INVARIANT DISTRIBUTIONS SUPPORTED ON
THE NILPOTENT CONE OF A SEMISIMPLE LIE ALGEBRA

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Abstract. Let \( g \) be a semisimple complex Lie algebra with adjoint group \( G \) and \( \mathcal{D}(g) \) be the algebra of differential operators with polynomial coefficients on \( g \). If \( g_0 \) is a real form of \( g \), we give the decomposition of the semisimple \( \mathcal{D}(g)_G \)-module of invariant distributions on \( g_0 \) supported on the nilpotent cone.

0. Introduction

Let \( g \) be a semisimple complex Lie algebra with adjoint group \( G \). Choose a Cartan subalgebra \( h \) of \( g \) and let \( W \) be the associated Weyl group. Denote by \( W(b) \) the set of isomorphism classes of irreducible \( W \)-modules and by \( H(h) \) the graded vector space of \( W \)-harmonic polynomials on \( h \). For \( \chi \in W^- \), set
\[
b(\chi) = \inf \{ j \in \mathbb{N} : [H^j(h^*) : \chi] \neq 0 \}
\]
and choose a \( W \)-submodule \( V_\chi \subset H^{b(\chi)}(h^*) \) in the class of \( \chi \). Denote by \( d(\chi) \) the dimension of \( V_\chi \).

Let \( S(g^*) \) be the algebra of polynomial functions on \( g \) and \( \mathcal{D}(g) \) be the algebra of differential operators on \( g \), with coefficients in \( S(g) \). The group \( G \) acts on \( g \), via the adjoint action, and hence has an induced action on \( S(g^*) \), \( S(g) \) and \( \mathcal{D}(g) \). Denote the differential of this action by \( \tau : g \to \mathcal{D}(g) \). Let \( S_+(g^*)^G \) and \( S_+(g^*)^S \) be the set of invariant elements without constant term. Recall that \( N(g) \), the nilpotent cone of \( g \), is the variety of zeroes of the ideal \( S_+(g^*)^G \).

Let \( g_0 \) be a real form of \( g \) with adjoint group \( G_0 \subset G \). Denote by \( \mathcal{D}(g_0) \) the \( \mathcal{D}(g) \)-module of distributions on \( g_0 \). Then, the subspace of invariant distributions \( \mathcal{D}(g_0)_G \) as a \( \mathcal{D}(g)_G \)-module, containing the submodule of invariant distributions supported on the nilpotent cone
\[
\mathcal{D}(g_0)_G^{\text{nil}} = \{ \Theta \in \mathcal{D}(g_0)_G : \text{Supp} \Theta \subset N(g_0) \}
\]
where \( N(g_0) = N(g) \cap g_0 \) is the nilpotent cone of \( g_0 \). The structure of \( \mathcal{D}(g_0)_G^{\text{nil}} \) as a vector space is well understood, see, for example, [1][5]. Let \( [h_1], \ldots, [h_r] \) be the conjugacy classes of Cartan subalgebras of \( g_0 \). For each \( j \), let \( \varepsilon_{i,j} : W(h_j) \to \{ \pm 1 \} \) be

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the imaginary signature of the real Weyl group \( W(\mathfrak{h}) \). Then \cite[Proposition 6.1.1]{5} there exists a vector space isomorphism

\((*)\)

\[
\bigoplus_{j=1}^{r} S(\mathfrak{h}_j, \mathfrak{c})^{\varepsilon_{I,j}} \cong D\mathfrak{b}(\mathfrak{g}_0)^{G_0}_{\text{nil}}
\]

where \( S(\mathfrak{h}_j, \mathfrak{c})^{\varepsilon_{I,j}} \) is the isotypic component of type \( \varepsilon_{I,j} \) in the \( W(\mathfrak{h}_j) \)-module \( S(\mathfrak{h}_j, \mathfrak{c}) \).

One aim of this note is to give a complete description of the \( D(\mathfrak{g})^G \)-module \( D\mathfrak{b}(\mathfrak{g}_0)^{G_0}_{\text{nil}} \). This description is given in terms of the simple summands of the equivariant holonomic \( D(\mathfrak{g}) \)-module

\[
\mathcal{M} = D(\mathfrak{g})/(D(\mathfrak{g})\tau(\mathfrak{g}) + D(\mathfrak{g})S_+ (\mathfrak{g}^*)^G).
\]

By \cite[15]{9}, \cite[12]{15} or \cite{13}, it is known that we have a decomposition

\[
\mathcal{M} = \bigoplus_{\chi \in W^\times} d(\chi) \mathcal{M}_\chi
\]

where the \( \mathcal{M}_\chi \) are pairwise non-isomorphic simple \( D(\mathfrak{g}) \)-modules. Moreover, the support (in \( \mathfrak{g} \)) of \( \mathcal{M}_\chi \) is the closure of a nilpotent orbit and \( \mathcal{M}_\chi^G \) is a simple \( D(\mathfrak{g})^G \)-module. Then we have, see Corollary \cite[40]{4}

**Theorem A.** The \( D(\mathfrak{g})^G \)-module \( D\mathfrak{b}(\mathfrak{g}_0)^{G_0}_{\text{nil}} \) decomposes as

\[
D\mathfrak{b}(\mathfrak{g}_0)^{G_0}_{\text{nil}} \cong \bigoplus_{\chi \in W^\times} m_\chi \mathcal{M}_\chi^G
\]

where \( m_\chi = \sum_{j=1}^{r} \text{dim} V_\chi^{\varepsilon_{I,j}} \).

This theorem is proved by combining the isomorphism \((*)\) and the properties, established in \cite{15} \cite{11} \cite{12} \cite{13}, of the Harish-Chandra homomorphism

\[
\delta : D(\mathfrak{g})^G \longrightarrow D(\mathfrak{h})^W.
\]

In the particular case where \( \mathfrak{g}_0 \) is a complex Lie algebra \( \mathfrak{g}_1 \) (viewed as a real Lie algebra), Theorem A was proved by N. Wallach \cite{15}. In this case, \( \mathfrak{g} \cong \mathfrak{g}_1 \times \mathfrak{g}_1, \ W \cong W_1 \times W_1 \) where \( W_1 \) is the Weyl group of \( \mathfrak{g}_1 \). Then, each \( \mathcal{M}_\chi \) occurring in the decomposition of \( D\mathfrak{b}(\mathfrak{g}_0)^{G_0}_{\text{nil}} \) is of the form \( \mathcal{M}_\phi \boxtimes \mathcal{M}_\phi \) with \( \chi = \phi \boxtimes \phi, \phi \in W_1^\times \), and one has \( m_\chi = 1 \). Hence \( D\mathfrak{b}(\mathfrak{g}_0)^{G_0}_{\text{nil}} \cong \bigoplus_{\phi \in W_1^\times} \mathcal{M}_\phi^{G_1} \boxtimes \mathcal{M}_\phi^{G_1} \) as a \( D(\mathfrak{g})^G \)-module.

The next corollary is an easy consequence of Theorem A

**Corollary B.** Let \( \chi \in W^\times \), then, \( \mathcal{M}_\chi \cong D(\mathfrak{g})\Theta \) for some \( \Theta \in D\mathfrak{b}(\mathfrak{g}_0) \) if, and only if, \( V_\chi^{\varepsilon_{I,j}} \neq 0 \) for some \( j \in \{1, \ldots, r\} \).

In Remark 3.7 we apply this result to give examples of modules \( \mathcal{M}_\chi \) which cannot be generated by a distribution on any real form of \( \mathfrak{g} \).

1. **Preliminary results**

   We retain the notation of the introduction. Denote by \( \Delta \) the root system of \( \mathfrak{h} \) in \( \mathfrak{g} \) and fix a system \( \Delta^+ \) of positive roots. Set \( n = \text{dim} \mathfrak{g}, \ell = \text{dim} \mathfrak{h} \) and \( \nu = \#\Delta^+ \), hence \( n = 2\nu + \ell \). Let \( \pi \) be the product of positive roots and recall that \( x \in \mathfrak{g} \) is called generic if \( \pi(x) \neq 0 \). If \( a \subset \mathfrak{g} \), we denote by \( a' \) the set of generic elements in \( a \).

