RANDOM VARIABLE DILATION EQUATION AND MULTIDIMENSIONAL PRESCALE FUNCTIONS

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ABSTRACT. A random variable $Z$ satisfying the random variable dilation equation $MZ = Z + G$, where $G$ is a discrete random variable independent of $Z$ with values in a lattice $\Gamma \subseteq \mathbb{R}^d$ and weights $\{c_k\}_{k \in \Gamma}$ and $M$ is an expanding and $\Gamma$-preserving matrix, if absolutely continuous with respect to Lebesgue measure, will have a density $\varphi$ which will satisfy a dilation equation

$$\varphi(x) = |\det M| \sum_{k \in \Gamma} c_k \varphi(Mx - k).$$

We have obtained necessary and sufficient conditions for the existence of the density $\varphi$ and a simple sufficient condition for $\varphi$’s existence in terms of the weights $\{c_k\}_{k \in \Gamma}$. Wavelets in $\mathbb{R}^d$ can be generated in several ways. One is through a multiresolution analysis of $L^2(\mathbb{R}^d)$ generated by a compactly supported prescale function $\varphi$. The prescale function will satisfy a dilation equation and its lattice translates will form a Riesz basis for the closed linear span of the translates. The sufficient condition for the existence of $\varphi$ allows a tractable method for designing candidates for multidimensional prescale functions, which includes the case of multidimensional splines. We also show that this sufficient condition is necessary in the case when $\varphi$ is a prescale function.

1. INTRODUCTION

Multiresolution analysis on $\mathbb{R}^d$ is one possible framework for construction of wavelet bases. Let $\Gamma$ be a lattice in $\mathbb{R}^d$ and let $M : \mathbb{R}^d \to \mathbb{R}^d$ be an expansive linear transformation, that is, all eigenvalues of $M$ have modulus greater than 1, such that $\Gamma M \subseteq \Gamma$. Then $m = |\det M|$ is an integer, greater than one, equal to the order of the group $\Gamma / \Gamma M$. A multiresolution analysis associated to $\Gamma$ and $M$ with prescale function $\varphi$ is an increasing sequence of subspaces of $L^2(\mathbb{R}^d)$, $V_0 \subseteq V_1 \subseteq \cdots$ satisfying the following four conditions:

(i) $\bigcup_j V_j$ is dense in $L^2(\mathbb{R}^d)$;

(ii) $\bigcap_j V_j = \{0\}$;

(iii) $f(\cdot) \in V_j \iff f(M^{-j}(\cdot)) \in V_0$;

(iv) $\{\varphi(\cdot - \gamma)\}_{\gamma \in \Gamma}$ is a Riesz basis for $V_0$.

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A wavelet basis associated to the multiresolution analysis is an orthonormal basis for $L^2(\mathbb{R}^d)$ of the form \[ \{m^{j/2}\psi_k(M^j \cdot -\gamma): j \in \mathbb{Z}, \gamma \in \Gamma, 1 \leq k \leq m\} \] where \[ \psi_k(x) = \sum_{\gamma \in \Gamma} a_k(\gamma) \varphi(Mx - \gamma) \] and \{a_k(\gamma)\}_{\gamma \in \Gamma} is square summable for $1 \leq k \leq m$. The functions \{\psi_k\}_{k=1}^m are called the wavelet generators. When the lattice translates of $\varphi$ form an orthonormal basis of $V_0$ we take $\psi_1 := \varphi$.

Conditions (iii) and (iv) together imply that the set \(\{\varphi(M \cdot -\gamma)\}_{\gamma \in \Gamma}\) is a Riesz basis for the subspace $V_1$. Since $\varphi \in V_0 \subseteq V_1$, we can write

\begin{equation}
\varphi(x) = \sum_{\gamma \in \Gamma} a(\gamma) \varphi(Mx - \gamma);
\end{equation}

equation (1.1) is called a dilation equation.

One way to understand (1.1) is through a probabilistic approach. Consider a discrete random variable $G$ with values in a subset $\Gamma_1$ of $\Gamma$ and a random variable $Z$, independent of $G$, with values in $\mathbb{R}^d$, both defined on a complete probability space $(\Omega, \mathcal{F}, P)$, which satisfy

\begin{equation}
MZ \overset{d}{=} Z + G.
\end{equation}

Here, $\overset{d}{=} \text{ denotes equality of the corresponding laws. Assume that } Z \text{ is absolutely continuous with respect to Lebesgue measure and denote its density by } \varphi. \text{ Equation (1.2) implies that } \varphi \text{ satisfies the dilation equation (1.1) with } a(\gamma) = |\det M| P(G = \gamma). \text{ Our approach to constructing candidates for prescale functions comes from understanding the structure of the solution of this random variable dilation equation.}"

In the one-dimensional case with $M = 2$, Gundy and Zhang [6] proved that $Z$ is absolutely continuous with respect to Lebesgue measure if and only if the fractional part of $Z$ is uniform. They also gave a sufficient condition for the uniformity of the fractional part. In the higher dimensional case, we show that the statements of Gundy and Zhang hold true when a proper notion of the “fractional” part of a random variable is introduced. We have found the theory of self-affine tilings of $\mathbb{R}^d$ and use of the digit representation of the fractional part of $Z$ to be the correct framework for the higher dimensional case. The major difficulty in generalizing the results to higher dimensions comes from the fact that $M$ may not be merely an expansion but may include a rotation. Such an $M$ causes a tile to have, in general, a fractal boundary. The boundary difficulties called for some new techniques of proofs beyond those used in [6].

In Section 2 we introduce notation needed to express an explicit solution $Z$ to (1.2). Definitions of the “fractional” and “integer” parts of an $\mathbb{R}^d$-valued random variable $Z$ are given based on concepts of self-affine tilings. We also give some basic results regarding the fractional part of $Z$. In Section 3 we give necessary and sufficient conditions under which the random variable $Z$ will have a density, in terms of the fractional part of $Z$. In Section 4 we give a simple sufficient condition on the weights on the values of $G$ which guarantee absolute continuity of $Z$. In Section 5 we give examples of density functions obtained using these results. In Section 6 we show that the sufficient condition of Section 4 is also necessary when $\varphi$ is a prescale function.
2. Basic properties of a random variable
dilation equation solution

In order to write an explicit solution of (1.2), some definitions are needed. Let
\( G_1, G_2, \ldots \) be an i.i.d. sequence of random variables defined on the space \((\Omega, \mathcal{F}, P)\),
with \( G_1 \overset{d}{=} G \). Recall that \( G \) is discrete with values in the lattice. Assume
\[
\sum_{j=1}^{\infty} M^{-j} G_j < \infty \text{ a.s.}
\]
Then the sequence \( \{Z_k\} \) defined by
\[
Z_k = \sum_{j=1}^{\infty} M^{-j} G_{j+k} \text{ for } k = 1, 2, \ldots
\]
is a sequence of random variables. Note that the following two properties hold:
\[
MZ_k = M(\sum_{j=1}^{\infty} M^{-j} G_{j+k}) = G_{k+1} + \sum_{j=2}^{\infty} M^{-j+1} G_{j+k} = G_{k+1} + Z_{k+1},
\]
and
\[
Z_0 \overset{d}{=} Z_k, \text{ and } G_k \text{ is independent of } Z_k.
\]
Therefore for any \( k \), \( Z_k \) solves the dilation equation (1.2).

The fractional part of \( Z \) will play an essential role in what follows. In order to
define the fractional part of \( Z \), we first invoke some basic facts about self-affine
 tilings. Let \( \Gamma_0 \) denote a set of coset representatives of \( \Gamma/\mathbb{M} \), and without loss of
generality, we assume \( 0 \in \Gamma_0 \). A self-affine tiling of \( \mathbb{R}^d \) consists of a closed set \( T \)
with nonempty interior such that
\[
\bigcup_{\gamma \in \Gamma} (T + \gamma) = \mathbb{R}^d \text{ and } \bigcup_{\gamma \in \Gamma_0} (T + \gamma) = M T.
\]
Clearly a tiling depends on the choice of \( \Gamma_0 \). In dimensions \( d = 2 \) and \( 3 \), one can
always find a \( \Gamma_0 \) that admits a self-affine tiling, and in higher dimensions it can be
done for \( m = |\text{det } M| > d \) [10]. For the remainder of the paper, we will assume
that \( \Gamma_0 \) admits a self-affine tiling.

The lattice translates of the interior of \( T \) are disjoint and \( \text{int } T \neq \emptyset \), so if
\( x \in \bigcup_{\gamma \in \Gamma} (\text{int } T + \gamma) \), then \( x \in \text{int } T + \gamma_x \) where \( \gamma_x \) denotes the unique element of \( \Gamma \)
giving the location of the point \( x \). If \( x \notin \bigcup_{\gamma \in \Gamma} (\text{int } T + \gamma) \), then we say \( x \) is a boundary
point and note that \( x \in \bigcap_{\gamma \in \Gamma_1} (T + \gamma) \), for some finite \( \Gamma_1 \subseteq \Gamma \). The fact that \( \Gamma_1 \) is
finite follows from the compactness of \( T \).

