SHELLABILITY IN REDUCTIVE MONOIDS

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Abstract. The purpose of this paper is to extend to monoids the work of Björner, Wachs and Proctor on the shellability of the Bruhat-Chevalley order on Weyl groups. Let $M$ be a reductive monoid with unit group $G$, Borel subgroup $B$ and Weyl group $W$. We study the partially ordered set of $B \times B$-orbits (with respect to Zariski closure inclusion) within a $G \times G$-orbit of $M$. This is the same as studying a $W \times W$-orbit in the Renner monoid $R$. Such an orbit is the retract of a ‘universal orbit’, which is shown to be lexicographically shellable in the sense of Björner and Wachs.

Introduction

The combinatorial concept of shellability of a simplicial complex provides a powerful link between algebra, topology of geometry, [3], [6], [21]. A shellable complex has the homotopy type of a wedge of $r$-spheres and its Stanley-Reisner ring is Cohen-Macaulay. Björner [1] and Björner and Wachs [2] have introduced the stronger concept of lexicographic shellability of a poset. It has been shown in [2], [13] that the Bruhat-Chevalley order on a Weyl group is lexicographically shellable. This in turn has connections to the geometry of Schubert varieties [8], [9]. In this paper we apply the Björner-Wachs approach to reductive monoids.

Reductive monoids are Zariski closures of reductive groups. They arise naturally connection with embeddings of some symmetric spaces [7], the behaviour at infinity of a Lie group [22] and Schur algebras [10]. They have been studied for the last 20 years by Lex Renner and the author. There is a monograph [15] on the earlier work. There is also an excellent expository paper by Solomon [20].

Our focus in this paper is on the Bruhat decomposition for reductive monoids [17], where the Weyl group $W$ is replaced by the Renner monoid $R$. The Bruhat-Chevalley order on $R$, first studied by Renner [17], [15], remains quite mysterious. We studied this order in detail in an earlier paper [12] (with Pennell and Renner). In particular, we obtained an algebraic description of the order. The main purpose of the present paper is to study this order on the $W \times W$-orbits of $R$. We show that such an orbit is isomorphic to a nicely constructed poset $W_{I,K}$, where $I$ is a set of simple reflections and $K$ is a union of some components of $I$. $W_{I,K}$ is a retract of a universal orbit $W_I = W_{I,G}$ (which arise as maximal orbits of some Renner monoid). Making use of the methods of Björner and Wachs, we show that a universal orbit $W_I$ and its dual are lexicographically shellable, Eulerian posets. In particular their Stanley-Reisner rings are Gorenstein. The question of whether the Bruhat-Chevalley order is shellable on $R$, remains open.

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1. Reductive monoids

Let $k$ be an algebraically closed field. By a reductive monoid $M$ we will mean an irreducible linear algebraic monoid $M$ defined over $k$ such that the unit group $G$ is reductive. Let $T$ be a maximal torus contained in a Borel subgroup $B$ of $G$. Let $W = N_G(T)/T$ denote the Weyl group of $G$ and let $S$ denote the generating set of simple reflections of $W$. Then $G$ has the Bruhat decomposition:

\[ G = \bigsqcup_{w \in W} B w B. \]

By the theory of torus embeddings [11], the diagonal idempotents (i.e. the idempotents in $T$) form a finite lattice that is isomorphic to the face lattice of a rational polytope $P$. We have shown in [14] that there is a diagonal idempotent cross-section of $G$ and $G$-orbits of $M$ such that for all $e, f \in \Lambda$,

\[ e \leq f \iff e \in M f M. \]

Here as usual [5], $e \leq f$ means that $ef = e = fe$. $\Lambda$ is a finite lattice called the cross-section lattice of $M$. $\Lambda$ may also be viewed as the quotient of the face lattice of $P$ by the action of $W$. All maximal chains of $\Lambda$ have the same length. $\Lambda$ is unique up to conjugacy by an element of $W$. We note that in the case of the multiplicative monoid $M_n(k)$ of all $n \times n$ matrices,

\[ \Lambda = \left\{ \begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right\} \mid 0 \leq r \leq n \]

is the usual set of idempotent representatives of matrices of different ranks. In general, determining the possible lattices $\Lambda$ (in terms of face lattices of polytopes) remains a difficult open problem. However, when $M$ is the Zariski closure of the image of an irreducible representation of a reductive group, the problem has been solved in [16].

Example 1.1. The table in Figure [1] lists the cross-section lattice $\Lambda$ and the polytope $P$ when $M$ is the closure of the image of a representation of $M_4(k)$.

