REPRESENTATION TYPE OF HECKE ALGEBRAS OF TYPE A

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Abstract. In this paper we provide a complete classification of the representation type for the blocks for the Hecke algebra of type $A$, stated in terms of combinatorial data. The computation of the complexity of Young modules is a key component in the proof of this classification result.

1. Introduction

1.1. Let $R$ be an integral domain and $q$ be a unit in $R$ with $q \neq 1$. Let us regard the symmetric group $\Sigma_r$ as the set of bijections of $\{1, \ldots, r\}$. The Hecke algebra $\mathcal{H}_q(r) = \mathcal{H}_{R,q}(\Sigma_r)$ is the free $R$-module with basis $\{T_w : w \in \Sigma_r\}$. The multiplication in the algebra is defined by the rule

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w & \text{otherwise,} \end{cases}$$

where $s = (i, i+1) \in \Sigma_r$ is a basic transposition and $w \in \Sigma_r$. The function $\ell : \Sigma_r \to \mathbb{N}$ is the usual length function. Here we will assume that $R = k$ is a field; in this case the Hecke algebra $\mathcal{H}_q(r)$ is a finite-dimensional self-injective algebra.

Let $l$ be the multiplicative order of $q$ in $k$, so that either $q$ is a primitive $l$th root of unity, or $l = 1$.

In [DJ2], Dipper and James gave a necessary and sufficient condition for the semisimplicity of $\mathcal{H}_q(r)$. For $q \neq 1$, this is the case if and only if $l > r$. So the case when $l$ is finite (and $l \leq r$) is most interesting here.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition of $r$. That is, $|\lambda| = \sum_{i=1}^t \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0$. A partition $\lambda$ is $l$-regular if there does not exist an $i$ such that $\lambda_{i+1} = \cdots = \lambda_{i+l} > 0$. It is well-known that the simple modules for the Hecke algebra $\mathcal{H}_q(r)$ are naturally indexed by $l$-regular partitions of $r$. For each partition $\lambda$ of $r$ there is a $q$-Specht module of the Hecke algebra $\mathcal{H}_q(r)$, denoted by $S^\lambda$. If $\lambda$ is $l$-regular, then $S^\lambda$ has a unique simple quotient denoted by $D^\lambda$.

The blocks of the Hecke algebra can be described via the Nakayama Rule, that is $D^\lambda$ and $D^\mu$ belong to the same $l$-block if and only if $\lambda$ and $\mu$ have the same $l$-core. The $l$-core is the partition whose Young diagram is obtained by removing as many rim $l$-hooks from Young diagram corresponding to the original partition. For details we refer to [Jam, DJ2] or [JM]. Equivalently, $S^\lambda$ and $S^\mu$ belong to the same $l$-block if and only if $\lambda$ and $\mu$ have the same $l$-core. Let $B_\lambda$ denote the block of $\mathcal{H}_q(r)$ containing $S^\lambda$. Given a partition $\lambda$ of $r$, the weight $w := w(\lambda)$ of the block
$\mathcal{B}_\lambda$ is given by $|\gamma| + lw = r$ where $\gamma$ is the $l$-core corresponding to $\mathcal{B}_\lambda$. The block containing the trivial module is called principal block, its core is the partition $(t)$ where $0 \leq t < l - 1$ and $t \equiv r \pmod{l}$.

A finite-dimensional algebra $A$ has finite representation type if and only if there are only finitely many indecomposable modules in $\text{mod}(A)$. Otherwise, $A$ has infinite representation type. Algebras of infinite representation type fall into two mutually exclusive categories—ones which are either tame or wild. An algebra $A$ has tame representation type if for each dimension $d$ there are finitely many one parameter families of modules which contain all $d$-dimensional indecomposable modules. In this situation there is a good chance one may be able to classify all indecomposable representations. On the other hand, algebras of wild representation type are those algebras whose representation theory is as complicated as that of the free associative algebra $k\langle x, y \rangle$.

1.2. Main result. The main purpose of this paper is to classify the representation type of the blocks of the Hecke algebra $\mathcal{H}_q(r)$. This classification is given in terms of the “weight” of the corresponding partition $\lambda$ for the block $\mathcal{B}_\lambda$.

Theorem. Let $\mathcal{B}_\lambda$ be a block for $\mathcal{H}_q(r)$.

