DIOPHANTINE APPROXIMATION, BESSEL FUNCTIONS AND RADIALLY SYMMETRIC PERIODIC SOLUTIONS OF SEMILINEAR WAVE EQUATIONS IN A BALL

J. BERKOVITS AND J. MAWHIN

ABSTRACT. The aim of this paper is to consider the radially-symmetric periodic-Dirichlet problem on $[0, T] \times B^n[a]$ for the equation

$$u_{tt} - \Delta u = f(t, |x|, u),$$

where Δ is the classical Laplacian operator, and $B^n[a]$ denotes the open ball of center 0 and radius a in $\mathbb{R}^n$. When $\alpha = a/T$ is a sufficiently large irrational with bounded partial quotients, we combine some number theory techniques with the asymptotic properties of the Bessel functions to show that 0 is not an accumulation point of the spectrum of the linear part. This result is used to obtain existence conditions for the nonlinear problem.

1. INTRODUCTION

The arithmetical properties of the ratio $\alpha = a/T$ play an important role in the solvability of the periodic-Dirichlet problem over $[0, T] \times [0, a]$ for the semilinear wave equation

$$u_{tt} - u_{xx} = f(t, x, u),$$

or the one of the radially-symmetric periodic-Dirichlet problem over $[0, T] \times B^n[a]$, for the equation

$$u_{tt} - \Delta u = f(t, |x|, u),$$

where Δ is the classical Laplacian operator in $\mathbb{R}^n$, and $B^n[a]$ denotes the open ball of center 0 and radius a in $\mathbb{R}^n$. The main reason is that the nature of the spectrum of the corresponding linear problem depends in an essential way upon the arithmetical nature of α. This was already noticed by Borel in 1895 [7]. Such a spectrum is made of isolated eigenvalues when α is rational and can be the real line for some irrational values of this ratio. References on this question can be found in [16].

In particular, it was proved in [5], using the asymptotic behavior of the Bessel functions, that if $\alpha = 1/4$ and n is even, the spectrum of the radially symmetric periodic-Dirichlet problem over $[0, T] \times B^n[a]$ for the wave operator $D_{tt}^2 - \Delta$ is made of eigenvalues with finite multiplicity, which accumulate only at $-\infty$ and

$+\infty$. When α has the same value and n is odd, there may be eigenvalues having infinite multiplicity (for example $\lambda = 0$ has infinite multiplicity when $n = 1$ or 3).

It has been proved in [3] that if $\alpha = 1/4$, $n > 3$ is odd and if we set

$$q_n = \frac{1}{\pi^2}(n-1)(n-3),$$

then every element of the spectrum outside of $[-2\pi q_n, -q_n]$ is an isolated eigenvalue with finite multiplicity, and that 0 is not in the spectrum. Moreover, it follows from the methods of this paper that there exists an accumulation point of the spectrum on the interval $[-2\pi q_n, -q_n]$. Similar results hold when α is an arbitrary rational number [3].

In the case where α is irrational, little is known about the corresponding spectrum. A class of irrationals α for which the resolvent of the one-dimensional periodic-Dirichlet problem over $[0, T] \times [0, a]$ contains a neighbourhood of the origin was defined independently and differently in [14] and [17], the equivalence of their results being proved in [4]. This class, namely the irrational numbers with bounded partial quotients [21], is studied in Section 2.

The aim of this paper is to consider the case of the radially-symmetric periodic-Dirichlet problem on $[0, T] \times B^n[a]$, when α belongs to the same class of irrationals, and to combine the number theory techniques used in [4] and the asymptotic properties of the Bessel functions used in [5] and [3] to show in Theorem 1 of Section 3 that, for sufficiently large α with bounded partial quotients, 0 is not an accumulation point of the spectrum.

This result is combined with nonlinear techniques introduced in [15] to obtain, in Section 4, some existence conditions for the nonlinear problem (Theorems 2 to 5), which provide extensions of the results of [14] and [17] to radially symmetric solutions of the periodic-Dirichlet problem for semilinear wave equations on a ball.

2. Irrational numbers with bounded partial quotients

Let

$$\alpha = [a_0, a_1, a_2, \dots]$$

be the continued fraction decomposition of the real number α [11, 18, 19]. Recall that it is obtained as follows: put $a_0 = [\alpha]$, where $[\cdot]$ denotes the integer part. Then

$$\alpha = a_0 + \frac{1}{\alpha_1}$$

with $\alpha_1 > 1$, and we set $a_1 = [\alpha_1]$. If $a_0, a_1, \dots, a_{n-1}$ and $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$ are known, then

$$\alpha_{n-1} = a_{n-1} + \frac{1}{\alpha_n},$$

with $\alpha_n > 1$ and we set $a_n = [\alpha_n]$. It can be shown [18] that this process does not terminate if and only if α is irrational. The integers $a_0, a_1, \dots$ are the *partial quotients* of α; the numbers $\alpha_1, \alpha_2, \dots$ are the *complete quotients* of α and the rationals

$$\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n] = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\dots\frac{1}{a_n},$$

with p_n, q_n relatively prime integers, called the *convergents* of α, are such that

$$p_n/q_n \to \alpha \quad \text{as} \quad n \to \infty.$$

It is well known that the p_n, q_n are recursively defined by the relations

$$p_0 = a_0, q_0 = 1, p_1 = a_0 a_1 + 1, q_1 = a_1,$$

$$p_n = a_n p_{n-1} + p_{n-2}, q_n = a_n q_{n-1} + q_{n-2}.$$

The following lemmas are proved, for example, in [4].

