ON ARITHMETIC MACAULAYIFICATION
OF NOETHERIAN RINGS

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Abstract. The Rees algebra is the homogeneous coordinate ring of a blowing-up. The present paper gives a necessary and sufficient condition for a Noetherian local ring to have a Cohen-Macaulay Rees algebra: A Noetherian local ring has a Cohen-Macaulay Rees algebra if and only if it is unmixed and all the formal fibers of it are Cohen-Macaulay. As a consequence of it, we characterize a homomorphic image of a Cohen-Macaulay local ring. For non-local rings, this paper gives only a sufficient condition. By using it, however, we obtain the affirmative answer to Sharp’s conjecture. That is, a Noetherian ring having a dualizing complex is a homomorphic image of a finite-dimensional Gorenstein ring.

1. Introduction

Let \( A \) be a commutative ring with identity and \( \mathfrak{b} \) an ideal in \( A \). The Rees algebra of \( \mathfrak{b} \) is the graded ring

\[
R(\mathfrak{b}) = \bigoplus_{n \geq 0} (\mathfrak{b}T)^n,
\]

where \( T \) is an indeterminate. We often regard \( R(\mathfrak{b}) \) as an \( A \)-subalgebra \( A[\mathfrak{b}T] \) of the polynomial ring \( A[T] \). The Rees algebra is an important object of Algebraic Geometry and Commutative Algebra because the canonical morphism \( \text{Proj} R(\mathfrak{b}) \to \text{Spec} A \) is the blowing-up of \( \text{Spec} A \) along the closed subscheme \( \text{Spec} A/\mathfrak{b} \).

In the present paper, we consider the existence of Cohen-Macaulay Rees algebras. A Rees algebra \( R(\mathfrak{b}) \) is said to be an arithmetic Macaulayfication of \( A \) if it is Cohen-Macaulay and \( \mathfrak{b} \) is of positive height. The main theorem of this paper is the following.

Theorem 1.1. Let \( A \) be a Noetherian local ring of positive dimension. Then the following statements are equivalent:

(A) \( A \) has an arithmetic Macaulayfication;
(B) \( A \) is unmixed and all the formal fibers of \( A \) are Cohen-Macaulay.

Here a Noetherian local ring \( A \) is said to be unmixed if \( \dim \hat{A}/\mathfrak{p} = \dim \hat{A} \) for every associated prime \( \mathfrak{p} \) of the completion \( \hat{A} \). The formal fibers of \( A \) are the fiber rings of the natural homomorphism \( A \to \hat{A} \).
The studies in the Cohen-Macaulay property of Rees algebras started from Barry's paper \[5\]. He gave the defining ideal of \( R(b) \) and its free resolution if \( b \) is generated by a regular sequence. He also showed that \( R(b) \) is Cohen-Macaulay if \( A \) is also and if \( b \) is generated by a regular sequence. Around 1980, Goto and Shimoda studied several properties of \( R(b) \) in the case where \( A \) is a Buchsbaum local ring and \( b \) a parameter ideal. See \[9\], \[10\], \[11\], and \[31\]. Summarizing these investigations, Goto and Yamagishi \[12\] established the theory of unconditioned strong \( d \)-sequences. Their theory contains the existence of an arithmetic Macaulayfication in the case where \( A \) is unmixed and \( \text{Spec} A \) is Cohen-Macaulay except for the closed point. See also Brodmann \[7\] and Schenzel \[27\]. Recently Kurano \[19\] proved that a Noetherian local ring \( A \) containing a finite field has an arithmetic Macaulayfication if the non-\( F \)-rational locus of \( A \) is of dimension 1. Independently this was also done by Aberbach \[1\]. Motivated by Kurano’s work, the author \[18\] also gave some sufficient conditions for \( A \) to have an arithmetic Macaulayfication. Theorem 1.1 gives a necessary and sufficient condition for an arithmetic Macaulayfication to exist.

If the Rees algebra \( R(b) \) is a Cohen-Macaulay ring, then the projective scheme \( \text{Proj} R(b) \) is Cohen-Macaulay. However, the converse is not true in general. The author \[17\] gave an ideal \( b \) such that \( \text{Proj} R(b) \) is a Cohen-Macaulay scheme for fairly general Noetherian local rings. Theorem 1.1 gives another proof of the result in \[17\].

In our arithmetic Macaulayfication \( R(b) \), the ideal \( b \) is generated by monomials of a certain system of parameters, named a \( p \)-standard system of parameters. Sections 2 and 3 are devoted to discussing the existence and properties of a \( p \)-standard system of parameters. Theorems 2.5 and 3.6 are improvements of Theorems 2.7 and 3.1 of \[17\], respectively. We give a proof of Theorem 1.1 in Section 4. In our proof the theory of multigraded Rees algebras, which was introduced by Herrmann, Hyry, and Ribbe \[15\], plays a key role. Our ideal \( b \) is very complicated. However, their theory makes the proof of Theorem 1.1 simple.

In section 5 we give a consequence of Theorem 1.1.

Corollary 1.2. A Noetherian local ring is a homomorphic image of a Cohen-Macaulay local ring if and only if it is universally catenary and all the formal fibers of it are Cohen-Macaulay. An excellent local ring is a homomorphic image of a Cohen-Macaulay excellent local ring.

However, there exists no analogy with the Gorenstein property. In fact, Ogoma \[22\] Example 1] gave an example of an acceptable local ring which is not a homomorphic image of a Gorenstein ring.

For non-local rings, this paper gives only a sufficient condition for an arithmetic Macaulayfication to exist.

Theorem 1.3. Let \( B \) be a Noetherian ring possessing a dualizing complex. If the codimension function is a constant on the associated primes of \( B \), then \( B \) has an arithmetic Macaulayfication.

We refer the readers to Section 5 for the definition of the codimension function. By using Theorem 1.3, we give an affirmative answer to Sharp’s conjecture \[30\] Conjecture 4.4.

Corollary 1.4. A Noetherian ring has a dualizing complex if and only if it is a homomorphic image of a finite-dimensional Gorenstein ring.
This is a simple criterion for a dualizing complex to exist. Several authors gave partial answers. See [2], [3], [4], [22], and [23]. We give proofs of Theorem 1.3 and Corollary 1.4 in Section 6.

Throughout this paper, \( A \) denotes a Noetherian local ring with maximal ideal \( \mathfrak{m} \). We assume that the dimension of \( A \) is positive. We refer the reader to [13], [14], and [20], for unexplained terminology.

2. A \( p \)-standard system of parameters, I

In this section, we give the definition of a \( p \)-standard system of parameters and discuss the existence of it. For a finitely generated \( A \)-module \( M \), let \( a^p(M) \) denote the annihilator of the \( p \)-th local cohomology module \( H^p_{\mathfrak{m}}(M) \) of \( M \) and let \( a(M) = \prod_{p<\dim A} a^p(M) \).

**Definition 2.1.** Let \( M \) be a finitely generated \( A \)-module of dimension \( d > 0 \), \( x_1, \ldots, x_d \) a system of parameters for \( M \) and \( s \) an integer such that \( 0 \leq s < d \). We say that \( x_1, \ldots, x_d \) is a \( p \)-standard system of parameters of type \( s \) for \( M \) if

1. \( x_{s+1}, \ldots, x_d \in a(M) \);
2. \( x_i \in a(M/(x_{i+1}, \ldots, x_d)M) \) for \( 1 \leq i \leq s \).

This notion was given by N. T. Cuong [8]. He showed that there exists a \( p \)-standard system of parameters of type \( d-1 \) for \( M \) whenever \( A \) possesses a dualizing complex. We improve his result. For a finitely generated \( A \)-module \( M \), let \( \text{NCM}(M) \) denote the non-Cohen-Macaulay locus of \( M \), that is, \( \text{NCM}(M) = \{ p \in \text{Spec } A \mid M_p \) is not a Cohen-Macaulay \( A_p \)-module\}. By modifying the proof of [29, Theorem 3.3], we obtain the following lemma.

**Lemma 2.2.** Let \( B \) and \( C \) be Noetherian rings and \( B \to C \) a faithfully flat ring homomorphism. We assume that \( C_p/pC_p \) is a Cohen-Macaulay ring for every prime ideal \( p \) in \( B \). Let \( M \) be a finitely generated \( B \)-module. If there exists an ideal \( \mathfrak{c} \) in \( C \) such that \( \text{NCM}(M \otimes_B C) = V(\mathfrak{c}) \), then \( \text{NCM}(M) = V(\mathfrak{c} \cap B) \).

We need the following propositions to choose a \( p \)-standard system of parameters.

**Proposition 2.3.** Assume that \( A \) is universally catenary and that all the formal fibers of \( A \) are Cohen-Macaulay. Let \( M \) be a finitely generated \( A \)-module of dimension \( d > 0 \). If \( M \) is equidimensional, then \( \text{NCM}(M) = V(a(M)) \). In particular, \( \dim A/a(M) < d \).

**Proof.** If \( A \) has a dualizing complex, then the assertion was given by Schenzel [20, p. 52]. Assume that \( A \) has no dualizing complex. The completion \( \hat{A} \) of \( A \) has a dualizing complex and is a faithfully flat \( A \)-algebra. Since \( A \) is formally catenary, \( M \otimes \hat{A} \) is also equidimensional. Therefore the non-Cohen-Macaulay locus of \( M \otimes A \) is

\[
V(a(M \otimes \hat{A})) = V(a^0(M \otimes \hat{A}) \cap \cdots \cap a^{d-1}(M \otimes \hat{A})).
\]

By using Lemma 2.2 we find that the non-Cohen-Macaulay locus of \( M \) is

\[
V(a^0(M \otimes \hat{A}) \cap \cdots \cap a^{d-1}(M \otimes \hat{A}) \cap M) = V(a^0(M) \cap \cdots \cap a^{d-1}(M)).
\]

The right-hand side of the equation above is equal to \( V(a(M)) \). Since \( \text{NCM}(M) \) contains no minimal prime of \( M \), \( \dim A/a(M) = \dim \text{NCM}(M) < d \).
Corollary 2.4. Assume that $A$ is universally catenary and that all the formal fibers of $A$ are Cohen-Macaulay. Let $M$ be a finitely generated $A$-module of dimension $d > 0$. If $\dim A/p = d$ for every associated prime ideal $p$ of $M$, then $\dim A/a(M) < d - 1$.

Proof. Let $p$ be a prime ideal of $A$ such that $\dim A/p = d - 1$ and $M_p \neq 0$. Then the one-dimensional $A_p$-module $M_p$ is Cohen-Macaulay because $M_p$ has no embedded prime. Therefore $\dim A/a(M) = \dim NCM(M) < d - 1$. \hfill $\square$

The following theorem assures us of the existence of the $p$-standard system of parameters.

Theorem 2.5. Assume that $A$ is universally catenary and that all the formal fibers of $A$ are Cohen-Macaulay. Let $M$ be a finitely generated $A$-module of dimension $d > 0$. If $M$ is equidimensional and $s$ an integer such that $\dim A/a(M) \leq s < d$, then there exists a $p$-standard system of parameters of type $s$ for $M$.

Proof. Since $d - \dim A/a(M) \geq d - s$, there exist $d - s$ elements $x_{s+1}, \ldots, x_d$ in $a(M)$ such that $\dim M/(x_{s+1}, \ldots, x_d)M = s$. If $s$ elements $x_{i_1}, \ldots, x_{i_s}$ in $A$ such that $\dim M/(x_{i_1}, \ldots, x_{i_s})M = i$ are given, then $M/(x_{i_1}, \ldots, x_{i_s})M$ is also equidimensional. Therefore $\dim A/M/(x_{i_1}, \ldots, x_{i_s})M < i$ and hence there exists an element $x_i$ in $a(M/(x_{i_1}, \ldots, x_{i_s})M)$ such that $\dim M/(x_i, \ldots, x_{i_s})M = i - 1$. \hfill $\square$


In this section, we give some properties of a $p$-standard system of parameters. First we recall the definition of $d$-sequences and the one of unconditioned strong $d$-sequences.

Definition 3.1. Let $M$ be an $A$-module. A sequence $x_1, \ldots, x_d$ of elements in $A$ is said to be a $d$-sequence on $M$ if

$$(x_1, \ldots, x_{i-1})M : x_i x_j = (x_1, \ldots, x_{i-1})M : x_j$$

for any $1 \leq i \leq j \leq d$. Here we set $(x_1, \ldots, x_{i-1}) = (0)$ if $i = 1$.