   For \( g \in S(\mathfrak{g}) \), let \( \partial(g) \in D(\mathfrak{g}) \) be the corresponding differential operator with constant coefficients. Let \( \{e_i\}_{1 \leq i \leq n} \) be an orthonormal basis of \( \mathfrak{g} \) with respect to the Killing form \( \kappa \) such that \( \{e_i\}_{1 \leq i \leq \ell} \) is a basis of \( \mathfrak{h} \). Denote by \( x_i \in S(\mathfrak{g}^*) \),
1 \leq i \leq n$, the associated coordinate functions; thus $\partial(e_i)$ identifies with the partial derivative $\frac{\partial}{\partial x_i}$. Denote the Euler vector fields on $\g$ and $\h$ by $E_\g = \sum_{i=1}^n x_i \partial_i$ and $E_\h = \sum_{i=1}^\ell x_i \partial_i$.

We now give some notation and results from \cite{??? [12]} \cite{[13]}. Recall first that the algebra homomorphism, defined by Harish-Chandra,

$$\delta : D(\g) \to D(\h)^W$$

extends the Chevalley isomorphisms $S(\g) \cong S(\h)$ and $S(\g^*) \cong S(\h^*)$. The map $\delta$ is surjective and its kernel is $I = (D(\g)\tau(\g))$. This enables one to identify, through $\delta$, modules over $A(\g) := D(\g)^G / I$ with $D(\h)^W$-modules.

**Lemma 1.1.** Let $D \in D(\g)^G$. Then $D = P + Q$ with $P \in \mathbb{C} \langle S(\g)^G, S(\g^*)^G \rangle$ and $Q \in I$.

**Proof.** By \cite{??? [11]}, we know that $D(\h)^W = \mathbb{C} \langle S(\h)^W, S(\h^*)^W \rangle$. The lemma is therefore a consequence of the properties of $\delta$ previously recalled. \hfill $\square$

Recall that the $(D(\h)^W, W)$-module $S(\h^*)$ decomposes as

$$S(\h^*) \cong \bigoplus_{\chi \in W^*} V^\chi \otimes \mathbb{C} V_\chi$$

where $V^\chi = \text{Hom}_W(V_\chi, S(\h^*))$ is a simple $D(\h)^W$-module. Let $\{e_1^\chi, \ldots, e_{d(\chi)}^\chi\}$ be a basis of $V_\chi$, then $V^\chi \cong D(\h)^W v_1^\chi$ for all $j$ and \cite{??? [11]} implies that

$$S(\h^*) = \bigoplus_{\chi \in W^*} \bigoplus_{j=1}^{d(\chi)} D(\h)^W v_j^\chi.$$

Now, set $N = D(\g) / (D(\g)\tau(\g) \otimes A(\g) S(\h^*)$ and $N_\chi = D(\g) / (D(\g)\tau(\g) \otimes A(\g) V^\chi$. We have

$$N = D(\g) / (D(\g)\tau(\g) + D(\g)\tau(\g) S(\g)^G)$$

and, using \cite{??? [11]},

$$N = \bigoplus_{\chi \in W^*} N_\chi \otimes \mathbb{C} V_\chi.$$

Then each $N_\chi$ is a simple (holonomic) $D(\g)$-module \cite{??? [13]} and, therefore, $N$ is a semisimple $D(\g)$-module (see also \cite{??? [9]}). Let $\mathcal{C}(N)$ be the full subcategory of finitely generated $D(\g)$-modules of the form $\bigoplus_{\chi \in W^*} m_\chi N_\chi, m_\chi \in \mathbb{N}$. From \cite{??? [13]} we know that the category $\mathcal{C}(N)$ is equivalent to the category $W$-mod (of finite dimensional $W$-modules) via the functor

$$\text{Sol} : \mathcal{C}(N) \to W\text{-mod}, \quad \text{Sol}(N) = \text{Hom}_{D(\h)^W}(N^G, S(\h^*))$$

where $W$ acts on $\text{Sol}(N)$ through its natural action on $S(\h^*)$.

The Killing form $\kappa$ induces a $G$-isomorphism $\g \cong \g^*$ and an algebra automorphism $\kappa$ of $D(\g)$, defined by $\kappa(\partial(v)) = \kappa(v, -)$, $\kappa(v, v) = -\partial(v)$, for all $v \in \g$. Hence, in coordinates, $\kappa(\partial_j) = x_j$, $\kappa(x_j) = -\partial_j$. Set $i = \sqrt{-1} \in \mathbb{C}$ and denote by $i$ the automorphism of $D(\g)$ given by $i(\partial_j) = -i\partial_j$, $i(x_j) = ix_j$. Define then the “Fourier transformation” $F_\g \in \text{Aut} D(\g)$ by $F_\g = i \circ \kappa = \kappa \circ i^{-1}$; thus $F_\g(x_j) = ix_j$, $F_\g(\partial_j) = i\partial_j$. One easily checks that $\kappa(\tau(x)) = F_\g(\tau(x)) = \tau(x)$
for all $x \in \mathfrak{g}$; moreover, $\alpha$ and $F_\mathfrak{g}$ are $G$-equivariant. Similarly, since $\kappa$ is non-degenerate and $W$-invariant on $\mathfrak{h}$, one can define $W$-equivariant automorphisms $\alpha$ and $F_\mathfrak{h} = 1 \circ \alpha$ in Aut $D(\mathfrak{h})$.

**Lemma 1.2.** One has $\delta \circ F_\mathfrak{h} = F_\mathfrak{h} \circ \delta$.

**Proof.** A direct computation shows that $\delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P))$ when $P$ belongs to $S(\mathfrak{g})^G$ or $S(\mathfrak{g}^*)^G$. Since $\delta$ is a homomorphism, it follows that $\delta(F_\mathfrak{h}(P)) = F_\mathfrak{h}(\delta(P))$ for all $P \in \mathbb{C}(S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G)$. Now, let $D \in D(\mathfrak{g})^G$ and write $D = P + Q$ as in Lemma 1.1. Then, since $F_\mathfrak{h}(I) = I$, we obtain $\delta(F_\mathfrak{h}(D)) = \delta(F_\mathfrak{g}(P)) = F_\mathfrak{h}(\delta(P)) = F_\mathfrak{h}(\delta(D))$. \hfill $\square$

Recall that $\mathcal{H}(\mathfrak{h}^*)$ is the vector space of $W$-harmonic polynomials on $\mathfrak{h}$. Hence

$$\mathcal{H}(\mathfrak{h}^*) = \{ f \in S(\mathfrak{h}^*) : \partial(q).f = 0 \text{ for all } q \in S_+(\mathfrak{h})^W \}$$

and, as a $W$-module, $\mathcal{H}(\mathfrak{h}^*)$ identifies with the regular representation of $W$. The vector space $\mathcal{H}(\mathfrak{h}^*)$ is a graded subspace of $S(\mathfrak{h}^*)$ and we set $\mathcal{H}_j(\mathfrak{h}^*) = S_j(\mathfrak{h}^*) \cap \mathcal{H}(\mathfrak{h}^*)$, $0 \leq j \leq \nu$. Define the harmonic elements of $S(\mathfrak{h})$ by $\mathcal{H}(\mathfrak{h}) = F_\mathfrak{h}(\mathcal{H}(\mathfrak{h}^*)) = \bigoplus_{j=0}^\nu \mathcal{H}_j(\mathfrak{h})$. (We could as well have set $\mathcal{H}(\mathfrak{h}) = \alpha(\mathcal{H}(\mathfrak{h}^*))$, since $\mathcal{H}_1(\mathfrak{h}^*)$ is stable under $\alpha$.)