In [17] the Bruhat decomposition (1) is extended to $M$ as

\[ M = \bigsqcup_{\sigma \in R} B \sigma B \]

where $R = N_G(T)/T$ is the Renner monoid. $W$ is the unit group of $R$. If $W(e) = WeW$, $e \in \Lambda$, then

\[ R = \bigsqcup_{e \in \Lambda} W(e). \]

By a maximal $W \times W$-orbit, we will mean an orbit maximal in $R\backslash W$. If $R$ has a zero, then by a minimal $W \times W$-orbit we will mean an orbit minimal in $R\backslash \{0\}$. We note that for $M_n(k)$, $W$ is the symmetric group of permutation matrices, $R$ is the symmetric inverse semigroup of all partial permutation matrices, and a $W \times W$-orbit $W(e)$ consists of partial permutation matrices of a particular rank.
Bruhat-Chevalley order

The Bruhat-Chevalley order on the Weyl group $W$, first studied in the 1950s by Chevalley [4], is defined as

$$x \leq y \quad \text{if} \quad BxB \subseteq ByB. \quad (5)$$

As is well known, this is equivalent to $x$ being a subword of a reduced expression $y = s_1 \cdots s_m$, $s_1, \ldots, s_m \in S$. The length $\ell(y)$ is defined to be $m$. If $w_1, \ldots, w_n \in$
Lemma 2.1. Let \( w, x, y, z \in W \). Then:

(i) If \( x \leq y \) and \( xw = x \cdot w \), then \( x \cdot w \leq y \cdot u \) for some \( u \leq w \).

(ii) If \( x \cdot w \leq x \cdot z \), then \( w \leq z \).

Proof. (i) We proceed by induction on \( \ell(w) \). If \( \ell(w) = 0 \), this is clear. So let \( \ell(w) > 0 \). Then \( w = w_1 \cdot s, s \in S \). By induction hypothesis, \( xw_1 = x \cdot w_1 \leq y \cdot u_1 \) for some \( u_1 \leq w_1 \). Since \( xw = xw_1 \cdot s \), we see that either \( xw \leq yu_1 \) or else \( xw \leq yu_1 \cdot s = y \cdot u_1 s \).

(ii) We proceed by induction on \( \ell(x) \). If \( \ell(x) = 0 \), this is clear. So let \( \ell(x) > 0 \). Then \( x = s \cdot x_1, s \in S \). Then either \( x \cdot w \leq x_1 \cdot z \) or else \( x_1 \cdot w \leq x_1 \cdot z \). In either case \( x \cdot w \leq x_1 \cdot z \). By induction hypothesis, \( w \leq z \).

For \( I \subseteq S \), let \( W_I \) denote the parabolic subgroup of \( W \) generated by \( I \) and let

\[
D_I = \{ x \in W \mid xw = x \cdot w \quad \text{for all } w \in W_I \}.
\]

Let \( w_0, w_0 \) denote the longest elements of \( W_I \) and \( W \) respectively. Then for \( x \in D_I \), \( w_0xw_0 \in D_I \) and

\[
x \leq y \iff w_0xw_0 \leq w_0xw_0 \quad \text{for all } x, y \in D_I.
\]

Moreover for all \( x \in D_I \),

\[
\ell(w_0xw_0) = \ell(w_0) - \ell(x) = \ell(w_0w_0) - \ell(x).
\]

Lemma 2.2. Let \( x, y \in D_I \), \( w, u \in W_I \) such that \( xw \leq yw \). Then \( w = w_1 \cdot w_2 \) with \( xw_1 \leq y \) and \( w_2 \leq u \).

Proof. We proceed by induction on \( \ell(u) \). If \( \ell(u) = 0 \), this is clear. So let \( \ell(u) > 0 \). Then \( u = u_1 \cdot s, s \in I \). If \( xw \leq yu_1 \), then we are done by the induction hypothesis. Otherwise, since \( yu = yu_1 \cdot s, xws < xw \). Since \( xws = x \cdot ws \) and \( xw = x \cdot w \), we see by Lemma 2.1 (ii) that \( ws < w \). So \( w = w' \cdot x \) and \( xw' \leq yu_1 \). By induction hypothesis \( w' = w_1 \cdot w_2 \) with \( xw_1 \leq y \) and \( w_2 \leq u_1 \). Then \( w = w_1 \cdot w_2 \cdot s \) and \( w_2 \cdot s \leq u \).

For \( K \subseteq I \subseteq S \), we will write \( K \triangleleft I \) if \( K \) is a union of some components of \( I \) (including the possibility that \( K = \emptyset \)). In such a case

\[
W_I = W_K \times W_{I \setminus K}, \quad D_K = D_I W_{I \setminus K}.
\]

Now for monoids. The order (12) on \( W \) extends naturally to \( R \) if we define

\[
\sigma \leq \theta \quad \text{if } B\sigma B \subseteq B\theta B.
\]

Renner [17] has shown that all the maximal chains in \( R \) have the same length.

Example 2.3. For \( M = M_2(k) \), the poset \( (R, \leq) \) is given in Figure 2.