(a) $\mathcal{B}_\lambda$ is semisimple if and only if $w(\lambda) = 0$;
(b) $\mathcal{B}_\lambda$ has finite representation type (and is not semisimple) if and only if $w(\lambda) = 1$;
(c) $\mathcal{B}_\lambda$ has tame representation type if and only if $w(\lambda) = 2$ and $q$ is a primitive 2nd root of unity;
(d) $\mathcal{B}_\lambda$ has wild representation type if and only if $q$ is a primitive $l$th root of unity, $w(\lambda) \geq 2$ and $l \geq 3$, or $l = 2$ and $w(\lambda) \geq 3$.

For blocks of group algebras of finite groups, the representation type may be characterized in terms of the defect groups of the block; see the introduction in [273] for a discussion. The defect groups for blocks of symmetric groups are iterated wreath products of cyclic groups of order $p$, see [12.6.2.39]. This gives a characterization of the representation type for blocks of symmetric groups which is analogous to our theorem; but the proof is quite different.

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2. Complexity

2.1. Complexity and representation type. Let $A$ be a finite-dimensional algebra over a field $k$ and $M$ be a finite-dimensional $A$-module. Let $\Omega^0(M) = M$ and $P_0$ be the projective cover of $\Omega^0(M)$. Now let $\Omega^1(M)$ be the kernel of the map $P_0 \to \Omega^0(M)$. Recursively, one can now let $\Omega^{r+1}(M)$ be the kernel of map $P_r \to \Omega^r(M)$ where $P_1$ is the projective cover of $\Omega^1(M)$. In this process we have constructed a minimal projective resolution of $M$ in $\text{mod}(A)$

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

For $\mathcal{V} = \{V_t : t \in \mathbb{N}\}$ a sequence of $k$-vector spaces, let $r(\mathcal{V})$ be the smallest integer $s \geq 0$ such that $\dim_k V_t \leq C \cdot t^{r-1}$ for some constant $C \geq 0$ and all $t \in \mathbb{N}$. If no such integer exists we say that $r(\mathcal{V})$ is infinite. The complexity $c_A(M)$ of $M$ in $\text{mod}(A)$ is then equal to $r(\{P_t : t \in \mathbb{N}\})$. 
There are some elementary properties about the complexity of modules which are presented below.

(2.1.1) \( c_A(M) = r(\{ \text{Ext}^t_A(M, S) : t \in \mathbb{N} \}) \) where \( S \) is the direct sum of all simple \( A \)-modules.

(2.1.2) If \( B \) is a subalgebra of \( A \) such that \( A \) is a free (left) \( B \)-module, then \( c_B(M) \leq c_A(M) \) for \( M \in \text{mod}(A) \) and \( c_A(U \otimes_B A) \leq c_B(U) \) for \( U \in \text{mod}(B) \).

(2.1.3) If \( N \) is a direct summand of \( M \) in \( \text{mod}(A) \), then \( c_A(N) \leq c_A(M) \).

The following proposition demonstrates that the notion of complexity is is related to the representation type of the algebra provided that the algebra \( A \) is self-injective.

**Proposition.** Let \( A \) be a finite-dimensional self-injective algebra and \( M \in \text{mod}(A) \) be indecomposable.

(a) If \( c_A(M) \geq 1 \), then \( A \) is not semisimple.
(b) If \( c_A(M) \geq 2 \), then \( A \) has infinite representation type.
(c) If \( c_A(M) \geq 3 \), then \( A \) has wild representation type.

**Proof.** (a) Observe that if \( A \) is self-injective, then \( c_A(M) = 0 \) occurs precisely when \( M \) is a projective \( A \)-module. Therefore, \( A \) is semisimple if and only if \( c_A(M) = 0 \) for all \( M \in \text{mod}(A) \). (b) The modules \( \Omega^i(M) \) are indecomposable and there are infinitely many non-isomorphic ones. Thus \( A \) has infinite representation type. (c) Let \( T \) be a composition factor of \( M \) in \( \text{mod}(A) \). Then \( c_A(T) \geq c_A(M) \geq 3 \). Therefore, by [Ric] Thm. 2] \( A \) has wild representation type.