Lemma 1. *To each irrational number α corresponds a unique (extended) number $M(\alpha) \in [\sqrt{5}, \infty]$ having the following properties.*

   i. *For each positive number $\mu < M(\alpha)$ there exist infinitely many pairs (p_i, q_i) with $q_i \neq 0$, such that*

$$\left| \alpha - \frac{p_i}{q_i} \right| \leq \frac{1}{\mu q_i^2}.$$

   ii. *If $M(\alpha)$ is finite, then, for each $\mu > M(\alpha)$, there are only finitely many pairs (p_i, q_i) satisfying the inequality*

$$\left| \alpha - \frac{p_i}{q_i} \right| \leq \frac{1}{\mu q_i^2}.$$

The extended real number $M(\alpha)$ is called the *Lagrange* or the *Markoff constant* of α. If we set

$$\mathcal{M}(\alpha) = \left\{ M \in \mathbb{R}_0^+ : \text{ infinitely many } (p_i, q_i) \text{ satisfy } \left| \alpha - \frac{p_i}{q_i} \right| \leq \frac{1}{M q_i^2} \right\},$$

then $\mathcal{M}(\alpha)$ is an interval and Lemma 1 clearly states that $M(\alpha) = \sup \mathcal{M}(\alpha)$.

Lemma 2. $M(\alpha)$ *is finite if and only if the sequence $(a_i)_{i \in \mathbb{N}}$ of partial quotients of α is bounded.*

Now let us define the set $\mathcal{N}(\alpha)$ by

$$\left\{ M \in \mathbb{R}_0^+ : \text{ infinitely many } (p, q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \text{ satisfy } \left| \alpha - \frac{p}{q} \right| \leq \frac{1}{M q^2} \right\}.$$

Clearly $\mathcal{N}(\alpha)$ is an interval and $\mathcal{N}(\alpha) \supset \mathcal{M}(\alpha)$. It is known [18, 20] that if $M > 2$ and $M \in \mathcal{N}(\alpha)$, then $M \in \mathcal{M}(\alpha)$, and that $\sqrt{5} \in \mathcal{M}(\alpha)$. Thus,

$$M(\alpha) = \sup \mathcal{M}(\alpha) = \sup \mathcal{N}(\alpha).$$

Any α for which the sequence $(a_i)_{i \in \mathbb{N}}$ of partial quotients of α is bounded is said to have *bounded partial quotients.* The reader can consult the interesting survey [21] on real numbers with bounded partial quotients which 'appear in many different fields of mathematics and computer science : Diophantine approximation, fractal geometry, transcendental number theory, ergodic theory, numerical analysis, pseudo-random number generation, dynamical systems, and formal language theory'. The set of irrational numbers with bounded partial quotients coincide with the set of numbers of *constant type,* which are the numbers α such that

$$q \| q\alpha \| \geq \frac{1}{r}$$

for some real number $r \geq 1$ and all integers $q \geq 0$, where $\|\theta\|$ denotes the distance between the irrational number θ and the closest integer. Also,

$$\frac{1}{M(\alpha)} = \liminf_{q \to \infty} q \| q\alpha \|.$$

For a proof of the equivalence, see e.g. [13].

By a classical theorem of Lagrange (see e.g. [11]), all real quadratic irrationals have bounded partial quotients. In particular, $M(\Phi) = \sqrt{5}$, where $\Phi = \frac{1+\sqrt{5}}{2}$ is the *golden section.* It follows from results of Borel [7, 8] and Bernstein [6] that the set of irrational numbers having bounded partial quotients is a dense uncountable and null subset of the real line, and from a result of Jarnik [12], that any intersection of this set with a bounded interval has Hausdorff dimension one. Examples of transcendental numbers with bounded partial quotients are given by

$$f(n) = \sum_{i=0}^{\infty} \frac{1}{n^{2^i}},$$

for $n \geq 2$ an integer [21], and by continued fractions of the form

$$[0, 1, 2, 2, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, \ldots],$$

where each group of 2^j partial quotients 1 is followed by a group of 2^j partial quotients 2 $(j = 0, 1, 2, \ldots)$ [2].

If α is an irrational number, we need some properties on the behavior of the function $M(\alpha)$ under the action of the group of transformations T defined by

(1) $$\beta = T(\alpha) = \frac{a\alpha + b}{c\alpha + d},$$

where $a, b, c, d \in \mathbb{Z}$ are such that $ad - bc \neq 0$. Notice that then

(2) $$\alpha = T^{-1}(\beta) = \frac{-d\beta + b}{c\beta - a},$$

and

(3) $$(-d)(-a) - bc = ad - bc.$$

Lemma 3. *If* $\beta = \frac{a\alpha+b}{c\alpha+d}$, *for some* $a, b, c, d \in \mathbb{Z}$ *such that* $ad - bc \neq 0$, *then*

(4) $$M(\alpha) \leq |ad - bc|M(\beta), \quad M(\beta) \leq |ad - bc|M(\alpha).$$

Proof. Let $M \in \mathcal{N}(\alpha)$. Then we can find a sequence $\left(\frac{r_j}{s_j}\right)_{j\in\mathbb{Z}_+}$ of rational numbers such that, for each $j \in \mathbb{N}$, one has

$$\left| \alpha - \frac{r_j}{s_j} \right| \leq \frac{1}{Ms_j^2}.$$

Now

$$\left| \beta - \frac{ar_j + bs_j}{cr_j + ds_j} \right| = |ad - bc| \frac{|\alpha - \frac{r_j}{s_j}|}{\left| c\frac{r_j}{s_j} + d \right| |c\alpha + d|}.$$

Let $\varepsilon > 0$. Then there exists $j_\varepsilon \in \mathbb{N}$ such that

$$\frac{1}{|c\alpha + d|} \leq \frac{1 + \varepsilon}{\left| c\frac{r_j}{s_j} + d \right|}$$

whenever $j \geq j_\varepsilon$. Therefore,

$$\left| \beta - \frac{ar_j + bs_j}{cr_j + ds_j} \right| \leq \frac{(1 + \varepsilon)|ad - bc|}{M \left(c\frac{r_j}{s_j} + d \right)^2 s_j^2} = \frac{(1 + \varepsilon)|ad - bc|}{M} \frac{1}{(cr_j + ds_j)^2},$$

whenever $j \geq j_\varepsilon$. Consequently, $\frac{M}{|ad-bc|(1+\varepsilon)} \in \mathcal{N}(\beta)$ for each $\varepsilon > 0$, which implies, letting $\varepsilon \to 0+$, that

$$\frac{M}{|ad - bc|} \leq M(\beta),$$

for all $M \in \mathcal{N}(\alpha)$, and hence that

$$\frac{M(\alpha)}{|ad - bc|} \leq M(\beta).$$

The second inequality in (4) follows from the first one and relations (2) and (3). □

Lemma 3 has a few immediate consequences.