Define \( [\cdot ] : \mathbb{R}^d \to \Gamma \) by
\[
[x] = \begin{cases} 
\gamma_x & \text{if } x \in \bigcup_{\gamma \in \Gamma} (\text{int } T + \gamma), \\
\max_{\gamma \in \Gamma_1} \gamma & \text{if } x \text{ is a boundary point,}
\end{cases}
\]
where “max” is meant in the sense of the dictionary ordering of \( \mathbb{R}^d \).

**Proposition 1.** \( [\cdot ] \) is Borel-measurable.
Proof. We only need to consider \( \{ x \mid [x] = \gamma \} \) for a fixed \( \gamma \in \Gamma \). Since \( T \) is compact and \( \Gamma \) is countable,
\[
[\gamma_1]^{-1} = (\text{int} \, T + \gamma_1) \cup \bigcup_{\gamma \in \Gamma} (T + \gamma) \cap (T + \gamma_1)
\]
is a Borel set.

For any \( x \in \mathbb{R}^d \) we will call \([x]\) the integer part of \( x \) and \((x) = x - [x]\) the fractional part of \( x \). By Proposition 1, \([Z]\) is a random variable and therefore so is \((Z) = Z - [Z]\). Notice that \((Z)\) takes values in the tile \( T \).

A point \( t \in \mathbb{R}^d \) is in \( T \) if and only if
\[
t = \sum_{j=1}^{\infty} M^{-j} \gamma_j,
\]
where for all \( j, \gamma_j \in \Gamma_0 \). Based on the expansion (2.3), define functions \( \xi_j : \Omega \to \Gamma_0, j = 1, 2, \ldots, \) by

\[
(Z_0) = \sum_{j=1}^{\infty} M^{-j} \xi_j;
\]

that is, \( \xi_j (\omega) \) is the element of \( \Gamma_0 \) which appears in the \( j \)th term of the tile expansion of \((Z_0)(\omega)\). If there is more than one expansion for a tile point, simply choose one of them.

**Proposition 2.** Assume that \( P((Z_0) \in \partial T) = 0 \). Then \( \{ \xi_j \}_{j=1}^{\infty} \) is a sequence of random variables and for each \( k \)
\[
(Z_k) = \sum_{j=1}^{\infty} M^{-j} \xi_{j+k} \quad \text{a.s.}
\]

Proof. From the dilation equation (1.2) and from the decomposition of \( Z_0 \) into its fractional and integer parts, we obtain
\[
M [Z_0] + M (Z_0) = M Z_0 = G_1 + Z_1 = G_1 + [Z_1] + (Z_1).
\]
Using (2.4) it follows that
\[
M [Z_0] + \xi_1 + \sum_{j=1}^{\infty} M^{-j} \xi_{j+1} = G_1 + [Z_1] + (Z_1).
\]
The definition of a lattice tiling implies \( (\gamma + T) \cap (\gamma' + \text{int} \, T) = \emptyset \) if and only if \( \gamma \neq \gamma' \). So, if \((Z_1) \in \text{int} \, T\), then by (2.5), we have
\[
M [Z_0] + \xi_1 = G_1 + [Z_1] \quad \text{and} \quad (Z_1) = \sum_{j=1}^{\infty} M^{-j} \xi_{j+1}.
\]
Since \( P((Z_0) \in \partial T) = 0 \) and since \( Z_1 \equiv Z_0 \), it follows that \( P((Z_1) \in \text{int} \, T) = 1 \), and therefore
\[
(Z_1) = \sum_{j=1}^{\infty} M^{-j} \xi_{j+1} \quad \text{a.s.}
\]
By (2.6) \( \xi_1 = G_1 + [Z_1] - M [Z_0] \) almost surely and so \( \xi_1 \) is a random variable.

The proof is completed by induction on \( k \). \qed
Define $h : \Gamma \to \Gamma_0$ to be the map which assigns to each element of $\Gamma$ its coset representative.

**Proposition 3.** Suppose $P \left((Z_0) \in \partial T\right) = 0$. Then for
\[ k = 1, 2, \ldots, \quad \xi_k = h([Z_k] + G_k) \text{ a.s.} \]

**Proof.** $P \left((Z_0) \in \partial T\right) = 0$ implies that (2.6) holds. So $\xi_1 = h(G_1 + [Z_1])$ since coset representatives are unique.

Proposition 2, the fact that $Z_k = [Z_k] + (Z_k)$, and the dilation equation (1.2) together lead to
\[ M[Z_k] + \xi_{k+1} + (Z_{k+1}) = G_{k+1} + [Z_{k+1}] + (Z_{k+1}) \text{ a.s.} \]

This implies that $M[Z_k] + \xi_{k+1} = G_{k+1} + [Z_{k+1}]$ a.s. since $P \left((Z_{k+1}) \in \partial T\right) = 0$. The uniqueness of coset representatives ensures $\xi_{k+1} = h(G_{k+1} + [Z_{k+1}])$ a.s. □

Define $g : (\mathbb{R}^d)^{\infty} \to \mathbb{R}^d$ by
\[ g(x_1, x_2, \ldots) = x_1 + \left[ \sum_{j=1}^{\infty} M^{-j} x_{j+1} \right]. \]

The measurability of $g$ follows from Proposition 1 and from the fact that the projection map is a measurable function.

**Proposition 4.** Let $Y_k := (h \circ g)(G_k, G_{k+1}, \ldots)$. Then $Y_1, Y_2, \ldots$ is a stationary and ergodic sequence of random variables.

**Proof.** The proof follows from the fact that $h \circ g$ is measurable and $\{G_k\}_{k=1}^{\infty}$ is i.i.d. □

**Corollary 1.** If $P \left((Z_0) \in \partial T\right) = 0$, the sequence $\xi_1, \xi_2, \ldots$ is stationary and ergodic.

**Proof.** If $P \left((Z_0) \in \partial T\right) = 0$, Proposition 4 implies $\xi_k = Y_k$ a.s. □

### 3. Necessary and Sufficient Conditions for Absolute Continuity of $Z$

Throughout this section let $\lambda_T := \frac{\lambda}{\lambda(T)}$ denote Lebesgue measure normalized by the measure of the tile $T$ (if $\Gamma = \mathbb{Z}^d$, then $\lambda(T) = 1$).

**Theorem 1.** Let $M, \Gamma, \Gamma_0$ and random variables $G, Z$ and $\xi_k$ be as defined in the previous sections. Suppose $G$ has values in a finite set $\Gamma_1$ such that $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma$.

Then the following are equivalent:
1) The law of $(Z)$ is $\lambda_T$ on $T$;
2) The $\xi_k$ are independent and uniformly distributed on $\Gamma_0$;
3) The law of $Z$ is absolutely continuous with respect to $\lambda$.

**Proof.** (1$\Rightarrow$3) Since $G$ is bounded, so is $Z$, and therefore $[Z]$ takes on only finitely many values. Let $\Gamma_2$ be the range of $[Z]$. One solution of equation (1.2) is $Z \overset{d}{=} \sum_{k=1}^{\infty} M^{-k} G_k$. Jessen and Wintner's theorem [8] implies that the law of $Z$ must be either purely discrete, purely singular, or purely absolutely continuous. We will rule out the discrete and singular cases.
First, suppose $Z$ is purely discrete. Then $P(Z = z) > 0$ for some $z$. Now,

$$0 < P(Z = z) = P([Z] + (Z) = z) = P([Z] = z) P([Z] = \gamma) P([Z] = \gamma)$$

implies that there exists a $\gamma \in \Gamma_1$ such that

$$P((Z) = z - \gamma \mid [Z] = \gamma) P([Z] = \gamma) > 0,$$

contradicting the assumption that $(Z)$ is uniform.

Second, suppose $Z$ is purely singular with respect to Lebesgue measure. Then there exists $B$ such that $P(Z \in B) = 1$ and $\lambda_T(B) = 0$. So

$$P([Z] + (Z) \in B) = \sum_{\gamma \in \Gamma_2} P([Z] + (Z) \in B \mid [Z] = \gamma) P([Z] = \gamma) = 1,$$

which implies that there exists a $\gamma \in \Gamma_2$ such that

$$P((Z) \in B - \gamma \mid [Z] = \gamma) P([Z] = \gamma) \geq \frac{1}{|\Gamma_2|}.$$

But under the assumption that $(Z)$ is uniform, $P((Z) \in B - \gamma) = \lambda_T(B - \gamma) = \lambda_T(B) = 0$, a contradiction.

Next, (2) $\Rightarrow$ 1). This proof will be broken into three main steps:

(i) assumption 2) implies $P((Z_0) \in \partial T) = 0$;
(ii) $\nu := \ell (Z)$ and $\lambda_T$ agree on sets of the type $M^{-k}T + M^{-k}\gamma, \gamma \in \Gamma$;
(iii) $\nu$ and $\lambda_T$ agree on all closed balls.