In general the order \( \leq \) on \( R \) is much more subtle than on \( W \). We have studied this order in [12] (with Pennell and Renner). In particular we found an algebraic
description of this order that we now describe. For \( e \in \Lambda \), we have the parabolic subgroups,

\[
W(e) = \{ w \in W \mid we = ew \},
\]

\[
W_e = \{ w \in W \mid we = e = ew \}
\]
of \( W \). Then by (9),

\[
W(e) = W_e \times \widetilde{W(e)}
\]
from some parabolic subgroup \( \widetilde{W(e)} \) of \( W \). If \( W(e) = W_I \) and \( W_e = W_K \), then let

(11)

\[
D(e) = D_I, \quad D_e = D_K.
\]

Then by (9),

(12)

\[
D_e \cap W(e) = \widetilde{W(e)}, \quad D_e = D(e)\widetilde{W(e)}.
\]

We note that for \( M_n(k) \), if \( e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \), then \( W(e) \) consists of permutation matrices of the form \( \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \), \( W_e \) consists of permutation matrices of the form \( \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \) and \( \widetilde{W(e)} \) consists of permutation matrices of the form \( \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \).

If \( \sigma \in R \), then

(13)

\[
\sigma = xey \quad \text{for unique } e \in \Lambda, \ x \in D_e, \ y \in D(e)^{-1}.
\]

We call this the standard form of \( \sigma \). Let \( \sigma, \theta \in R \). Let \( \sigma = xey, \sigma' = x'e'y' \) in standard form. Then by (12),

(14)

\[
\sigma \leq \sigma' \iff e \leq e', x \leq x'w, w^{-1}y' \leq y \quad \text{for some } w \in W(e')W_e.
\]

Fix \( e \in \Lambda \). Our interest is in the poset \( \mathcal{W}(e) = W_eW \). Then for \( \sigma = xey, \sigma' = x'e'y' \) in standard form, (12) simplifies to

(15)

\[
\sigma \leq \sigma' \iff x \leq x'w, w^{-1}y' \leq y \quad \text{for some } w \in W(e).
\]
Let $W(e) = W_I$. We call $I$ the type of the $W \times W$-orbit $W(e)$. If $W_e = W_K$, $K \subset I$, then we call $K$ the subtype of $W(e)$. Thus the $W \times W$-orbits $W(e)$ are classified first according to type, and then according to subtype. We will call $W(e)$ universal if $W_e = 1$ and fundamental if $W_e = W(e)$. Let $v_0, w_0$ denote respectively the longest elements of $W_I$ and $W$. Then $w_0e, ev_0w_0$ are respectively the maximum and minimum elements of $W(e)$. If $\sigma = xey \in W(e)$ in standard form, then any maximal chain from $\sigma$ to $ev_0w_0$ has length

$$\ell(\sigma) = \ell(x) - \ell(y) + \ell(v_0w_0) = \ell(x) + \ell(v_0yw_0).$$

This agrees with the definition of length $\ell(\sigma)$ given by Solomon [19] and Renner [18].

**Example 2.4.** Let $M = M_3(k)$. The poset of rank 2 elements of $R$ is given in Figure 3 and the poset of rank 1 elements is given in Figure 4.
We now proceed to obtain a more useful description of the \( W \times W \)-orbits \( W(e) \). Let \( I \subseteq S, K \triangleleft I \). Let \( D_I \) be as in (13) and set
\[
W_{I,K} = D_I \times W_{I \setminus K} \times D_I^{-1}.
\]
Let \( v_0 \) denote the longest element of \( W_I \) and let
\[
\bar{w} = v_0 w v_0, \quad w \in W_I.
\]
For \( \sigma = (x, w, y), \sigma' = (x', w', y') \in W_{I,K} \), define
\[
\sigma \leq \sigma' \quad \text{if} \quad w = w_1 * w_2 * w_3 \quad \text{with} \quad xw_1 \leq x', w_2 \leq w', \bar{w}_3 y \leq y'.
\]
Also define the length
\[
\ell(\sigma) = \ell(x) + \ell(w) + \ell(y).
\]
We call
\[
W_I = W_{I,0}, \quad W_I^* = W_{I,I}
\]
respectively the universal and fundamental orbit of type \( I \). Clearly
\[
W_I^* = D_I \times D_I^{-1} \cong D_I \times D_I.
\]
Let \( W_{I,K}^* = W_{I,K} \) as sets. For \( \sigma = (x, w, y), \sigma' = (x', w', y') \in W_{I,K}^* \), define
\[
\sigma \leq \sigma' \quad \text{if} \quad w = w_1 * w_2 * w_3 \quad \text{with} \quad xw_1 \leq x', w_2 \leq w', w_3 y \leq y'.
\]
Note the subtle difference between (29) an (20).