Since $V_\chi \subset \mathcal{H}^b(\chi)(\mathfrak{h}^*)$, we have $(E_\mathfrak{h} - b(\chi)).v_\chi^j = 0$. For all $d \in L := \text{ann}_{D(\mathfrak{h})^W}(v_\chi^j)$, we have $(E_\mathfrak{h} - b(\chi), d) = [E_\mathfrak{h}, d] \in L$. It follows that $L = \bigoplus_{k \in \mathbb{Z}} L \cap D^k(\mathfrak{h})^W$, where $D^k(\mathfrak{h}) = \{ d \in D(\mathfrak{h}) : [E_\mathfrak{h}, d] = kd \}$. Equivalently, $L$ is stable under the $\mathbb{C}^*$-action on $D(\mathfrak{h})$ given by $f \mapsto \lambda f$, $\partial(v) \mapsto \lambda^{-1} \partial(v)$, $f \in \mathfrak{h}^*$, $v \in \mathfrak{h}$. In particular, we see that $F_\mathfrak{h}(L) = \alpha(L)$.

Let $R$ be a ring and $\alpha \in \text{Aut}(R)$. If $M$ is an $R$-module, we define the $R$-module $M^\alpha$ to be the abelian group $M$ with action of $a \in R$ on $x \in M$ given by $a \cdot x = \alpha(a)x$. This applies to the modules $\mathcal{N}$, $\mathcal{N}_\chi$ and the automorphism $\alpha = F_\mathfrak{h}^{-1}$. Define

$$M = N_{F_\mathfrak{h}^{-1}}, \quad M_\chi = N_{F_\mathfrak{h}^{-1}}^\chi.$$

Thus, from (1.2) and (1.3), we obtain

$$M = D(\mathfrak{g}) / (D(\mathfrak{g}) \tau(\mathfrak{g}) + D(\mathfrak{g}) S_+(\mathfrak{g}^*)^G) \cong \bigoplus_{\chi \in W^*} M_\chi \otimes_{\mathbb{C}} V_\chi.$$

**Remark.** In (1.3) one defines $M_\chi$ to be $N_{F_\mathfrak{h}^{-1}}^\chi$, but the two definitions agree. Indeed, let $V_\chi \cong D(\mathfrak{h})^W.v_\chi^j = D(\mathfrak{h})^W/L$ be as above. Then,

$$N_\chi \cong D(\mathfrak{h}) / J, \quad J = D(\mathfrak{g}) \tau(\mathfrak{g}) + D(\mathfrak{g}) S_+(\mathfrak{g}^*)^G + D(\mathfrak{g}) \delta^{-1}(L).$$

Write $N_\chi = D(\mathfrak{g}).(I \otimes A(\mathfrak{g}) v_\chi^j)$, where $I$ is the canonical generator of $D(\mathfrak{g}) / D(\mathfrak{g}) \tau(\mathfrak{g})$. From $\delta(E_\mathfrak{g}) = E_\mathfrak{h} - \nu$, we get that $(E_\mathfrak{g} - (b(\chi) - \nu)).(I \otimes A(\mathfrak{g}) v_\chi^j) = 0$. It follows (as above) that $J$ is stable under the natural $\mathbb{C}^*$-action on $D(\mathfrak{g})$. Hence, $F_\mathfrak{h}(J) = \alpha(J)$ and we have $N_{F_\mathfrak{h}^{-1}}^\chi = N_{F_\mathfrak{h}^{-1}}^\chi$.

We can define the category $\mathbb{C}(M)$ similar to $\mathbb{C}(N)$. We clearly have $M \in \mathbb{C}(M)$ if, and only if, $N = M_{F_\mathfrak{h}^{-1}} \in \mathbb{C}(N)$. Moreover, by (1.3), this is equivalent to saying that $M$ is a $G$-equivariant finitely generated $D(\mathfrak{g})$-module such that $M = D(\mathfrak{g}) M_\chi^G$ and $\text{Supp} M \subset N(\mathfrak{g})$. This is also equivalent to: $N$ is a $G$-equivariant finitely generated $D(\mathfrak{g})$-module such that $N = D(\mathfrak{g}) N^G$ and $N$ is $S_+$-finite (meaning that each $v \in N$ is killed by a power of $S_+(\mathfrak{g})^G$).

Recall that $N_\chi^G \cong V_\chi$ through the identification of $A(\mathfrak{g})$ with $D(\mathfrak{h})^W$. 

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Lemma 1.3. One has $M^G \simeq (\mathcal{V}^\times)^{F_\theta^{-1}}$.

Proof. Write $N_\chi = \mathcal{D}(g)/J$. Then, $M_\chi = \mathcal{D}(g)/F_\theta(J)$ and $M^G = \mathcal{D}(g)^G/F_\theta(J^G)$. By Lemma [12, Lemma 1.2], $(\mathcal{D}(g)^G(J^G)) = F_\theta(\delta(J^G))$, therefore $M^G \simeq \mathcal{D}(h)^W/F_\theta(\delta(J^G))$. Since $\mathcal{V}^\times \cong \mathcal{D}(h)^W/F_\theta(\delta(J^G))$, the lemma follows.

Let $g_0$ be a real form of $g$ with adjoint group $G_0 \subset G$. There exists a natural action of $\mathcal{D}(g)$ on $\text{Db}(g_0)$ defined by

$$\langle \theta(v)\cdot T, f \rangle = \langle T, -\partial(v) \cdot f \rangle, \quad \langle \xi, T, f \rangle = \langle T, \xi f \rangle$$

for all $T \in \text{Db}(g_0)$, $f \in C_c^\infty(g_0)$, $v \in g$, $\xi \in g^*$. This induces a structure of $\mathcal{D}(g)^G$-module on $\text{Db}(g_0)^{G_0}$. From $\mathcal{I}_T \text{Db}(g_0)^{G_0} = 0$, we obtain a natural $A(g)$-module structure on $\text{Db}(g_0)^{G_0}$.

Fix a basis $\{u_1, \ldots, u_n\}$ of $g_0$ such that $\kappa(u_j, u_k) = \pm \delta_{jk}$ and denote by $dy$ the Lebesgue measure associated to this choice. Let $\mathcal{S}(g_0)$ be the Schwartz space on $g_0$. Define, as in [13, Appendix 1], the Fourier transform of $f \in \mathcal{S}(g_0)$ by

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{g_0} f(y)e^{-i\kappa(y, x)} dy.$$

Let $T$ be a tempered distribution on $g_0$. The Fourier transform of $T$ is defined by $\langle \hat{T}, f \rangle = \langle T, \hat{f} \rangle$ for $f \in C_c^\infty(g_0)$. Then we have

$$(1.4) \quad \forall D \in \mathcal{D}(g), \quad \forall T \in \text{Db}(g_0), \quad \hat{\mathcal{D}}_T = F_\theta(D) \hat{T}.$$

Recall [2] that $T \in \text{Db}(g_0)$ is said to be homogeneous of degree $d$ if, for all $f \in C_c^\infty(g_0)$, $t \in \mathbb{R}^*$, $\langle T, f_t \rangle = t^d \langle T, f \rangle$, where $f_t(v) = t^n f(t^{-1}v)$. Then, a homogeneous distribution of degree $d$ is tempered and satisfies $E_gT = dtT$. We will need the following well-known result:

Lemma 1.4. Let $T \in \text{Db}(g_0)$ be tempered and set $M = \mathcal{D}(g).T$. Then $M^{F_\theta} \cong \mathcal{D}(g) \hat{T}$.

Proof. By [13], we have $\text{ann}_{\mathcal{D}(g)}(\hat{T}) = F_\theta^{-1}(\text{ann}_{\mathcal{D}(g)}(T))$. Hence the result.

Let $N(g_0)$ be the set of nilpotent elements of $g_0$. Define $\mathcal{D}(g)$-submodules of $\text{Db}(g_0)$ by

$$\text{Db}(g_0)_{\text{nil}} = \{\Theta \in \text{Db}(g_0) : \text{Supp}\Theta \subset N(g_0)\},$$

$$\text{Db}(g_0)_{S_+} = \{T \in \text{Db}(g_0) : \exists k \in \mathbb{N}, S_+(g)^k T = 0\}.$$

The elements of $\text{Db}(g_0)_{S_+}$ are called $S_+$-finite. Observe that $\text{Db}(g_0)_{\text{nil}}^{G_0}$ and $\text{Db}(g_0)_{S_+}^{G_0}$ are $\mathcal{D}(g)^G$-modules. The next theorem is a consequence of the results proved in [18].