**Remark.** The first step is trivial in one dimension. For example, if $M = 2, \Gamma = \mathbf{Z}$ and $\Gamma_0 = \{0, 1\}$, then $T = [0, 1]$ and

$$P \left( \sum_{k=1}^{\infty} 2^{-k} \xi_k \in \partial T \right) = P(\xi_k = 0 \text{ for all } k \text{ or } \xi_k = 1 \text{ for all } k) = 0.$$

i) For each $n = 0, 1, 2, \ldots$ let

$$W_n = \sum_{k=1}^{\infty} M^{-k} \xi_{k+n}.$$

Notice that the range of $W_n$ is in $T$ and since the sequence $\{\xi_k\}_{k=1}^{\infty}$ is i.i.d., $W_n \overset{d}{=} W_0, n = 1, 2, \ldots$.

**Claim.** $P(W_0 \in \text{int } T) > 0$.

**Proof.** Since $\text{int } T \neq \emptyset$, let $B(x; r) \subset \text{int } T$ be an open ball centered at $x$ with radius $r$. Then $x = \sum_{i=1}^{\infty} M^{-i} \gamma_i(x)$, where $\gamma_i(x) \in \Gamma_0$ for all $i$ [5]. Choose $k$ large enough so that

$$\sum_{i=k}^{\infty} \|M^{-i}\| \max \{\|\gamma\| \mid \gamma \in \Gamma_0\} < \frac{r}{2}.$$

Let $y = \sum_{i=1}^{k-1} M^{-i} \gamma_i(x)$. Note that $y \in B \left( x; \frac{r}{2} \right)$. Let

$$S = \{ t \in T \mid \gamma_i(t) = \gamma_i(x) \text{ for } i = 1, 2, \ldots, k - 1 \}.$$
Therefore, (3.2) becomes
\[ P(W_0 \in S) = P(\xi_1 = \gamma_1(x), \ldots, \xi_{k-1} = \gamma_{k-1}(x)) = \frac{1}{m^{k-1}}. \]
So \( P(W_0 \in S) > 0 \), which together with \( S \subset \text{int } T \) implies \( P(W_0 \in \text{int } T) > 0 \).

One property of a tiling is that distinct tiles may only intersect on their boundaries. If we set \( \Gamma_\delta = \{ \gamma \in \Gamma \setminus \{0\} \mid T \cap (T + \gamma) \neq \emptyset \} \), then
\[ \partial T = \bigcup_{\gamma \in \Gamma_\delta} (T \cap (T + \gamma)). \]

**Claim.** \( \{W_n \in \partial T\} \subseteq \{W_{n+1} \in \partial T\} \) for \( n = 0, 1, 2, \ldots \).

**Proof.** Suppose \( \omega \in \{W_0 \in \partial T\} \); that is,
\[ \sum_{k=1}^{\infty} M^{-k} \xi_k(\omega) \in \partial T. \]
Applying \( M \) to both sides and using properties of tiles yields
\[ W_1(\omega) = \sum_{k=1}^{\infty} M^{-k} \xi_{k+1}(\omega) \in \partial MT - \xi_1(\omega). \]
Set \( \gamma_1 := \xi_1(\omega) \). By the self-affine property of the tiling, \( \partial MT \subseteq \bigcup_{\gamma \in \Gamma_0} (\gamma + \partial T) \).

Therefore, (3.2) becomes
\[ W_1(\omega) \in \bigcup_{\gamma \in \Gamma_0} ((\gamma - \gamma_1) + \partial T), \]
implying that for at least one \( \gamma \in \Gamma_0 \), \( W_1(\omega) \in (\gamma - \gamma_1) + \partial T \). So
\[ W_1(\omega) \in ((\gamma - \gamma_1) + \partial T) \cap T. \]
If \( \gamma = \gamma_1 \), then \( W_1(\omega) \in \partial T \); if \( \gamma \neq \gamma_1 \), then \( \text{int } T \cap (\gamma - \gamma_1 + \text{int } T) = \emptyset \), so \( W_1(\omega) \in \partial T \). We have shown that \( \{W_0 \in \partial T\} \subseteq \{W_1 \in \partial T\} \). By the same argument, \( \{W_n \in \partial T\} \subseteq \{W_{n+1} \in \partial T\} \) for each \( n \).

**Claim.** \( P(W_0 \in \partial T) = 0 \).

Suppose not. Set \( B_k = \{W_k \in \partial T\} \) and \( B = \bigcup_{k=0}^{\infty} B_k \). Notice that since the \( B_k \) are nested, \( B \in \bigcap_{n=1}^{\infty} \sigma(\xi_n, \xi_{n+1}, \ldots) \). By the Kolmogorov 0-1 law for independent random variables \( P(B) = 1 \), because \( \{W_0 \in \partial T\} \subseteq B \) and \( P(W_0 \in \partial T) > 0 \).

Furthermore,
\[ 1 = P(B) = \lim_{k \to \infty} P(W_k \in \partial T) = P(W_0 \in \partial T), \]
with the last equality following from the fact that the sequence \( \xi_1, \xi_2, \ldots \) is i.i.d. But this is a contradiction of the fact that \( P(W_0 \in \text{int } T) > 0 \). So \( P(W_0 \in \partial T) = 0 \).

Since \( W_0 = (Z_0) \) almost surely we have shown that \( P((Z_0) \in \partial T) = 0 \), concluding the first step.

(ii) To begin the second step of the proof, fix \( \gamma \in \Gamma \) and \( k \in \mathbb{N} \). Then
\[ \lambda(M^{-k}T + M^{-k}\gamma) = \lambda(M^{-k}T) = \frac{\lambda(T)}{m^k}. \]
By Proposition 2 and (i) $(Z_k) = \sum_{i=1}^{\infty} M^{-i} \xi_{i+k}$ a.s. Now,

$$P \left( (Z_0) \in M^{-k}T + M^{-k}\gamma \right) = P \left( M^k (Z_0) \in T + \gamma \right) = P \left( \sum_{i=1}^{\infty} M^{-i} \xi_i \in T + \gamma \right) = P \left( \sum_{j=1-k}^{0} M^{-j} \xi_{j+k} + \sum_{j=1}^{\infty} M^{-j} \xi_{j+k} \in T + \gamma \right) = P \left( L(k) + (Z_k) \in T + \gamma \right),$$

where $L(k) := \sum_{j=1-k}^{0} M^{-j} \xi_{j+k}$. Notice that $L(k)$ is a function of finitely many $\xi_i$ and has values in the lattice; therefore,

$$P \left( L(k) + (Z_k) \in T + \gamma \right) = \sum_{\gamma'} P \left( (Z_k) \in T + \gamma - \gamma', L(k) = \gamma' \right) = P \left( (Z_k) \in T, L(k) = \gamma \right).$$

The last equality follows since all the terms in the sum are zero except when $\gamma' = \gamma$ as a consequence of $P \left( (Z_k) \in \partial T \right) = 0$. Furthermore,

$$P \left( (Z_k) \in T, L(k) = \gamma \right) = P \left( L(k) = \gamma \right)$$

(3.3) becomes

$$P \left( (Z_k) \in T, L(k) = \gamma \right) = P \left( M^{k-1} \xi_1 + \cdots + M \xi_{k-1} + \xi_k = \gamma \right) = P \left( M \left( M^{k-2} \xi_1 + \cdots + \xi_{k-1} \right) + \xi_k = \gamma \right).$$

Since each $\gamma \in \Gamma$ has a unique representation $\gamma = \gamma_0 + M\gamma''$, (3.3) becomes

$$P \left( (Z_k) \in T, L(k) = \gamma \right) = P \left( (Z_k) \in T, L(k) = \gamma \right) = P \left( \xi_k = \gamma_0, M^{k-2} \xi_1 + \cdots + \xi_{k-1} = \gamma'' \right)$$

$$= P \left( \xi_k = \gamma_0, \xi_{k-1} = \gamma_1, M^{k-3} \xi_1 + \cdots + \xi_{k-2} = \gamma'' \right)$$

$$= P \left( \xi_k = \gamma_0, \xi_{k-1} = \gamma_1, \ldots, \xi_1 = \gamma_{k-1} \right)$$

$$= \prod_{i=1}^{k} P \left( \xi_i = \gamma_{k-i} \right) = \frac{1}{m^k}.$$

So $L \left( (Z) \right)$ and $\frac{\lambda_T}{\lambda(T)}$ are equal on sets of the type $M^{-k}T + M^{-k}\gamma, \gamma \in \Gamma$ and $k \in N$.

(iii) We now show that $L \left( (Z) \right)$ and $\lambda_T$ agree on all closed balls in $\mathbb{R}^d$.

Set $\nu := L \left( (Z) \right)$, and suppose there is a closed ball $B(x, r)$ on which the measures do not agree. Assume first that $\nu \left( B(x, r) \cap T \right) > \lambda_T \left( B(x, r) \cap T \right)$. There exists $\eta > 0$, such that $\nu \left( B(x, r) \cap T \right) > \lambda_T \left( B(x, r+\eta) \cap T \right)$. Choose $k_0$ such that $\text{diam} \left( M^{-k_0}T \right) < \frac{\eta}{2}$. Set

$$D = \cup \left\{ M^{-k_0}T + M^{-k_0} \gamma | \gamma \in M^{k_0}B \left( x, r + \frac{\eta}{2} \right) \right\}.$$

Claim. $B(x, r) \subseteq D \subseteq B(x, r+\eta)$.