**Theorem 2.5.** (i) $W_{I,K}^*$ is isomorphic to the dual of $W_{I,K}$.
(ii) $W_{I,K}$ is a retract of $W_I$ and $W_{I'}^*$ is a retract of $W_{I,K}$ with the fibres being isomorphic respectively to $W_K$ and $W_{I,K}$.
(iii) The orbit $W(\epsilon)$ is isomorphic to $W_{I,K}$ if $I$, $K$ are respectively the type and subtype of $W(\epsilon)$.
(iv) Any maximal orbit $W(\epsilon)$ is universal and if $R$ has a zero, then any minimal orbit is fundamental.

**Proof.** Let $u_0, v_0, w_0$ denote respectively the longest elements of $W_{I \setminus K}, W_I$ and $W$.
For $w \in W_{I \setminus K}$, let $\bar{w} = v_0wv_0 = u_0w_0$.
(i) For $\sigma = (x, w, y) \in W_{I,K}$, let
\[ \Phi(\sigma) = (w_0xv_0, u_0w, v_0yw_0) \in W_{I,K}^* . \]
Let $\sigma = (x, w, y), \sigma' = (x', w', y') \in W_{I,K}$ such that $\sigma \leq \sigma'$. Then $w = w_1 \cdot w_2 \cdot w_3$ with $xw_1 \leq x', w_2 \leq w', w_3y \leq y'$.
By computing the lengths, we see that $w_3^{-1} \cdot v_0y'w_0 \leq v_0yw_0$.
From $xw_1 \leq x'$, we deduce that $w_0x'v_0 \cdot w_0 = w_0xv_0 = w_0xv_0 \cdot v_0w_0$.
Since $v_0 = \bar{w}_1^{-1} \cdot v_0w_1$, we see by Lemma 2.1 (ii) that $w_0x'v_0 \cdot \bar{w}_1^{-1} \leq w_0xv_0$.
Similarly from $w_3y \leq y'$, we deduce that $w_3^{-1} \cdot v_0y'w_0 \leq v_0yw_0$.
Thus
\begin{equation}
(24) \quad w_0x'v_0 \cdot \bar{w}_1^{-1} \leq w_0xv_0, \quad w_3^{-1} \cdot v_0y'w_0 \leq v_0yw_0 .
\end{equation}
Since $w_2 \leq w'$ and $w = w_1 \cdot w_2 \cdot w_3$, we see by Lemma 2.1 (i) that
\begin{equation}
(25) \quad w \leq w'_1 \cdot w' \cdot w'_3 \quad \text{for some } w'_1 \leq w_1, w'_3 \leq w_3 .
\end{equation}
Let
\[ w''_1 = u_0(w'_1)^{-1}u_0, \quad w''_2 = u_0w'_1w'w'_3, \quad w''_3 = (w'_3)^{-1} . \]
By computing the lengths, we see that $u_0w' = w''_1 \cdot w''_2 \cdot w''_3$. By (24), (25)
\[ w_0x'v_0 \cdot w''_1 \leq w_0xv_0, \quad w''_2 \leq u_0w, \quad w''_3 \cdot v_0y'w_0 \leq v_0yw_0 .
\]
Hence $\Phi(\sigma') \leq \Phi(\sigma)$. Similarly $\Phi(\sigma') \leq \Phi(\sigma)$ implies that $\sigma \leq \sigma'$.
(ii) Clearly $W_{I,K} \subseteq W_I$. The natural map from $W_I$ to $W_{I,K}$ in (9), yields the retracts from $W_I$ to $W_{I,K}$. Clearly the retraction from $W_I$ to $W_{I,K}$ factors through $W_{I,K}$.
(iii) Define $\Phi : W_{I,K} \rightarrow W(\epsilon)$ as $\Phi(x, w, y) = xwev_0yw_0$. Let $\sigma(x, w, y), \sigma' = (x', w', y') \in W_{I,K}$. First suppose that $\sigma \leq \sigma'$. Then $w = w_1 \cdot w_2 \cdot w_3$ with $xw_1 \leq x', w_2 \leq w', w_3y \leq y'$. Then $xw_1w_3 \leq x'w'$. By Lemma 2.1 (i),
\begin{equation}
(26) \quad xw = xw_1w_2 \cdot w_3 \leq x'w' \cdot u \quad \text{for some } u \leq w_3 .
\end{equation}
So
\[ \bar{w}y \leq \bar{w}_3yu \leq y' .
\]
So
\[ v_0 \cdot v_0y'w_0 = y'w_0 \leq \bar{w}_3yv_0 = v_0u \cdot v_0yw_0 .
\]
Since $v_0 = v_0u \cdot u^{-1}$, Lemma 2.1 (ii) implies that $u^{-1} \cdot v_0y'w_0 \leq v_0yw_0$. Combined with (29), we see by (13) that $\Phi(\sigma) \leq \Phi(\sigma')$. 