Theorem 1.5. (1) $\text{Db}(g_0)_{\text{nil}}^{G_0} = \{\Theta \in \text{Db}(g_0)^{G_0} : \mathcal{D}(g).\Theta \subset C(M)\}$.

(2) $\text{Db}(g_0)_{S_+}^{G_0} = \{T \in \text{Db}(g_0)^{G_0} : \mathcal{D}(g).T \subset C(N)\}$.

(3) $\Theta \in \text{Db}(g_0)_{\text{nil}}^{G_0} \iff \hat{\Theta} \in \text{Db}(g_0)_{S_+}^{G_0}$.

Proof. (1) follows from [18, Theorem 6.1], since $\mathcal{D}(g).\Theta \subset C(M)$ is equivalent to $\mathcal{D}(g)^G.\Theta \cong \bigoplus_{\chi \in W^-} m_\chi M^G_{\chi}$. (2) and (3) are consequences of (1) and Lemma 1.4.
Lemma 2.1. Let $V$ be a system of roots in $\mathbb{R}^n$, and one can define a character of $V$ by

$$\langle T, f \rangle = \int_{\mathfrak{g}_0} F_T(y)f(y)dy$$

for some analytic function $F_T$ on $\mathfrak{g}_0'$, locally integrable on $\mathfrak{g}_0$.

2. The distributions $\Theta_{u,\Gamma}$ and $T_{\varphi,\Gamma}$

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, with adjoint group $G_0$, $\mathfrak{h}_0$ a Cartan subalgebra and let $H_0$ be the associated Cartan subgroup. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_0$ and adopt the notation of [1].

Denote by $W(\mathfrak{h}_0)$ the real Weyl group, i.e. $W(\mathfrak{h}_0) = N_{G_0}(\mathfrak{h}_0)/Z_{G_0}(\mathfrak{h}_0)$. Define

$$\Delta_R = \{\alpha \in \Delta : \alpha(\mathfrak{h}_0) \subset \mathbb{R}\} \quad \text{(the real roots)},$$

$$\Delta_I = \{\alpha \in \Delta : \alpha(\mathfrak{h}_0) \subset i\mathbb{R}\} \quad \text{(the imaginary roots)}.$$  

A root which is neither real nor imaginary is called complex. Let $\Delta^+_I$ be a positive system of roots in $\Delta_I$ and set $\pi_I = \prod_{\alpha \in \Delta^+_I} \alpha$. Then each $w \in W(\mathfrak{h}_0)$ permutes the imaginary roots and one can define a character of $W(\mathfrak{h}_0)$, the imaginary signature, by

$$\varepsilon_I : W(\mathfrak{h}_0) \to \{\pm 1\}, \quad w.\pi_I = \varepsilon_I(w)\pi_I.$$  

If $V$ is a $W(\mathfrak{h}_0)$-module we denote by $V^{\varepsilon_I}$ the isotypic component of type $\varepsilon_I$ in $V$.

In the sequel, we adopt the notation of [5] with the minor difference that we use $e^{-i\alpha(x,y)}$ in the definition of the Fourier transform.

Let $h \in \mathfrak{h}_0'$ and $f \in \mathcal{C}^\infty(\mathfrak{g}_0)$. Define [5] [3.1] the distribution $\mu_{G_0,h}$ by

$$\langle \mu_{G_0,h}, f \rangle = \left| \det \text{ad}_{\mathfrak{g}_0/\mathfrak{h}_0}(h) \right|^\frac{1}{2} \int_{G_0/H_0} f(\dot{g}.h)dg.$$ 

Then one defines the function $J_{\mathfrak{g}_0}(f)$, or simply $J(f)$, on $\mathfrak{h}_0'$ by

$$J_{\mathfrak{g}_0}(f) = \{ h \mapsto \langle \mu_{G_0,h}, f \rangle \}.$$ 

Set $\mathfrak{h}_0^{\text{reg}} = \{ h \in \mathfrak{h}_0 : \pi_I(h) \neq 0 \}$ and fix a connected component $\Gamma$ of $\mathfrak{h}_0^{\text{reg}}$. Let $u \in S(\mathfrak{h})$; Harish-Chandra has shown, see [17] [8.1, p. 123], that one can define a tempered $G_0$-invariant distribution on $\mathfrak{g}_0$ by

$$\forall f \in \mathcal{C}^\infty_c(\mathfrak{g}_0), \quad \langle \Theta_{u,\Gamma}, f \rangle = \lim_{h \in \mathfrak{g}_0^{\text{reg}}} [\partial(u).J(f)](h).$$

Furthermore $\Theta_{u,\Gamma} \in \mathcal{D}(\mathfrak{g}_0)_{S_+}^{G_0}$ and, when $u \in S^b(\mathfrak{h})$, $\Theta_{u,\Gamma}$ is homogeneous of degree $-b - \nu - \ell$.

Now let $p \in S(\mathfrak{h}^*)$ and define $T \in \mathcal{D}(\mathfrak{g}_0)_{S_+}^{G_0}$ by

$$T_{\varphi,\Gamma} = \Theta_{F_t(p),\Gamma} = \left\{ f \mapsto \lim_{h \in \mathfrak{g}_0^{\text{reg}}} [\partial(F_t(h)).J(f)](h) \right\}.$$ 

Then, $T_{\varphi,\Gamma}$ is tempered and is homogeneous of degree $b - \nu$ when $p \in S^b(\mathfrak{h}^*)$.

Lemma 2.1. (1) Let $\varphi \in S(\mathfrak{g}^*)^G$. Then, $\varphi T_{\varphi,\Gamma} = T_{\delta(\varphi),p,\Gamma}$.

(2) Let $q \in S(\mathfrak{g})^G$. Then, $\partial(q).T_{\varphi,\Gamma} = T_{\partial(\delta(q)),p,\Gamma}$. 

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Proof. Set \( u = F_\theta(p) \), \( \phi = \delta(\varphi) \in S(\mathfrak{h}^*)_W \) and \( s = \delta(q) \in S(\mathfrak{h})^W \). Let \( f \in \mathcal{S}_c(\mathfrak{g}_0) \).

(1) By definition, see (2.2), \( \langle \varphi_{T_{p,\Gamma}}, f \rangle = \lim_{h \to 0} [\partial(u).J(\varphi f)](h) \). But, \[ \tag{1.4} \]
Lemma 3.2.7, p. 38, (1.4) and Lemma 1.2 imply that \( J(\varphi f) = \partial(F_\theta(\phi)).J(\hat{f}) \).

Hence,

\[
\langle \varphi_{T_{p,\Gamma}}, f \rangle = \lim_{h \to 0} [\partial(u).\partial(F_\theta(\phi)).J(\hat{f})](h) = \lim_{h \to 0} [\partial(F_\theta(\phi)).J(\hat{f})](h)
\]

as desired.