Corollary 1. *If* $\beta = \frac{a\alpha+b}{c\alpha+d}$ *with* $a, b, c, d \in \mathbb{Z}$ *such that* $ad - bc \neq 0$*, then* β *has bounded partial quotients if and only if* α *has bounded partial quotients.*

Corollary 2. *If* p *and* $q \in \mathbb{Z}$*, with* $p, q \neq 0$*, then*

$$M\left(\frac{p}{q}\alpha\right) \leq |pq|M(\alpha).$$

The *modular group* is the group of transformations defined by (4) with $|ad-bc| = 1$. Lemma 3 shows that $M(\alpha)$ is invariant under the action of the modular group. In particular, the Lagrange-Markoff constant is invariant under translations through integers so that, if $\{\alpha\} = \alpha - [\alpha]$, one has

$$M(\alpha) = M(\{\alpha\}).$$

Two real numbers α and β are called *equivalent* if relation (1) holds with $|ad - bc| = 1$. Equivalent numbers have the same Lagrange-Markoff constant.

3. The linear radially symmetric periodic-Dirichlet problem for the wave equation

The problem consists in finding the conditions for the existence of weak radially symmetric solutions for the linear periodic-Dirichlet problem on a ball

$$\begin{aligned} u_{tt} - \Delta u - \lambda u &= h(t, |x|), \quad (t, x) \in \mathbb{R} \times B^n[a], \\ u(t, x) &= 0, \quad (t, x) \in \mathbb{R} \times S_a^{n-1}, \\ u(0, x) - u(T, x) &= u_t(0, x) - u_t(T, x) = 0, \quad x \in B^n[a]. \end{aligned}$$

Here, $S_a^{n-1} = \{x \in \mathbb{R}^n, |x| = a\}$, and $h(\cdot, |\cdot|) \in L^2([0, T] \times B^n[a])$. The above problem can be written in the equivalent form, letting $r = |x|$,

$$\begin{aligned} u_{tt} - u_{rr} - \frac{n-1}{r}u_r - \lambda u &= h(t, r), \quad (t, r) \in]0, T[\times]0, a[, \\ u_r(t, 0) = u(t, a) &= 0, \quad t \in]0, T[, \\ u(0, r) - u(T, r) &= u_t(0, r) - u_t(T, r) = 0, \quad r \in]0, a[. \end{aligned} \tag{5}$$

By a solution of (5) we mean, like in [22], a *weak solution* in the following sense. Let D denote the class of radially symmetric functions $\varphi \in C^\infty([0, T] \times B^n[a], \mathbb{R})$ which are T-periodic in time for each $x \in B^n[a]$, and have compact support in $B^n[a]$ for each $t \in [0, T]$. Let H denote the vector space of radially symmetric functions $u \in L^2([0, T] \times B^n[a], \mathbb{R})$. Equipped with the usual L^2-norm $\| \cdot \|$ and

inner product $\langle \cdot, \cdot \rangle$, H is a Hilbert space. We say that $u \in H$ is a *weak solution* of (5) provided that

$$\int_0^T \int_0^a \left[u \left(\varphi_{tt} - \varphi_{rr} - \frac{n-1}{r} \varphi_r - \lambda \varphi \right) - h\varphi \right] r^{n-1} \, dr dt = 0$$

for every $\varphi \in D$.

Let us first reproduce, for the reader's convenience, some results of [22] for the case where $h = 0$, i.e. for the linear eigenvalue problem

$$
\begin{aligned}
(6) \qquad u_{tt} - u_{rr} - \frac{n-1}{r} u_r - \lambda u &= 0, \quad (t, r) \in]0, T[\times]0, a[, \\
u_r(t, 0) = u(t, a) &= 0, \quad t \in]0, T[, \\
u(0, r) - u(T, r) &= u_t(0, r) - u_t(T, r) = 0, \quad r \in]0, a[.
\end{aligned}
$$

By a classical method of separation of variables, we set $\varphi(t, r) = \tau(t)\rho(r)$ and derive that ρ must satisfy the equation

$$
\begin{aligned}
(7) \qquad r^2 \rho'' + (n-1) r \rho' + r^2 \mu^2 \rho &= 0, \quad 0 < r < a, \\
\rho(a) &= 0, \\
\rho \text{ bounded} \quad &\text{on} \quad [0, a],
\end{aligned}
$$

where $\mu^2 = \lambda + \left(\frac{2k\pi}{T} \right)^2$ for any integer $k \geq 0$, the corresponding functions τ_k being linear combinations of $\cos(2k\pi t/T)$ and $\sin(2k\pi t/T)$. The change of variables $\psi(r) = r^{\frac{n-2}{2}} \rho(r)$ transforms (7) into

$$(8) \qquad r^2 \psi'' + r\psi' + \left[\mu^2 r^2 - \left(\frac{n-2}{2} \right)^2 \right] \psi = 0, \quad 0 < r < a,$$

$$\psi(a) = 0, \quad \psi(r) = 0(r^{\frac{n-2}{2}}) \quad \text{as} \quad r \to 0^+.$$

This is the classical eigenvalue problem for the Bessel equation of order

$$\nu = \frac{n-2}{2}.$$

If $J_\nu(x)$ denote the Bessel function of the first kind of order ν, then $y = J_\nu(x)$ satisfies

$$x^2 y'' + xy' + (x^2 - \nu^2) y = 0, \quad x > 0,$$
$$J_\nu(x) = 0(x^\nu) \quad as \quad x \to 0^+$$

(cf. [1], [23]). Consequently, $\psi(r) = J_\nu(\mu r)$, with μ such that $J_\nu(\mu a) = 0$, which gives the eigenvalues

$$\mu_{n,j}^2 = \left(\frac{\alpha_{n,j}}{a} \right)^2, \quad (j \geq 1),$$

where

$$\alpha_{n,j} := x_{\nu,j}$$

is the j-th positive zero of J_ν, and the corresponding eigenfuctions

$$\psi_{n,j}(r) = J_\nu \left(\frac{\alpha_{n,j} r}{a} \right), \quad (j \geq 1).$$