2.2. **Young modules.** Let \( \lambda \) be a composition of \( r \) and let \( \Sigma_\lambda \) be the corresponding Young subgroup of \( \Sigma_r \), so \( \Sigma_\lambda \cong \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots \). Let

\[ x_\lambda = \sum_{w \in \Sigma_\lambda} T_w. \]

For any partition \( \lambda \), let \( \mathcal{H}_q(\lambda) \) be the subalgebra generated by \( T_w, w \in \Sigma_\lambda \); this is isomorphic to the tensor product \( \bigotimes \mathcal{H}_q(\lambda_i) \) of Hecke algebras. It is important to note that \( \mathcal{H}_q(r) \) is free as a module over \( \mathcal{H}_q(\lambda) \) of rank equal to the index of \( \Sigma_\lambda \) in \( \Sigma_d \). The element \( x_\lambda \) above lies in this algebra, and \( \langle x_\lambda \rangle \) is a module for \( \mathcal{H}_q(\lambda) \) which is isomorphic to the trivial module. Set \( M^\lambda := x_\lambda \mathcal{H}_q(r) \cong \langle x_\lambda \rangle \otimes_{\mathcal{H}_q(\lambda)} \mathcal{H}_q(r) \). The module \( M^\lambda \) is often called the \( q \)-permutation module.

The module \( M^\lambda \) has a unique submodule isomorphic to the Specht module \( S^\lambda \), and there is a unique indecomposable direct summand of \( M^\lambda \) containing \( S^\lambda \); this is the Young module \( Y^\lambda \). In particular, \( Y^\lambda, S^\lambda \) belong to the same block \( B_\lambda \). Furthermore, \( Y^\lambda \cong Y^\mu \) if and only if \( \lambda = \mu \). Each \( M^\lambda \) is a direct sum of Young modules \( Y^\mu \) (see [Mar]).

In [DDu] it is proved that indecomposable \( \mathcal{H}_q(r) \)-modules have “vertices” and “sources”, where vertices are certain parabolic subalgebras of the form \( \mathcal{H}_q(\mu) \). Moreover it is proved that the vertices of Young modules are \((l - p)\)-parabolics given as follows. Fix a partition \( \lambda \) and take the \((l - p)\)-adic decomposition of \( \lambda \) as

\[ \lambda = \lambda(0) + \sum_{i=0}^{m} \mu(i)lp^i \]

where \( \lambda(0) \) is an \( l \)-restricted partition and the \( \mu(i) \) are \( p \)-restricted partitions. This decomposition is unique. Define the partition

\[ \rho = (l^a, lp^b, (lp)^c, \cdots, (lp^m)^p) \]

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where \( a = |\lambda(0)| \) and \( b_i = |\mu(i)| \). Then \( Y^\lambda \) has Young vertex \( H_q(\rho) \), and its source is the trivial module for \( H_q(\rho) \). So we have that \( Y^\lambda \) is a direct summand of \( M^\rho \) and the restriction of \( Y^\lambda \) to \( H_q(\rho) \) has \( k \) as a direct summand. The following result describes the complexity of Young modules.

**Theorem.** Let \( H_q(\rho) \) be a Young vertex of \( Y^\lambda \). Then \( c_{H_q(\rho)}(Y^\lambda) = c_{H(\rho)}(k) \).

**Proof.** Recall that \( H := H_q(\rho) \) is free as a module over \( L := H(\rho) \). Therefore, induction and restriction are exact and projective resolutions of \( L \)-modules induce to projective resolutions. By (2.1.2), it follows that \( c_L(k) \geq c_H(k \otimes_L H) \). Since \( Y^\lambda \) is a direct summand of \( k \otimes_L H \), we also have that \( c_H(k \otimes_L H) \geq c_H(Y^\lambda) \) by (2.1.3), thus \( c_L(k) \geq c_H(Y^\lambda) \).

On the other hand, \( c_H(Y^\lambda) \geq c_L(Y^\lambda) \) by (2.1.2). By [DD], \( k \) is a direct summand of \( Y^\lambda \) as an \( L \)-module, thus \( c_L(Y^\lambda) \geq c_L(k) \) and \( c_H(Y^\lambda) \geq c_L(k) \).

2.3. Let \( A \) and \( B \) be algebras and let \( F \) be a functor from \( \text{mod}(A) \) to \( \text{mod}(B) \). We say that the functor \( F \) is bounded if for all \( d \geq 1 \) there is a constant \( c = c(d) \) such that for all \( M \in \text{mod}(A) \) of length \( \leq d \), the module \( F(M) \) has length \( \leq c(d) \).

**Proposition.** Let \( A \) and \( B \) be algebras and \( F : \text{mod}(A) \to \text{mod}(B) \) be a bounded functor. Moreover, let \( G : \text{mod}(B) \to \text{mod}(B) \) be a functor such that \( M \) is a direct summand of \( GF(M) \) for all \( M \in \text{mod}(A) \). If \( A \) is wild then so is \( B \).

