_6(\ell, u) &= \xi_6, \\
\ell_7(\ell, u) &= \frac{1}{8\xi_2}u^3 + \frac{2\xi_1 - \xi_2^2 - \xi_3^2}{2\xi_2}u + \frac{\xi_3 + 4\xi_2^2\xi_6 - 3\xi_2\xi_3^2}{3\xi_2^2}, \\
\ell_8(\ell, u) &= \frac{\xi_3}{\xi_2}u + \frac{\xi_3^2 - \xi_2^2}{\xi_2^2}.
\end{align*}
\]

Thus, this gives us rational functions and then $H$-invariant polynomial functions that are written in terms of the variables $\xi_i$. The elements of $\mathcal{U}(g, h, \beta)$ obtained by symmetrization are: $X_1, \, X_2, \, 2X_2X_4 - X_3^2, \, X_5, \, X_3^3 + 3X_2^2X_5 - 3X_2X_3X_4$ and $X_2X_7 - X_3X_6$. We have $[X_3^3 + 3X_2^2X_5 - 3X_2X_3X_4, \, X_6] = 3X_2^3 \not\in \mathcal{U}(g)\alpha_\beta$ and $[X_2X_7 - X_3X_6, \, 2X_2X_4 - X_3^2] = 2X_2^3 \not\in \mathcal{U}(g)\alpha_\beta$. The left action of these elements on $C^\infty(G, H, \beta)$ gives elements of the algebra $D(G, H, \beta)$, which is not commutative.

**Example 2.** Put $\mathfrak{h} = \mathbb{R}X_1 \oplus \mathbb{R}X_6$ with $\beta = \xi_1X_1^* + \xi_6X_6^*$ where $\xi_6 \neq 0$. Since $\mathfrak{h}$ is commutative, we apply Theorem 2. The condition $\xi_6 \neq 0$ ensures the coincidence of the fundamental and generic layers. Analogous calculations as those of Example 1 give the following elements of $\mathcal{U}(g, h, \beta)$ whose images under $L$ belong to $D(G, H, \beta)$: $X_6, \, X_1, \, X_3, \, X_2X_7 - X_3X_6X_6$ and $2X_4X_6 - 2X_3X_6X_7 + X_2X_7^2$.

As in the previous example, the algebra $D(G, H, \beta)$ is not commutative, since $[X_2X_7 - X_3X_6, \, 2X_4X_6^2 - 2X_3X_6X_7 + X_2X_7^2] = 2X_2^3X_6^2 \not\in \mathcal{U}(g)\alpha_\beta$.

**Example 3.** Take $\mathfrak{h} = \mathbb{R}X_1 \oplus \mathbb{R}X_6$ with $\beta = \xi_1X_1^* + \xi_6X_6^*$ where $\xi_6 = 0$. Analogous calculations as those of Example 1 give the following elements of $\mathcal{U}(g, h, \beta)$: $X_6, \, X_1, \, X_7, \, X_2$ and $2X_2X_4 - X_3^2$.

Since $[2X_2X_4 - X_3^2, \, X_7] = -2X_2^3 \not\in \mathcal{U}(g)\alpha_\beta$, the algebra $D(G, H, \beta)$ is not commutative. Here it is interesting to note that our situation is degenerated. However, we observe that in this example the previous constructions give a non-commutative family of elements in $D(G, H, \beta)$.

**References**


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