(2) By (1.4), \( \partial(s).T_{p,\Gamma} \) is the Fourier transform of \( F_\theta^{-1}(q)\Theta_{\mu,\Gamma} \), hence

\[
\partial(s).T_{p,\Gamma} = \lim_{h \to 0} [\partial(u).J(F_\theta^{-1}(q)\hat{f})](h).
\]

Set \( g = J(\hat{f}) \). From \[ \tag{1.7} \]
Lemma 3.2.7, p. 38 and Lemma 1.2 we obtain that \( J(F_\theta^{-1}(q)\hat{f}) = F_\theta^{-1}(s)g \). Therefore

\[
\partial(s).T_{p,\Gamma} = \lim_{h \to 0} [\partial(u).J(F_\theta^{-1}(q)\hat{f})](h).
\]

Recall (see \[ \tag{1.11} \]) that we have chosen a coordinate system \( \{ x_j; e_j \}_{1 \leq j \leq \ell} \). With standard notation, we write \( x^\alpha = \prod_{k=1}^\ell x_k^{\alpha_k}, \ e^\mu = \prod_{k=1}^\ell e_k^{\mu_k} \) and

\[
p = \sum_{\alpha \in \mathbb{N}^\ell} p_\alpha x^\alpha, \ s = \sum_{\mu \in \mathbb{N}^\ell} s_\mu e^\mu.
\]

Set \( \partial^\mu = \prod_j \partial(e_j)^{\mu_j} \) thus \( \partial(s) = \sum_{\mu \in \mathbb{N}^\ell} s_\mu \partial^\mu \). Order \( \mathbb{N}^\ell \) by saying that \( \mu \leq \alpha \) if \( \mu_j \leq \alpha_j \) for all \( j \). Set \( \alpha! = \prod_j \alpha_j! \) and \( \binom{\alpha}{\mu} = \prod_j \binom{\alpha_j}{\mu_j} \), when \( \mu \leq \alpha \). Then,

\[
\partial^\mu(x^\alpha) = \begin{cases} 0 & \text{if } \mu \nleq \alpha, \\ \binom{\alpha}{(\alpha - \mu)} x^{\alpha - \mu} & \text{if } \mu \leq \alpha. \end{cases}
\]

Now we have \( u = F_\theta(p) = \sum_\alpha p_\alpha i^{|\alpha|} \partial^\alpha \) and \( F_\theta^{-1}(s) = \sum_\mu q_\mu i^{-|\mu|} x^\mu \). Therefore, using the Leibniz formula, we get that

\[
\partial(u).J(F_\theta^{-1}(s)g) = \sum_\alpha p_\alpha i^{|\alpha|} \partial^\alpha (F_\theta^{-1}(s)g)
\]

\[
= \sum_\alpha \sum_{\beta \leq \alpha} \sum_\mu p_\alpha s_\mu i^{|\alpha| - |\mu|} \binom{\alpha}{\beta} \partial^\beta(x^\mu) \partial^{\alpha - \beta}(g).
\]

But \( \lim_{h \to 0} \partial^\beta(x^\mu)(h) = 0 \) unless \( \beta = \mu \), hence

\[
\lim_{h \to 0} [\partial(u).J(F_\theta^{-1}(s)g)](h) = \sum_\alpha \sum_{\mu \leq \alpha} p_\alpha s_\mu i^{|\alpha| - |\mu|} \binom{\alpha}{\mu} \mu! \lim_{h \to 0} [\partial^{\alpha - \mu}(g)](h).
\]

On the other hand, we have

\[
\langle T_{\partial(s), p, \Gamma}, f \rangle = \lim_{h \to 0} [\partial(F_\theta(\partial(s).p)).J(\hat{f})](h).
\]

Since \( \partial(s).p = \sum_\alpha \sum_{\mu \leq \alpha} \frac{\alpha!}{(\alpha - \mu)!} s_\mu p_\alpha x^{\alpha - \mu} \), we obtain that

\[
\langle T_{\partial(s), p, \Gamma}, f \rangle = \sum_\alpha \sum_{\mu \leq \alpha} \frac{\alpha!}{(\alpha - \mu)!} s_\mu p_\alpha i^{|\alpha| - |\mu|} \lim_{h \to 0} [\partial^{\alpha - \mu}(g)](h).
\]

This proves the desired equality. \( \square \)
Theorem 2.2. Let \( p \in S(\mathfrak{h}^*) \) and \( D \in \mathcal{D}(\mathfrak{g})^G \). Then, \( D.T_{p,\Gamma} = T_{\delta(D),p,\Gamma} \).

Proof. Since \( T_{p,\Gamma} \) is \( G_0 \)-invariant, we have \( \mathcal{I}.T_{p,\Gamma} = 0 \). Let \( P \in \mathbb{C}(S(\mathfrak{g})^G, S(\mathfrak{g}^*)^G) \); by Lemma 2.1 and an obvious induction, we obtain that \( P.T_{p,\Gamma} = T_{\delta(P),p,\Gamma} \). The theorem then follows from Lemma 1.1. \( \blacksquare \)

Recall, see Remark 1.6 that \( \hat{\Theta}_{u,\Gamma} \in \text{Db}(\mathfrak{g}_0)^{G_0}_{\mathcal{S}_u} \) is determined by a locally integrable function on \( \mathfrak{g}_0 \). We still denote this function by \( \hat{\Theta}_{u,\Gamma} \).

Lemma 2.3. ([5 Lemme 6.1.2]) There exists \( c_T \in \mathbb{C}^* \), such that

\[
a_{\Delta_T^+}(h)|\det\mu_{\mathfrak{g}_0/\mathfrak{h}_0}(h)|^{\frac{1}{2}}\hat{\Theta}_{F_k(p),\Gamma}(h) = c_T p(h)
\]

for all \( p \in S(\mathfrak{h}^*)^{G_1} \) and \( h \in \mathfrak{g}_0^{\text{reg}} \). \( \blacksquare \)

Remark. In the notation of the lemma, if \( u = F_b(p) \), the function \( \hat{u}(ih) \) of [5] is replaced here by \( p(h) \) since we are using \( e^{-i\xi(x,y)} \) in the definition of the Fourier transform.

Theorem 2.4. Let \( p \in S(\mathfrak{h}^*)^{G_1} \). There exists a bijective map

\[
\rho : \mathcal{D}(\mathfrak{g})^G.T_{p,\Gamma} \rightarrow \mathcal{D}(\mathfrak{h})^W.p, \quad \rho(D.T_{p,\Gamma}) = \delta(D).p
\]

which, through \( \delta \), yields an isomorphism

\[
\rho : A(\mathfrak{g}).T_{p,\Gamma} \xrightarrow{\sim} \mathcal{D}(\mathfrak{h})^W.p.
\]

Proof. We first need to show that \( \rho \) is well defined. Let \( D \in \mathcal{D}(\mathfrak{g})^G \); by Theorem 2.2, we have

\[
(D.T_{p,\Gamma}) = T_{\delta(D),p,\Gamma} = \hat{\Theta}_{F_k(p),\Gamma}. \quad \tag{1}
\]

Suppose that \( D.T_{p,\Gamma} = 0 \). Then, the analytic function associated to \( T_{\delta(D),p,\Gamma} \in \text{Db}(\mathfrak{g}_0)^{G_0}_{\mathcal{S}_u} \) vanishes on \( \mathfrak{g}_0^{\text{reg}} \). Notice that, since \( \delta(D) \) is \( W \)-invariant, \( \delta(D).p \in S(\mathfrak{h}^*)^{G_1} \). Therefore Lemma 2.3 gives \( \delta(D).p = 0 \) on \( \mathfrak{g}_0^{\text{reg}} \). Thus \( \delta(D).p = 0 \) on \( \mathfrak{g} \) and \( \rho \) is well defined.

Now, it follows easily from (1) that \( \rho \) is a linear bijection. Since \( \mathcal{I}.T_{p,\Gamma} = 0 \), the last assertion is clear. \( \blacksquare \)

Recall that we denote by \( V_\chi \subset \mathcal{H}(\mathfrak{h}^*)(\mathfrak{h}^*)^{G_1} \) a simple \( W \)-module in the class of \( \chi \in W^- \).