A sequence $x_1, \ldots, x_d$ of elements in $A$ is said to be an unconditioned strong $d$-sequence (for short, a $u.s.d$-sequence) on $M$ if $x_{i_1}^{n_1}, \ldots, x_{i_d}^{n_d}$ is a $d$-sequence on $M$ for any positive integers $n_1, \ldots, n_d$ and in any order.

The following is one of the important properties of $d$-sequences. It was first given by Goto and Shimoda [11 Lemma 4.2] for the system of parameters for a Buchsbaum local ring, which is a typical example of $d$-sequences.

Proposition 3.2 ([12 Theorem 1.3]). Let $M$ be an $A$-module and $x_1, \ldots, x_d$ a $d$-sequence on $M$. If we put $q = (x_1, \ldots, x_d)$, then

$$(x_1, \ldots, x_{i-1})M : x_i \cap q^n M = (x_1, \ldots, x_{i-1})q^{n-1}M$$

for any $n > 0$ and $1 \leq i \leq d$.

A $p$-standard system of parameters has several nice properties. The following two properties are given in [17].

Proposition 3.3 ([17 Proposition 2.8]). Let $M$ be a finitely generated $A$-module of dimension $d > 0$ and $x_1, \ldots, x_d$ a $p$-standard system of parameters of type $s$ for $M$. Then $x_{s+1}, \ldots, x_d$ is a $u.s.d$-sequence on $M/(y_1, \ldots, y_u)M$ where $y_1, \ldots, y_u$ is a subsystem of parameters for $M/(x_{s+1}, \ldots, x_d)M$. 

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Proposition 3.4 ([17] Theorem 2.9]). Let $M$ be a finitely generated $A$-module of dimension $d > 0$, $x_1, \ldots, x_d$ a $p$-standard system of parameters of type $s$ for $M$, and $y_1, \ldots, y_u$ a subsystem of parameters for $M/(x_1, \ldots, x_d)M$ where $2 \leq i \leq d$ and $1 \leq u < i$. If $y_u \in \mathfrak{a}(M)$ or $y_u \in \mathfrak{a}(M/(x_1, \ldots, x_d)M)$, then

$$(y_1, \ldots, y_{u-1}, \{x_{\lambda} \mid \lambda \in \Lambda\})M : y_u = (y_1, \ldots, y_{u-1}, \{x_{\lambda} \mid \lambda \in \Lambda\})M : y_u$$

for any $1 \leq v \leq u$ and $\Lambda \subseteq \{i, \ldots, d\}$.

The next proposition is not in [17] but we need it to prove Theorem 1.1. The author is inspired by [8] Theorem 2.6.

Proposition 3.5. Let $M$ be a finitely generated $A$-module of dimension $d > 0$, $x_1, \ldots, x_d$ a $p$-standard system of parameters of type $s$ for $M$ and $y_1, \ldots, y_u$ a subsystem of parameters for $M/(x_1, \ldots, x_d)M$ where $1 \leq i \leq d$ and $1 \leq u < i$. Then $x_i, \ldots, x_j$ is a $d$-sequence on $M/(y_1, \ldots, y_u, x_{j+1}, \ldots, x_d)M$ for any $i \leq j \leq d$.

Proof. Let $i \leq l \leq j$ be an integer. By applying Proposition 3.4 to a subsystem of parameters $y_1, \ldots, y_u, x_1, \ldots, x_l$ for $M/(x_{l+1}, \ldots, x_d)M$ and a subset $\{j+1, \ldots, d\}$ of $\{l+1, \ldots, d\}$, we obtain

$$(y_1, \ldots, y_u, x_1, \ldots, x_{l-1}, x_{j+1}, \ldots, x_d)M : x_k x_l = (y_1, \ldots, y_u, x_1, \ldots, x_{l-1}, x_{j+1}, \ldots, x_d)M : x_l$$

for any $i \leq k \leq l$. \hfill $\Box$

The following theorem and corollaries are improvements of Theorem 3.1, Corollaries 3.2 and 3.3 of [17], respectively. The old theorems require that all $n_i, \ldots, n_j$ are positive but new ones require only that all $n_i, \ldots, n_j$ are nonnegative.

Theorem 3.6. Let $M$ be a finitely generated $A$-module of dimension $d > 0$ and $x_1, \ldots, x_d$ a $p$-standard system of parameters of type $s$ for $M$. We put $q_i = (x_1, \ldots, x_d)$ for all $1 \leq i \leq d$. Then, for any integers $1 \leq i \leq j \leq d$ and $n_i, \ldots, n_j \geq 0$, the following statements hold:

$(A_{ij})$ If $y_1, \ldots, y_u$ is a subsystem of parameters for $M/q_i M$ and if $n_k > 0$ for some integer $i \leq k \leq j$, then

$$(y_1, \ldots, y_u, x_i, \ldots, x_{i-1})M : x_i \cap [(y_1, \ldots, y_u)M + q_i^{n_i}\cdots q_j^{n_j}M]$$

$$(y_1, \ldots, y_u)M + (x_1, \ldots, x_{i-1})q_i^{n_i}\cdots q_k^{n_k-1}\cdots q_j^{n_j}M$$

for arbitrary integer $k \leq l \leq d$.

$(B_{ij})$: If $y_1, \ldots, y_u$ is a subsystem of parameters for $M/q_i M$ and if $n_k > 0$ for some integer $i \leq k \leq j$, then

$$[(y_1, \ldots, y_{u-1})M + (x_1, \ldots, x_l)q_i^{n_i}\cdots q_j^{n_j}M] : y_u$$

$$= (y_1, \ldots, y_u)M + (x_1, \ldots, x_l)q_i^{n_i}\cdots q_j^{n_j}M : y_u + (y_1, \ldots, y_{u-1})M : y_u$$

for arbitrary integer $k \leq l \leq d$. In particular, by letting $l = d$, we have

$$[(y_1, \ldots, y_{u-1})M + q_i^{n_i}\cdots q_k^{n_k+1}\cdots q_j^{n_j}M] : y_u$$

$$= q_k [(y_1, \ldots, y_{u-1})M + q_i^{n_i}\cdots q_j^{n_j}M] : y_u + (y_1, \ldots, y_{u-1})M : y_u.$$
(C$_i$): If $y_1, \ldots, y_u$ is a subsystem of parameters for $M/q_iM$ and if $n_i > 0$, then

\begin{equation}
[(y_1, \ldots, y_{u-1})M + q_i^{n_i} \cdots q_j^{n_j}M]: y_u \\
\subseteq (y_1, \ldots, y_{u-1})M: y_u + q_i^{n_i-1} \cdots q_j^{n_j}M.
\end{equation}

(D$_i$): If $y_1, \ldots, y_u$ is a subsystem of parameters for $M/q_iM$ and if $n_i > 0$, then

\begin{equation}
[(y_1, \ldots, y_{u-1})M + q_i^{n_i} \cdots q_j^{n_j}M]: y_u \\
\subseteq (y_1, \ldots, y_{u-1})M: y_u + q_i^{n_i-1} \cdots q_j^{n_j}M.
\end{equation}

(E$_i$): Let $y_1, \ldots, y_u$ be a subsystem of parameters for $M/q_kM$ where $2 \leq k \leq i$ and $1 \leq u < k$. If $y_u \in a(M/q_kM)$ or $y_u \in a(M)$ and if $n_i > 0$, then

\begin{equation}
[(y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i} \cdots q_j^{n_j}M]: y_u \\
= [(y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i} \cdots q_j^{n_j}M]: y_u
\end{equation}

for any $1 \leq v \leq u$ and $\Lambda \subseteq \{k, \ldots, i-1\}$.

**Proof.** We work by induction on $j - i$. First we assume that $i = j$.

(A$_i$): Since $x_1, \ldots, x_d$ is a d-sequence on $M/(y_1, \ldots, y_u)M$, (3.6.1) coincides with Proposition 3.2.

(B$_i$): Let $a$ be an element in the left-hand side of (3.6.2) and put $y_ua = xib + c$ with $b \in q_i^{n_i}M$ and $c \in (y_1, \ldots, y_{u-1})M + (x_i, \ldots, x_{i-1})q_i^{n_i}M$. By using (A$_ii$), we obtain

$$b \in (y_1, \ldots, y_u, x_i, \ldots, x_{i-1})M: x_i \cap q_i^{n_i}M$$

$$\subseteq (y_1, \ldots, y_u)M + (x_i, \ldots, x_{i-1})q_i^{n_i-1}M.$$

Let $b = y ua' + c'$ with $c' \in (y_1, \ldots, y_{u-1})M + (x_i, \ldots, x_{i-1})q_i^{n_i-1}M$. Then $a' \in [(y_1, \ldots, y_{u-1})M + q_i^{n_i}M]: y_u$ and

$$a - xa' \in [(y_1, \ldots, y_{u-1})M + (x_i, \ldots, x_{i-1})q_i^{n_i}M]: y_u.$$ 

By induction on $l$, we find that $a$ is in the right-hand side of (3.6.2). The opposite inclusion is obvious.

(C$_i$): By using (B$_ii$) repeatedly, we have

$$[(y_1, \ldots, y_{u-1})M + q_i^{n_i}M]: y_u = (y_1, \ldots, y_{u-1})M: y_u$$

$$+ q_i^{n_i-1} [(y_1, \ldots, y_{u-1})M + q_iM]: y_u$$

$$\subseteq (y_1, \ldots, y_{u-1})M: y_u + q_i^{n_i-1}M.$$ 

(D$_i$): If $n_i = 1$, then the right-hand side of (3.6.3) equals $(y_1, \ldots, y_{u-1}, x_i)M$ and hence contains the left-hand side.

Assume that $n_i > 1$. Let $a$ be an element in $M$ such that $x_ia$ is in the left-hand side of (3.6.4). Then

$$y_uax_ia \in [(y_1, \ldots, y_{u-1})M + q_i^{n_i}M] \cap (y_1, \ldots, y_{u-1}, x_i)M$$

$$= (y_1, \ldots, y_{u-1})M: x_iq_i^{n_i-1}M$$

because of (A$_ii$). Hence

$$x_ia \in [(y_1, \ldots, y_{u-1})M + x_iq_i^{n_i-1}M]: y_u$$

$$= (y_1, \ldots, y_{u-1})M: y_u + x_i[(y_1, \ldots, y_{u-1})M + q_i^{n_i-1}M]: y_u.$$
Here we used \((B_{ii})\). By applying Proposition 3.4 to a subsystem of parameters \(y_1, \ldots, y_a, x_i\) for \(M/q_{i+1}M\), we have
\[
(y_1, \ldots, y_{a-1})M : y_ux_i = (y_1, \ldots, y_{a-1})M : x_i
\]
and hence
\[
(y_1, \ldots, y_{a-1})M : y_u \cap x_iM = x_i[(y_1, \ldots, y_{a-1})M : y_ux_i] \subseteq (y_1, \ldots, y_{a-1})M.
\]
Therefore
\[
x_i^{a} \in x_i \{(y_1, \ldots, y_{a-1})M + q_i^{n_i}M : y_u\} + (y_1, \ldots, y_{a-1})M : y_u \cap x_iM \subseteq (y_1, \ldots, y_{a-1})M + q_i^{n_i}M : y_u + (y_1, \ldots, y_{a-1})M.
\]

\((E_{ii})\): By using \((B_{ii})\), we have
\[
[(y_1, \ldots, y_{a-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i}M : y_vy_u] = (y_1, \ldots, y_{a-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_vy_u + q_i^{n_i-1}[(y_1, \ldots, y_{a-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_iM : y_vy_u].
\]
Applying Proposition 3.4 to a subsystem of parameters \(y_1, \ldots, y_a\) for \(M/q_kM\) and two subsets of \(\{k, \ldots, d\}\): \(\Lambda\) and \(\Lambda \cup \{i, \ldots, d\}\), we obtain
\[
(y_1, \ldots, y_{a-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_vy_u + q_i^{n_i-1}[(y_1, \ldots, y_{a-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_iM : y_vy_u] = (y_1, \ldots, y_{a-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_vy_u + q_i^{n_i-1}[(y_1, \ldots, y_{a-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_iM : y_vy_u] = [(y_1, \ldots, y_{a-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i}M : y_u.
\]
Thus (3.6.6) is shown.

Next we assume that \(j > i\) and prove \((A_{ij})-(E_{ij})\). If \(n_i = 0\), then \((A_{ij})\) and \((B_{ij})\) are contained in \((A_{i+1,j})\) and \((B_{i+1,j})\), respectively. Therefore we may assume that \(n_i > 0\). Similarly we may also assume that \(n_j > 0\).