Proof. Let $y \in B(x, r)$. Since $\mathbb{R}^d = \cup_{\gamma \in \Gamma} \left( M^{-k_0}T + M^{-k_0} \gamma \right)$, there is a $\gamma \in \Gamma$ such that $y \in M^{-k_0}T + M^{-k_0} \gamma$. So $y = z + M^{-k_0} \gamma$, for some $z \in M^{-k_0}T$. If $z \in M^{-k_0}T$, then $\|z\| \leq \text{diam} \left( M^{-k_0}T \right)$ since $0 \in T$. Now

$$\|M^{-k_0} \gamma - x\| \leq \|y - x\| + \|z\| \leq r + \text{diam} \left( M^{-k_0}T \right) \leq r + \frac{\eta}{2},$$

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that is,
\[ M^{-k_0} \gamma \in B \left( x, r + \frac{\eta}{2} \right), \]
which means \( y \in D \).

Now suppose that \( y \in D \). Then \( y = z + M^{-k_0} \gamma \) for some \( z \in M^{-k_0} T \) and \( \gamma \in B \left( x, r + \frac{\eta}{2} \right) \), and
\[ \| y - x \| \leq \| M^{-k_0} \gamma - x \| + \| z \| \leq r + \eta; \]
so \( y \in B \left( x, r + \eta \right) \). This completes the proof of the claim.

Thus \( \lambda_T \left( B \left( x, r + \eta \right) \cap T \right) \geq \lambda_T \left( D \cap T \right) \) and \( \nu \left( D \cap T \right) \geq \nu \left( B \left( x, r \right) \cap T \right) \). If we can show that \( \lambda_T \left( D \cap T \right) = \nu \left( D \cap T \right) \), we will obtain a contradiction. To see this, recall that by the self-affine property of the tiling, we can write

\[ T = \bigcup_{\gamma \in \Gamma_{k_0}} M^{-k_0} T + M^{-k_0} \gamma, \tag{3.4} \]

where \( \Gamma_{k_0} = \Gamma_0 + M \Gamma_0 + \cdots + M^{k_0-1} \Gamma_0 \). If \( \gamma \in \Gamma_{k_0} \), then \( M^{-k_0} T + M^{-k_0} \gamma \subset T \), so \( \text{int}(M^{-k_0} T + M^{-k_0} \gamma) \subset T \). If \( \gamma \notin \Gamma_{k_0} \), then \( T \cap \text{int}(M^{-k_0} T + M^{-k_0} \gamma) = \emptyset \). If not, there is an \( x \) in \( T \cap \text{int}(M^{-k_0} T + M^{-k_0} \gamma) \). Since \( x \in T \), \( x \) is in one of the sets in the right-hand side of (3.4); that is, \( x \in M^{-k_0} T + M^{-k_0} \gamma' \), where \( \gamma' \in \Gamma_{k_0} \). So
\[ x \in (M^{-k_0} T + M^{-k_0} \gamma') \cap \text{int}(M^{-k_0} T + M^{-k_0} \gamma), \]
which implies \( M^{k_0} x \in (T + \gamma') \cap \text{int}(T + \gamma) \). This contradicts the fact that distinct translates of \( T \) are disjoint except at the boundary. So,
\[ \text{either } \text{int}(M^{-k_0} T + M^{-k_0} \gamma) \subset T \text{ or } \text{int}(M^{-k_0} T + M^{-k_0} \gamma) \subset T^c. \]
Set \( C = \Gamma_{k_0} \cap M^{k_0} B \left( x, r + \frac{\eta}{2} \right) \) and \( C' = (\Gamma \setminus \Gamma_{k_0}) \cap M^{k_0} B \left( x, r + \frac{\eta}{2} \right). \) Then
\[ D \cap T = \left( \bigcup_{\gamma \in C} M^{-k_0} T + M^{-k_0} \gamma \right) \cap T \bigcup \left( \bigcup_{\gamma \in C'} M^{-k_0} T + M^{-k_0} \gamma \right) \cap T. \]
The second intersection consists only of boundary points of \( T \). Since \( \nu(\partial T) = 0 \), then
\[ \nu(D \cap T) = \nu \left( \left( \bigcup_{\gamma \in C} M^{-k_0} T + M^{-k_0} \gamma \right) \cap T \right). \]
and $\nu(\partial (M^{-k_0}T + M^{-k_0}\gamma)) = 0$. The Lebesgue measure of $\partial T$ is zero [10], so $\lambda(\partial M^{-k_0}T) = 0$. Thus we have

$$\lambda_T (D \cap T) = \lambda_T \left( \bigcup_{\gamma \in \Gamma_{k_0}} M^{-k_0}T + M^{-k_0}\gamma \right) = \sum_{\gamma \in \Gamma_{k_0}} \lambda_T (M^{-k_0}T + M^{-k_0}\gamma) = \sum_{\gamma \in \Gamma_{k_0}} \nu(M^{-k_0}T + M^{-k_0}\gamma) = \nu \left( \bigcup_{\gamma \in \Gamma_{k_0}} M^{-k_0}T + M^{-k_0}\gamma \right) = \nu(D \cap T).$$

As mentioned above, the fact that $\lambda_T(D \cap T) = \nu(D \cap T)$ implies

$$\lambda_T(B(x, r + \eta) \cap T) \geq \nu(B(x, r) \cap T)$$

which contradicts $\nu(B(x, r) \cap T) > \lambda_T(B(x, r) \cap T)$. So we conclude that $\lambda_T \leq \nu$ on all closed balls. Repeating the proof with the roles of $\nu$ and $\lambda_T$ reversed yields that $\nu$ and $\lambda_T$ agree on all closed balls.

Hoffmann-Jörgensen proved that Radon probabilities which agree on all closed balls in $\mathbb{R}^d$ agree on all Borel sets. (Corollary 5 in [7], which completes the proof that 2) $\Rightarrow$ 1).

In order to prove 3) $\Rightarrow$ 2), we need a version of the Kakutani Dichotomy for stationary ergodic sequences.

**Lemma 1.** Let $\{\xi_k'\}_{k=1}^{\infty}$ be a stationary, ergodic sequence, such that each $\xi_k'$ is uniform with values in $\Gamma_0$. Let $\{\xi_k\}_{k=1}^{\infty}$ be a stationary, ergodic sequence, such that each $\xi_k$ has values in $\Gamma_0$, but is not uniform. Then $\mu = L(\xi_1, \xi_2, \ldots)$ and $\mu' = L(\xi'_1, \xi'_2, \ldots)$ are mutually singular.

**Proof.** Let $\mu = \mu_a + \mu_s$, where $\mu_a << \mu'$ and $\mu_s \perp \mu'$. Suppose $\mu_a(\Omega) > 0$.

Since $\mu \neq \mu'$, there must be a cylindrical set $A$ such that $\mu_a(A) \neq \mu'(A)$. (If not, then $\mu_a = \mu'$, which implies $\mu = \mu'$, contradicting the assumption that $\mu \neq \mu'$.) Let $f = 1_A$, then we get

$$\int_\Omega f(x_1, \ldots, x_n)d\mu_a(x) \neq \int_\Omega f(x_1, \ldots, x_n)d\mu'(x),$$

$$E_{\mu_a}(f) \neq E_{\mu'}(f).$$

Set $c = E_{\mu_a}(f)$ and $c' = E_{\mu'}(f)$. The fact that $\{\xi_k\}_{k=1}^{\infty}$ and $\{\xi'_k\}_{k=1}^{\infty}$ are ergodic sequences means that the shift operator is an ergodic operator for $(\Omega, S, \mu)$ and $(\Omega, S, \mu')$ respectively, where $\Omega = \Gamma_0^{\infty}$. Applying the Ergodic Theorem (with $f$) and the fact that the sequences are stationary, it follows that

1) $\frac{1}{k} \sum_{i=0}^{k-1} f(x_{1+i}, \ldots, x_{n+i}) \xrightarrow{k \to \infty} c$ a.s. $\mu_a$,

2) $\frac{1}{k} \sum_{i=0}^{k-1} f(x_{1+i}, \ldots, x_{n+i}) \xrightarrow{k \to \infty} c'$ a.s. $\mu'$.

So, 1) is true for all $\{x_i\}_{i=1}^{\infty} \in \Omega \setminus N$, where $\mu_a(N) = 0$ and 2) is true for all $\{x_i\}_{i=1}^{\infty} \in \Omega \setminus N'$, where $\mu'(N') = 0$. 
Define $M := N \cup N'$. Notice that $\mu_a(M) \leq \mu_a(N) + \mu_a(N') = \mu_a(N')$. Since $\mu_a << \mu'$ and $\mu'(N') = 0$, we have $\mu_a(N') = 0$ and so $\mu_a(M) = 0$. We have assumed that $\mu_a(\Omega) > 0$; therefore, $\mu_a(M) = 0$ implies that $\mu_a(\Omega \setminus M) > 0$; that is, $\mu_a((\Omega \setminus N) \cap (\Omega \setminus N')) > 0$, which means that there is a sequence $\{x_i\}_{i=1}^\infty \in (\Omega \setminus N) \cap (\Omega \setminus N')$ such that

$$\frac{1}{k} \sum_{i=0}^{k-1} f(x_{1+i}, \ldots, x_{n+i}) \xrightarrow{k \to \infty} c \text{ and } \frac{1}{k} \sum_{i=1}^{k-1} f(x_{1+i}, \ldots, x_{n+i}) \xrightarrow{k \to \infty} c'.$$

This is a contradiction, since $c \neq c'$. Therefore, $\mu_a = 0$ and thus, $\mu \perp \mu'$.