Hence problem (6) has the eigenvalues and eigenfunctions

$$(9) \qquad \lambda_{j,k}^{(n)} = \left(\frac{\alpha_{n,j}}{a}\right)^2 - \left(\frac{2k\pi}{T}\right)^2 = \left(\frac{\pi}{4a}\right)^2 \left[\left(\frac{4\alpha_{n,j}}{\pi}\right)^2 - (8k\alpha)^2\right],$$

$$(10) \qquad \varphi_{j,k}^n(t,r) = \left\{ \begin{array}{c} \cos(2k\pi t/T) \\ \sin(2k\pi t/T) \end{array} \right\} r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}\left(\frac{\alpha_{n,j}\, r}{a}\right)$$

for $k \geq 0$ and $j \geq 1$, with $\alpha = \frac{a}{T}$. It is clear that for each $n \geq 1$, the sequence $\left\{\lambda_{j,k}^{(n)}\right\}$ is unbounded from above and below. We shall denote by

$$(11) \qquad \Sigma_\alpha^n = cl\left\{\left(\frac{\pi}{4a}\right)^2 \left[\left(\frac{4\alpha_{n,j}}{\pi}\right)^2 - (8k\alpha)^2\right] : j \geq 1,\ k \geq 0\right\},$$

the spectrum of (6).

We recall, for the reader's convenience, a number of properties of the zeros of Bessel functions which will be used in this paper (see [1, 22, 23]).

(B1) J_ν has an infinite sequence of distinct positive zeros $(x_{\nu,j})_{j=1}^\infty$ tending to infinity.

(B2) $x_{\nu,j+1} - x_{\nu,j} := d_{\nu,j} \to \pi$ as $j \to \infty$.

(B3) $x_{\nu,j} = b_{\nu,j} - \varepsilon_{\nu,j}$, where $b_{\nu,j} = (j + \frac{\nu}{2} - \frac{1}{4})\pi$, $(j \geq 1)$, and

$$\begin{array}{rcll}
\varepsilon_{\nu,j} & > & 0, \quad \lim_{j\to\infty}\varepsilon_{\nu,j} = 0, & (\nu > \frac{1}{2}), \\
\varepsilon_{\frac{1}{2},j} & = & 0, & \\
\varepsilon_{0,j} & < & 0, \quad \lim_{j\to\infty}\varepsilon_{0,j} = 0, & \\
\varepsilon_{-\frac{1}{2},j} & = & 0. &
\end{array}$$

(B4)

$$\begin{array}{ccccl}
\dfrac{\pi(\nu^2 - \frac{1}{4})}{2d_{\nu,j}x_{\nu,j+1}} & < & \varepsilon_{\nu,j} & < & \dfrac{\pi(\nu^2 - \frac{1}{4})}{x_{\nu,j}}, \qquad \left(j \geq 2, \nu > \frac{1}{2}\right), \\
\dfrac{1}{8x_{0,j+1}} & < & -\varepsilon_{0,j} & < & \dfrac{\pi}{4x_{0,j-1}}, \qquad \left(j \geq 2, \nu = 0\right).
\end{array}$$

Using (B4), we can obtain further estimates for $b_{\nu,j}\varepsilon_{\nu,j}$ if $\nu > \frac{1}{2}$, $j \geq 2$:

$$b_{\nu,j}\varepsilon_{\nu,j} < \frac{\pi\left(\nu^2 - \frac{1}{4}\right)}{x_{\nu,j-1}}b_{\nu,j} = \frac{\pi(n-1)(n-3)}{4}\frac{b_{\nu,j-1} + \pi}{b_{\nu,j-1} - \varepsilon_{\nu,j-1}}.$$

Therefore, given $\varepsilon > 0$, we can find $j_\varepsilon \in \mathbb{N}$ such that

$$(12) \qquad b_{\nu,j}\varepsilon_{\nu,j} \leq \frac{\pi(n-1)(n-3)}{4} + \varepsilon$$

whenever $j \geq j_\varepsilon$. Similarly,

$$b_{0,j}\left|\varepsilon_{0,j}\right| < \frac{\pi}{4}\frac{b_{0,j}}{x_{0,j-1}},$$

so that, given $\varepsilon > 0$, we can find $j_\varepsilon \in \mathbb{N}$ such that

$$(13) \qquad b_{0,j}\left|\varepsilon_{0,j}\right| \leq \frac{\pi}{4} + \varepsilon$$

whenever $j \geq j_\varepsilon$.

To simplify the notations, set, for $n \geq 1$, $j \geq 1$,

$$c_{n,j} := \frac{4}{\pi} b_{\nu,j} = 4j + n - 3,$$

$$\delta_{n,j} := \frac{4}{\pi} \varepsilon_{\nu,j}, \ (n \neq 2), \quad \delta_{2,j} := -\frac{4}{\pi} \varepsilon_{0,j}.$$

Then, for $n \geq 1$, $j \geq 1$, we have

$$\frac{4}{\pi} \alpha_{n,j} = c_{n,j} - \delta_{n,j}, \quad (n \neq 2), \qquad \frac{4}{\pi} \alpha_{2,j} = c_{2,j} + \delta_{2,j}.$$

Assume now that α is irrational and has a bounded sequence of partial quotients, so that, by the results of Section 2, $M(\alpha) < \infty$. Set

$$(14) \qquad\qquad m_\alpha = \min_{p,q \in \mathbb{Z}_+} pqM\left(\frac{8p}{q}\alpha\right).$$

Notice that it follows from Corollary 2 that

$$M\left(\frac{8p}{q}\alpha\right) \leq 8pqM(\alpha),$$

and hence

$$(15) \qquad\qquad m_\alpha \leq 8M(\alpha) = 8M(\{\alpha\}).$$

Theorem 1. *Assume that $\alpha = \frac{a}{T}$ is irrational, that $M(\alpha) < \infty$ and that*

$$(16) \qquad\qquad \alpha > \frac{|(n-1)(n-3)|}{2\pi} m_\alpha,$$

where m_α is defined in (14). Then 0 is not an accumulation point of Σ_α^n.