\((A_{ij})\): Let \(a\) be an element in the left-hand side of (3.6.1). If \(k = l = i\), then
\[
a \in (y_1, \ldots, y_a)M : x_i \cap (y_1, \ldots, y_a, x_i, \ldots, x_d)M = (y_1, \ldots, y_a)M.
\]

Otherwise, by using \((A_{i+1,j})\), we have
\[
a \in (y_1, \ldots, y_a, x_i, x_k, \ldots, x_{i-1})M : x_i \cap [(y_1, \ldots, y_a, x_i)M + q_i^{n_i+n_{i-1}+1} \cdots q_j^{n_j}M] = \begin{cases} (y_1, \ldots, y_a, x_i)M + (x_{i+1}, \ldots, x_{i-1})q_i^{n_i+n_{i-1}+1} \cdots q_j^{n_j}M & \text{if } k \leq i + 1, \\ (y_1, \ldots, y_a, x_i)M + (x_k, \ldots, x_{i-1})q_i^{n_i+n_{i-1}+1} \cdots q_k^{n_k-1} \cdots q_j^{n_j}M & \text{if } k > i + 1 \end{cases} = (y_1, \ldots, y_a, x_i)M + (x_k, \ldots, x_{i-1})q_i^{n_i} \cdots q_k^{n_k-1} \cdots q_j^{n_j}M.
\]
Taking the intersection with \((y_1, \ldots, y_a)M + q_i^{n_i} \cdots q_j^{n_j}M\), we obtain
\[
a \in (y_1, \ldots, y_a)M + (x_k, \ldots, x_{i-1})q_i^{n_i} \cdots q_k^{n_k-1} \cdots q_j^{n_j}M + x_iM \cap [(y_1, \ldots, y_a)M + q_i^{n_i} \cdots q_j^{n_j}M].
\]
Because of \((C_{i+1,j})\),
\[
x_i M \cap [(y_1, \ldots, y_u) M + q_i^{n_i} \cdots q_j^{n_j} M]
\]
\[
= x_i q_i^{n_i-1} \cdots q_j^{n_j} M
\]
\[
+ x_i M \cap [(y_1, \ldots, y_u) M + q_i^{n_i+n_{i+1}} \cdots q_j^{n_j} M]
\]
\[
= x_i q_i^{n_i-1} \cdots q_j^{n_j} M + x_i [(y_1, \ldots, y_u) M + q_i^{n_i+n_{i+1}} \cdots q_j^{n_j} M] x_i
\]
\[
\subseteq x_i q_i^{n_i-1} \cdots q_j^{n_j} M + x_i [(y_1, \ldots, y_u) M : x_i + q_i^{n_i+n_{i+1}} \cdots q_j^{n_j} M]
\]
\[
\subseteq (y_1, \ldots, y_u) M + x_i q_i^{n_i-1} \cdots q_j^{n_j} M.
\]
Therefore
\[
a \in (y_1, \ldots, y_u) M + (x_k, \ldots, x_{l-1}) q_i^{n_i} \cdots q_k^{n_k-1} \cdots q_j^{n_j} M
\]
\[
+ x_i q_i^{n_i-1} \cdots q_j^{n_j} M.
\]
If \(k = i\), then the proof is completed. If \(k > i\), then we work by induction on \(n_i\).
Let \(a = x_i b + c\) with \(b \in q_i^{n_i-1} \cdots q_j^{n_j} M\) and
\[
c \in (y_1, \ldots, y_u) M + (x_k, \ldots, x_{l-1}) q_i^{n_i} \cdots q_k^{n_k-1} \cdots q_j^{n_j} M.
\]
If we apply Proposition \((3.6.2)\) to a subsystem of parameters \(y_1, \ldots, y_u, x_k, \ldots, x_{l-1}, x_i, x_l\) for \(M/q_{i+1} M\), then we have
\[
b \in (y_1, \ldots, y_u, x_k, \ldots, x_{l-1}) M : x_i x_l = (y_1, \ldots, y_u, x_k, \ldots, x_{l-1}) M : x_l.
\]
If \(n_i = 1\), then \((A_{i+1,j})\) says that
\[
b \in (y_1, \ldots, y_u, x_k, \ldots, x_{l-1}) M : x_l \cap q_i^{n_i+1} \cdots q_j^{n_j} M
\]
\[
\subseteq (y_1, \ldots, y_u) M + (x_k, \ldots, x_{l-1}) q_i^{n_i+1} \cdots q_k^{n_k-1} \cdots q_j^{n_j} M
\]
and hence \(a = x_i b + c\) is in the right-hand side of \((3.6.1)\). If \(n_i > 1\), then we obtain
\[
b \in (y_1, \ldots, y_u, x_k, \ldots, x_{l-1}) M : x_l \cap q_i^{n_i-1} \cdots q_j^{n_j} M
\]
\[
\subseteq (y_1, \ldots, y_u) M + (x_k, \ldots, x_{l-1}) q_i^{n_i-1} \cdots q_k^{n_k-1} \cdots q_j^{n_j} M
\]
by the induction hypothesis. Thus \(a = x_i b + c\) is also in the right-hand side of \((3.6.1)\).
\((B_{ij})\): Let \(a\) be an element in the left-hand side of \((3.6.2)\) and put \(y_u a = x_i b + c\) with \(b \in q_i^{n_i} \cdots q_j^{n_j} M\) and \(c \in (y_1, \ldots, y_u-1) M + (x_k, \ldots, x_{l-1}) q_i^{n_i} \cdots q_j^{n_j} M\). Then
\[
b \in (y_1, \ldots, y_u, x_k, \ldots, x_{l-1}) M : x_l \cap q_i^{n_i} \cdots q_j^{n_j} M
\]
\[
\subseteq (y_1, \ldots, y_u) M + (x_k, \ldots, x_{l-1}) q_i^{n_i} \cdots q_k^{n_k-1} \cdots q_j^{n_j} M.
\]
Here we used \((A_{ij})\). If we put \(b = y_u a' + c'\) with
\[
c' \in (y_1, \ldots, y_u-1) M + (x_k, \ldots, x_{l-1}) q_i^{n_i} \cdots q_k^{n_k-1} \cdots q_j^{n_j} M,
\]
then \(a' \in [(y_1, \ldots, y_u-1) M + q_i^{n_i} \cdots q_j^{n_j} M] : y_u\) and
\[
a - x_i a' \in [(y_1, \ldots, y_u-1) M + (x_k, \ldots, x_{l-1}) q_i^{n_i} \cdots q_j^{n_j} M] : y_u.
\]
By induction on \(l\), we find that \(a\) is in the right-hand side of \((3.6.2)\). The opposite inclusion is obvious.
\((C_{ij})\): We first show that
\[
(y_1, \ldots, y_u-1, x_i) M : y_u \cap (y_1, \ldots, y_u-1, x_i, \ldots, x_l) M
\]
\[
= (y_1, \ldots, y_u-1, x_i) M
\]
for all $i \leq l \leq d$. We work by induction on $l$. If $l = i$, then there exists nothing to prove. Assume that $l > i$ and let $a$ be an element in the left-hand side of (3.6.7).

If we put $a = x_ib + c$ with $c \in (y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1})M$, then

$$b \in (y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1})M : y_u x_i = (y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1})M : x_i.$$  

Here we applied Proposition 3.3 to a subsystem of parameters $y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1}, y_u, x_i$ for $M/q_{l+1}M$. Thus we obtain

$$a = x_ib + c \in (y_1, \ldots, y_{u-1}, x_i)M : y_u \cap (y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1})M$$

$$= (y_1, \ldots, y_{u-1}, x_i)M$$

by the induction hypothesis.

Next we show (3.6.3). By using (B_{ij}), we may assume that $n_i = 1$. Let $a$ be an element in the left-hand side of (3.6.3). Then

$$a \in [(y_1, \ldots, y_{u-1}, x_i)M : y_u + q_{i+1}^{n_j+1} \cdots q_j^n M] : y_u$$

$$\subseteq (y_1, \ldots, y_{u-1}, x_i)M : y_u + q_{i+1}^{n_j+1} \cdots q_j^n M$$

because of $(C_{i+1,j})$. On the other hand, since $n_j > 0$, we obtain

$$a \in [(y_1, \ldots, y_{u-1})M + q_i^2 M] : y_u$$

$$\subseteq (y_1, \ldots, y_{u-1})M : y_u + q_i M.$$  

Here we used $(C_u)$. Hence

$$a \in [(y_1, \ldots, y_{u-1}, x_i)M : y_u + q_{i+1}^{n_j+1} \cdots q_j^n M] \cap [(y_1, \ldots, y_{u-1})M : y_u + q_i M]$$

$$= (y_1, \ldots, y_{u-1})M : y_u + q_{i+1}^{n_j+1} \cdots q_j^n M + (y_1, \ldots, y_{u-1}, x_i)M : y_u \cap q_i M$$

$$= (y_1, \ldots, y_{u-1})M : y_u + q_{i+1}^{n_j+1} \cdots q_j^n M + x_i M.$$

Here we used (3.6.7). Taking the intersection with

$$[(y_1, \ldots, y_{u-1})M + q_{i+1}^{n_j+1} \cdots q_j^n M] : y_u,$$

we obtain

$$a \in (y_1, \ldots, y_{u-1})M : y_u + q_{i+1}^{n_j+1} \cdots q_j^n M$$

$$+ x_i \{[(y_1, \ldots, y_{u-1})M + q_{i+1}^{n_j+1} \cdots q_j^n M] : y_u x_i\}.$$  

By applying $(E_{i+1,j})$ to a subsystem of parameters $y_1, \ldots, y_u, x_i$ for $M/q_{i+1}M$, we have

$$[(y_1, \ldots, y_{u-1})M + q_{i+1}^{n_j+1} \cdots q_j^n M] : y_u x_i = [(y_1, \ldots, y_{u-1})M + q_{i+1}^{n_j+1} \cdots q_j^n M] : x_i.$$  

Therefore $a \in (y_1, \ldots, y_{u-1})M : y_u + q_{i+1}^{n_j+1} \cdots q_j^n M$.

$(D_{ij})$: Let $a$ be an element in $M$ such that $x_i a$ is in the left-hand side of (3.6.4). Then

$$y_u x_i a \in x_i M \cap [(y_1, \ldots, y_{u-1})M + q_i^{n_j} \cdots q_j^n M]$$

$$\subseteq (y_1, \ldots, y_{u-1})M + x_i q_i^{n_j-1} \cdots q_j^n M.$$  

Here we used $(A_{ij})$. We put $y_u x_i a = x_i b + c$ with $b \in q_i^{n_j-1} \cdots q_j^n M$ and $c \in (y_1, \ldots, y_{u-1})M$. Then

$$b \in (y_1, \ldots, y_u)M : x_i \cap q_j M$$

$$\subseteq (y_1, \ldots, y_u)M : x_i \cap q_i M$$

$$\subseteq (y_1, \ldots, y_u)M.$$
because $n_j > 0$ and $x_i, \ldots, x_d$ is a $d$-sequence on $M/(y_1, \ldots, y_u)M$. If we put $b = y_u a' + c'$ with $c' \in (y_1, \ldots, y_{u-1})M$, then

$$a' \in [(y_1, \ldots, y_{u-1})M + q_i^{n_{i-1}} \cdots q_j^{n_j} M] : y_u$$

and

$$x_i(a - a') \in (y_1, \ldots, y_{u-1})M : y_u \cap x_i M \subseteq (y_1, \ldots, y_{u-1})M.$$ 

Here we used \((3.6.6)\) again. Therefore

$$x_i a \in (y_1, \ldots, y_{u-1})M + x_i \{[(y_1, \ldots, y_{u-1})M + q_i^{n_{i-1}} \cdots q_j^{n_j} M] : y_u \}.$$ 

\((E_{ij})\): We may assume that $n_i = 1$ in the same way as the proof of \((E_{i})\). We divide the proof into two cases.