**Proof.** Drozd has proved that a finite-dimensional algebra is either tame or wild and not both (Dr). Assume that \( B \) is not wild; then by Drozd’s theorem \( B \) is tame, and we will show that then \( A \) is also tame. By [DIP] it suffices to show that \( A \) is weakly tame, which is to say that for every \( d \geq 1 \) there are finitely many \( k[T] - A \)-bimodules \( M_i \) which are finitely generated free as \( k[T] \)-modules such that every indecomposable \( A \)-module of dimension \( \leq d \) is a direct summand of some \( S \otimes_{k[T]} M_i \) where \( S \) is a simple \( k[T] \)-module. (That is, the weakening consists of allowing direct summands, instead of taking the whole module.)

Let \( d \geq 1 \), consider \( A \)-modules \( M \) of dimension \( \leq d \). Since \( B \) is tame, there are finitely many \( k[T] - B \)-bimodules \( M_i \) such that every \( B \)-module \( X \) of dimension \( \leq c(d) \) is of the form \( X \cong S \otimes M_i \). Now take \( X = F(M) \), and let \( X = F(M) \cong S \otimes M_i \). Then \( G(X) = GF(M) \) which is \( S \otimes M_i \) restricted to \( A \) has \( M \) as a direct summand by our hypothesis. Hence, \( A \) is weakly tame.

3. Blocks of the Hecke algebra

3.1. If \( q \) is not a root of unity then the Hecke algebra is semisimple and \( w(\lambda) = 0 \) for all partitions \( \lambda \). So assume that \( q \) is a primitive \( l \)-root of unity, and let \( B_\lambda \) be a block of \( H_q(r) \) of weight \( w(\lambda) \). It is known that the block \( B_\lambda \) is semisimple if and only if \( w(\lambda) = 0 \), and the block is of finite type if \( w(\lambda) = 1 \) (see also §3.2). Therefore, Theorem 1.2 (main result) will be established by proving the following theorem.

**Theorem.** Let \( B_\lambda \) be a block of \( H_q(r) \).

(a) If either \( w(\lambda) \geq 2 \) and \( l \geq 3 \), or if \( l = 2 \) and \( w(\lambda) \geq 3 \), then \( B_\lambda \) is wild.

(b) If \( w(\lambda) = 2 \) and \( l = 2 \), then \( B_\lambda \) is tame.

The remainder of this section will be devoted to the proof of this theorem. We will proceed by proving the following steps.
(3.1.1) If $w(\lambda) \geq 3$ then $B_\lambda$ is wild. Also for $l \geq 3$ the principal block of $H_q(2l)$ is wild.

(3.1.2) If $w(\lambda) = 2$ and $l \geq 3$ then $B_\lambda$ is wild.

(3.1.3) If $w(\lambda) = 2$ and $l = 2$ then $B_\lambda$ is tame.

The statement (3.1.2) is verified by investigating $\{2 : t\}$-pairs of blocks. The definition of these pairs of blocks is explained in detail at the beginning of Section 3.4. For (3.1.3) we determine the basic algebra of $B_\lambda$ by quiver and relations.

3.2. Quiver and relations for blocks of weight one. These were determined in [U]. Also by [Jo], any such block is Morita equivalent to the principal block $A$ say, of $H_q(l)$, which can be described as follows.

There are $l - 1$ simple modules in the block, $X_j := D^l$ where $\lambda = (l - j, 1^j)$ for $j = 0, 1, 2, \ldots, l - 2$. The projective covers for these simple modules have the following structures.

\[
\begin{array}{cccc}
P^{(l)} & P^{(j, 1^{l-j})} : j = l - 1, \ldots, 3 & P^{(2, 1^{l-2})} \\
D^{(l)} & D^{(j, 1^{l-j})} & D^{(2, 1^{l-2})} \\
D^{(l-1)} & D^{(j+1, 1^{l-j-1})} & D^{(3, 1^{l-3})} \\
D^{(l)} & D^{(j+1, 1^{l-j-1})} & D^{(2, 1^{l-2})}
\end{array}
\]

The quiver for the algebra is Figure 1 below with relations $\alpha_j \alpha_{j+1} = 0 = \beta_{j+1} \beta_j$, and $\beta_j \alpha_j = \alpha_{j+1} \beta_{j+1}$ for $j = 1, 2, \ldots, l - 3$. The only indecomposable modules of radical length three are the projective modules. Therefore, $A$ has the same representation type as the algebra $A/J^2$ ($J = \text{Rad } A$). But, the quiver for $A/J^2$ which is the same as for $A$ separates into a union of two quivers of type $A_{l-1}$. Hence, the principal block of $H_q(l)$ is of finite representation type, by Gabriel’s Theorem.