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Assume conversely that $\Phi(\sigma) \leq \Phi(\sigma')$. Then by (15), there exists $u \in W_I$ such that
\[ xw \leq x'w'u, \quad u^{-1}v_0y'w_0 \leq v_0yw_0. \tag{27} \]
Since $x, x' \in D_I$, $w, w'u \in W_I$, we see by Lemma 2.2 that $w = w_1 * w_2 * w_3$ with $xw_1 \leq x'$, $w_2 \leq w'$, $w_3 \leq u$. Hence by (27)
\[ w_3^{-1} * v_0y'w_0 \leq u^{-1} * v_0y'w_0 \leq v_0yw_0. \tag{28} \]
By comparing lengths we see that $v_0 = w_3^{-1}v_0 \equiv w_3$. Hence by (28),
\[ w_3^{-1}v_0 \equiv w_3y = v_0y \leq w_3^{-1}v_0 \equiv y'. \]
By Lemma 2.1 (ii), $w_3y \leq y'$. Hence $\sigma \leq \sigma'$.

(iv) If $W(e)$ is maximal, then by [15] Chapter 10, $W_e = 1$. Hence $W(e)$ is universal. If $W(e)$ is minimal, then by [15] $W_e = W(e)$ and $W(e)$ is fundamental. This completes the proof.

Remark 2.6. (i) the isomorphism in Theorem 2.5 (iii) preserves length as defined in (10), (29).

(ii) Let $G_0$ be a semisimple group with Weyl group $W$ and let $K \trianglelefteq I \subseteq S$. Let $\theta$ be an irreducible representation of $G_0$ such that $K$ is the set of simple reflections fixing the highest weight vector. Let $M = k\theta(G_0)$. By [15], there exists $e \in A$ such that $W(e)$ is of type $I$ and subtype $K$. Hence by Theorem 2.5 (iii), $W(e) \cong W_{I,K}$. Thus each of the synthetically constructed posets $W_{I,K}$ arise naturally. For this reason we call $W_{I,K}$, $W \times W$-orbits.

(iii) The orbit in Figure 3 is universal while the orbit in Figure 4 is fundamental.

(iv) If $M$ is a canonical monoid (unique minimal orbit and this orbit is of type $\emptyset$), then by (15), every $W \times W$-orbit is universal.

(v) If $M$ is a dual canonical monoid (unique maximal orbit and this orbit is of type $\emptyset$), then every $W \times W$-orbit is fundamental.

(vi) By (7), (22), any fundamental orbit is isomorphic to its dual.

(vii) If no component of $W(e) = W_{I,K}$ is of type $A_{\ell}$ ($\ell > 1$), $D_{\ell}$ ($\ell$ odd) or $E_6$, then $\bar{w} = w$ in (13), for all $w \in W_{I,K}$. Hence [15], (23) are identical in this situation and $W_{I,K} = W_{I,K}^*$. Hence by Theorem 2.5 $W(e)$ is isomorphic to its dual. This is what is happening in Example 2.4.

Example 2.7. Let $M = M_4(k)$. The universal $W \times W$-orbit of rank 3 partial permutations is not isomorphic to its dual. This is most easily seen using Theorem 2.5. Let $s_1 = (12), s_2 = (23), s_3 = (34), I = \{s_1, s_2\}$. We claim that $W_I \not\cong W_I^*$. The elements of length 1 are
\[ (s_3, 1, 1), (1, s_1, 1), (1, s_2, 1), (1, 1, s_3). \]
In $W_I^*$, $(1, s_2, 1)$ is covered by 6 elements:
\[ (s_2s_3, 1, 1), (s_3, s_2, 1), (1, s_1s_2, 1), (1, s_2s_1, 1), (1, 1, s_3s_2), (1, s_2, s_3). \]
In $W_I$, $(s_3, 1, 1)$ is covered by
\[ (s_2s_3, 1, 1), (s_3, s_1, 1), (s_3, s_2, 1), (s_3, 1, s_3) \]
$(1, s_1, 1)$ is covered by
\[ (s_3, s_1, 1), (1, s_1s_2, 1), (1, s_2s_1, 1), (1, s_1, s_3), (1, 1, s_3s_2) \]
Let $P$ be a finite partially ordered set with a maximum element $1$ and minimum element $0$, and so that all maximal chains have the same length. If $a, b \in P$, write $a \rightarrow b$ if $a$ covers $b$ (i.e., $a > b$ and there is no $c$ such that $a > c > b$). For an $a \in P$, let $\ell(a)$ denote the length of a maximal chain from $a$ to $0$. $P$ is said to be Eulerian (cf. [21]) if for $a \leq b$, the Möbius function $\mu(a, b) = (-1)^{\ell(a)+\ell(b)}$. Of much importance in the study of $P$ has been the topological concept of shellability of the order complex of all chains in $P$. We now briefly review the stronger concept of lexicographic shellability introduced by Björner and Wachs [2]. The edges of $P$ are labeled recursively starting from the top, whereby for $a \rightarrow b$ the label depends on the choice of a maximal chain from 1 to $a$. Fix $a > b$ and a maximal chain from 1 to $a$, the labeling must be such that there is a unique maximal chain from $a$ to $b$ with increasing labels and so that this chain is lexicographically less than any other maximal chain from $a$ to $b$.