First we assume that $n_{i+1} + \cdots + n_j = 1$, that is, $n_{i+1} = \cdots = n_{j-1} = 0$ and $n_j = 1$. We show that

\[(3.6.8)\] \[[(y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\})M + (x_1, \ldots, x_{l-1}, y_j, \ldots, x_d)q_iM] : y_v y_u \]

for all $i \leq l \leq j$ by descending induction on $l$. If $l = j$, then \((3.6.8)\) coincides with \((E_v)\). Assume that $l < j$ and let $a$ be an element in the left-hand side of \((3.6.8)\). The induction hypothesis says that

$$a \in [(y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\})M + (x_1, \ldots, x_j, \ldots, x_d)q_iM] : y_u.$$ 

We put $y_u a = x_j b + c$ with $b \in q_i M$ and

$$c \in (y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\})M + (x_1, \ldots, x_{l-1}, y_j, \ldots, x_d)q_i M.$$ 

On the other hand, Proposition \(3.4\) says that

$$a \in (y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\}, x_1, \ldots, x_{l-1}, y_j, \ldots, x_d)M : y_v y_u$$

$$= (y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\}, x_1, \ldots, x_{l-1}, x_j, \ldots, x_d)M : y_u.$$ 

Hence

$$b \in (y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\}, x_1, \ldots, x_{l-1}, x_j, \ldots, x_d)M : x_i \cap q_i M$$

$$\subseteq (y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\}, x_1, \ldots, x_{l-1}, x_j, \ldots, x_d)M$$

because $x_1, \ldots, x_{j-1}$ is a $d$-sequence on $M/(y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\}, x_j, \ldots, x_d)M$.

Therefore

$$y_u a = x_j b + c \in (y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\})M + (x_1, \ldots, x_{l-1}, x_j, \ldots, x_d)q_i M.$$ 

Thus \((3.6.8)\) is proved. If we put $l = i$, then we obtain

$$[(y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\})M + q_i q_j M] : y_v y_u$$

$$= [(y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\})M + q_i q_j M] : y_u.$$ 

Next we assume that $n_{i+1} + \cdots + n_j > 1$. Let

$$a \in [(y_1, \ldots, y_{v-1}, \{x_\lambda | \lambda \in \Lambda\})M + q_i q_{i+1}^{n_{i+1}} \cdots q_j^{n_j} M] : y_v y_u.$$
Then \((E_{i+1,j})\) says that
\[
    a \in [(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\}, x_i)M + q_i^{n_i+1} \cdots q_j^n M] : y_v y_u
\]
\[
= [(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\}, x_i)M + q_i^{n_i+1} \cdots q_j^n M] : y_u.
\]

Therefore
\[
y_u a \in [(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i q_i^{n_i+1} \cdots q_j^n M] : y_v
\]
\[
\cap [(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\}, x_i)M + q_i^{n_i+1} \cdots q_j^n M]
\]
\[
= (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^n M
\]
\[
+ [(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i q_i^{n_i+1} \cdots q_j^n M] : y_v \cap x_i M
\]
\[
= (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^n M
\]
\[
+ x_i [(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i q_i^{n_i+1} \cdots q_j^n M] : y_v.
\]

Here we used \((D_{ij})\) to show the second equality. We put \(y_u a = x_i b + c\) with
\[(3.6.9)\]
\[
b \in [(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^n M] : y_v
\]
and
\[
c \in (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + a q_i^{n_i+1} \cdots q_j^n M.
\]

By applying \((C_{i+1,j})\) to a subsystem of parameters \(y_1, \ldots, y_{v-1}, y_u, \{x_\lambda \mid \lambda \in \Lambda\}, x_i\) for \(M/q_i+1 M\), we obtain
\[(3.6.10)\]
\[
b \in [(y_1, \ldots, y_{v-1}, y_u, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^n M] : x_i
\]
\[
\subseteq (y_1, \ldots, y_{v-1}, y_u, \{x_\lambda \mid \lambda \in \Lambda\})M : x_i + q_i q_i^{n_i+1} \cdots q_j^n M.
\]

On the other hand, since \(n_i+1 + \cdots + n_j > 1\), we have
\[(3.6.11)\]
\[
b \in [(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^2 M] : y_v
\]
\[
\subseteq (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v + q_i+1 M
\]
by using \((C_{i+1,i+1})\).

Furthermore, by applying Proposition \[3.3\] to a subsystem of parameters \(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\}, y_v, x_i\) for \(M/q_i+1 M\), we obtain
\[(3.6.12)\]
\[
(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v
\]
\[
\subseteq (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v x_i
\]
\[
= (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : x_i.
\]

Hence, by taking the intersection of \[(3.6.10)\] and \[(3.6.11)\], we have
\[
b \in (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v + q_i q_i^{n_i+1} \cdots q_j^n M
\]
\[
+ (y_1, \ldots, y_{v-1}, y_u, \{x_\lambda \mid \lambda \in \Lambda\})M : x_i \cap q_i+1 M
\]
\[
\subseteq (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v + y_v M + q_i q_i^{n_i+1} \cdots q_j^n M.
\]

Here we apply Proposition \[3.22\] to a \(d\)-sequence \(x_i, \ldots, x_d\) on
\[
M/(y_1, \ldots, y_{v-1}, y_u, \{x_\lambda \mid \lambda \in \Lambda\})M.
\]
Taking the intersection with (3.6.3), we obtain
\[ b \in (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v + q_i^{n_{i+1}} \cdots q_j^{n_j}M \]
+ \[((y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_{i+1}} \cdots q_j^{n_j}M) : y_v \cap y_uM \]
= \((y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v + q_i^{n_{i+1}} \cdots q_j^{n_j}M \)
+ \[ y_u((y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_{i+1}} \cdots q_j^{n_j}M) : y_vy_u \]
= \((y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v + q_i^{n_{i+1}} \cdots q_j^{n_j}M \).

Here we used \((E_{i+1,j})\) to show the last equality. By using (3.6.12) again, we find that
\[ y_ua = x_ib + c \in (y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_{i+1}} \cdots q_j^{n_j}M. \]

That is,
\[ a \in [(y_1, \ldots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_{i+1}} \cdots q_j^{n_j}M] : y_u. \]

The opposite inclusion is obvious. The proof is completed.

**Corollary 3.7.** With the same notation as Theorem 3.6, we have
\[ [(y_1, \ldots, y_u)M + q_i^{n_i} \cdots q_j^{n_j}M] : x_{i-1}^{n_{i-1}} \subseteq [(y_1, \ldots, y_u)M + q_i^{n_i} \cdots q_j^{n_j}M] : q_{i-1} \]
for any integers \(2 \leq i \leq j \leq d, n_{i-1} > 0, n_i, \ldots, n_j \geq 0\) and for any subsystem of parameters \(y_1, \ldots, y_u\) for \(M/q_{i-1}M\).

**Proof.** If \(n_i = \cdots = n_j = 0\), then the equality is trivial. Therefore we may assume that one of \(n_i, \ldots, n_j\) is positive. We may also assume that \(n_{i-1} = 1\) by using Theorem 3.6.\((E_{i,j})\). Then we have
\[ [(y_1, \ldots, y_u)M + q_i^{n_i} \cdots q_j^{n_j}M] : x_{i-1} \subseteq (y_1, \ldots, y_u)M : x_{i-1} + q_i^{n_{i-1}} \cdots q_j^{n_j}M \]
by applying Theorem 3.6.\((C_{i,j})\) to a subsystem of parameters \(y_1, \ldots, y_u, x_{i-1}\) for \(M/q_iM\). Since \(x_{i-1}, \ldots, x_d\) is a d-sequence on \(M/(y_1, \ldots, y_u)M\),
\[ (y_1, \ldots, y_u)M : x_{i-1} \subseteq (y_1, \ldots, y_u)M : q_{i-1}. \]

Therefore
\[ q_{i-1}[(y_1, \ldots, y_u)M + q_i^{n_i} \cdots q_j^{n_j}M] : x_{i-1} \subseteq (y_1, \ldots, y_u)M + q_i^{n_i} \cdots q_j^{n_j}M. \]

The opposite inclusion is trivial.

**Corollary 3.8.** With the same notation of Theorem 3.6, we let \(k\) be an integer such that \(1 \leq k \leq d\) and \(y_1, \ldots, y_u\) a subsystem of parameters for \(M/q_kM\). Assume that
\[ [(y_1, \ldots, y_{u-1})M + q_kM] : y_u = (y_1, \ldots, y_{u-1})M + q_kM. \]

Then
\[ (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_u = (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M \]
for any \(\Lambda \subseteq \{k, \ldots, d\}\). Furthermore
\[ [(y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i} \cdots q_j^{n_j}M] : y_u = (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i} \cdots q_j^{n_j}M \]
for any integers \(k \leq i \leq j, n_i, \ldots, n_j \geq 0\), and \(\Lambda \subseteq \{k, \ldots, i-1\}\).
Proof. We first show (3.8.1) by descending induction on the number of elements in $\Lambda$. If $\Lambda = \{k, \ldots, d\}$, then there exists nothing to prove. Assume that $\Lambda \neq \{k, \ldots, d\}$ and let $l$ be an element in $\{k, \ldots, d\} \setminus \Lambda$. Let $a$ be an element in the left-hand side of (3.8.1). Then

$$a \in (y_1, \ldots, y_{u-1}, x_l, \{x_\lambda \mid \lambda \in \Lambda\})M : y_u = (y_1, \ldots, y_{u-1}, x_l, \{x_\lambda \mid \lambda \in \Lambda\})M$$

because of the induction hypothesis. We put $a = xb + c$ with

$$c \in (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M.$$

Since $x_l \in a(M)$ or $x_l \in a(M/q_{i+1}M)$, we obtain

$$b \in (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_u x_l = (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : x_l$$

by using Proposition 3.3. Therefore $a = xb + c \in (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M$.

Next we show that (3.8.2). If $n_i = \cdots = n_j = 0$, then the equality is trivial. We assume that $n_i, n_j > 0$ and we work by induction on $j - i$. If $i = j$, then

$$((y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i}M : y_u$$

$$= (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_u$$

$$+ q_i^{n_i-1}[(y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_iM : y_u]$$

$$= (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i}M.$$

Here we used Theorem 3.6 (B.1) and (3.8.1). Assume that $j > i$. We may assume that $n_i = 1$ by using Theorem 3.6 (B.1). Let $a$ be an element of the left-hand side of (3.8.2). The induction hypothesis says that

$$[(y_1, \ldots, y_{u-1}, x_i, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^{n_j}M] : y_u$$

$$= (y_1, \ldots, y_{u-1}, x_i, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^{n_j}M.$$

Therefore

$$a \in [(y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^{n_j}M : y_u$$

$$\cap [(y_1, \ldots, y_{u-1}, x_i, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^{n_j}M]$$

$$= (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^{n_j}M + [(y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^{n_j}M : y_u \cap x_iM$$

$$\subseteq (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^{n_j}M + x_i[(y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^{n_j}M : y_u]$$

$$= (y_1, \ldots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + q_i^{n_i+1} \cdots q_j^{n_j}M.$$

Here we used Theorem 3.6 (D.1) and the induction hypothesis.

4. THE PROOF OF THEOREM 1.1

Before the proof of Theorem 1.1, we give some statements on $\mathbb{Z}^r$-graded rings. Let $R = \bigoplus_{n_r \geq 0} R(n_1, \ldots, n_r)$ be a Noetherian $\mathbb{Z}^r$-graded ring. For such a ring, let $R_+ = \bigoplus_{(n_1, \ldots, n_r) \neq (0, \ldots, 0)} R(n_1, \ldots, n_r)$.

Proposition 4.1. Let $M$ be a finitely generated graded $R$-module and $b$ an ideal in $R(0, \ldots, 0)$. Then there exists an integer $n$ such that

$$[H^p_{bR_R}(M)](n_1, \ldots, n_r) = 0$$

unless $n_1, \ldots, n_r < n$ for all $p \geq 0$. 

Proof. If $b = (0)$, then we can prove the assertion in the same way as [28, no. 66 Théorème 2]. The spectral sequence $E_2^{pq} = H_p^b R_q^b (-) \Rightarrow H_{b+R^b}^q (-)$ says that the assertion holds in general.

Let $\varphi: \mathbb{Z}' \to \mathbb{Z}^r$ be a group homomorphism satisfying $\varphi(N') \subseteq N^r$. We put

$$R^\varphi = \bigoplus_{m_1, \ldots, m_r \geq 0} \left( \bigoplus_{n_1, \ldots, n_r = (m_1, \ldots, m_r)} R_{(n_1, \ldots, n_r)} \right),$$

which is a $\mathbb{Z}^r$-graded ring. For a graded $R$-module $M$, let

$$M^\varphi = \bigoplus_{m_1, \ldots, m_r \in \mathbb{Z}} \left( \bigoplus_{n_1, \ldots, n_r = (m_1, \ldots, m_r)} M_{(n_1, \ldots, n_r)} \right),$$

which is a graded $R^\varphi$-module. We know that $[H_{b+R^b}^p(M)]^\varphi = H_{b+R^b+R^\varphi}^p(M^\varphi)$ for any ideal $b$ in $R(0, \ldots, 0)$. See Lemma 1.1 of [15].