![Figure 1](attachment:image.png)

3.3. From the previous section the complexity of $k$ as an $H_q(l)$-module is 1. Namely, $H_q(l)$ is a self-injective Brauer tree algebra, and $k$ is a non-projective periodic module. This implies that the complexity of $k$ as a module for $L := H_q(l) \otimes H_q(l) \otimes \cdots \otimes H_q(l)$ ($m$ factors) is $m$, by the Knöth formula. Now suppose that $A = H_q(\rho)$ contains $L$ as a parabolic subalgebra and that $A$ is a free $L$-module. Then by (2.1.2), $c_{H_q(\rho)}(k) \geq m$. We can now verify (3.1.1).

**Proposition (A).** Let $B_\lambda$ be a block of $H_q(r)$ of weight $w(\lambda) \geq 3$. Then $B_\lambda$ is wild.

**Proof.** Let $b$ be the $l$-core of $B_\lambda$, and we may assume that $\lambda$ is the largest partition in this block, that is

\[
\lambda = b + (lw) = (b_1 + lw, b_2, \ldots).
\]
Any $l$-core is $l$-restricted and hence $\lambda$ has $(l-p)$-adic expansion
\[
\lambda = b + \sum_i (w_i)lp^i
\]
where $w = \sum w_ip^i$ is the $p$-adic expansion of the integer $w$. Hence, $Y^\lambda$ has Young
vertex $A := \mathcal{H}_q(\rho)$ where $\rho$ has $|b|$ parts equal to 1, $w_0$ parts equal to $l$, $w_1$ parts
equal to $lp$ and so on. We see that $A$ contains a subalgebra $L = \mathcal{H}_q(l) \otimes \mathcal{H}_q(l) \otimes
\mathcal{H}_q(l)$ so that $A$ is free over $L$. Consequently, $c_A(k) \geq 3$, and by Theorem 2.2,
$c_{\mathcal{H}_q(r)}(Y^\lambda) \geq 3$. Hence, $B_\lambda$ is wild, by Proposition 2.1.

**Proposition (B).** The principal block of $\mathcal{H}_q(2l)$ is wild when $l \geq 3$.

*Proof.* It suffices to show that $\mathcal{H}_q(2l)$ is wild, since all non-principal blocks have
finite type; these blocks have weight less than or equal to one. For this, it suffices by
Proposition 2.3 to show that $L := \mathcal{H}_q(l) \otimes \mathcal{H}_q(l)$ is wild because the induction
functor is bounded and it is well-known that for any $L$-module $M$, $M$ is a direct
summand of $M \otimes_L \mathcal{H}_q(2l)$ restricted to $L$.

The principal block of $\mathcal{H}_q(l)$ has quiver as Figure 1 with $l - 1$ vertices. Hence the
quiver of $L$ has "lattice form", with $(l - 1)^2$ vertices. If $l > 3$ then this has a proper
subquiver which is a circuit, so by [EN, Prop. 2.3(B)] it must be wild. This leaves
us with the case $l = 3$. Here we need to look at the projectives in the principal
block of $L$. We see easily that the module category of the connected quiver in [Ri,
p.168] with three vertices and three arrows (and no relations) can be embedded into
the module category of $L$. This is wild (see [Ri]) and hence the algebra $L$ is wild
as well. In order to obtain this embedding, note that the algebra $L$ has four simple
modules, $S_i \otimes S_j$ where $S_1, S_2$ are the two simple modules for the principal block
$\mathcal{H}_q(l)$. The projective cover of the $\mathcal{H}_q(l)$-module $S_i$ is uniserial of length three, with
factors $S_i, S_j, S_k$. It follows that the third Loewy layer of $P(S_1 \otimes S_1)$ is the direct
sum of two copies of $S_1 \otimes S_1$ and one copy of $S_2 \otimes S_2$. So label the vertices of the
above wild quiver with $S_2 \otimes S_2$ and $S_1 \otimes S_1$ twice (we take the separated quiver of
the block).

3.4. For the study of blocks of weight $w = 2$, we will work with the set-up as in
[Sc1] and [Jo]. For general $w$ and $t$ they define $[w : t]$-pairs of blocks $\mathcal{B}$, $\mathcal{B}$ where
$\mathcal{B}$ is a block of $\mathcal{H}_q(r-t)$ and $\mathcal{B}$ is a block of $\mathcal{H}_q(r)$, both of weight $w$. Every block
of weight $w$ of some Hecke algebra (over the same field) is related to the principal
block of $\mathcal{H}_q(lw)$ by a finite sequence of $[w : t]$-pairs, where $t$ varies. In [Jo] it was
proved that for $t \geq w$ the blocks $\mathcal{B}$ and $\mathcal{B}$ are Morita equivalent.