It is shown in [2] that $D_I$ is lexicographically shellable. It therefore follows from [22] that the fundamental orbit $W_I^f$ is lexicographically shellable and hence that its Stanley-Reisner ring is Cohen-Macaulay. Figure 4 shows that in general $W_I^f$ is not Eulerian and the Stanley-Reisner ring is not Gorenstein.

We proceed to show that $W_I$ is an Eulerian lexicographically shellable poset. Let $v_0, w_0$ denote the longest elements of $W_I$, $W$, respectively. For $w \in W_I$, let $\tilde{w} = v_0 w v_0 \in W_I$. Let $\sigma = (x, w, y) \in W_I$. Let $\sigma' \in W_I, \sigma \rightarrow \sigma'$. We will say that the edge is of type 1 if

$$\sigma' = (x', u \ast w, y), \quad x \rightarrow x' u \text{ in } W.$$  

(29)

We will say that the edge is of type 2 if

$$\sigma' = (x, w', y), \quad w \rightarrow w' \text{ in } W_I.$$  

(30)

We will say that the edge is of type 3 if

$$\sigma' = (x, w \ast \bar{v}, y'), \quad y \rightarrow vy' \text{ in } W.$$  

(31)

We see by (19), (20) that exactly one of these cases occurs.

Let $s_1, \ldots, s_m, s_{m+1}, \ldots, s_n \in S$. Then

$$s_1 \cdots s_m = x w_1, \quad x \in D_I, w_1 \in W_I,$$

$$s_{m+1} \cdots s_n = w_2 y, \quad y \in D_I^{-1}, w_2 \in W_I.$$  

(32)

Let

$$\Phi(s_1 \cdots s_m; s_{m+1} \cdots s_n) = (x, w_1 \bar{w}_2, y) \in W_I.$$  

(33)

Let $\sigma(x, w, y), w = w_1 \bar{w}_2$. Then by (20),

$$\ell(\sigma) = \ell(x) + \ell(w) + \ell(y) \leq \ell(x) + \ell(w_1) + \ell(w_2) + \ell(y) \leq n.$$
We will say that \( s_1 \cdots s_m; s_{m+1} \cdots s_n \) is an expression for \( \sigma \). If \( m = 0 \), we write the expression as 1; \( s_1 \cdots s_n \). If \( m = n \), we write the expression as \( s_1 \cdots s_n; 1 \). If \( \ell(\sigma) = n \), we will say that the expression is reduced. By (33), this happens if and only if

\[
\ell(xw_1) = m, \quad \ell(w_2y) = n - m \quad \text{and} \quad w = w_1 * \tilde{w}_2.
\]

Lemma 3.1. Let \( s_1 \cdots s_m; s_{m+1} \cdots s_n \) be a reduced expression for \( \sigma \) and let \( \sigma \to \sigma' \). Then for some \( 1 \leq i \leq n, s_1 \cdots \hat{s}_i \cdots s_m; s_{m+1} \cdots s_n \) or \( s_1 \cdots s_m; s_{m+1} \cdots \hat{s}_i \cdots s_n \) is a reduced expression for \( \sigma' \). Moreover, \( i \) is unique.

Proof. The existence of the reduced expression for \( \sigma' \) follows from (32), (33), and (34). We claim that \( i \) is unique. So suppose that \( \sigma' \) is obtained by deleting either \( s_i \) or \( s_j \) from the expression for \( \sigma \). First suppose that the \( \sigma \to \sigma' \) is of type 1. Let \( \sigma' \) be as in (29). Then \( i,j \leq m, \ell(x'w_1) = m - 1 \) and

\[
x'w_1 = s_1 \cdots \hat{s}_i \cdots s_m = s_1 \cdots \hat{s}_j \cdots s_m
\]

which implies \( i = j \). Next assume that the \( \sigma \to \sigma' \) is of type 2 and that \( \sigma' \) is as in (30). Now \( w = w_1 * \tilde{w}_2 \to w' \). So

\[
w' = w'_1 * \tilde{w}_2, w_1 \to w'_1 \quad \text{or} \quad w' = w_1 * \tilde{w}'_2, w_2 \to w'_2
\]

with the two cases being exclusive. In the first case \( i, j \leq m, \ell(xw'_1) = m - 1 \), and