The following proposition is contained in the proof of [15, Theorem 2.2].

Proposition 4.2. Let $M = \bigoplus_{n_1, \ldots, n_r \geq 0} M_{(n_1, \ldots, n_r)}$ be a finitely generated graded $R$-module and $b$ an ideal in $R(0, \ldots, 0)$. We put

$$S = \bigoplus_{n_1, \ldots, n_r+1 \geq 0} R_{(n_1, \ldots, n_r-1, n_r+n_r+1)}$$

and

$$N = \bigoplus_{n_1, \ldots, n_r+1 \geq 0} M_{(n_1, \ldots, n_r-1, n_r+n_r+1)}.$$

Then $S$ is a Noetherian $\mathbb{Z}^{r+1}$-graded ring and $N$ a finitely generated graded $S$-module.

If there exists an integer $p_0$ such that

(4.2.1) $H_{b+R^b}^p(M) = 0$ for all $p > p_0$,

then

$$H_{b+S+S^+}^p(N) = 0 \text{ for all } p > p_0 + 1.$$

If

(4.2.2) $[H_{b+R^b}^p(M)]_{(n_1, \ldots, n_r)} = 0$ unless $n_1, \ldots, n_r < 0$

for all $p$, then

$$[H_{b+S+S^+}^p(N)]_{(n_1, \ldots, n_r+1)} = 0 \text{ unless } n_1, \ldots, n_r+1 < 0$$

for all $p$. If, in addition, there exist integers $p_0 > 0$ and $n_0 < 0$ such that

(4.2.3) $[H_{b+R^b}^p(M)]_{(n_1, \ldots, n_r)} = 0$ whenever $n_1 + \cdots + n_r \leq n_0$

for all $p < p_0$, then

$$[H_{b+S+S^+}^p(N)]_{(n_1, \ldots, n_r+1)} = 0 \text{ whenever } n_1 + \cdots + n_r+1 \leq n_0$$

for all $p < p_0 + 1$. 

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Proof. It is easy to show that $S$ is a $\mathbb{Z}^{r+1}$-graded ring and $N$ a graded $S$-module. First we show that $S$ is Noetherian. To do this, we may assume that $r = 1$ without loss of generality. Since $R$ is Noetherian, $R_0$ is also and $R$ is generated by finitely generated $R_0$-modules $R_1, \ldots, R_k$ over $R_0$. Then $S = S(0, 0)[S(n_1, n_2) \mid n_1 + n_2 \leq k]$. Indeed, if $i + j > k$, then $R_{i+j} = R_iR_{i+j-1} + \cdots + R_kR_{i+j-k}$. Therefore

$$S_{(i,j)} = \begin{cases} \sum_{l=1}^{k} S_{(l,0)}S_{(i-l,j)}, & \text{if } i \geq k; \\ \sum_{l=1}^{k} S_{(l,0)}S_{(i-l,j)} + \sum_{m=1}^{k-i} S_{(i,m)}S_{(0,j-m)}, & \text{if } i < k. \end{cases}$$

We can show that $S_{(i,j)} \subset S(0, 0)[S(n_1, n_2) \mid n_1 + n_2 \leq k]$ by induction on $i + j$. Similarly we can prove that $N$ is a finitely generated $S$-module.

Next we consider local cohomology modules. Let

$$I = \bigoplus_{n_1, \ldots, n_r \geq 0, n_{r+1} > 0} R_{(n_1, \ldots, n_r-1, n_r+n_{r+1})}$$

and

$$L_1 = \bigoplus_{n_1, \ldots, n_r \geq 0, n_{r+1} > 0} M_{(n_1, \ldots, n_r-1, n_r+n_{r+1})}.$$ 

If we put $\varphi(n_1, \ldots, n_r) = (n_1, \ldots, n_r, 0)$, then $S/I \cong R^\varphi$ and $N/L_1 \cong M^\varphi$. Therefore

$$[H_{bS+S_+}^p (N/L_1)]_{(n_1, \ldots, n_{r+1})} = \begin{cases} [H_{bR+R_+}^p (M)]_{(n_1, \ldots, n_r)}, & \text{if } n_{r+1} = 0; \\ 0, & \text{otherwise} \end{cases}$$

for all $p$. Similarly we put

$$L_2 = \bigoplus_{n_1, \ldots, n_r-1, n_{r+1} \geq 0, n_r > 0} M_{(n_1, \ldots, n_r-1, n_r+n_{r+1})}.$$

Then

$$[H_{bS+S_+}^p (N/L_2)]_{(n_1, \ldots, n_{r+1})} = \begin{cases} [H_{bR+R_+}^p (M)]_{(n_1, \ldots, n_r-1, n_{r+1})}, & \text{if } n_r = 0; \\ 0, & \text{otherwise} \end{cases}$$

for all $p$.

There exist two long exact sequences of local cohomology modules

$$\cdots \to H_{bS+S_+}^{p-1} (N/L_i) \to H_{bS+S_+}^p (L_i) \to H_{bS+S_+}^p (N) \to H_{bS+S_+}^p (N/L_i) \to \cdots$$

for $i = 1$ and 2. On the other hand, $L_1 \cong L_2(0, \ldots, 0, 1, -1)$.

Assume that (4.2.1) holds. If $p > p_0 + 1$, then

$$[H_{bS+S_+}^p (N)]_{(n_1, \ldots, n_{r+1})} \cong [H_{bS+S_+}^p (L_1)]_{(n_1, \ldots, n_{r+1})} \cong [H_{bS+S_+}^p (L_2)]_{(n_1, \ldots, n_{r-1}, n_r+1, n_{r+1}-1)} \cong [H_{bS+S_+}^p (N)]_{(n_1, \ldots, n_{r-1}, n_r+1, n_{r+1}-1)} \cong \cdots = 0.$$ 

Here we used Proposition (4.1).
Next we assume that (12.2) holds for all \( p \). Unless \( n_1, \ldots, n_r < 0 \), then
\[
[H_{bS+S_+}^p(N)](n_1, \ldots, n_{r+1}) \cong [H_{bS+S_+}^p(L_1)](n_1, \ldots, n_{r+1})
\cong [H_{bS+S_+}^p(L_2)](n_1, \ldots, n_{r-1}, n_r + 1, n_{r+1}-1)
\cong [H_{bS+S_+}^p(N)](n_1, \ldots, n_{r-1}, n_r + 1, n_{r+1}-1)
\cong \cdots = 0.
\]

We can also show that \([H_{bS+S_+}^p(L)](n_1, \ldots, n_{r+1}) = 0\) if \( n_{r+1} \geq 0 \). In addition, we also assume that (12.3) holds for all \( p < p_0 \). If \( p < p_0 + 1, n_1 + \cdots + n_{r+1} \leq n_0, \) and \( n_1, \ldots, n_{r-1} < 0 \), then
\[
[H_{bS+S_+}^p(N)](n_1, \ldots, n_{r+1}) \cong [H_{bS+S_+}^p(L_1)](n_1, \ldots, n_{r+1})
\cong [H_{bS+S_+}^p(L_2)](n_1, \ldots, n_{r-1}, n_r + 1, n_{r+1}-1)
\subseteq [H_{bS+S_+}^p(N)](n_1, \ldots, n_{r-1}, n_r + 1, n_{r+1}-1)
\cong \cdots = 0.
\]

The proof is completed. \( \Box \)

Let \( b_1, \ldots, b_r \) be ideals in \( A \). The multigraded Rees algebra of \( A \) (for short, the multi-Rees algebra) with respect to them is defined to be
\[
R(b_1, \ldots, b_r) = A[b_1T_1, \ldots, b_rT_r],
\]
where \( T_1, \ldots, T_r \) are indeterminates. If \( b_1, \ldots, b_r \) are of positive height, then \( \dim R(b_1, \ldots, b_r) = \dim A + r \). See Proposition 1.17 of [15]. For an \( A \)-module \( M \), let \( R_M(b_1, \ldots, b_r) \) denote the \( R(b_1, \ldots, b_r) \)-module
\[
\bigoplus_{n_1, \ldots, n_r \geq 0} b_1^{n_1} \cdots b_r^{n_r} MT_1^{n_1} \cdots T_r^{n_r}.
\]

Recently Hyry gives the following theorem.

**Theorem 4.3 (10 Corollary 2.10).** Let \( b_1, \ldots, b_r \) be ideals in \( A \) of positive height. If the multi-Rees algebra \( R(b_1, \ldots, b_r) \) is Cohen-Macaulay, then the ordinary Rees algebra \( R(b_1 \cdot b_r) \) is also Cohen-Macaulay.

We start to prove Theorem 4.1.

**Theorem 4.4.** Let \( M \) be a finitely generated \( A \)-module and \( x_1, \ldots, x_d \) elements in \( A \). We fix integers \( t \leq s + 1 < d, \alpha_1, \ldots, \alpha_s > 0, \) and \( \alpha_{s+1} \geq d - s - 1 \). Let \( q_i = (x_i, \ldots, x_d) \) for all \( t \leq i \leq s + 1 \). We put
\[
S = A[q_1T_{t,1}, \ldots, q_{i}T_{t,\alpha_i}, q_{i+1}T_{t+1,1}, \ldots, q_{s}T_{s,\alpha_s}, q_{s+1}T_{s+1,1}, \ldots, q_{s+1}T_{s+1,\alpha_{s+1}}]
\]
and \( N \) the \( S \)-module \( R_M(q_1, \ldots, q_{s+1}) \). If the sequence \( x_1, \ldots, x_d \) satisfies the following six conditions:

1. the sequence \( x_1, \ldots, x_d \) is a \( d \)-sequence on \( M/(x_\lambda^n | \lambda \in \Lambda)M \) for all \( t \leq i \leq s + 1, n_t, \ldots, n_{i-1} > 0, \) and \( \Lambda \subseteq \{t, \ldots, i-1\} \);
2. the sequence \( x_1, \ldots, x_{d-1} \) is a \( d \)-sequence on \( M/(\{x_\lambda | \lambda \in \Lambda, x_d\})M \) for all \( t < i \leq s + 1, n_t, \ldots, n_{i-1} > 0, \) and \( \Lambda \subseteq \{t, \ldots, i-1\} \);
3. the sequence \( x_{i+1}, \ldots, x_d \) is a u.d.\( d \)-sequence on \( M/(x_\lambda^n | \lambda \in \Lambda)M \) for all \( n_t, \ldots, n_s > 0 \) and \( \Lambda \subseteq \{t, \ldots, s\} \);
(4) the equality
\[
\{(x_\lambda^n \mid \lambda \in \Lambda), x_k, \ldots, x_{l-1}\} M : x_l \cap [(x_\lambda^n \mid \lambda \in \Lambda) M + q_i^n \cdots q_{s+1}^n M] = (x_\lambda^n \mid \lambda \in \Lambda) M + (x_k, \ldots, x_{l-1}) q_i^n \cdots q_{s+1}^n M
\]
holds for any integers \( t \leq i \leq k \leq s + 1, k \leq l \leq d, n_i, \ldots, n_{i-1}, n_k > 0, n_i, \ldots, n_{k-1}, n_{k+1}, \ldots, n_{s+1} \geq 0, \) and \( \Lambda \subseteq \{t, \ldots, i-1\} \); 

(5) the equality
\[
[(x_\lambda^n \mid \lambda \in \Lambda) M + q_i^n \cdots q_{s+1}^n M] : x_{i-1} = [(x_\lambda^n \mid \lambda \in \Lambda) M + q_i^n \cdots q_{s+1}^n M] : q_{i-1}
\]
holds for any \( t < i \leq s + 1, n_i, \ldots, n_{i-1} > 0, n_i, \ldots, n_{s+1} \geq 0, \) and \( \Lambda \subseteq \{t, \ldots, i-2\} \); 

(6) \( 0 : M x_d \subseteq 0 : M x_t \),

then
\[
H^p_{q_1 S + S_+} (N) = 0 : x_d,
\]
(4.4.2)
\[
H^p_{q_1 S + S_+} (N) = 0 \text{ for } p \neq 0, d-t+1 + \alpha_t + \cdots + \alpha_{s+1},
\]
and
(4.4.3)
\[
[H^d-{t+1+\alpha_t+\cdots+\alpha_{s+1}}_{q_1 S + S_+} (N)](n_1, \ldots, n_{s+1}, \alpha_{s+1}) = 0,
\]
unless \( n_t, \ldots, n_{s+1}, \alpha_{s+1} < 0 \).