For the set-up in [Sc1], [Jo] one needs to use the abacus display of partitions.
Let $\mathcal{B}$ be a block of $\mathcal{H}_q(r)$ with $l$-core $b = (b_1, \cdots, b_s)$, $b_s \neq 0$, and $l$-weight $w$. The
partition $b$ is a partition of $r - lw$. Display the $(r + lw)$-element $\beta$-set of $b$ on an
abacus with $l$ runners. If there exists an $i$, $2 \leq i \leq l$, such that the $i$th runner
contains $t$ more beads than the $(i - 1)$th runner, we can swap these two runners
to obtain the abacus display of a partition $\hat{b}$ of $r - t - lw$. Let $\mathcal{B}$ be the block of
$\mathcal{H}_q(r - t)$ which has $l$-core $\hat{b}$ (and therefore $l$-weight $w$). We then say that $\mathcal{B}$ and $\mathcal{B}$
form a $[w : t]$-pair.

We assume that $w = 2$ and $t = 1$. Then the functors used in [Sc1] and [Jo] are
easy to describe. Take the pair of adjoint functors $(G, F)$ where $G : \text{mod}(\mathcal{B}) \to
\text{mod}(\mathcal{B})$ and $F : \text{mod}(\mathcal{B}) \to \text{mod}(\mathcal{B})$, where $G$ is restriction to $\mathcal{H}_q(r - 1)$ followed by
taking the block component, and $F$ is (co)induction to $\mathcal{H}_q(r)$ followed by taking the
block component. This can be described by taking the $B$-$B$-bimodule $W = eH_q(r)e$ where $e$ is the block idempotent for $B$ and $e \in H_q(r-1)$ is the block idempotent for $\bar{B}$. Then

$$G \cong (-) \otimes_B W \quad \text{and} \quad F \cong \text{Hom}_B(W, -).$$

The results in \cite{Se2} for symmetric groups generalize to Hecke algebras: the proofs use only the abacus combinatorics and the functor set-up, and these work the same way for Hecke algebras (see \cite{Ho}). The only other ingredient is column removal, but this is also known for Hecke algebras in \cite[4.2(9), (15)]{D}. From \cite{Se2} there are three partitions each in $\bar{B}$, $B$ respectively which are “exceptional” denoted by $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ and $\alpha, \beta, \gamma$ respectively. If $\lambda \neq \alpha$ and $\lambda$ in the block $B$ is $l$-regular, then $G(D^\lambda)$ is simple, and if $\lambda$ in the block $\bar{B}$ with $\lambda \neq \tilde{\alpha}$ is $l$-regular, then $\bar{F}(D^\lambda)$ is simple. The partitions $\alpha$ and $\tilde{\alpha}$ are always $l$-regular. Moreover $G(D^\alpha)$ is not simple but has a simple socle and top isomorphic to $D^\alpha$ and precisely two composition factors isomorphic to $D^\alpha$ and $\bar{F}(D^\alpha)$ is not simple but has a simple socle and top isomorphic to $D^\alpha$ and precisely two composition factors isomorphic to $D^\alpha$. The projectives $P(\alpha)$ and $P(\tilde{\alpha})$ have Specht filtrations with $S^\alpha, S^\beta, S^\gamma$; and with $S^\tilde{\alpha}, S^\tilde{\beta}$ and $S^\tilde{\gamma}$ respectively. So $P(\alpha)$ has precisely three composition factors isomorphic to $D^\alpha$ and similarly $P(\tilde{\alpha})$ has three composition factors isomorphic to $D^\alpha$, by reciprocity.

Note that the $B$-module $W$ is a projective generator. Namely it is projective since it is a block component of $eH_q(r)$. To see that it is a generator, observe that $\text{Hom}_B(W, D^\lambda) \neq 0$ for all $l$-regular partitions $\lambda$ in $B$ (which is just $\bar{F}(D^\lambda)$).

Lemma. The module $G(\bar{F}(D^\tilde{\alpha}))$ is isomorphic to $D^\tilde{\alpha} \oplus P(\tilde{\alpha})$.

Proof. Let $X = G(\bar{F}(D^\tilde{\alpha}))$. By the adjointness we get from the above that $\text{soc}_B(X)$ and $\text{top}_B(X)$ are both the direct sum of two copies of $D^\tilde{\alpha}$. Also, in total $X$ has four composition factors isomorphic to $D^\tilde{\alpha}$.