\[
xw'_1 = s_1 \cdots \hat{s}_i \cdots s_m = s_1 \cdots \hat{s}_j \cdots s_m
\]

which implies \( i = j \). In the second case, \( i, j > m, \ell(w'_2y) = n - m - 1 \) and

\[
w'_2y = s_{m+1} \cdots \hat{s}_i \cdots s_n = s_{m+1} \cdots \hat{s}_j \cdots s_n
\]

and this too implies \( i = j \). Finally assume that \( \sigma \to \sigma' \) is of type 3 and that \( \sigma' \) is as in (31). Then \( i, j > m, \ell(w_2y_1') = n - m - 1 \) and

\[
w_2y_1' = s_{m+1} \cdots \hat{s}_i \cdots s_n = s_{m+1} \cdots \hat{s}_j \cdots s_n
\]

which implies \( i = j \). Thus \( i \) is unique.

The maximum element of \( W_I \) is \( 1 = (w_0v_0, v_0, v_0w_0) \). Fix a reduced expression

\[
1 = \Phi(s_1 \cdots s_m; s_{m+1} \cdots s_n)
\]

for \( 1 \). Let \( \sigma \in W_I \) and consider a maximal chain from \( 1 \) to \( \sigma \). By Lemma 8.1 this leads uniquely to a reduced expression

\[
\sigma = \Phi(s_{i_1} \cdots s_{i_p}; s_{i_{p+1}} \cdots s_{i_q})
\]

of \( \sigma \). If \( \sigma \to \sigma' \), then by Lemma 3.1, a reduced expression for \( \sigma' \) is obtained by deleting some \( s_{i_j} \) from the reduced expression for \( \sigma \). We attach the label \( i_j \) to the edge. We proceed to show that this labeling process leads to lexicographic shelling.

Let \( \sigma', \sigma \in W_I, \sigma' \prec \sigma \). Fix a maximal chain from \( 1 \) to \( \sigma \) resulting in the reduced expression \( t_1 \cdots t_{p'}; t_{p+1} \cdots t_{q'} \) for \( \sigma \) where \( t_j = s_{i_j} \). We will need the following analogue of [2, Lemma 4.3].

Lemma 3.2. Let \( \sigma_1 \in [\sigma', \sigma] \) such that \( \ell(\sigma_1) = \ell(\sigma) - 2 \). Then the open interval \((\sigma_1, \sigma) = \{\sigma_2, \sigma_3\}\) such that:

(i) \( \sigma \to \sigma_2 \to \sigma_1 \) has increasing labels.
(ii) \( \sigma \to \sigma_3 \to \sigma_1 \) has decreasing labels.
(iii) The label for \( \sigma \to \sigma_2 \) is less than the label for \( \sigma \to \sigma_3 \).
Proof. Let \( \sigma = (x, w, y), \sigma_1 = (x', w', y') \). Then by (32),
\[
(37) \quad w = w_1 \ast \bar{w}_2 \quad \text{with} \quad xw_1 = t_1 \cdots t_p, w_2y = t_{p+1} \cdots t_q.
\]

Suppose first that \( x = x' \) and \( y = y' \). Then \( w' < w \), \( \ell(w) - \ell(w') = 2 \). If \( (w', w) = \{u_1, u_2\} \), then \( (\sigma_1, \sigma) = \{ (x, u_1, y), (x, u_2, y) \} \). For \( i = 1, 2 \),
\[
\begin{align*}
xw &= t_1 \cdots t_p \ast \bar{w}_2 \rightarrow xu_i \rightarrow xw', \\
\bar{wy} &= \bar{w}_1 \ast t_{p+1} \cdots t_q \rightarrow \bar{u}_i y \rightarrow \bar{w}'y.
\end{align*}
\]

Fix reduced expressions for \( \bar{w}_1 \) and \( \bar{w}_2 \). A deletion in \( \bar{w}_2 \) in the first sequence corresponds to deleting some \( t_\mu, \mu > p \), in the second sequence. A deletion in \( \bar{w}_1 \) in the second sequence corresponds to deleting \( t_\mu, \mu \leq p \), in the first sequence. Thus applying \([2, \text{Lemma 4.3}] \) to (38), we see that the lemma is valid.