Proof. a) $n \neq 2$. Assume that 0 is an accumulation point of Σ_α^n. Then we can find a sequence $\left\{\lambda_{j_l,k_l}^{(n)}\right\}_{l=1}^{\infty}$ of eigenvalues such that $\lambda_{j_l,k_l}^{(n)} \to 0$ if $l \to \infty$. In other terms,

$$c_{n,j_l}^2 + \delta_{n,j_l}^2 - 2c_{n,j_l}\delta_{n,j_l} - (8k_l\alpha)^2 \to 0,$$

if $l \to \infty$, which, because of $\lim_{l \to \infty} \delta_{n,j_l} = 0$, is equivalent to

$$(17) \qquad (c_{n,j_l} - 8|k_l|\alpha)(c_{n,j_l} + 8|k_l|\alpha) - 2c_{n,j_l}\delta_{n,j_l} \to 0,$$

if $l \to \infty$. If we write (17) in the form

$$|k_l|\left(\frac{c_{n,j_l}}{|k_l|} - 8\alpha\right)(c_{n,j_l} + 8\alpha|k_l|) - 2c_{n,j_l}\delta_{n,j_l}$$

$$(18) \qquad = k_l^2\left(\frac{c_{n,j_l}}{|k_l|} - 8\alpha\right)\left(\frac{c_{n,j_l}}{|k_l|} + 8\alpha\right) - 2c_{n,j_l}\delta_{n,j_l} \to 0,$$

and observe that, by (12), $2c_{n,j_l}\delta_{n,j_l}$ is bounded, and that $c_{n,j_l} + 8\alpha|k_l|$ is bounded below, then necessarily we have

$$|k_l|\left(\frac{c_{n,j_l}}{|k_l|} - 8\alpha\right) \to 0,$$

and hence also

$$\frac{c_{n,j_l}}{|k_l|} \to 8\alpha,$$

if $l \to \infty$. Consequently, writing now (18) in the form

$$k_l^2 \left(\frac{c_{n,j_l}}{|k_l|} - 8\alpha \right) \cdot 16\alpha - 2c_{n,j_l}\delta_{n,j_l} + k_l^2 \left(\frac{c_{n,j_l}}{|k_l|} - 8\alpha \right)^2 \to 0$$

if $l \to \infty$, we deduce that

$$k_l^2 \left(\frac{c_{n,j_l}}{|k_l|} - 8\alpha \right) \cdot 8\alpha - c_{n,j_l}\delta_{n,j_l} \to 0,$$

when $l \to \infty$. Consequently, for each $p, q \in \mathbb{Z}_+$, we have

$$\frac{1}{pq}(k_l q)^2 \left(\frac{pc_{n,j_l}}{q|k_l|} - \frac{8p\alpha}{q} \right) \cdot 8\alpha - c_{n,j_l}\delta_{n,j_l} \to 0,$$

when $l \to \infty$. Let $\varepsilon > 0$. There exists $l_\varepsilon \in \mathbb{N}$ such that

$$(k_l q)^2 \left| \frac{pc_{n,j_l}}{q|k_l|} - \frac{8p\alpha}{q} \right| \leq \frac{pq}{8\alpha}c_{n,j_l}\delta_{n,j_l} + \varepsilon = \frac{2pq}{\alpha\pi^2}b_{\nu,j_l}\varepsilon_{\nu,j_l} + \varepsilon,$$

whenever $l \geq l_\varepsilon$. Using (12), we see that there exists $l_\varepsilon' \in \mathbb{N}$ such that

$$(k_l q)^2 \left| \frac{pc_{n,j_l}}{q|k_l|} - \frac{8p\alpha}{q} \right| \leq \frac{pq(n-1)(n-3)}{2\pi\alpha} + 2\varepsilon,$$

whenever $l \geq l_\varepsilon'$. Combining this result with the definition of the function $M(\alpha)$, we see that

$$\frac{1}{\frac{pq(n-1)(n-3)}{2\pi\alpha} + 2\varepsilon} \leq M \left(\frac{8p\alpha}{q} \right),$$

for each $\varepsilon > 0$, and hence

$$\frac{2\pi\alpha}{pq(n-1)(n-3)} \leq M \left(\frac{8p\alpha}{q} \right).$$

In other terms,

$$\alpha \leq \frac{(n-1)(n-3)}{2\pi}pqM \left(\frac{8p\alpha}{q} \right),$$

for all $p, q \in \mathbb{Z}_+$, and hence

$$\alpha \leq \frac{(n-1)(n-3)}{2\pi}m_\alpha,$$

a contradiction.

b) $n = 2$. Assume again by contradiction that 0 is an accumulation point of Σ_α^n. Then we can find a sequence $\left\{ \lambda_{j_l,k_l}^{(2)} \right\}_{l=1}^\infty$ of eigenvalues such that $\lambda_{j_l,k_l}^{(2)} \to 0$ if $l \to \infty$. In other terms,

$$(c_{2,j_l} + \delta_{2,j_l})^2 - (8k_l\alpha)^2 \to 0,$$

if $l \to \infty$, which is equivalent to

(19) $$(c_{2,j_l} - 8|k_l|\alpha)(c_{2,j_l} + 8|k_l|\alpha) + 2c_{2,j_l}\delta_{2,j_l} \to 0,$$

if $l \to \infty$. Proceeding as in the first part of the proof, we deduce that

$$k_l^2 \left(\frac{c_{2,j_l}}{|k_l|} - 8\alpha \right) \cdot 8\alpha + c_{2,j_l}\delta_{2,j_l} \to 0,$$

when $l \to \infty$. Consequently, for each $p, q \in \mathbb{Z}_+$, we have

$$\frac{1}{pq}(k_l q)^2 \left(\frac{pc_{2,j_l}}{q|k_l|} - \frac{8p\alpha}{q} \right) \cdot 8\alpha + c_{2,j_l}\delta_{2,j_l} \to 0,$$

when $l \to \infty$. Let $\varepsilon > 0$. There exists $l_\varepsilon \in \mathbb{N}$ such that