It suffices to show that $\text{soc}_B(X) \subseteq \text{rad}_B(X)$. If so then the simple $D^\pi$ is a direct summand. The remaining part has a simple socle and top and has three composition factors $D^\tilde{\alpha}$, and the only module with this property is the indecomposable projective.

So suppose that $\text{soc}_B(X) \subseteq \text{rad}_B(X)$. The projective $P(\tilde{\alpha})$ has precisely three composition factors $D^\tilde{\alpha}$. Therefore there is precisely one module (up to isomorphism) which is not simple or projective which has a simple socle and top isomorphic to $D^\tilde{\alpha}$, and hence this is $G(D^\alpha)$.

We have an epimorphism $\pi : X \to G(D^\alpha)$. Let $x \in X$ such that $\pi(x)$ generates the image of $\pi$. Then $\langle x \rangle$ is a submodule of $X$ which has simple top $D^\alpha$ and its socle has only composition factors $D^\tilde{\alpha}$. It cannot have socle of length two because the projective $P(\tilde{\alpha})$ does not have such a quotient. Therefore, the socle is simple and it follows by the uniqueness that $\langle x \rangle$ is isomorphic to $G(D^\alpha) \cong \text{Im}(\pi)$. It follows that $\pi$ is split. Moreover we have an embedding of $G(D^\alpha)$ into the kernel of $\pi$ and it follows that $X$ is the direct sum of two copies of $G(D^\alpha)$.

We will now get a contradiction by considering stable homomorphisms. For modules $Y, Z$ let $\text{Hom}(Y, Z)$ be the quotient of $\text{Hom}(Y, Z)$ factored out by maps which factor through a projective module. First we have $\text{Hom}(X, D^\tilde{\alpha})$ is isomorphic to $\text{Hom}(X, D^\alpha)$ since $X$ has no projective summand and $D^\tilde{\alpha}$ is simple, and this is 2-dimensional.
On the other hand, the adjointness is compatible with stable homomorphisms, see [DJ]. So we have

\[ \text{Hom}(X, D^\alpha) \cong \text{Hom}(F(D^\alpha), F(D^\alpha)). \]

Clearly \( \text{Hom}(F(D^\alpha), F(D^\alpha)) \) is 2-dimensional. So we will get a contradiction by showing that there are non-zero maps in this space which factor through a projective module.

Let \( Z = F(D^\alpha) \); this has a simple socle and top isomorphic to \( D^\alpha \), so we have an inclusion \( j : Z \to P(\alpha) \) and an epimorphism \( \pi : P(\alpha) \to Z \). Since \( D^\alpha \) occurs precisely once in the heart of \( P(\alpha) \) it follows that \( \pi \circ j \) cannot be zero and its image must be the socle of \( Z \). So \( \pi \circ j \) is a non-zero endomorphism of \( Z \) which factors through a projective.

The following result allows us to relate blocks with weight two to blocks with weight two for smaller Hecke algebras.

**Theorem.** Let \( (\tilde{\mathcal{B}}, \mathcal{B}) \) be a \([2 : 1]\)-pair. Then there are (bounded) functors \( G : \text{mod}(\mathcal{B}) \to \text{mod}(\tilde{\mathcal{B}}) \) and \( F : \text{mod}(\tilde{\mathcal{B}}) \to \text{mod}(\mathcal{B}) \) such that for any \( \tilde{\mathcal{B}} \)-module \( M \) we have

\[ GF(M) \cong M \oplus P, \]

where \( P \) is a projective \( \tilde{\mathcal{B}} \)-module.

**Proof.** We proceed now by induction on the composition length. Note that by the adjointness we have a natural transformation \( \epsilon : GF(-) \to \text{id} \) which is “evaluation”. Since the \( \tilde{\mathcal{B}} \)-module \( W \) is a projective generator, \( \epsilon \) is onto.

If \( M \) is simple then the claim follows by Lemma 3.4. So assume now that \( M \) is not simple, then there is a commutative diagram with exact rows, where \( E \) is simple in \( \tilde{\mathcal{B}} \):

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & GF(N) & \longrightarrow & GF(M) & \longrightarrow & GF(E) & \longrightarrow & 0 \\
& & \downarrow \epsilon_N & & \downarrow \epsilon_M & & \downarrow \epsilon_E & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & E & \longrightarrow & 0
\end{array}
\]

By the above remark, \( \epsilon_N \) and \( \epsilon_E \) are onto and then by inductive hypothesis, they are split epimorphisms. Now use the Snake Lemma and the fact that projectives are injective, to complete the proof.