Suppose next that \( x = x' \) and \( y \neq y' \). Then \( w'y' < wy, \ell(\bar{wy}) - \ell(\bar{w}'y) = 2 \). So
\[
(\bar{w}'y', \bar{wy}) = \{u_1y_1, u_2y_2\}, \quad u_1, u_2 \in W_I, y_1, y_2 \in D_I^{-1}.
\]

Correspondingly \( (\sigma_1, \sigma) = \{ (x, \bar{u}_1, y_1), (x, \bar{u}_2, y_2) \} \). So for \( i = 1, 2 \),
\[
\bar{wy} = \bar{w}_1t_{p+1} \cdots t_q \rightarrow u_iy_1 \rightarrow \bar{w}'y'.
\]

Fix a reduced expression for \( \bar{w}_1 \). A deletion in \( \bar{w}_1 \) corresponds to deleting some \( t_\mu, \mu \leq p \), in \( xw_1 = t_1 \cdots t_p \). Since \( y \neq y' \), not both the deletions in (39) can be from \( \bar{w}_1 \). Again applying \([2, \text{Lemma 4.3}] \) to (39) yields the lemma. The case when \( x \neq x' \) and \( y = y' \) is handled similarly.

Finally, let \( x' \neq x \) and \( y' \neq y \). Then
\[
w' = w'_1 \ast w \ast \bar{w}'_2, \quad x \rightarrow x'w'_1, \quad y \rightarrow w'_2y'.
\]

Then by (37),
\[
xw_1 = t_1 \cdots t_p \rightarrow x'w'_1 \ast w_1, \quad w_2y = t_{p+1} \cdots t_q \rightarrow w_2 \ast w'_2y'.
\]

So we see that the lemma is valid with
\[
\sigma_2 = (x', w'_1 \ast w, y), \quad \sigma_3 = (x, w \ast \bar{w}'_2, y').
\]

Hence the lemma is valid in all cases. \( \square \)

It follows from Lemma 3.2 and induction that the maximal chain of \([\sigma', \sigma]\) with lexicographically minimal labeling has increasing labels. Let \( \sigma = (x, w, y) \) be as in (37) and suppose that there are two maximal chains
\[
\begin{align*}
\sigma &= \sigma_r \rightarrow \sigma_{r-1} \rightarrow \cdots \rightarrow \sigma_1 \rightarrow \sigma', \\
\sigma &= \theta_r \rightarrow \theta_{r-1} \rightarrow \cdots \rightarrow \theta_1 \rightarrow \sigma',
\end{align*}
\]
both with increasing labels. Suppose \( \sigma' \) is obtained from \( \sigma_1 \) by deleting \( t_\beta \) and that \( \sigma' \) is obtained from \( \theta_1 \) by deleting \( t_\alpha \). Let \( \alpha \leq \beta \). Let \( \sigma = (x', w', y) \) and let \( w' = w'_1 \ast \bar{w}'_2 \) analogous to (37). If \( \beta \leq p \), then \( w'_2 = w_2 \) and (40) yields two maximal chains from \( xw_1 \) to \( x'w'_1 \) with increasing labels. So by \([2, \text{Theorem 5.1}] \), the two chains are identical. So assume \( \beta > p \). Then \( \sigma_1 = (x', w'_1, y_1) \) and
\[
\tilde{w}_1y_1 = \tilde{w}'y' \cdot t_q \cdots t_{\beta+1} \cdot \beta \cdots t_q.
\]

So if \( \alpha < \beta \), then since \( \sigma' \) is also obtained from \( \theta_1 \) by deleting \( t_\alpha \),
\[
\tilde{w}'y' \cdots t_{\beta-1} \cdot \beta \cdots t_q,
\]
and hence \( \ell(\tilde{w}_1y_1) < \ell(\tilde{w}'y') \). This implies that \( \ell(\sigma_1) < \ell(\sigma) \), a contradiction. Hence \( \alpha = \beta \) and \( \theta_1 = \sigma_1 \). By induction, the two maximal chains in (40) are
identical. Hence $W_I$ is lexicographically shellable. Our proof shows that there is also a unique maximal chain in $[\sigma^i, \sigma]$ with decreasing labels. Hence by [2, Theorem 3.4], $W_I$ is Eulerian and its Stanley-Reisner ring is Gorenstein. Using Theorem 2.5 (i) and an almost identical (and slightly easier) argument as above, we see that $W_I^*$ is also an Eulerian lexicographically shellable poset. Hence we have proved

**Theorem 3.3.** The universal orbit $W_I$ and its dual $W_I^*$ are Eulerian, lexicographically shellable posets. In particular, their Stanley-Reisner rings are Gorenstein.

Many problems remain open.

**Problem 3.4.** Is the Renner monoid $R$ shellable with respect to the Bruhat-Chevalley order?

**Problem 3.5.** Is the cross-section lattice always shellable?

**Remark 3.6.** Suppose $\Lambda \setminus \{e\}$ has a minimum element $e$. This happens when $M$ is the Zariski closure of the image of an irreducible representation of a reductive group. Let $I$ be the type of $W e W$. Then by [16, Theorem 4.16], $\Lambda \setminus \{e\}$ is isomorphic to the poset (with respect to inclusion) of all subsets of $S$ with no components contained in $I$. Being closed under taking unions, this poset is easily seen to be a semimodular lattice. Hence by [1], $\Lambda$ is shellable.

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**References**


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