Now we are ready to show that 3) $\Rightarrow$ 2).

First, we note that $L(Z) << \lambda_T$ implies that $L((Z)) << \lambda_T$. To see this, observe that for $E \in \mathcal{B}(\mathbb{R}^d)$,

$$P((Z) \in E) = P(Z - [Z] \in E) = \sum_{\gamma \in T} P(Z \in E + \gamma; [Z] = \gamma)$$

$$\leq \sum_{\gamma \in T} P(Z \in E + \gamma).$$

If $\lambda_T(E) = 0$, then $\lambda_T(E + \gamma) = 0$ and so $P(Z \in E + \gamma) = 0$ for all $\gamma \in T$ by the assumption of absolute continuity of $L(Z)$. Then \[ \eqref{eq:3.5} \] implies $P((Z) \in E) = 0$. So $L((Z)) << \lambda_T$.

Since $\lambda(\partial T) = 0$, $L((Z)) << \lambda_T$ implies that $P((Z) \in \partial T) = 0$. Therefore, if we define $s : \Gamma_0^\infty \to \mathbb{R}$ by

$$s(x_1, x_2, ...) := \sum_{i=1}^\infty M^{-i}x_i,$$

If $\Gamma_0^\infty$ is equipped with the product topology, $s$ is continuous. By Proposition \[ \eqref{prop:2} \] for every Borel set $F$ the following holds true:

$$L((Z))(F) = L\left(\sum_{i=1}^\infty M^{-i}\xi_i\right)(F) = P(s(\xi_1, \xi_2, ...) \in F)$$

$$= P((\xi_1, \xi_2, ...) \in s^{-1}(F)) = \mu(s^{-1}(F)),$$

where $\mu = L(\xi_1, \xi_2, ...)$. Corollary \[ \eqref{corollary:1} \] assures that the sequence $\{\xi_k\}_{k=1}^\infty$ is stationary and ergodic. Let $\mu' = L(\xi_1', \xi_2', ...)$, where $\{\xi_k'\}_{k=1}^\infty$ is an i.i.d. sequence with $\xi_1'$ uniform on $\Gamma_0$. Suppose that $\mu \neq \mu'$. Then by Lemma \[ \eqref{lemma:1} \] $\mu \perp \mu'$. So there is a set $B \subset \mathcal{B}(\Gamma_0^\infty)$ such that $\mu(B) = 1$ and $\mu'(B) = 0$. Set $A = s(B)$. Since $\Gamma_0^\infty$ is a Polish space and $s$ is continuous, $A$ being the continuous image of a Borel set, is an analytic set. As such, $A$ is universally measurable \[ \eqref{universal-meas} \]. Let $C$ and $D$ be Borel sets so that $C \subseteq A \subseteq D$ and $\lambda(C) = \lambda(A) = \lambda(D)$. Since the Lebesgue measure of boundary of a tile is 0, we may assume that $C$ does not contain any points on the boundary of tiles (the union of the tiles boundaries is a Borel set). This implies that $s^{-1}(C) \subseteq B$. From the proof of 2) $\Rightarrow$ 1) it follows that $L(s(\xi_1', \xi_2', ...)) = \lambda_T$. Now

$$0 = \mu'(B) = P((\xi_1', \xi_2', \ldots) \in B) \geq P((\xi_1', \xi_2', \ldots) \in s^{-1}(C))$$

$$= P(s(\xi_1', \xi_2', \ldots) \in C) = \lambda_T(C) = \lambda_T(A).$$
We also have that
\[ 1 = \mu(B) = P((\xi_1, \xi_2, \ldots) \in B) \leq P(s(\xi_1, \xi_2, \ldots) \in D) = \mathcal{L}((Z))(A) \]
where the last equality follows from (3.4). This contradicts the fact that \( \mathcal{L}((Z)) < \lambda_T \). Therefore \( \mu = \mu' \), i.e. \( \xi_i, i = 1, 2, \ldots \), are i.i.d. and \( \xi_1 \) is uniform on \( \Gamma_0 \). This completes the proof of 3) \( \Rightarrow 2) \) and thus of Theorem 1.

4. Conditions for independence of \( \{\xi_k\} \)

In Theorem 1 the existence of a density of the solution \( Z \) to (1.2) is equivalent to the fact that the stationary, ergodic sequence \( \{\xi_k\}_{k=1}^\infty \) is a sequence of independent random variables and that \( \xi_1 \) is uniform on \( \Gamma_0 \). In this section we first investigate the effects of uniformity of \( \xi_1 \) on the distributions of \( G_1 \) and \( [Z_1] \); the results are then summarized in Theorem 2. In Theorem 3 we give a sufficient condition on \( G_1 \) for the independence and uniformity of the sequence \( \{\xi_k\}_{k=1}^\infty \).

By Proposition 1, \( \xi_k = h(G_k + [Z_k]) \) a.s., provided that \( P([Z_0] \in \partial T) = 0 \). In order to describe the effects of uniformity of \( \xi_k \), it suffices to consider the relationship between \( G_1, [Z_1] \) and \( \xi_1 \).

Let \( p_i = P([Z_i] \cong \gamma_i) \) and \( q_i = P(G_1 \cong \gamma_i) \) for \( i = 0, 1, \ldots, m-1 \), where \( \gamma \cong \gamma_i \) means that the lattice point \( \gamma \) is in the coset represented by \( \gamma_i \). Recalling that \( G_1 \) and \( Z_1 \) are independent we have
\[
P(\xi_1 = \gamma_k) = P(h(G_1 + [Z_1]) = \gamma_k)
= \sum_{\gamma_i + \gamma_j \cong \gamma_k} P(G_1 \cong \gamma_i, [Z_1] \cong \gamma_j)
= \sum_{\gamma_i + \gamma_j \cong \gamma_k} P(G_1 \cong \gamma_i) P([Z_1] \cong \gamma_j)
= \sum_{\gamma_i + \gamma_j \cong \gamma_k} q_i p_j.
\]

Due to the uniqueness of equivalence class representatives, there are exactly \( m \) terms in the right-hand side of the equation. Now if \( p_i = \frac{1}{m} \) for all \( i \) or \( q_i = \frac{1}{m} \) for all \( i \) then \( \xi_i \) are uniform. Assuming that \( \xi_1 \) is uniform on \( \Gamma_0 = \{\gamma_0, \ldots, \gamma_{m-1}\} \) we have,
\[
\frac{1}{m} = q_0 p_0 + q_1 p_1 + \cdots + q_{m-1} p_{m-1},
\frac{1}{m} = q_{m-1} p_0 + q_0 p_1 + \cdots + q_{m-2} p_{m-1},
\frac{1}{m} = q_{m-2} p_0 + q_{m-1} p_1 + \cdots + q_{m-3} p_{m-1},
\cdots
\frac{1}{m} = q_{1} p_0 + q_2 p_1 + \cdots + q_{0} p_{m-1},
\]
or, in matrix form, \( QX = \frac{1}{m} [1 \ 1 \ \cdots \ 1]^T \), where \( X = [p_0 \ p_1 \ \cdots \ p_{m-1}]^T \). Notice that the rows as well as the columns of \( Q \) sum to 1. Without loss of generality, we may assume that \( q_0 \geq q_1 \geq \cdots \geq q_{m-1} \); if not, just reindex \( \Gamma_0 \) so that this ordering holds. It is obvious that \( p_i = \frac{1}{m}, i = 0, \ldots, m-1, \) is a solution of the system; we will show that it is unique by showing that the eigenvalues of the matrix \( Q \) are different from zero.
Let $\alpha_k$ be the $k$th root of $z^m = 1$. Direct computation shows that the eigenvalues of $Q$ are $\eta_k = \sum_{j=0}^{m-1} q_j \alpha_k^j$ and the associated eigenvectors are

$$v_k = \begin{bmatrix} 1 & \alpha_k & \alpha_k^2 & \cdots & \alpha_k^{m-1} \end{bmatrix}^T,$$

for $k = 0, 1, \ldots, m - 1$.

**Remark.** If $k \in \{0, 1, \ldots, m - 1\}$ and $m$ are relatively prime, then

$$\left\{e^{\frac{2\pi i j k}{m}} \mid j = 0, 1, \ldots, m - 1\right\}$$

is equal to the set of distinct roots of $z^m = 1$.

**Definition 1.** We say that the set $\{q_j\}_{j=0}^{m-1}$ has a cycle of length $r$ if

$$q_0 = q_1 = \cdots = q_{r-1},$$

$$q_r = q_{r+1} = \cdots = q_{2r-1},$$

$$\ldots,$$

$$q_{m-r} = \cdots = q_{m-1}.$$ 

The trivial case $r = 1$ is excluded.