$$(k_l q)^2 \left| \frac{pc_{2,j_l}}{q|k_l|} - \frac{8p\alpha}{q} \right| \leq \frac{pq}{8\alpha}c_{2,j_l}\delta_{2,j_l} + \varepsilon = \frac{2pq}{\alpha\pi^2}b_{\nu,j_l}\varepsilon_{\nu,j_l} + \varepsilon,$$

whenever $l \geq l_\varepsilon$. Using (13), we see that there exists $l'_\varepsilon \in \mathbb{N}$ such that

$$(k_l q)^2 \left| \frac{pc_{n,j_l}}{q|k_l|} - \frac{8p\alpha}{q} \right| \leq \frac{pq}{2\pi\alpha} + 2\varepsilon,$$

whenever $l \geq l'_\varepsilon$. Reasoning as in the first part of the proof, we see that

$$\frac{2\pi\alpha}{pq} \leq M\left(\frac{8p\alpha}{q} \right),$$

for all $p, q \in \mathbb{Z}_+$, and hence

$$2\pi\alpha \leq m_\alpha,$$

a contradiction. $\square$

Remark 1. We know that

$$\lambda_{j,k}^{(1)} = \left(\frac{\pi}{2a} \right)^2 \left[(2j-1)^2 - (4k\alpha)^2 \right],$$

and

$$\lambda_{j,k}^{(3)} = \left(\frac{\pi}{2a} \right)^2 \left[(2j)^2 - (4k\alpha)^2 \right].$$

As α is irrational, we see that, when $n = 1$ or $n = 3$, 0 is not an eigenvalue.

Remark 2. By the above proof, we can see that if 0 is an eigenvalue, its multiplicity is finite.

4. Application to the radially symmetric semilinear wave equation

Now let $J = [0, T] \times [0, a]$ and $g : J \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(\cdot, \cdot, u)$ is measurable on J for each $u \in \mathbb{R}$ and $g(t, r, \cdot)$ is continuous on $\mathbb{R}$ for a.e. $(t, r) \in J$. Moreover, assume that g satisfies the linear growth condition

$$|g(t, r, u)| \leq c_0|u| + h_0(t, x), \quad (t, x) \in J, u \in \mathbb{R},$$

where $c_0 \geq 0$ and $h_0 \in H$. As before, $\alpha = a/T$ is assumed to be irrational.

We consider the weak radially symmetric solutions of the semilinear wave equation on a ball

$$\begin{aligned}
u_{tt} - u_{rr} - \frac{n-1}{r}u_r - g(t, r, u) &= h(t, r), \quad (t, r) \in {]0, T[} \times {]0, a[}, \\
(20) \qquad\qquad u_r(t, 0) = u(t, a) &= 0, \quad t \in {]0, T[}, \\
u(0, r) - u(T, r) &= u_t(0, r) - u_t(T, r) = 0, \quad r \in {]0, a[},
\end{aligned}$$

where $h \in H$. We recall that $u \in H$ is a weak solution of this problem, provided

$$\int_0^T \int_0^a \left[u \left(\varphi_{tt} - \varphi_{rr} - \frac{n-1}{r}\varphi_r \right) - (g(\cdot, \cdot, u) + h)\varphi \right] r^{n-1} dr dt = 0$$

for every $\varphi \in D$. We define the abstract realization L in H of the radial symmetric wave operator with the periodic-Dirichlet conditions on $[0,T] \times B^n[a]$ as follows. Each $u \in H$ can be writen as the Fourier series

$$u \sim \sum_{k \geq 0,\, j \geq 1} u_{j,k} \varphi_{j,k}^n,$$

where $u_{j,k} = \langle u, \varphi_{j,k}^n \rangle$, and the $\varphi_{j,k}^n$ are the eigenfunctions defined in (10). Let

$$D(L) = \left\{ u \in H : \sum_{k \geq 0,\, j \geq 1} \left[\left(\frac{\alpha_{n,j}}{a} \right)^2 - \left(\frac{2k\pi}{T} \right)^2 \right]^2 |u_{j,k}|^2 < \infty \right\},$$

and

$$L : D(L) \to H, \quad u \to Lu = \sum_{k \geq 0,\, j \geq 1} \left[\left(\frac{\alpha_{n,j}}{a} \right)^2 - \left(\frac{2k\pi}{T} \right)^2 \right] u_{j,k} \phi_{j,k}^n.$$

Then L is a self-adjoint operator in H, with spectrum $\sigma(L) = \Sigma_\alpha^n$ given by (11). If we call N the Nemytski operator generated in H by g, we get the equivalent abstract equation

(21) $$Lu - N(u) = h, \qquad u \in D(L).$$

In view of Theorem 1 we can now apply the abstract results obtained in [4]. For any $l \notin \sigma(L)$, we denote by d_l the distance of l to $\sigma(L)$. As L is self-adjoint, we have

$$\| (L - lI)^{-1} \| = d_l^{-1}.$$

Now Theorem 1 states that there are open (maximal) intervals $]l_-, 0[$ and $]0, l_+[$, which have empty intersection with the spectrum $\sigma(L)$. When $n \neq 1$ or 3, the point $l = 0$ may be an eigenvalue, but, in that case, it follows from Remark 2 that its multiplicity is finite. Our first existence result for (20) is on the line of Theorem 2 in [4].

Theorem 2. *Assume that n is different from 1 and 3, that $\alpha = a/T$ is irrational, has a bounded sequence of partial quotients and satisfies condition*

(22) $$\alpha > \frac{|(n-1)(n-3)|}{2\pi} m_\alpha,$$

where m_α is defined in (14). Assume also that there exist constants β_0, β_1, γ and some $h_1 \in H$ such that the following assumptions are satisfied.

(i) $\beta_0 \leq \frac{g(t,r,u)-g(t,r,v)}{u-v} \leq \beta_1,$

(ii) $\left| g(t,r,u) - \frac{l_-}{2} u \right| \leq \gamma |u| + h_1(t,r),$

for a.e. $(t,r) \in [0,T] \times [0,a]$, all $u \in \mathbb{R}$ and all $v \neq u \in \mathbb{R}$. If

$$l_- \leq \beta_0 \leq \beta_1 \leq 0 \quad and \quad 0 \leq \gamma < \frac{-l_-}{2},$$

then problem (20) has at least one weak solution for each $h \in H$.