Corollary 2.5. Let \( p \in S(\mathfrak{h}^*)^{G_1} \) such that \( CW.p \) is simple. Then there exists \( \chi \in W^- \) such that \( V_\chi \neq 0 \). We have

1. \( \mathcal{D}(\mathfrak{g}).T_{p,\Gamma} \xrightarrow{\sim} N_\chi \) and \( \mathcal{D}(\mathfrak{g})^G.T_{p,\Gamma} \xrightarrow{\sim} V_\chi \);
2. \( \mathcal{D}(\mathfrak{g}).\Theta_{F_k(p),\Gamma} \xrightarrow{\sim} M_\chi \) and \( \mathcal{D}(\mathfrak{g})^G.\Theta_{F_k(p),\Gamma} \xrightarrow{\sim} (V_\chi)^{F_k^{-1}}. \)

Proof. The first assertion follows from \( \mathcal{H}(\mathfrak{h}^*) \cong CW \). Then, 1 and 2 are consequences of \( V_\chi \cong \mathcal{D}(\mathfrak{h})^W.p \), Lemma 1.3 and Theorem 2.4. \( \blacksquare \)

Remark 2.6. Let \( \chi \in W^- \) be such that \( V_\chi \neq 0 \). It follows obviously from the previous corollary that

\[
N_\chi \cong \mathcal{D}(\mathfrak{g}).T_{p,\Gamma}, \quad M_\chi \cong \mathcal{D}(\mathfrak{g}).\Theta_{u,\Gamma}
\]

where \( 0 \neq p \in V_\chi^{G_1} \subset \mathcal{H}(\mathfrak{h}^*)(\mathfrak{h}^*)^{G_1} \) and \( u = F_p(p) \in \mathcal{H}(\mathfrak{h}^*)(\mathfrak{h})^{G_1} \).
3. The decomposition of $\mathcal{Db}(\mathfrak{g}_0)_{\mathbb{C}}^{G_0}$ and $\mathcal{Db}(\mathfrak{g}_0)_{\mathbb{R}}^{G_0}$

Fix a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and let $[\mathfrak{h}_1],\ldots,[\mathfrak{h}_r]$ be the conjugacy classes of Cartan subalgebras in $\mathfrak{g}_0$. For each $j=1,\ldots,r$ we denote by

$$\mathfrak{h}_j, \mathfrak{c}_j = \mathfrak{h}_j \otimes \mathbb{C}, \quad W_j = W(\mathfrak{g},\mathfrak{h}_j,\mathfrak{c}_j), \quad \Delta_{\mathfrak{c}_j}^+$$

d a set of positive imaginary roots, $\varepsilon_{1,j} : W(\mathfrak{h}_j) = W(\mathfrak{g}_0,\mathfrak{h}_j) \to \{\pm 1\}$ the imaginary signature associated to $\mathfrak{h}_j$.

For each $j$ we fix a connected component $\Gamma_j$ of $\mathfrak{h}_j^{\text{reg}}$. The results of [2] then apply to $\mathfrak{h}_0 = \mathfrak{h}_j, \Gamma = \Gamma_j$ etc.

**Remark 3.1.** Recall that the $\mathfrak{h}_j, \mathfrak{c}_j$ are $G$-conjugate. Therefore, if $1 \leq j, k \leq r$, the algebras $\mathcal{D}(\mathfrak{h}_j, \mathfrak{c}_j)^{W_j}$ and $\mathcal{D}(\mathfrak{h}_k, \mathfrak{c}_k)^{W_k}$ are naturally isomorphic. Denote this isomorphism by $\gamma_{jk}$ and let $\delta_j$ be the Harish-Chandra isomorphism from $A(\mathfrak{g})$ onto $\mathcal{D}(\mathfrak{h}_j, \mathfrak{c}_j)^{W_j}$. One can check that $\delta_k = \gamma_{jk} \circ \delta_j$. Therefore, we can choose an “abstract” Cartan subalgebra $\mathfrak{h}$ and identify $\delta_j$ with the homomorphism $\delta : \mathcal{D}(\mathfrak{g})^{G} \to \mathcal{D}(\mathfrak{g})^{W_j}$, where $W = W(G, \mathfrak{h})$. Then, if $\chi \in W^\ast$, we have an irreducible $W$-module $V_{\chi} \subset \mathcal{H}^{\chi}(\mathfrak{h}^\ast)$ and a simple $\mathcal{D}(\mathfrak{h})^W$-module $V_{\chi}$.

For each $\chi \in W^\ast$, choose a simple $W$-module $V_{\chi,j} \subset \mathcal{H}^{\chi}(\mathfrak{h}_j^\ast, \mathfrak{c}_j^\ast)$, $V_{\chi,j} \cong V_{\chi}$. Write $V_{\chi,j} = V_{\chi,j}^{\varepsilon_{1,j}} \oplus E_{\chi,j}$ with $E_{\chi,j}$ stable under $W(\mathfrak{h}_j)$. Let $\{v^k_{\chi,j}\}_{1 \leq k \leq d(\chi)}$ be a basis of $V_{\chi,j}$ such that

$$V_{\chi,j}^{\varepsilon_{1,j}} = \bigoplus_{k=1}^{d(\chi)} \mathbb{C} v^k_{\chi,j}, \quad E_{\chi,j} = \bigoplus_{k=n(\chi)+1}^{d(\chi)} \mathbb{C} v^k_{\chi,j}$$

(hence $n(\chi) = \dim V_{\chi,j}^{\varepsilon_{1,j}}$).

**Lemma 3.2.** The $\mathcal{D}(\mathfrak{h}_j, \mathfrak{c}_j)^{W_j}$-module $S(\mathfrak{h}_j^\ast)^{\varepsilon_{1,j}}$ decomposes as

$$S(\mathfrak{h}_j^\ast)^{\varepsilon_{1,j}} = \bigoplus_{\chi \in W^\ast} n(\chi) \bigoplus_{k=1}^{d(\chi)} \mathcal{D}(\mathfrak{h}_j, \mathfrak{c}_j)^{W_j} v^k_{\chi,j}$$

with $\mathcal{D}(\mathfrak{h}_j, \mathfrak{c}_j)^{W_j} v^k_{\chi,j} \cong V_{\chi}$.  

**Proof.** Clearly, we can drop the index $j$ and write $\mathfrak{h}_0 = \mathfrak{h}_j$, $\mathfrak{h} = \mathfrak{h}_j, \mathfrak{c}_j, \mathfrak{c}_j = v^k_{\chi,j}$ etc. Since $\mathcal{D}(\mathfrak{h})^W v^k_{\chi} \subset S(\mathfrak{h}^\ast)^{\varepsilon_{1}}$ for $1 \leq k \leq n(\chi) = \dim V_{\chi}^{\varepsilon_{1}}$, one has

$$S(\mathfrak{h}^\ast)^{\varepsilon_{1}} \supset \bigoplus_{\chi \in W^\ast} n(\chi) \bigoplus_{k=1}^{d(\chi)} \mathcal{D}(\mathfrak{h})^W v^k_{\chi}.$$  

Recall from [2] that $S(\mathfrak{h}^\ast) = \bigoplus_{\chi} S(\mathfrak{h}^\ast)[\chi]$ with $S(\mathfrak{h}^\ast)[\chi] = \bigoplus_{k=1}^{d(\chi)} \mathcal{D}(\mathfrak{h})^W v^k_{\chi}$. Write $S(\mathfrak{h}^\ast)[\chi] = E_1 \oplus E_2$, where $E_1 = \bigoplus_{k=1}^{n(\chi)} \mathcal{D}(\mathfrak{h})^W v^k_{\chi}$ and $E_2 = \bigoplus_{k=n(\chi)+1}^{d(\chi)} \mathcal{D}(\mathfrak{h})^W v^k_{\chi}$. Notice that $E_1, E_2$ are stable under $W(\mathfrak{h}_0)$ and that we have $S(\mathfrak{h}^\ast)[\chi]^{\varepsilon_{1}} = E_1 \oplus E_2^{\varepsilon_{1}}$.

We now show that $E_2^{\varepsilon_{1}} = 0$. This will prove that

$$S(\mathfrak{h}^\ast)^{\varepsilon_{1}} = \bigoplus_{\chi \in W^\ast} n(\chi) \bigoplus_{k=1}^{d(\chi)} \mathcal{D}(\mathfrak{h})^W v^k_{\chi}.$$  

Let $D \in \mathcal{D}(\mathfrak{h})^W$ and $v \in V_{\chi}$. Notice first that if $D.v \neq 0$, the operator $D$ yields an isomorphism of $W$-modules $V_{\chi} \cong D.V_{\chi}$. Therefore, if $V_{\chi} = \bigoplus_k S_k$ with an
$S_k$ irreducible $W(\mathfrak{h}_0)$-module, we get that $D.V_\chi = \bigoplus_k D.S_k$, $D.S_k \cong S_k$. It follows that if $v \in E_\chi$ (the $W(\mathfrak{h}_0)$-stable complement of $V_\chi^{\varepsilon_I}$), then $D.v \in D.E_\chi$ with $D.E_\chi \cap S(h_j^*)^{\varepsilon_I} = 0$. Let $p = \sum_{k=n(\chi)+1}^{d(\chi)} D_k v^k_\chi \in E_2$. Then, $CW(\mathfrak{h}_0).p \subset \sum_{k>n(\chi)} CW(\mathfrak{h}_0).\langle D_k.v^k_\chi \rangle$ and, by the previous remarks, $(CW(\mathfrak{h}_0).\langle D_k.v^k_\chi \rangle)^{\varepsilon_I} = 0$. Thus $(CW(\mathfrak{h}_0).p)^{\varepsilon_I} = 0$, which shows that $E_2^{\varepsilon_I} = 0$.