So, for example, the set $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}\right\}$ has a cycle of length 2 while the set $\left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}\right\}$ has no cycle. Note that if $\{q_j\}_{j=0}^{m-1}$ has a cycle of length $r$, then $r$ divides $m$.

**Lemma 2.** Zero is an eigenvalue of $Q$ if and only if $\{q_j\}_{j=0}^{m-1}$ has a cycle.

**Proof.** ($\Rightarrow$) The case of a cycle of length $m$ is trivial.

Now consider a cycle of length $r$, where $r < m$. Denote the greatest common divisor of $m$ and $r$ by $(m, r)$. We claim that $\eta_m = 0$ (recall that $r$ divides $m$).

Since $\{q_j\}_{j=0}^{m-1}$ has a cycle of length $r$ and $\eta_m = 2\pi i r$, we have

$$\eta_m = \sum_{j=0}^{r-1} q_j e^{\frac{2\pi i j}{m}} + \sum_{j=r}^{2r-1} q_j e^{\frac{2\pi i j}{m}} + \cdots + \sum_{j=(m, r)-1}^{m-1} q_j e^{\frac{2\pi i j}{m}}$$

$$= q_0 \sum_{j=0}^{r-1} e^{\frac{2\pi i j}{m}} + q_r \sum_{j=r}^{2r-1} e^{\frac{2\pi i j}{m}} + \cdots + q_{(m, r)-1} \sum_{j=(m, r)-1}^{m-1} e^{\frac{2\pi i j}{m}}$$

$$= 0.$$

($\Rightarrow$) The assumption that there is no cycle implies $q_0 > q_{m-1}$. First we will show that $q_k \neq 0$ in the case that $(k, m) = 1$. Define $l_0 := \max\{k : q_k = q_0\}$ and inductively $l_{i+1} := \max\{k : q_k = q_{l_i+1}\}$, $i = 0, 1, 2, \ldots, n - 2$, that is, there are $n$ different values in the set of $q$’s. Notice that $l_{n-1} = m - 1$ and $q_0 \neq 0$. Now,

$$\eta_k \left(1 - e^{\frac{2\pi i k}{m}}\right)$$

$$= q_0 + (q_1 - q_0) e^{\frac{2\pi i k}{m}} + \cdots + (q_{m-1} - q_{m-2}) e^{\frac{2\pi i (m-1)}{m}} - q_{m-1} e^{\frac{2\pi i (m-1)}{m}}$$

$$= (q_0 - q_{m-1}) + (q_1 - q_0) e^{\frac{2\pi i k}{m}} + \cdots + (q_{m-1} - q_{m-2}) e^{\frac{2\pi i (m-1)}{m}}$$

$$= (q_0 - q_{l_{n-1}}) + (q_{l_0} - q_0) e^{\frac{2\pi i l_0}{m}} (l_0 + 1) + (q_{l_1} - q_1) e^{\frac{2\pi i l_1}{m}} (l_1 + 1)$$

$$+ \cdots + (q_{l_{n-2}} - q_{l_{n-3}}) e^{\frac{2\pi i l_{n-2}}{m}} (l_{n-2} + 1).$$
If we set
\[ z_0 = (q_0 - q_{0+1}) e^{\frac{2\pi ik}{m}(l_0+1)} + \cdots + (q_{n-2} - q_{n-2+1}) e^{\frac{2\pi ik}{m}(l_{n-2}+1)}, \]
we have \( \eta_k \left( 1 - e^{\frac{2\pi ik}{m}} \right) = q_0 - q_{m-1} - z_0 \), and
\[ \left| \eta_k \left( 1 - e^{\frac{2\pi ik}{m}} \right) \right| \geq q_0 - q_{m-1} - |z_0|. \]
We claim \( |z_0| < q_0 - q_{m-1} \). Observe
\[ |z_0| \leq q_0 - q_{0+1} + \left| (q_1 - q_{1+1}) e^{\frac{2\pi ik}{m}(l_1+1)} + \cdots + (q_{n-2} - q_{n-2+1}) e^{\frac{2\pi ik}{m}(l_{n-2}+1)} \right| \leq q_0 - q_{n-4+1} + \left| (q_{n-3} - q_{n-3+1}) + (q_{n-2} - q_{n-2+1}) e^{\frac{2\pi ik}{m}(l_{n-2} - l_{n-3})} \right|.
\]
Since \( k > 0 \), and \( k \) and \( m \) are relatively prime, \( e^{\frac{2\pi ik}{m}(l_{n-2} - l_{n-3})} \) has a nonzero imaginary part and so \( (q_{n-2} - q_{n-2+1}) e^{\frac{2\pi ik}{m}(l_{n-2} - l_{n-3})} \) cannot be a positive scalar multiple of \( q_{n-3} - q_{n-3+1} \). Therefore
\[ |z_0| < q_0 - q_{n-4+1} + q_{n-3} - q_{n-3+1} + \left| (q_{n-3} - q_{n-3+1}) + (q_{n-2} - q_{n-2+1}) e^{\frac{2\pi ik}{m}(l_{n-2} - l_{n-3})} \right| = q_0 - q_{m-1}.
\]
So \( \eta_k \left( 1 - e^{\frac{2\pi ik}{m}} \right) \geq q_0 - q_{m-1} - |z_0| > 0 \), which completes the case \((k, m) = 1\).

Suppose now that \((k, m) > 1\). Set
\[ m_1 := \frac{m}{(k, m)}, k_1 := \frac{k}{(k, m)} \text{ and } q^j_i = \sum_{\ell \equiv j} q_i, \]
where \( i \equiv j \) means \( i = j \mod m_1 \). Notice that \( q_0' \geq q_1' \geq \cdots \geq q_{m_1-1}' \) and \((k_1, m_1) = 1\). Rewriting \( \eta_k \) as
\[ \eta_k = \sum_{j=0}^{m-1} q_j e^{\frac{2\pi ij k}{m}} = \sum_{j=0}^{m_1-1} q_j' e^{\frac{2\pi ij k}{m_1}}, \]
we may apply the previous case because the absence of a cycle implies \( q_0' > q_{m_1-1}' \).

We can summarize the above in the following theorem: (Recall that \( p_i = P([Z_1] \equiv \gamma_i) \) and \( q_i = P(G_1 \equiv \gamma_i) \).)

**Theorem 2.** The random variable \( \xi_1 \) is uniform on \( \Gamma_0 \) if and only if one of the following two statements holds:

(i) if \( \{q_j\}^m_{j=1} \) has no cycles, then \( p_i = \frac{1}{m} \) for \( i = 0, \ldots, m - 1 \), or

(ii) if \( \{q_j\}^m_{j=1} \) has no cycles, then \( q_i = \frac{1}{m} \) for \( i = 0, \ldots, m - 1 \).

The next theorem gives a condition on the distribution of \( G_1 \) which will guarantee the independence of the sequence \( \{\xi_k\}^\infty_{k=1} \).

**Theorem 3.** If \( P(G_1 \equiv \gamma_i) = \frac{1}{m} \) for \( i = 1, \ldots, m \), then \( \xi_1, \xi_2, \ldots \) are independent and \( \xi_1 \) is uniform.
Proof. Uniformity of $\xi_1$ follows from Theorem \[\text{(2)}\]. Let $k_1 < k_2 < \cdots < k_n$. We proceed by induction on $n$.

Suppose $n = 2$. Then

\[
P(\xi_{k_1} = \gamma_{k_1}, \xi_{k_2} = \gamma_{k_2}) = P(G_{k_1} + [Z_{k_1}] \cong \gamma_{k_1}, \xi_{k_2} = \gamma_{k_2})
= \sum_{i=0}^{m-1} P(G_k \cong \gamma_i, [Z_k] \cong \gamma_{j(i)}, \xi_{k_2} = \gamma_{k_2})
= \sum_{i=0}^{m-1} P(G_k \cong \gamma_i) P([Z_k] \cong \gamma_{j(i)}, \xi_{k_2} = \gamma_{k_2})
\]

where $\gamma_i + \gamma_{j(i)} \cong \gamma_{k_1}$, and the last equality is due to the fact that $G_{k_1}$ is independent of $[Z_{k_1}]$ and of $\xi_{k_2}$. Notice that when $\gamma_i$ runs through $\Gamma_0$, so does $\gamma_{j(i)}$, and since $P(G_k \cong \gamma_i) = \frac{1}{m}$, we obtain

\[
P(\xi_{k_1} = \gamma_{k_1}, \xi_{k_2} = \gamma_{k_2}) = \frac{1}{m} \sum_{i=0}^{m-1} P([Z_k] \cong \gamma_{j(i)}, \xi_{k_2} = \gamma_{k_2})
= \frac{1}{m} P(\xi_{k_2} = \gamma_{k_2})
= P(\xi_{k_1} = \gamma_{k_1}) P(\xi_{k_2} = \gamma_{k_2}).
\]