Moreover, if condition (i) holds, together with inequalities,

$$l_- < \beta_0 \leq \beta_1 < 0,$$

then problem (20) has, for each $h \in H$, a unique weak solution which can be obtained, from any $u_0 \in D(L)$, by the iterative process defined by

$$(23) \qquad Lu_{k+1} - \frac{l_-}{2}u_{k+1} = h + N(u_k) - \frac{l_-}{2}u_k, \quad (k \in \mathbb{N}).$$

Proof. It follows from assumptions (i) and (ii) that one has

$$\frac{l_-}{2} \leq \frac{g(t,r,u) - \frac{l_-}{2}u - \left(g(t,r,v) - \frac{l_-}{2}v\right)}{u - v} \leq -\frac{l_-}{2},$$

for a.e. $(t,r) \in [0,T] \times [0,a]$ and all $u \neq v \in \mathbb{R}$. Hence,

$$\left| g(t,r,u) - \frac{l_-}{2}u - \left(g(t,r,v) - \frac{l_-}{2}v\right) \right| \leq -\frac{l_-}{2}|u - v|,$$

for a.e. $(t,r) \in [0,T] \times [0,a]$ and all $u, v \in \mathbb{R}$. On the other hand, we have $d_l = -\frac{l_-}{2}$ and the first part of the result follows from the first part of Lemma 1 in [4]. The second part follows in a similar way from the second part of Lemma 1 in [4]. ☐

Of course one can state and prove a similar result based upon l_+ instead of l_-.

Theorem 3. *Assume that n is different from 1 and 3, that $\alpha = a/T$ is irrational, has a bounded sequence of partial quotients and satisfies condition (22). Assume also that there exist constants β_0, β_1, γ and some $h_1 \in H$ such that the following assumptions are satisfied.*

(i) $\beta_0 \leq \frac{g(t,r,u) - g(t,r,v)}{u - v} \leq \beta_1$,

(ii) $\left| g(t,r,u) - \frac{l_+}{2}u \right| \leq \gamma|u| + h_1(t,r)$,

for a.e. $(t,r) \in [0,T] \times [0,a]$, all $u \in \mathbb{R}$ and all $v \neq u \in \mathbb{R}$. If

$$0 \leq \beta_0 \leq \beta_1 \leq l_+ \quad and \quad 0 \leq \gamma < \frac{l_+}{2},$$

then problem (20) has at least one weak solution for each $h \in H$.
Moreover, if condition (i) holds together with inequalities

$$0 < \beta_0 \leq \beta_1 < l_+,$$

then problem (20) has, for each $h \in H$, a unique weak solution which can be obtained, from any $u_0 \in D(L)$, by the iterative process defined in (23).

The same technique, taking into account that 0 is not an eigenvalue, can be used to prove a better result when $n = 1$ or 3.

Theorem 4. *Assume that $n = 1$ or 3, that $\alpha = a/T$ is irrational and has a bounded sequence of partial quotients. Assume also that there exist constants β_0, β_1, γ and some $h_1 \in H$ such that the following assumptions are satisfied.*

(i) $\beta_0 \leq \frac{g(t,r,u) - g(t,r,v)}{u - v} \leq \beta_1$,

(ii) $\left| g(t,r,u) - \frac{(l_- + l_+)}{2}u \right| \leq \gamma|u| + h_1(t,r)$,

for a.e. $(t,r) \in [0,T] \times [0,a]$, all $u \in \mathbb{R}$ and all $v \neq u \in \mathbb{R}$. If

$$l_- \leq \beta_0 \leq \beta_1 \leq l_+ \quad and \quad 0 \leq \gamma < \frac{(l_- + l_+)}{2},$$

then problem (20) has at least one weak solution for each $h \in H$.

Moreover, if condition (i) holds together with inequalities

$$l_- < \beta_0 \leq \beta_1 < l_+,$$

then problem (20) has, for each $h \in H$, a unique weak solution which can be obtained, from any $u_0 \in D(L)$, by the iterative process defined by

$$Lu_{k+1} - \frac{l_- + l_+}{2}u_{k+1} = h + N(u_k) - \frac{l_- + l_+}{2}u_k, \quad (k \in \mathbb{N}).$$

For $n = 1$ or 3 and some irrational α, one can determine explicitly the optimal values of l_- and l_+. For example, let us consider the special case of (20) when $n = 3$ and $\alpha = 1/\sqrt{2}$. Clearly, α is a quadratic irrational and hence has bounded partial quotients. It follows from (9) and property (B3) of Bessel functions that the eigenvalues of L are given by

$$\lambda_{j,k}^{(3)} = \left(\frac{\pi}{a}\right)\left(j^2 - 2k^2\right), \quad (j \geq 1, \ k \geq 0).$$

This immediately implies that 0 is not an eigenvalue, and, using classical results on the Pell equations

$$j^2 - 2k^2 = \pm 1,$$

one sees that the largest negative eigenvalue is $-\left(\frac{\pi}{a}\right)^2$ and the smallest positive eigenvalue is $\left(\frac{\pi}{a}\right)^2$, both having infinite multiplicity (see e.g. [11, 18]). Thus

$$l_\pm = \pm\left(\frac{\pi}{a}\right)^2,$$

and Theorem 4 takes the following form, which can be seen as an extension to $n = 3$ of results for $n = 1$ contained in Theorem 3 and the end of Section 3 of [14], and in Theorem 1 of [17].

Corollary 3. *Assume that $n = 3$, $\alpha = 1/\sqrt{2}$, and that there exist a positive constant*

(24) $$\gamma < \left(\frac{\pi}{a}\right)^2$$

and some $h_1 \in H$ such that the following assumptions are satisfied.

  (i) $|g(t,r,u) - g(t,r,v)| \leq \left(\frac{\pi}{a}\right)^2 |u - v|$,
 (ii) $|g(t,r,u)| \leq \gamma|u| + h_1(t,r)$,

for a.e. $(t,r) \in [0,T] \times [0,a]$ and all u, $v \in \mathbb{R}$. Then problem (20) has at least one weak solution for each $h \in H$.