Proof. We show that (4.4.1), (4.4.3) by descending induction on \( t \). First we note that \( d-s \geq 2 \) because of the assumption. Furthermore \( 0 : M x_t \subseteq \cdots \subseteq 0 : M x_d \) because \( x_t, \ldots, x_d \) is a \( d \)-sequence on \( M \). Therefore (1) and (6) say that \( 0 : M x_t = \cdots = 0 : M x_d \). Without loss of generality, we may assume that \( 0 : M x_d = 0 \). Indeed, assumptions (1)–(6) hold on \( \overline{M} = M/0 : M x_d \). For example,
\[
[(x_\lambda^n \mid \lambda \in \Lambda), x_k, \ldots, x_{l-1}] M + 0 : x_l : x_l = (x_\lambda^n \mid \lambda \in \Lambda), x_k, \ldots, x_{l-1}) M : x_l^2 = (x_\lambda^n \mid \lambda \in \Lambda), x_k, \ldots, x_{l-1}) M : x_l
\]
because \( 0 : M x_t \subseteq 0 : M x_t \). Hence
\[
(x_\lambda^n \mid \lambda \in \Lambda), x_k, \ldots, x_{l-1}) M : x_l \cap [(x_\lambda^n \mid \lambda \in \Lambda) M + q_i^n \cdots q_{s+1}^n M + 0 : x_l]
\]
\[
= (x_\lambda^n \mid \lambda \in \Lambda), x_k, \ldots, x_{l-1}) M : x_l \cap [(x_\lambda^n \mid \lambda \in \Lambda) M + q_i^n \cdots q_{s+1}^n M + 0 : x_l]
\]
\[
= (x_\lambda^n \mid \lambda \in \Lambda) M + (x_k, \ldots, x_{l-1}) q_i^n \cdots q_{s+1}^n M + 0 : x_l.
\]
Thus (4) holds on \( \overline{M} \). Similarly we can show that (1)–(3) and (5) hold on \( \overline{M} \). Of course \( 0 : \overline{M} x_t = 0 : \overline{M} x_d = 0 \). On the other hand, if \( \overline{N} \) denotes the \( S \)-module \( R_{\overline{M}}(q_1, \ldots, q_{s+1}) \), then there exists an exact sequence of \( S \)-modules
\[
0 \rightarrow 0 : x_t \rightarrow N \rightarrow \overline{N} \rightarrow 0.
\]
Since \( 0 : M x_t \) is annihilated by \( q_1 S + S_+ \),
\[
0 \rightarrow 0 : x_t \rightarrow H^0_{q_1 S + S_+} (N) \rightarrow H^0_{q_1 S + S_+} (\overline{N}) \rightarrow 0
\]
is exact and
\[ H^p_{q,t+S+S_+}(N) \cong H^p_{q,t+S_+}(N) \quad \text{for all } p > 0. \]
Thus if the assertion holds for \( M \), then the one holds for \( M \).

From now on we assume that \( 0 :_M x_t = \cdots = 0 :_M x_d = 0 \). Because of Proposition 4.2, we may assume that \( \alpha_t = \cdots = \alpha_s = 1 \) and \( \alpha_{s+1} = d - s - 1 \). For the simplicity, we write \( T_t = T_{t+1}, \ldots, T_{s+1} = T_{s+1,1}, T_{s+2} = T_{s+1,2}, \ldots, T_{d-1} = T_{s+1,d-s-1} \).

Assume that \( t = s + 1 \) and put \( R = \text{Ann}(T_{s+1}T_{s+1}) \). Then we know that
\[ [H^p_{q+t+1,R+R_+}(R_M(q_{s+1}))]_{n+1} = 0 \quad \text{unless } 2 - p \leq n \leq -1 \]
for all \( p < d - s + 1 \),
\[ [H^d_{d-s+1,R+R_+}(R_M(q_{s+1}))]_{n} = 0 \quad \text{unless } n < 0, \]
and
\[ H^p_{q+t+1,R+R_+}(R_M(q_{s+1})) = 0 \quad \text{for all } p > d - s + 1. \]

See [12, Theorem 4.1]. By using Proposition 4.2 repeatedly, we find that
\[ H^p_{q+t+1,S+S_+}(N) = 0 \quad \text{for } p \neq 2d - 2s - 1 \]
and
\[ [H^{2d-2s-1}_{q+t+1,S_+}(N)]_{n} = 0 \quad \text{unless } n_{s+1}, \ldots, n_{d-1} < 0. \]
Thus we obtain [14.1]–[14.3].

Next we assume that \( t < s + 1 \). Then \( x_{m}M : x_{t+1} = x_{t}M : x_{d} \) for any \( m > 0 \). Indeed, if \( a \in x_{m}M : x_{d} \) and we put \( x_{ad} = x_{b} \), then \( b \in x_{d}M : x_{m} \subseteq x_{d}M : x_{t+1} \) because of (2). Let \( x_{t+1}b = x_{ab} \). Then \( x_{t+1}x_{ad} = x_{t+1}x_{b} = x_{t}x_{t+1}c \). Therefore \( x_{t+1}a - x_{p}c \in 0 :_M x_{d} = 0 \) and hence \( a \in x_{p}M : x_{t+1} \). Thus the sequence \( x_{t+1}, \ldots, x_{d} \) satisfies (1)–(6) on \( M \) and on \( M/x_{t}M \) for any \( m > 0 \).

Let \( R = \text{Ann}(T_{t+1}T_{t+1}, \ldots, q_{s+1}T_{s+1}, \ldots, q_{s+1}T_{d-1}) \) and
\[ Y = \bigoplus_{n_{t+1}, \ldots, n_{d-1} \geq 0} [q_{n_{t+1}}^{n_{t+1}} \cdots q_{n_{d-1}}^{n_{d-1}} M : q_{t}]T_{t+1}^{n_{t+1}} \cdots T_{d-1}^{n_{d-1}}. \]
Then assumption (5) gives an exact sequence of \( R \)-modules
\[ 0 \to Y \xrightarrow{x_{t}} R_{M}(q_{t+1}, \ldots, q_{s+1}) \to R_{M/x_{t}M}(q_{t+1}, \ldots, q_{s+1}) \to 0 \]
and hence \( Y \) is finitely generated over \( R \). The induction hypothesis says that
\[ H^p_{q+t,R+R_+}(R_M(q_{t+1}, \ldots, q_{s+1})) = 0 \quad \text{for } p \neq 2d - 2t - 1, \]
\[ [H^{2d-2t-1}_{q+t,R+R_+}(R_M(q_{t+1}, \ldots, q_{s+1}))]_{n_{t+1}, \ldots, n_{d-1}} = 0 \]
unless \( n_{t+1}, \ldots, n_{d-1} < 0 \),
\[ H^p_{q+t+1,R+R_+}(R_{M/x_{t}M}(q_{t+1}, \ldots, q_{s+1})) = 0 \quad \text{for } p \neq 0, 2d - 2t - 1, \]
and
\[ [H^{2d-2t-1}_{q+t+1,R+R_+}(R_{M/x_{t}M}(q_{t+1}, \ldots, q_{s+1}))]_{n_{t+1}, \ldots, n_{d-1}} = 0 \]
unless \( n_{t+1}, \ldots, n_{d-1} < 0 \). The spectral sequence
\[ E^2_{2} = H^p_{x_{t}}H^q_{q+t,R+R_+}(-) \Rightarrow H^{p+q}_{q,R+R_+}(-) \]
gives a short exact sequence

$$0 \to H^1_{x_t}H^{p-1}_{q_t+1,R_t+R_+}(-) \to H^p_{q_t,R_t+R_+}(-) \to H^0_{x_t}H^p_{q_t+1,R_t+R_+}(-) \to 0.$$  

By using it, we obtain

$$H^p_{q_t,R_t+R_+}(R_M(q_{t+1},\ldots,q_{s+1})) = 0 \quad \text{for } p \neq 2d - 2t - 1, 2d - 2t,$$

$$[H^{2d-2t}_{q_t,R_t+R_+}(R_M(q_{t+1},\ldots,q_{s+1}))](n_{t+1},\ldots,n_{d-1}) = 0$$

unless $n_{t+1}, \ldots, n_{d-1} < 0$,

$$H^p_{q_t,R_t+R_+}(R_{M/x^p_t M}(q_{t+1},\ldots,q_{s+1})) = 0 \quad \text{for } p \neq 0, 2d - 2t - 1,$$

and

$$[H^{2d-2t}_{q_t,R_t+R_+}(R_{M/x^p_t M}(q_{t+1},\ldots,q_{s+1}))](n_{t+1},\ldots,n_{d-1}) = 0$$

unless $n_{t+1}, \ldots, n_{d-1} < 0$. Therefore

$$H^p_{q_t,R_t+R_+}(Y) = 0 \quad \text{for } p \neq 1, 2d - 2t - 1, 2d - 2t,$$

$$[H^{2d-2t}_{q_t,R_t+R_+}(Y)](n_{t+1},\ldots,n_{d-1}) = 0 \quad \text{unless } n_{t+1}, \ldots, n_{d-1} < 0,$$

and

$$0 \to H^{2d-2t-1}_{q_t,R_t+R_+}(Y) \to H^{2d-2t-1}_{q_t,R_t+R_+}(R_M(q_{t+1},\ldots,q_{s+1}))$$

is exact. We show that $H^{2d-2t-1}_{q_t,R_t+R_+}(Y) = 0$. Let $E = H^{2d-2t-1}_{q_t,R_t+R_+}(R_M(q_{t+1},\ldots,q_{s+1}))$.

Because of (5),

$$q_1 Y \subseteq R_M(q_{t+1},\ldots,q_{s+1}) \subseteq Y.$$

Therefore

$$H^p_{q_t,R_t+R_+}(Y/R_M(q_{t+1},\ldots,q_{s+1})) \cong H^p_{R_+}(Y/R_M(q_{t+1},\ldots,q_{s+1})).$$

Let

$$f_{2t+2} = x_{t+1}T_{t+1},$$

$$f_{2t+3} = x_{t+2}T_{t+1},$$

$$f_{2t+4} = x_{t+3}T_{t+1} + x_{t+2}T_{t+2},$$

$$\vdots$$

$$f_{d+t+1} = x_dT_{t+1} + x_{d-1}T_{t+2} + \cdots,$$

$$f_{d+t+2} = x_dT_{t+1} + \cdots,$$

$$\vdots$$

$$f_{2d-2} = x_dT_{d-2} + x_{d-1}T_{d-1},$$

$$f_{2d-1} = x_dT_{d-1}.$$

Then $\sqrt{R_+} = \sqrt{(f_{2t+2},\ldots,f_{2d-1})R}$. The proof is quite similar to [11, Lemma 3.2]. We omit it. Therefore

$$H^p_{q_t,R_t+R_+}(Y/R_M(q_{t+1},\ldots,q_{s+1})) = 0 \quad \text{for } p > 2d - 2t - 2$$

and hence

$$H^{2d-2t-2}_{q_t,R_t+R_+}(Y/R_M(q_{t+1},\ldots,q_{s+1})) \to E \to H^{2d-2t-1}_{q_t,R_t+R_+}(Y) \to 0$$

is exact. Thus

$$(4.4.4) \quad H^{2d-2t-1}_{q_t,R_t+R_+}(Y/R_M(q_{t+1},\ldots,q_{s+1})) \to E \xrightarrow{\delta} E$$
is exact. Since the first term of (4.4.4) is annihilated by \( x_t \), we obtain 0;\( E x_t^m = 0;E x_t \). Therefore \( x_t E = 0 \) and hence \( H^{2d-2t-1}_R(x_t^m) = 0 \) because \( E = \bigcup_{m > 0} 0;E x_t^m \).