Now assume $P(\xi_{k_1} = \gamma_{k_1}, \ldots, \xi_{k_n} = \gamma_{k_n}) = \prod_{i=1}^{n} P(\xi_{k_i} = \gamma_{k_i})$. Consider

\[
P(\xi_{k_1} = \gamma_{k_1}, \ldots, \xi_{k_n} = \gamma_{k_n}, \xi_{k_{n+1}} = \gamma_{k_{n+1}})
= P(\xi_{k_1} = \gamma_{k_1}, \ldots, G_{k_n} + [Z_{k_n}] \cong \gamma_{k_n}, \xi_{k_{n+1}} = \gamma_{k_{n+1}})
= \sum_{i=0}^{m-1} P(\xi_{k_1} = \gamma_{k_1}, \ldots, G_{k_n} \cong \gamma_i, [Z_{k_n}] \cong \gamma_{j(i)}, \xi_{k_{n+1}} = \gamma_{k_{n+1}}),
\]

where $\gamma_i + \gamma_{j(i)} \cong \gamma_{k_n}$. Now, $G_{k_n}$ is independent of $[Z_{k_n}]$ and of $\xi_{k_{n+1}}$; by the inductive hypothesis, $G_{k_n}$ is also independent of $\xi_{k_1}, \ldots, \xi_{k_{n-1}}$. Thus

\[
\sum_{i=0}^{m-1} P(\xi_{k_1} = \gamma_{k_1}, \ldots, G_{k_n} \cong \gamma_i, [Z_{k_n}] \cong \gamma_{j(i)}, \xi_{k_{n+1}} = \gamma_{k_{n+1}})
= \sum_{i=0}^{m-1} P(G_{k_n} \cong \gamma_i) P(\xi_{k_1} = \gamma_{k_1}, \ldots, [Z_{k_n}] \cong \gamma_{j(i)}, \xi_{k_{n+1}} = \gamma_{k_{n+1}})
= \frac{1}{m} \sum_{i=0}^{m-1} P(\xi_{k_1} = \gamma_{k_1}, \ldots, [Z_{k_n}] \cong \gamma_{j(i)}, \xi_{k_{n+1}} = \gamma_{k_{n+1}})
= \frac{1}{m} P(\xi_{k_1} = \gamma_{k_1}, \ldots, \xi_{k_{n-1}} = \gamma_{k_{n-1}}, \xi_{k_{n+1}} = \gamma_{k_{n+1}})
= P(\xi_{k_n} = \gamma_{k_n}) \prod_{i \neq n-1}^{n+1} P(\xi_{k_i} = \gamma_{k_i})
\]

by the inductive hypothesis. So by induction, we have shown that $\{\xi_k\}_{k=1}^{\infty}$ is an independent sequence.

\[\square\]

Remark. Theorem \[\text{(2)}\] is symmetric in $[Z]$ and $G$, but Theorem \[\text{(3)}\] is not; that is, if $P([Z_k] = \gamma) = \frac{1}{m}$ for all $\gamma \in \Gamma_0$ but $P(G_k = \gamma_0) > \frac{1}{m}$ for some $\gamma_0 \in \Gamma_0$, the
sequence \( \{\xi_k\}_{k=1}^{\infty} \) is not necessarily independent. This is illustrated in the following example: \( M = 2, \Gamma = \mathbb{Z} \) (the integers) and \( \Gamma_0 = \{0, 1\} \). Let \( G \) be such that \( P(G = 0) = P(G = 1) = P(G = 2) = \frac{1}{3} \). So \( P(G \equiv 0) = P(G \text{ is even}) = \frac{2}{3} \) and \( P(G \equiv 1) = P(G \text{ is odd}) = \frac{1}{3} \). Then we have
\[
P([Z_1] \text{ even}) = P(0 \leq Z_1 < 1) + P(Z_1 = 2) = P(G_1 = 0) + P(G_1 = 1, G_2 = 0) + \cdots
\]
\[
= \frac{1}{3} \sum_{k=0}^{\infty} \left( \frac{1}{3} \right)^k = \frac{1}{2}.
\]
Therefore \( \xi_1 \) is uniform on \( \Gamma_0 \) by Theorem \( \text{[2]} \). However, the sequence \( \{\xi_k\}_{k=1}^{\infty} \) is not independent. Consider \( P(\xi_1 = 0, \xi_2 = 0) \):
\[
P(\xi_1 = 0, \xi_2 = 0)
= P(G_1 + [Z_1] \equiv 0, \xi_2 = 0)
= P(G_1 \equiv 0, [Z_1] \equiv 0, \xi_2 = 0) + P(G_1 \equiv 1, [Z_1] \equiv 1, \xi_2 = 0)
= \frac{2}{3} P([Z_1] \equiv 0, \xi_2 = 0) + \frac{1}{3} P([Z_1] \equiv 1, \xi_2 = 0).
\]
To compute the two remaining probabilities, note that
\[
[Z_1] = \left[ \sum_{k=1}^{\infty} 2^{-k} G_{k+1} \right] = \left[ \frac{G_2 + [Z_2]}{2} + \frac{[Z_2]}{2} \right].
\]
If \( \xi_2 = 0, G_2 + [Z_2] \) is even, so in this case, \( [Z_1] = \frac{G_2 + [Z_2]}{2} \), and if \( [Z_1] = \frac{G_2 + [Z_2]}{2} \), then \( \xi_2 = 0 \). This implies that
\[
\frac{2}{3} P([Z_1] \equiv 0, \xi_2 = 0) + \frac{1}{3} P([Z_1] \equiv 1, \xi_2 = 0)
= \frac{2}{3} P(G_2 + [Z_2] \equiv 0 \mod 4) + \frac{1}{3} P(G_2 + [Z_2] \equiv 2 \mod 4)
= \frac{2}{3} (P(G_2 = 0, [Z_2] = 0) + P(G_2 = 2, [Z_2] = 2))
+ \frac{1}{3} (P(G_2 = 0, [Z_2] = 2) + P(G_2 = 2, [Z_2] = 0) + P(G_2 = 1, [Z_2] = 1))
= \frac{2}{3} \left( \frac{1}{2} \cdot \frac{1}{3} + 0 \right) + \frac{1}{3} \left( 0 + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \right)
= \frac{2}{9} \neq \frac{1}{4} = P(\xi_1 = 0) P(\xi_2 = 0).
\]
Since the sequence \( \{\xi_k\}_{k=1}^{\infty} \) is not independent, by Theorem \( \text{[2]} \), \( \mathcal{L}(Z) \) is not absolutely continuous with respect to Lebesgue measure for this example. Thus, the assumption that \( \xi_1 \) is uniform does not necessarily imply the independence of the sequence \( \{\xi_k\}_{k=1}^{\infty} \).

If the range of \( G \) is \( \Gamma_0 \), then \( [Z] = 0 \). In this case \( G = \xi_1 \) and the application of Theorem \( \text{[2]} \) yields the following:
Corollary 2. Suppose that the range of $G$ is $\Gamma_0$. Then
\[ \varphi(x) = \sum_{\gamma \in \Gamma_0} c(\gamma) \varphi(Mx - \gamma). \]
has a functional solution if and only if $P(G = \gamma) = c(\gamma) = \frac{1}{m}$.

The result of Corollary 2 is known. It was first proved by Grochenig and Madych (Theorem 2 in [5]) using different methods. The solution of the dilation equation in this case is $\varphi = \frac{1}{\sqrt{A(M)}} 1_T$. Scaling functions that are indicator functions over the tile are used to construct “Haar-type” wavelet bases as discussed in detail in [3].

5. Examples

In this section we give several examples of density functions obtained by assigning probabilities so that the hypotheses of Theorem 3 are satisfied.

In most cases, there is no closed form for the density function [14]; those which cannot be computed explicitly can be numerically approximated by computing the function values on the points of \(\{M^{-k}\Gamma | k = 0, \ldots, k_0\}\) for some $k_0$, via the dilation equation. To obtain the approximation of the graph of $\varphi$, first the values of $\varphi$ at the integers are found by considering the vector of integer values as an eigenvector of eigenvalue 1 for a matrix of coefficients [14]. Then, using the scaling relation (1.1), the values of $\varphi$ can be found at all points in $M^{-1}\Gamma$. Repeatedly applying (1.1) $k_0$ times and plotting the results gives an approximation to the graph of $\varphi$. Questions of convergence of the approximations are discussed in [3].

For each of the following examples, the eigenvalue problem for a matrix corresponding to a set containing the support of $\varphi$ was solved to obtain the values at the lattice points. Then the above algorithm was applied, resulting in approximately 2000 points plotted for each graph approximation.

Example 1. Let $d = 1$, $M = 2$, $\Gamma = \mathbb{Z}$ and $\Gamma_0 = \{0, 1\}$. Suppose the range of $G$ is $\Gamma_1 = \{0, 1, 2, 3\}$ with the following weight assignments: $c(0) = .2$, $c(1) = .4$, $c(2) = .3$, $c(3) = .1$. Then the density function $\varphi$ is continuous [2] and is pictured in Figure 1 along with a four-coefficient spline function for comparison.