Recall the following result:

**Proposition 3.3** ([5] Proposition 6.1.1)], (1) The linear map

$$T: \bigoplus_{j=1}^r S(h^*_j,\mathcal{C})^{\varepsilon_I-j} \longrightarrow \text{Db}(\mathfrak{g}_0)^G_{S_k^+}, \quad T(p_1, \ldots, p_r) = \sum_{j=1}^r T_{p_j, \Gamma_j}$$

is an isomorphism of vector spaces.

(2) The map $T$ induces an isomorphism:

$$\bigoplus_{j=1}^r \mathcal{H}(h^*_j,\mathcal{C})^{\varepsilon_I-j} \cong \{T \in \text{Db}(\mathfrak{g}_0)^G_{S_k^+} : S_+(\mathfrak{g})^G.T = 0\}.$$

**Proof.** (2) follows from the proof of [5] Proposition 6.1.1].

**Theorem 3.4.** Set $T(h_j) = \sum_{p \in S(h^*_j,\mathcal{C})^{\varepsilon_I-j}} \mathcal{C}T_p, \Gamma_j$. Then we have the following decomposition of $D(\mathfrak{g})^G$-modules:

$$\text{Db}(\mathfrak{g}_0)^G_{S_k^+} = \bigoplus_{j=1}^r T(h_j)$$

with

$$T(h_j) = \bigoplus_{\chi \in \mathcal{W}} \bigoplus_{k=1}^{n_j(\chi)} D(\mathfrak{g})^G.T^{\varepsilon_I-j}_\chi, \Gamma_j$$

and $D(\mathfrak{g})^G.T^{\varepsilon_I-j}_\chi, \Gamma_j \cong N^G_\chi$.

**Proof.** The decomposition of $T(h_j)$, as a $D(\mathfrak{g})^G$-module, is consequence of Theorem 3.3 Lemma 322 (using the isomorphism $\delta : A(\mathfrak{g}) \cong D(h_j,\mathcal{C}^W)$) and Proposition 3.3. The decomposition of $\text{Db}(\mathfrak{g}_0)^G_{S_k^+}$ follows from Proposition 3.3.

Using the Fourier transform, we obtain the following:

**Corollary 3.5.** The $D(\mathfrak{g})^G$-module $\text{Db}(\mathfrak{g}_0)^G_{nil}$ decomposes as

$$\text{Db}(\mathfrak{g}_0)^G_{nil} = \bigoplus_{j=1}^r \bigoplus_{\chi \in \mathcal{W}} D(\mathfrak{g})^G.\Theta_{F_{\mathfrak{h}^{-1}}(v^k_\chi, \Gamma_j)} \cong N^G_\chi.$$  

The next corollary follows from Theorem 3.4 and Corollary 3.5.

**Corollary 3.6.** We have

$$\text{Db}(\mathfrak{g}_0)^G_{S_k^+} \cong \bigoplus_{\chi \in \mathcal{W}} m_\chi N^G_\chi, \quad \text{Db}(\mathfrak{g}_0)^G_{nil} \cong \bigoplus_{\chi \in \mathcal{W}} m_\chi N^G_\chi$$

where $m_\chi = \sum_{j=1}^r \dim V_\chi^{\varepsilon_I-j}$.
Remark 3.7. Let $\chi \in W^\land$. It is not always possible to “realize” the modules $N_\chi$ and $M_\chi$ as $\mathcal{D}(g).T$ for some $T \in \text{Db}(g_0)$, where $g_0$ is a real form of $g$. By the previous results, this statement is equivalent to the existence of a Cartan subalgebra $h_j \subset g_0$ such that $V^\chi_{i,j} \neq 0$. D. Renard has observed that, using the results of W. Rossmann [15], this can be translated to a question about centralizers of nilpotent elements. Fix a real form $g_\mathbb{R}$ of $g$ with adjoint group $G_\mathbb{R}$. If $x \in g_\mathbb{R}$ is nilpotent one defines a subgroup of the component group $A(G.x)$ (see [14] for notation) by

$$A(G_\mathbb{R}.x) = G_\mathbb{R}^x / G_\mathbb{R}^x \cap (G^x)^0.$$  

Recall that $\chi \in W^\land$ can be written $\sigma(O, \psi)$ via the Springer correspondence, where $O \subset g$ is a nilpotent orbit and $\psi : A(O) \rightarrow GL(E)$ is an irreducible representation. Then, by [15] Corollary 3.2 & Theorem 3.3, there exists a Cartan subalgebra $h_0 \subset g_\mathbb{R}$ such that $V^\chi_{i,j} \neq 0$ if, and only if, there exists a nilpotent element $x \in g_\mathbb{R}$ such that $O = G.x$ and $E(A(G_\mathbb{R}.x)) \neq 0$.

Let $g = \mathfrak{sp}(\ell, \mathbb{C})$ and let $\phi \in W^\land$ be the long sign character, i.e. $V_\phi = \mathbb{C}\pi_\ell$ where $\pi_\ell$ is the product of the long roots. Then, see [6], [13.3], $\phi = \sigma(O, \psi)$ where $O = G.x$ is the subregular nilpotent orbit with partition $[2\ell - 2, 2]$ and $\psi$ is the non-trivial character of $A(O) \equiv \{ \pm 1 \}$. The real forms of $g$ are $\mathfrak{sp}(\ell, \mathbb{R})$ and the $\mathfrak{sp}(p, q)$, $p+q = \ell$. Assume now that $\ell \geq 3$. By the classification of nilpotent orbits in $\mathfrak{sp}(p, q)$, see [7], Theorem 9.2.5, we know that $O \cap \mathfrak{sp}(p, q) = \emptyset$. Hence, by Rossmann’s results, $V^\chi_{i,j} \neq 0$ for each Cartan subalgebra $h_j \subset \mathfrak{sp}(p, q)$. On the other hand, if $G_\mathbb{R}$ is the adjoint group of $\mathfrak{sp}(\ell, \mathbb{R})$, one can show that $A(G_\mathbb{R}.x) = A(G.x)$. Thus, with the above notation, $E(A(G_\mathbb{R}.x)) = 0$ and it follows that $V^\chi_{i,j} = 0$ for each Cartan subalgebra $h_j \subset \mathfrak{sp}(\ell, \mathbb{R})$. For instance, when $g = \mathfrak{sp}(3, \mathbb{R})$ there are six conjugacy classes of Cartan subalgebras and one can directly verify (without using [15]) that $V^\chi_{i,j} = 0$ for $j = 1, \ldots, 6$. We thank D. Renard for showing the computation to us.

Let $x \in N(g_0)$ and denote by $\beta_x$ the Liouville (Kostant-Kirillov) measure on $G_0.x$. By [14] one can define $\Theta_x \in \text{Db}(g_0)^{G_0}_{\text{nil}}$ by $\langle \Theta_x, f \rangle = \int_{G_0.x} fd\beta_x$ for all $f \in C_c^\infty(g_0)$. Set $O = G.x$. Then, see [9], [10] or [18]; $\Theta_x$ is homogeneous of degree $\lambda_O = 1/2 \dim O - \dim g$ and satisfies

$$\mathcal{D}(g).\Theta_x \cong M^\chi_{\lambda_O}$$

for some $\chi_O \in W^\land$ such that $\lambda_O = \nu - n - b(\chi_O)$.