Since \( R = S/q_T S, Y \) is also an \( S \)-module and

\[
H^p_{q_t S + S_+}(Y) = 0 \quad \text{for} \quad p \neq 1, 2d - 2t, \quad [H^{2d-2t}_{q_t S + S_+}(Y)](n_t, \ldots, n_{d-1}) = 0 \quad \text{unless} \quad n_t = 0, n_{t-1}, \ldots, n_{d-1} < 0.
\]

Let \( S' = A[q_{t+1} T_t, q_{t+1} T_{t+1}, \ldots, q_s T_s, q_{s+1} T_{s+1}, \ldots, q_{s+1} T_{s+1}] \). Then the induction hypothesis says that

\[
H^p_{q_{t+1} S' + S'_+}(R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1})) = 0 \quad \text{for} \quad p \neq 0, 2d - 2t
\]

and

\[
[H^{2d-2t}_{q_{t+1} S' + S'_+}(R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1}))](n_t, \ldots, n_{d-1}) = 0
\]

unless \( n_t, \ldots, n_{d-1} < 0 \). Since \( S' \) is an \( A \)-subalgebra of \( S \), we can regard the \( S \)-module \( R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1}) \) as an \( S' \)-module and there exists an \( S' \)-isomorphism

\[
R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1}) \cong R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1}).
\]

Since \( (x_t, x_t T_t) R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1}) = 0 \),

\[
H^p_{q_{t+1} S' + S'_+}(R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1})) = 0
\]

for \( p \neq 0, 2d - 2t \) and

\[
[H^{2d-2t}_{q_{t+1} S' + S'_+}(R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1}))](n_t, \ldots, n_{d-1}) = 0 \quad \text{unless} \quad n_t, \ldots, n_{d-1} < 0.
\]

Let \( X \) be the kernel of the natural epimorphism \( N \to R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1}) \). Then there exists an exact sequence of \( S \)-modules

\[
0 \to X \to N \to R_{M/x_t M}(q_{t+1}, q_{t+1}, \ldots, q_{s+1}) \to 0.
\]

Since

\[
x_t M \cap q_{t+1}^{n_t+1} \cdots q_{s+1}^{n_s+1} M = x_t q_{t+1}^{n_t+1} \cdots q_{s+1}^{n_s+1} M
\]

if \( n_t > 0 \),

\[
\bigoplus_{n_t > 0} X(x_t, \ldots, x_n) = x_t T_t N
\]

and there exists an exact sequence

\[
0 \to N(-1, 0, \ldots, 0) \xrightarrow{x_t T_t} X \xrightarrow{x_t^{-1}} Y \to 0.
\]

Because of (4.4.3) and (4.4.4),

\[
0 \to H^p_{q_t S + S_+}(N)(-1, 0, \ldots, 0) \xrightarrow{x_t T_t} H^p_{q_t S + S_+}(N)
\]

is exact if \( 3 \leq p < 2d - 2t + 1 \) or \( p > 2d - 2t + 1 \). Since \( H^p_{q_t S + S_+}(N) \) is annihilated by some power of \( x_t T_t \) elementwise,

\[
H^p_{q_t S + S_+}(N) = 0 \quad \text{if} \quad 3 \leq p < 2d - 2t + 1 \text{ or } p > 2d - 2t + 1.
\]

Furthermore

\[
H^{2d-2t}_{q_t S + S_+}(Y) \to H^{2d-2t+1}_{q_t S + S_+}(N)(-1, 0, \ldots, 0) \to H^{2d-2t+1}_{q_t S + S_+}(X) \to 0
\]
and
\[ H_{q_{s}S + S_{s}}^{2d-2t}(R_{M, x_{1}, M}(q_{1}, \ldots, q_{s+1})) \to H_{q_{s}S + S_{s}}^{2d-2t+1}(X) \to H_{q_{s}S + S_{s}}^{2d-2t+1}(N) \to 0 \]
are exact. Unless \( n_{t}, \ldots, n_{d-1} < 0 \), then we obtain
\[ [H_{q_{s}S + S_{s}}^{2d-2t+1}(N)](n_{t}, \ldots, n_{d-1}) \cong [H_{q_{s}S + S_{s}}^{2d-2t+1}(X)](n_{t}, \ldots, n_{d-1}) \]
\[ \cong [H_{q_{s}S + S_{s}}^{2d-2t+1}(N)](n_{t}, \ldots, n_{d-1}) \cong \cdots = 0. \]
Thus (4.4.2) is proved.

Finally we show that \( x_{s}T_{s}, x_{s+1}T_{s+1}, x_{s+2} \) is a regular sequence on \( N \). Since \( x_{s} \) is regular on \( M \), \( x_{s}T_{s} \) is regular on \( N \).

Let \( aT_{s}^{n_{s}} \cdots T_{d-1}^{n_{d-1}} \in x_{s}T_{s}N:x_{s+1}T_{s+1} \). If \( n_{s} = 0 \), then \( x_{s+1}a = 0 \) and hence \( a = 0 \). If \( n_{s} > 0 \), then
\[ a \in x_{s}M:x_{s+1} \cap q_{s+1}^{n_{s+1}} \cdots q_{s+1}^{n_{s+1} + \cdots + n_{d-1}}M = x_{s}q_{s+1}^{n_{s+1}} \cdots q_{s+1}^{n_{s+1} + \cdots + n_{d-1}}M. \]
Here we used (4). Hence \( aT_{s}^{n_{s}} \cdots T_{d-1}^{n_{d-1}} \in x_{s}T_{s}N \).

Let \( aT_{s}^{n_{s}} \cdots T_{d-1}^{n_{d-1}} \in (x_{s}T_{s}, x_{s+1}T_{s+1})N:x_{s+2} \). If \( n_{s} = n_{s+1} = 0 \), then \( x_{s+2}a = 0 \) and hence \( a = 0 \). If \( n_{s} > 0 \) and \( n_{s+1} = 0 \), then \( a \in x_{s}M:x_{s+2} \). Because of (3), we have \( x_{s}M:x_{s+1} = x_{s}M:x_{s+2} \). Hence
\[ a \in x_{s}M:x_{s+1} \cap q_{s+1}^{n_{s+1}} \cdots q_{s+1}^{n_{s+1} + \cdots + n_{d-1}}M = x_{s}q_{s+1}^{n_{s+1}} \cdots q_{s+1}^{n_{s+1} + \cdots + n_{d-1}}M, \]
that is, \( aT_{s}^{n_{s}} \cdots T_{d-1}^{n_{d-1}} \in x_{s}T_{s}N \). If \( n_{s} = 0 \) and \( n_{s+1} > 0 \), then
\[ a \in x_{s+1}M:x_{s+2} \cap q_{s+1}^{n_{s+1}} \cdots q_{s+1}^{n_{s+1} + \cdots + n_{d-1}}M = x_{s+1}q_{s+1}^{n_{s+1}} \cdots q_{s+1}^{n_{s+1} + \cdots + n_{d-1}}M \]
and hence \( aT_{s}^{n_{s}} \cdots T_{d-1}^{n_{d-1}} \in x_{s+1}T_{s+1}N \). If \( n_{s}, n_{s+1} > 0 \), then
\[ a \in (x_{s}, x_{s+1})M:x_{s+2} \cap q_{s}^{n_{s}} \cdots q_{s}^{n_{s} + \cdots + n_{d-1}}M \]
\[ = (x_{s}, x_{s+1})q_{s}^{n_{s}} \cdots q_{s}^{n_{s} + \cdots + n_{d-1}}M \]
\[ = x_{s}q_{s}^{n_{s}} \cdots q_{s}^{n_{s} + \cdots + n_{d-1}}M + x_{s+1}q_{s}^{n_{s}} \cdots q_{s}^{n_{s} + \cdots + n_{d-1}}M. \]
Therefore \( aT_{s}^{n_{s}} \cdots T_{d-1}^{n_{d-1}} \in (x_{s}T_{s}, x_{s+1}T_{s+1})N \).
Thus we obtain
\[ H_{q_{s}S + S_{s}}^{p}(N) = 0 \quad \text{for } p < 3. \]
The proof is completed. \( \square \)

**Corollary 4.5.** Let \( A \) be a Noetherian local ring of dimension \( d \geq 2 \) and \( x_1, \ldots, x_d \) a \( p \)-standard system of parameters of type \( s \) for \( A \). We put \( q_i = (x_i, \ldots, x_d) \) for all \( 1 \leq i \leq s + 1 \). If \( s < d - 1 \) and \((0): x_d = 0 \), then the Rees algebra \( R(q_1 \cdots q_d S_{d-s}^{d-s-1}) \) is a Cohen-Macaulay ring. If, in addition, \( A/q_i \) is Cohen-Macaulay for some \( 1 < t \leq s + 1 \), then \( R(q_1 \cdots q_d S_{d-s}^{d-s-1}) \) is a Cohen-Macaulay ring.

**Proof.** In this case Propositions 3.6, 3.7, Theorem 3.6, and Corollary 3.7 say that \( x_1, \ldots, x_d \) satisfies assumptions (1)–(5) of Theorem 4.4. Moreover \((0): x_1 \geq \) \((0): x_1 = 0 \). Thus we find that \( A[q_1T_1, \ldots, q_sT_s, q_{s+1}T_{s+1}, \ldots, q_{s+1}T_{s+1}] \) is Cohen-Macaulay by using Theorem 4.1. By Hyr’s theorem says that \( R(q_1 \cdots q_d S_{d-s}^{d-s-1}) \) is Cohen-Macaulay.
Assume that $A/\mathfrak{q}$ is Cohen-Macaulay. That is, $x_1, \ldots, x_{t-1}$ is a regular sequence on $A/\mathfrak{q}$. We show that

$$(x_1, \ldots, x_i) : x_d = (x_1, \ldots, x_i) \text{ for } 1 \leq i \leq t - 1$$

by induction on $i$. If $i = 0$, then there exists nothing to prove. Assume that $i > 0$ and let $a \in (x_1, \ldots, x_i) : x_d$. If we put $x_da = b + x_ic$ with $b \in (x_1, \ldots, x_{i-1})$, then

$$c \in (x_1, \ldots, x_{i-1}, x_d) : x_i$$

$$(x_1, \ldots, x_{i-1}, x_d).$$

Here we used Corollary 3.8. Let $c = b' + x_da'$ with $b' \in (x_1, \ldots, x_{i-1})$. Then

$$a - x_ia' \in (x_1, \ldots, x_{i-1}) : x_d = (x_1, \ldots, x_{i-1})$$

because of the induction hypothesis. Therefore $a \in (x_1, \ldots, x_i)$. Thus $x_1, \ldots, x_d$ satisfies the assumptions of Theorem 4.4 on $\tilde{A} = A/(x_1, \ldots, x_{t-1})$. Therefore

$$\tilde{A}[q_1\tilde{A}t_1, \ldots, q_s\tilde{A}t_s, q_{s+1}\tilde{A}t_{s+1}, \ldots, q_{s+1}\tilde{A}t_{d-1}]$$

is a Cohen-Macaulay ring and hence $R(q_1 \cdots q_s q_{d-s+1}^{-1}A)$ is also. Corollary 3.8 also says that $x_1, \ldots, x_{t-1}$ is a regular sequence on $A$ and on $A/(q_1 \cdots q_s q_{d-s+1}^{-1})^n$ for all $n > 0$. Taking Koszul cohomology of a short exact sequence

$$0 \to R(q_1 \cdots q_s q_{d-s+1}^{-1}) \to A[T] \to \bigoplus_{n>0} (A/(q_1 \cdots q_s q_{d-s+1}^{-1})^n)T^n \to 0$$

with respect to $x_1, \ldots, x_{t-1}$, we obtain that

$$H^p(x_1, \ldots, x_{t-1}; R(q_1 \cdots q_s q_{d-s+1}^{-1})) = 0 \text{ for } p < t - 1$$

and

$$H^{t-1}(x_1, \ldots, x_{t-1}; R(q_1 \cdots q_s q_{d-s+1}^{-1})) \cong R(q_1 \cdots q_s q_{d-s+1}^{-1}A).$$

That is, $x_1, \ldots, x_{t-1}$ is a regular sequence on $R(q_1 \cdots q_s q_{d-s+1}^{-1})$ and

$$R(q_1 \cdots q_s q_{d-s+1}^{-1}A) \cong R(q_1 \cdots q_s q_{d-s+1}^{-1})/(x_1, \ldots, x_{t-1})R(q_1 \cdots q_s q_{d-s+1}^{-1}).$$

Therefore $R(q_1 \cdots q_s q_{d-s+1}^{-1})$ is a Cohen-Macaulay ring.

Proof of Theorem 1.1 Let $A$ be a Noetherian local ring of dimension $d > 0$. First we prove that (B) implies (A). Assume that $A$ satisfies (B). If $d = 1$, then $A$ is Cohen-Macaulay because $A$ has no embedded prime. Let $a$ be a system of parameters for $A$. Then $R(aA)$ is a polynomial ring over $A$ and hence Cohen-Macaulay.