Example 2. Suppose $d = 2$, $\Gamma = \mathbb{Z}^2$, $\Gamma_0 = \{(0,0), (1,0), (0,1), (1,1)\}$ and
\[ M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \]

Define $G$ to have values in $\Gamma_1 = [0, 2]^2 \cap \mathbb{Z}^2$ with the following probability distribution:
\[
\begin{align*}
    c((0,0)) &= c((2,0)) = c((0, 2)) = c((2,2)) = \frac{1}{16}, \\
    c((1,0)) &= c((0,1)) = c((2,1)) = c((1,2)) = \frac{1}{8}, \\
    c((1,1)) &= \frac{1}{4}.
\end{align*}
\]

Since $G$ is clearly the convolution of two independent copies of a uniform random variable on the unit square, $\varphi$ is continuous. The graph of the density function is pictured in Figure 2.
Example 3. Let $d = 2$, $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\Gamma = \mathbb{Z}^2$ and $\Gamma_0 = \{(0,0),(1,0)\}$. Define $G$ to have values in $\Gamma_1 = \{(0,0),(1,0),(2,0)\}$ with the following distribution: $c((0,0)) = \frac{1}{4}$, $c((1,0)) = \frac{1}{2}$, $c((2,0)) = \frac{1}{4}$. The graph of the density function is pictured in Figure 3. The density is a convolution of two indicator functions of the twin dragon tile and therefore it is continuous.
6. A NECESSARY CONDITION FOR MULTIDIMENSIONAL PRESCALE FUNCTIONS

Suppose \( \varphi \) is a functional solution of the dilation equation (1.1). If the lattice translates of \( \varphi \) form a Riesz basis, that is, for some positive constants \( C_1, C_2 \)

\[
C_1 \sqrt{\sum a(\gamma)^2} \leq \left\| \sum a(\gamma) \varphi(\cdot - \gamma) \right\|_{L^2(\mathbb{R}^d)} \leq C_2 \sqrt{\sum (a(\gamma))^2},
\]

then \( \varphi \) is said to be stable. We show that the condition \( \sum_{\gamma \in \delta} c(\gamma) = \frac{1}{m} \) for each \( \delta \in \Gamma_0 \), where \( m = |\det M| \), which was sufficient for the existence of a functional solution to (1.1), is necessary for the stability of \( \varphi \).

The Fourier transform version of the dilation equation (1.1) is

\[
\hat{\varphi}(\zeta) = \hat{\varphi}(M^*\zeta) A(M^*\zeta),
\]

where \( A(\zeta) = \sum_{\gamma \in \Gamma} c(\gamma) e^{-i\gamma\cdot\zeta} \). Stability of \( \varphi \) is equivalent to

\[
0 < C_1 \leq \sum_{k \in \Gamma} |\hat{\varphi}(\zeta + 2\pi k)|^2 \leq C_2 \text{ a.e.}
\]

In the case that the coefficient sequence \( c := \{c(\gamma)\}_{\gamma \in \Gamma} \) is finitely supported, the function in (6.2) is a polynomial [12] and so the inequality must hold everywhere. In the theorem below, which is known (see, for example, [9]), we will assume that the equation holds everywhere. This is not a restriction as proved in [11]. For completeness we include a short proof.

**Theorem 4.** Let \( \varphi \in L^2(\mathbb{R}^d) \) be a solution of the dilation equation (1.1). Suppose \( \varphi \) is stable and that equation (6.2) holds everywhere. Then \( \sum_{\gamma \in \Gamma} c(\gamma) = \frac{1}{m} \) for each \( \gamma_0 \in \Gamma_0 \).
Proof. Without loss of generality, we assume $\Gamma = \mathbb{Z}^d$. Since $\varphi$ is stable, (6.2) holds. Applying equation (6.1) we obtain

$$0 < C_1 \leq \sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\zeta + 2\pi k)|^2$$

$$= \sum_{\gamma \in \Gamma_0} |A(M^{*-1}\zeta + 2\pi M^{*-1}\gamma)|^2 \sum_{k' \in \mathbb{Z}^d} |\hat{\varphi}(M^{*-1}\zeta + 2\pi (M^{*-1}\gamma + k'))|^2.$$  

For $\zeta = 0$, we get

$$\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(2\pi k)|^2 = \sum_{\gamma \in \Gamma_0 \setminus \{0\}} |A(2\pi M^{*-1}\gamma)|^2 \sum_{k' \in \mathbb{Z}^d} |\hat{\varphi}(2\pi (M^{*-1}\gamma + k'))|^2$$

$$+ |A(0)|^2 \sum_{k' \in \mathbb{Z}^d} |\hat{\varphi}(2\pi k')|^2.$$  

Since by (3) $A(0) = 1$, and since $\sum_{k' \in \mathbb{Z}^d} |\hat{\varphi}(2\pi (M^{*-1}\gamma + k'))|^2 \geq C_1 > 0$, we have

$$\sum_{k \in \mathbb{Z}^d} c(k) e^{-i2\pi(M^{*-1}\gamma)k} = 0$$

for each $\gamma \in \Gamma_0 \setminus \{0\}$, which, after letting $k = \gamma_k + Mn_k$, $\gamma_k \in \Gamma_0$, $n_k \in \mathbb{Z}^d$ and setting $\sum_{k \in \delta} c(\gamma_k + Mn_k) = q_\delta$ leads to

$$(6.3) \quad 0 = \sum_{\delta \in \Gamma_0} e^{-i2\pi \gamma M^{-1}\delta} q_\delta.$$  

Claim. $\sum_{\delta \in \Gamma_0} e^{-i2\pi \gamma M^{-1}\delta} = 0$ for each $\gamma \in \Gamma_0 \setminus \{0\}$. \hfill \Box

Notice that the set $\left\{ e^{-i2\pi \gamma M^{-1}\delta} \mid \delta \in \Gamma_0 \right\}$ is a group on the unit circle. If

$$(6.4) \quad \sum_{\delta \in \Gamma_0} e^{-i2\pi \gamma M^{-1}\delta} = r \neq 0,$$  

then for every $\gamma \in \Gamma_0 \setminus \{0\}$ there is a $\delta \in \Gamma_0$ so that $e^{-i2\pi \gamma M^{-1}\delta} \neq 1$. (If not, $e^{-i2\pi \gamma M^{-1}\delta} = 1$ for all $\delta \in \Gamma_0$ implies that

$$0 = \sum_{\delta \in \Gamma_0} \sum_{k \in \Gamma_0} c(\gamma_k + Mn_k) = \sum_{\gamma \in \Gamma_0} c(\gamma),$$

contradicting $\sum c(\gamma) = 1$. ) Multiplying both sides of (6.4) by $e^{-i2\pi \gamma M^{-1}p}$ where $p \in \Gamma_0$ is such that $e^{-i2\pi \gamma M^{-1}p} \neq 1$, we obtain

$$\sum_{\delta \in \Gamma_0} e^{-i2\pi \gamma M^{-1}(\delta+p)} = re^{-i2\pi \gamma M^{-1}p}.$$  

Note that since $\delta + p = \delta' + M k$, where $\delta' \in \Gamma_0$ and $k \in \mathbb{Z}^d$, then $\sum_{\delta \in \Gamma_0} e^{-i2\pi \gamma M^{-1}(\delta+p)}$ includes all the elements of the group and nothing more, and therefore it is equal to $r$. So $r = re^{-i2\pi \gamma M^{-1}p}$, contradicting $e^{-i2\pi \gamma M^{-1}p} \neq 1$. \hfill \Box
The set of \( m - 1 \) equations \( \sum_{\delta \in \Gamma_0} e^{-i2\pi \gamma \cdot \delta} q_\delta = 0 \) for each \( \gamma \in \Gamma_0 \setminus \{0\} \), along with the constraint \( \sum_{\delta \in \Gamma_0} q_\delta = 1 \), comprises a system of \( m \) equations with \( m \) variables \( q_\delta \). Notice that \( q_\delta = \frac{1}{m} \) for each \( \delta \in \Gamma_0 \) is a solution. The coefficient matrix for this system is given by

\[
U = \left( e^{-i2\pi \gamma_i \cdot \delta_j} \right)_{0 \leq i,j \leq m-1}.
\]

By (6.3), \( UU^* = mL_m \), and so \( \det U \neq 0 \). Therefore \( q_\delta = \frac{1}{m} \) for all \( \delta \in \Gamma_0 \) is the unique solution of the system, which concludes the proof of the theorem.

If, as in the previous section, we let \( c(\gamma) = P(G = \gamma) \), the above theorem says that \( P(G = \gamma) = \frac{1}{m} \) for each \( \gamma \in \Gamma_0 \) is necessary in order for the density \( \varphi \) to be stable. However, this condition, which by Theorems 4 and 5 guarantees that \( \varphi \) is absolutely continuous, is not sufficient for the stability of \( \varphi \). Consider the following example: \( \Gamma = \mathbb{Z} \), \( M = 2 \) with the constants assigned as follows:

\[
c(0) = c(2) = c(3) = c(5) = \frac{1}{8},
\]

\[
c(1) = c(4) = \frac{1}{4}.
\]

Notice that the two cosets have equal weight and so by Theorems 4 and 5 the solution \( \varphi \) of the dilation equation will be a density function. However, it is shown in [11] that \( \varphi \) is not stable.

References


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