Moreover, if conditions (i) and (ii) are replaced by

(iii) $|g(t,r,u) - g(t,r,v)| \leq \gamma|u - v|$

for a.e. $(t,r) \in [0,T] \times [0,a]$, all u, $v \in \mathbb{R}$, and some positive γ satisfying inequality (24), then problem (20) has, for each $h \in H$, a unique weak solution which can be obtained, from any $u_0 \in D(L)$, by the iterative process defined by

$$Lu_{k+1} = h + N(u_k), \quad (k \in \mathbb{N}).$$

We close this paper with a result whose statement and proof, based on Corollary 1 in [15], is very similar to that of Theorem 5 of [4].

Theorem 5. *Assume again that $\alpha = a/T$ is irrational, has a bounded sequence of partial quotients and satisfies condition (22). Assume moreover that $g = g(u)$ and there exist real numbers a and b such that the following conditions hold.*

(i) $l_- < a \leq b < l_+$.

(ii) $a \leq \frac{g(u)-g(v)}{u-v} \leq b$ *for all* $u, v \in \mathbb{R}, u \neq v$.

(iii) $\left[\liminf_{|u| \to \infty} \frac{g(u)}{u}, \limsup_{|u| \to \infty} \frac{g(u)}{u} \right] \cap \sigma(L) = \emptyset$.

Then problem (20) has at least one weak solution for each $h \in H$.

Remark 3. It follows from inequality (15) that, in Theorems 2, 3, 5, condition (22) can be replaced by the weaker but more concrete inequality

$$\alpha > \frac{4|(n-1)(n-3)|}{\pi} M(\{\alpha\}).$$

References

[1] M. Abramowitz and I. Stegun (eds), *Handbook of Mathematical Functions*, Dover, New York, 1966. MR **34:**8606

[2] A. Baker, *Continued fractions of transcendental numbers,* Matematika **9** (1962), 1-8. MR **26:**2394

[3] A.K. Ben-Naoum and J. Berkovits, *On the existence of periodic solutions for semilinear wave equations on ball in* $\mathbb{R}^n$ *with the space dimension n odd,* Nonlinear Anal. TMA 24 (1995), 241-250. MR **95k:**35016

[4] A.K. Ben-Naoum and J. Mawhin, *The periodic-Dirichlet problem for some semilinear wave equations,* J. Differential Equations **96** (1992), 340-354. MR **93h:**35009

[5] A.K. Ben-Naoum and J. Mawhin, *Periodic solutions of some semilinear wave equations on balls and on spheres,* Topological Methods in Nonlinear Anal. **1** (1993), 113-137. MR **94c:**35019

[6] F. Bernstein, *Ueber eine Anwendung der Mengenlehre auf ein aus der Theorie der säkularen Störungen herrührendes Problem,* Math. Ann. **71** (1912), 417-439.

[7] E. Borel, *Sur les équations aux dérivées partielles à coefficients constants et les fonctions non analytiques,* C.R. Acad. Sci. Paris **121** (1895), 933-935.

[8] E. Borel, *Les probabilités dénombrables et leurs applications arithmétiques,* Rend. Circ. Mat. Palermo **27** (1909), 247-271.

[9] E. Borel, *Sur un problème de probabilités relatifs aux fractions continues,* Math. Ann. **72** (1912), 578-584.

[10] F.E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces,* Proc. Symp. Pure Math. No. 18-2, Amer. Math. Soc., Providence, 1976. MR **53:**8982

[11] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers,* 5th ed., Oxford Univ. Press, 1979. MR **81i:**10002

[12] V. Jarník, *Zur metrischen Theorie der diophantischen Approximationen,* Prace Mat.-Fiz. **36** (1928), 91-106.

[13] S. Lang, *Introduction to Diophantine Approximations,* New Enlarged Edition, Springer, New York, 1995. MR **96h:**11067

[14] J. Mawhin, *Solutions périodiques d'équations aux dérivées partielles hyperboliques non linéaires,* Mélanges Théodore Vogel, Rybak, Janssens, Jessel ed., Presses de l'ULB, Bruxelles, 1978, 301-315. MR **87d:**00007

[15] J. Mawhin, *Semilinear equations of gradient type in Hilbert spaces and applications to differential equations,* Nonlinear Differential Equations, Invariance, Stability and Bifurcation, Academic Press, New York, 1981, 269-282. MR **82i:**47095

[16] J. Mawhin, *Periodic solutions of some semilinear wave equations and systems : a survey,* Chaos, Solitons and Fractals **5** (1995), 1651-1669. MR **96m:**34074

[17] P.J. McKenna, *On solutions of a nonlinear wave equation when the ratio of the period to the length of the interval is irrational,* Proc. Amer. Math. Soc. **93** (1985), 59-64. MR **86f:**35017

[18] I. Niven and H.S. Zuckerman, *The Theory of Numbers,* 4th ed., Wiley, New York, 1980. MR **81g:**10001

[19] A.W. Rockett and P. Szüsz, *Continued Fractions,* World Scientific, Singapore, 1992. MR **93m:**11060

[20] G. Sell, *The prodigal integral,* Amer. Math. Monthly **84** (1977), 162-167. MR **55:**587

[21] J. Shallit, *Real numbers with bounded partial quotients : a survey,* Enseignement math. **38** (1992), 151-187. MR **93g:**11011

[22] M.W. Smiley, *Time periodic solutions of nonlinear wave equations in balls,* Oscillations, Bifurcation and Chaos, Toronto 1986, Canad. Math. Soc. Confer. Proc., 1987, 287-297. MR **89c:**35106

[23] G. Watson, *A Treatise on the Theory of Bessel Functions*, University Press, Cambridge, 1922.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OULU, OULU, FINLAND
E-mail address: `jberkovi@sun3.oulu.fi`

UNIVERSITÉ CATHOLIQUE DE LOUVAIN, INSTITUT MATHÉMATIQUE, B-1348 LOUVAIN-LA-NEUVE, BELGIUM
E-mail address: `mawhin@amm.ucl.ac.be`