Assume that $d \geq 2$. Since $A$ is unmixed, $\dim A/p = d$ for any associated prime $p$ of $A$. Thus $s = \dim A/a(A) < d - 1$ because of Corollary 2.3. Theorem 2.5 assures us that there exists a $p$-standard system of parameters $x_1, \ldots, x_d$ of type $s$ for $A$. Since $A$ is unmixed, $x_1, \ldots, x_d$ are non-zero divisors on $A$. Therefore Corollary 4.5 gives an arithmetic Macaulayfication of $A$.

Next we show that (A) implies (B). Let $b$ be an ideal in $A$ of positive height such that $R = A[bT]$ is a Cohen-Macaulay ring. Then $A$ is a homomorphic image of a Cohen-Macaulay local ring $R_{mR + R_s}$ and hence all the formal fibers of $A$ are Cohen-Macaulay. Next we show that $A$ is unmixed. By passing through the completion, we may assume that $A$ is complete. Since $b$ is of positive height, $\dim R = d + 1$. See [32, Corollary 1.6]. Let $p_1, \ldots, p_s$ be the associated primes of $A$. Then

$$p_i^* = p_iA[T] \cap R \text{ where } i = 1, \ldots, s$$
are the associated primes of $R$. Since $R$ is a Cohen-Macaulay ring of dimension $d + 1$, 
$\dim R/p_i^c = d + 1$ and hence $\dim A/p_i = d$; see [22] Corollary 1.6] again, for all $i$. 

To close this section, we give an example.

**Example 4.6.** Let $k$ be a field, $B$ an affine semigroup ring 

$$k[a, b, c, d, e^2, e^3, ade, bde, cde, d^2e]$$

and $n$ the homogeneous maximal ideal of $B$. Then $A = B_n$ is a Noetherian local 
ring of dimension 5. The sequence $x_1 = a^4, x_2 = b^4, x_3 = c^4, x_4 = d^4, x_5 = e^4$ is a 
$p$-standard system of parameter of type 3 for $A$. See [17] Appendix B].

Let $q_i = (x_1, \ldots, x_d)$ for $i = 1, \ldots, 4$. Then the proof of Corollary 4.5 says that the 
multi-Rees algebra $A[q_1T_1, \ldots, q_4T_4]$ is a Cohen-Macaulay ring of dimension 9. 
However, we can verify that it is a Cohen-Macaulay ring by using a computer [6].

Indeed the sequence $x_1, x_1T_1 + x_2, x_2T_1 + x_3, x_2T_2 + x_3T_1 + x_4, x_3T_2 + x_4T_1 + x_5, 
 x_3T_3 + x_4T_2 + x_5T_1, x_4T_3 + x_5T_2, x_4T_4 + x_5T_3, x_5T_4$ is a regular sequence on 
$A[q_1T_1, \ldots, q_4T_4]$ of length 9.

5. The proof of Corollary 1.2

Before proving Corollary 1.2 we state the definition of the codimension function.

**Definition 5.1.** Let $B$ be a Noetherian ring. An integer-valued function $t_B$ defined 
on Spec $B$ is said to be a codimension function of $B$ if

$$\text{ht } p_1/p_2 = t_B(p_1) - t_B(p_2)$$ whenever $p_1 \supseteq p_2$.

A codimension function of $B$ is not unique even if it exists. In fact, if $t(p)$ is a 
codimension function, then $t(p) + c$ is also a codimension function for any constant $c$. 
However, the codimension function is unique up to constant if Spec $B$ is connected.

**Proposition 5.2.** (1) A catenary local ring has a codimension function.

(2) A catenary integral domain has a codimension function.

(3) A Cohen-Macaulay ring has a codimension function even if it is neither a 
local ring nor an integral domain.

(4) If a Noetherian ring has a codimension function, then its homomorphic image 
does also.

(5) If a Noetherian ring has a codimension function, then its localization does 
also.

(6) A Noetherian ring possessing a dualizing complex has a codimension function.

**Proof.** Let $B$ be a Noetherian ring.

(1) Let $t(p) = - \dim B/p$. If $B$ is a catenary local ring, then $t(p)$ is a codimension 
function of $B$.

(2) Let $t(p) = \dim B_p$. If $B$ is a catenary integral domain, then $t(p)$ is a codimension 
function of $B$.

(3) Let $t(p) = \dim B_p$. Then $t(p)$ is the codimension function of $B$. See the proof 
of [20] Theorem 17.4(ii)].

(4) and (5) Obvious.

(6) See [14], Chapter 5, §7].

A Noetherian ring is catenary if it has a codimension function. But the converse 
is not necessarily true. Moreover the universally catenarity is independent of the 
existence of a codimension function.
Example 5.3.  (1) Ogoma [24, §5 I] gave a Noetherian, universally catenary ring with no codimension function.

(2) Nagata [21, Example 2] gave a two-dimensional local integral domain which is not quasi-unmixed. It has a codimension function but is not universally catenary.

If a Noetherian ring $B$ is universally catenary and has a codimension function, then the polynomial ring over $B$ does also.

Theorem 5.4. Let $B$ be a Noetherian, universally catenary ring and $C$ an essentially of finite type $B$-algebra. If $B$ has a codimension function, then $C$ does also.

Proof. We may assume that $C$ is a polynomial ring over $B$. Let $t_B$ be a codimension function. We put

$$ t_C(q) = t_B(p) + \text{ht } q/pC $$

for each prime ideal $q$ in $C$. Then $t_C$ is a codimension function of $C$.

The following is the key lemma for the proof of Corollary 1.2.

Lemma 5.5. Let $B$ be a Noetherian, universally catenary ring which has a codimension function. Then it is a homomorphic image of a finite type $B$-algebra $C$ such that the codimension function of $C$ is a constant on the associated primes of $C$. If, in addition, $B$ is a local ring, then there exists a maximal ideal $n$ of $C$ such that $B$ is a homomorphic image of $C_n$.

Proof. Let $t_B$ be a codimension function of $B$ and

$$(0) = q_1 \cap \cdots \cap q_s$$

the irredundant primary decomposition of $(0)$ in $B$. We may assume that

$$\sup\{t_B(\sqrt{q_i}) \mid i = 1, \ldots, s\} = 0.$$ 

We put $n = -\inf\{t_B(\sqrt{q_i}) \mid i = 1, \ldots, s\}$ and $n_i = -t_B(\sqrt{q_i})$ for all $i$. Then

$$C = B[T_1, \ldots, T_n] \amalg \bigcap_{i=1}^{s} (q_i, T_1, \ldots, T_n)B[T_1, \ldots, T_n]$$

has the required property. If $B$ is a local ring with maximal ideal $m$, then $n = mC + (T_1, \ldots, T_n)C$ has the required property.

Proof of Corollary 1.2. The only if part is obvious. We prove the if part. Let $A$ be a Noetherian, universally catenary local ring with maximal ideal $m$ and assume that all the formal fibers of $A$ are Cohen-Macaulay. If $\dim A = 0$, then $A$ itself is Cohen-Macaulay.

We assume that $\dim A > 0$. By modifying the proof of [29, Theorem 5.7], we find that all the formal fibers of an essentially of finite type $A$-algebra are Cohen-Macaulay. By using this fact and Lemma 5.5 we may assume that $\dim A/p = \dim A$ for each associated prime $p$ of $A$. It implies that $A$ is unmixed because $A$ is formally catenary and all the formal fibers of $A$ are Cohen-Macaulay. Theorem 1.1 says that there exists an arithmetic Macaulayfication $R$ of $A$. Thus $A$ is a homomorphic image of a Cohen-Macaulay local ring $R_{mR + R_n}$.

If $A$ is excellent, then any essentially of finite type $A$-algebra is also. Therefore we obtain the second assertion.
We should mention that Corollary 1.2 is not true for non-local rings. Indeed, all the formal fibers of all the localization of Ogoma’s example above are Cohen-Macaulay. But it is not a homomorphic image of a Cohen-Macaulay ring because it has no codimension function.

6. Non-local rings

First we prove Theorem 1.3. Let \( B \) be a Noetherian ring with dualizing complex \( D \). Then there exists a codimension function \( t \) of \( B \) such that

\[
H^p(\text{Hom}_B(B/p, D)_p) = 0 \quad \text{if } p \neq t(p)
\]

for each prime ideal \( p \) in \( B \). The following lemma is an analogue of Proposition 2.3 and Corollary 2.4. We can prove them by using the local duality theorem. Here \( \text{ann} M \) denotes the annihilator of a \( B \)-module \( M \).

**Lemma 6.1.** Let \( M \) be a finitely generated \( B \)-module and \( p \) a prime ideal in \( B \). Assume that \( t(q) = 0 \) for all minimal prime \( q \) of \( M \). Then \( M_p \) is Cohen-Macaulay if and only if \( p \supseteq \prod_{j>0} \text{ann} H^j(\text{Hom}(M, D)) \).

In particular, if \( p \supseteq \prod_{j>0} \text{ann} H^j(\text{Hom}(M, D)) \), then \( t(p) > 0 \). If \( t(q) = 0 \) for all associated prime \( q \) of \( M \), then \( p \supseteq \prod_{j>0} \text{ann} H^j(\text{Hom}(M, D)) \) implies that \( t(p) \geq 2 \).

We start the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Let \( d = \text{dim} B \) and assume that \( t(q) = 0 \) for all associated primes \( q \) of \( B \). Then \( s_0 = \inf \{t(p) \mid B_p \text{ is not Cohen-Macaulay}\} \geq 2 \). If \( s \) is an integer such that \( d - s_0 \leq s < d - 1 \), then there exist elements \( x_1, \ldots, x_d \) in \( B \) satisfying the following conditions:

1. If \( p \) is a minimal prime of \( B/(x_i, \ldots, x_d)B \), then \( t(p) = d - i + 1 \);
2. \( s_{i+1}, \ldots, x_d \in \prod_{j>0} \text{ann} H^j(D) \);
3. \( x_i \in \prod_{j>d-i} \text{ann} H^j(\text{Hom}(B/(x_{i+1}, \ldots, x_d), D)) \) for \( i \leq s \).

We note that (1) implies (0): \( x_d = 0 \). Let \( q_i = (x_i, \ldots, x_d) \) for \( 1 \leq i \leq s + 1 \) and \( R = R(q_1 \cdots q_{s+1}^{d-s-1}) \).

We show that \( R_p \) is Cohen-Macaulay for all prime ideal \( p \) in \( B \). If \( q_1 \cdots q_{s+1}^{d-s-1} \not\subseteq p \), then \( \prod_{j>0} \text{ann} H^j(D) \not\subseteq p \). Therefore \( R_p \) is a polynomial ring over a Cohen-Macaulay ring \( B_p \).

Assume that \( q_1 \cdots q_{s+1}^{d-s-1} \subseteq p \). Then \( x_1, \ldots, x_d \in p \) and \( x_{s+1} \notin p \) for some \( 1 \leq t \leq s + 1 \), where we put \( x_0 = 1 \). Taking localization of (1)–(3), we find that

1. \( \dim B_p/(x_t, \ldots, x_d) = \dim B_p - (d - t + 1) \);
2. \( x_{s+1}, \ldots, x_d \in \text{a}(B_p) \);
3. \( x_i \in \text{a}(B_p/(x_{i+1}, \ldots, x_d)) \) for \( t \leq i \leq s + 1 \);
4. \( \text{a}(B_p/(x_t, \ldots, x_d)) = B_p \) if \( t > 1 \).

Hence \( x_1, \ldots, x_d \) is a subsystem of a \( p \)-standard system of parameters for \( B_p \) and \( B_p/(x_t, \ldots, x_d) \) is Cohen-Macaulay if \( t > 1 \). We find that \( R_p = R(q_t \cdots q_{s+1}^{d-s-1}B_p) \) is Cohen-Macaulay by using Corollary 1.4. \( \square \)

Now Corollary 1.4 becomes trivial.

**Proof of Corollary 1.4.** Let \( B \) be a Noetherian ring with dualizing complex. We may assume that the codimension function of \( B \) is a constant on the associated primes of \( B \) because of [23, Theorem 3.5]. Then \( B \) has an arithmetic Macaulayfication \( R \). Since \( R \) also has a dualizing complex and is Cohen-Macaulay, \( R \) is
a homomorphic image of a finite-dimensional Gorenstein ring. See [25] and [30, Theorem 4.3]. Therefore $B$ is also.

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