Corollary 3.8. There exists $j \in \{1, \ldots, r\}$ and $u \in F_{h}^{-1}(V^\chi_{xO,j})^{\varepsilon_{i,j}}$ such that

$$\mathcal{D}(g)^G_0.\Theta_x \cong \mathcal{D}(g)^G.x_{\alpha_j,r_j}.$$  

Proof. Since $\mathcal{D}(g)^G_0.\Theta_x \cong M^G_{\chi_O}$ is a simple submodule of $\text{Db}(g_0)^{G_0}_{\text{nil}}$, the claim follows from Corollary 3.5.

Remark 3.9. It is proved in [11], see also [12], that $\Theta_x$ can be written as $\sum_{j=1}^r \Theta_{\alpha_j,r_j}$ with $\alpha_j \in H^0(\chi_O)(h_j,c)^{\varepsilon_{i,j}}$. It is easily seen that we may assume $\mathbb{C}W.a_j \cong V^\chi_{xO}$ for all $j$ such that $a_j \neq 0$. W. Rossmann [15] has given conditions to ensure that $\Theta_x = \Theta_{\alpha_j,r_j}$ for some $j$. 

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4. Example: The complex case

We assume in this section that \( g_0 = g_1^\mathbb{C} \) is a complex semisimple Lie algebra, \( g_1 \), viewed as a real Lie algebra. Then, \( g \) can be identified with \( g_1 \times g_1 \) and \( g_0 \) with the diagonal \( \{ (a,a) \in g_1 \times g_1 \} \). Let \( h_1 \) be a Cartan subalgebra of \( g_1 \). Recall the following well-known facts, see \cite{17} or \cite{18}:

- \( h_0 = \{ (a,a) : a \in h_1 \} \) is a Cartan subalgebra of \( h_0 \) and \( h = h_0 \otimes \mathbb{R} \mathbb{C} = h_1 \times h_1 \);
- \( W(g,h) = W_1 \times W_1 \), where \( W_1 = W(g_1,h_1) \), and \( W(h_0) = \{ (w,w) \in W \} \) is isomorphic to \( W_1 \);
- there is a unique conjugacy class \([h_0]\) of Cartan subalgebras and \( h_0' \) is connected;
- the roots in \( \Delta(g,h) \) are complex and, therefore, \( \varepsilon_I = 1 \);
- the irreducible representations of \( W \) are of the form \( \chi = \phi \otimes \mu, \phi, \mu \in W_1^* \);
- one has \( \phi = \phi^* \) for all \( \phi \in W_1^* \), where \( \phi^* \) is the dual representation.

Observe that \( D(g) = D(g_1) \boxtimes D(g_1) \) and \( D(g)^G = D(g_1)^{G_1} \boxtimes D(g_1)^{G_1} \).

**Lemma 4.1.** Let \( \chi \in W^\infty \). Then, the simple \( D(g) \)-module \( M_\chi \) is of the form \( \bigoplus \phi \in W_1^* M_\phi \otimes M_\mu \) for some \( \phi, \mu \in W_1^* \).

**Proof.** The claim follows easily from the definition of the category \( \mathsf{C}(M) \) and the decomposition of the \( W \)-module \( S(h^\ast) = S(h_1^\ast) \boxtimes S(h_1^\ast) \).

**Corollary 4.2.** (\cite{18} Theorem 6.11) We have

\[
\mathsf{Db}(g_0)_{nil}^G \cong \bigoplus_{\phi \in W_1^*} M_\phi \otimes M_\phi^* 
\]

as a \( D(g)^G \)-module.

**Proof.** Let \( \chi = \phi \otimes \mu \in W^\infty \). Then, \( V_\chi = (V_\phi \boxtimes V_\mu)^{W_1} \neq 0 \) if, and only if, \( \phi = \mu \) and therefore \( n(\chi) = 1 \). The assertion now follows from Corollary \text{3.3} \:

Recall the following general results from \cite{18}. Since the module \( M_\chi \) is irreducible and \( G \)-equivariant, its support is the closure of a nilpotent orbit \( O = G.x \). Furthermore, if \( i : O \hookrightarrow g \) is the inclusion, \( M_\chi \) is uniquely determined by its \((D\text{-module})\) inverse image \( L_\chi := i^!M_\chi \). The \( D_O \)-module \( L_\chi \) is an irreducible integrable connection associated to an irreducible representation \( \psi \) of the component group \( A(O) := G^\ast/(G^\ast)^0 \) (where \( (G^\ast)^0 \) is the connected component of the centralizer \( G^\ast \)). Therefore, since \( \chi \) is uniquely determined by \( O \) and \( \psi \), we set \( \chi = \sigma(O, \psi) \).

In our situation, i.e., in the complex case, we have \( O = O_1^1 \times O_2^1 \) with \( O_1^1 \) nilpotent orbits in \( g_1 \) for \( j = 1,2 \). Then, \( \chi = \sigma(O_1 \otimes O_2, \psi) = \phi_1 \otimes \phi_2 \), \( L_\chi = L_{\phi_1} \boxtimes L_{\phi_2} \), \( \phi_j = \sigma(O_1^j, \psi_j) \), \( \psi = \psi_1 \boxtimes \psi_2 \). Note that \( b(\chi) = b(\phi_1) + b(\phi_2) \) and \( \lambda_O = \lambda_{O_1} + \lambda_{O_2} \).

Let \( x \in N(g_0) \); set \( x = (x_1, x_1) \), \( x_1 \in N(g_1) \), \( O_1 = G_1.x_1 \), \( O = G.x = O_1 \times O_1 \). The inclusion \( i : O \hookrightarrow g \) is equal to \( i_1 \times i_1 \), where \( i_1 : O_1 \hookrightarrow g_1 \). By (3.1) and Corollary (4.2) there exist \( \chi \in W^\infty \), \( \chi_1 \in W_1^* \) such that \( \chi = \chi_1 \boxtimes \chi_1 \) and \( D(g) \Theta_x \cong M_\chi_1 \boxtimes M_\chi_1 \).

It is known (Harish-Chandra) that \( \Theta_x = \Theta_{u,h_0} \) for some \( u \in S(h_1) \boxtimes S(h_1) \). The following result has been proved by various authors; see \cite{2, 3} (when \( O_1 \) is “special”), \cite{8, 9, 10}.

**Theorem 4.3.** One has \( \chi_1 = \sigma(O_1, \text{triv}) \), and there exists \( p \in (V_{\chi_1} \boxtimes V_{\chi_1})^{W_1} \) such that \( \Theta_x = \Theta_{F_p(h_0), h_0'} \).
Proof. Recall from [9] or [10] that \( \chi = \chi_1 \otimes \chi_1 = \sigma(O, \text{triv}) \). This means that
\[
L_{\chi} = L_{\chi_1} \boxtimes L_{\chi_1} = O_{O_1} \boxtimes O_{O_1}
\]
(where we denote by \( O_{X} \) the structural sheaf of an algebraic variety \( X \)). This yields \( L_{\chi_1} = O_{O_1} \) and \( \chi_1 = \sigma(O_1, \text{triv}) \).

Set \( T_x = \hat{\Theta}_x \); then \( D(g).T_x = N_{X_1} \otimes N_{X_1} \) (see Lemma [4]). Since \( S_+^e(g^e)^G.\Theta_x = 0 \) we have \( S_+(g^e)^G.T_x = 0 \). It follows from Proposition [3,2] that we can write \( T_x = T_{p,b_1} \) for some \( p \in (H(b_1^e) \otimes H(b_1^e))^W \), or, equivalently, \( \Theta_x = \Theta_{F_p(b_1,b_1)} \). Now, by Theorem [24] \( D(h)^W.p = V_{\chi_1} \otimes V_{\chi_1} \) and therefore \( CW.p \cong V_{\chi_1} \otimes V_{\chi_1} \). Moreover, \( T_x = T_{p,b_1} \) is homogeneous of degree \( b(\chi_1) - 2\nu = 2b(\chi_1) - 2\nu = \deg p - 2\nu \). Thus \( \deg p = 2b(\chi_1) \) and, by definition of \( V_{\chi_1} \), \( p \in (V_{\chi_1} \otimes V_{\chi_1})^W \). \( \square \)

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