We know that for \( l \geq 3 \) the principal block of \( \mathcal{H}_q(2l) \) is wild by Proposition 3.3(B). From Theorem 3.4 and Proposition 2.3, if \( \tilde{\mathcal{B}} \) is wild then so is \( \mathcal{B} \). Since Morita equivalent blocks have the same representation type, we can conclude by that all blocks of weight \( w = 2 \) for \( l \geq 3 \) are wild which completes (3.1.2).

3.5. **Blocks of weight two for \( l = 2 \).** The goal of this section is to verify statement (3.1.3) that any block for the Hecke algebra of weight two when \( l = 2 \) is tame. By [Jo] there are only two Morita equivalence classes of such blocks, and they are represented by the principal blocks of \( \mathcal{H}_q(4) \) and \( \mathcal{H}_q(5) \). Note that here \( q = -1 \) and hence the trivial module is isomorphic to the alternating module.
Proposition (A). The Hecke algebra $\mathcal{H}_q(4)$ for $l = 2$ has quiver given in Figure 2 with relations

$$\beta \epsilon = 0 = \epsilon \alpha, \quad (\alpha \beta)^2 = \epsilon^2.$$ 

This algebra is tame.

![Figure 2.](image)

Proof. All partitions of 4 have empty 2-core, so there is only one block. The decomposition matrix for the Hecke algebra $\mathcal{H}_q(4)$ is

$$
\begin{bmatrix}
[S^{(4)} : D^\mu] \\
[S^{(3,1)} : D^\mu] \\
[S^{(2,2)} : D^\mu] \\
[S^{(2,1^2)} : D^\mu] \\
[S^{(1^4)} : D^\mu]
\end{bmatrix} = 
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
0 & 1 \\
1 & 1 \\
1 & 0
\end{pmatrix}.
$$

We consider Young modules; clearly $k = Y^{(4)}$ and from information from the decomposition matrix, $Y^{(3,1)}$ is uniserial of length three, with heart the other simple module which we call $E$. Moreover, we see that $Y^{(2,2)}$ has Specht quotients $S^{(3,1)}$ and $S^{(2,2)}$, and hence it is uniserial with socle and top $E$ and heart $k$. Clearly $S^{(1^4)} \cong k$.

So we have all decomposition numbers and the Cartan matrix. The projective $P(E)$ has a quotient $Y^{(2,2)}$ with kernel (necessarily) uniserial of length two and factors $E$ and $k$. It follows by self-duality that $P(E)$ is uniserial of length five, with “alternating” composition factors.

This shows that $\text{Ext}^1_{\mathcal{H}_q(4)}(E, k)$ is 1-dimensional. Next, observe that $P(k)$ has a quotient $Y^{(3,1)}$ with kernel of length three, and also $Y^{(3,1)}$ as a submodule. Using the fact that $\text{Ext}^1_{\mathcal{H}_q(4)}(E, k)$ is 1-dimensional, it follows that the heart of $P(k)$ is the direct sum of $k$ with a uniserial of length three, actually isomorphic to $Y^{(2,2)}$. Then it is straightforward to determine relations.

The algebra is special biserial and hence is tame. Furthermore, the algebra is of infinite type because the endomorphism ring of $P(k)$ is a Kronecker algebra of infinite type.

Proposition (B). The principal block of $\mathcal{H}_q(5)$ for $l = 2$ has quiver given in Figure 3 with relations

$$\epsilon \alpha = 0 = \beta \epsilon, \quad \alpha \gamma = 0 = \gamma \beta,$$

$$\epsilon^2 = \alpha \beta, \quad \gamma^2 = \beta \alpha.$$ 

This algebra is tame.
Proof. The partitions of five in the principal block are \((5), (3, 2), (3, 1^2)\) and their conjugates. The decomposition matrix is given by

\[
\begin{pmatrix}
[S^{(5)} : D^\alpha] & [S^{(3,2)} : D^\beta] & [S^{(3,1^2)} : D^\gamma] & [S^{(2^2,1)} : D^\delta] & [S^{(1^5)} : D^\epsilon] \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

From this information we have \(Y^{(3,2)} = D^{(3,2)} =: E\) say; and the Young module \(Y^{(3,1^2)}\) is indecomposable with Loewy length two, and both socle and top are \(k \oplus E\). The Cartan matrix has off-diagonal entries 1 and diagonal entries 3 and this together with self-injectivity and self-duality already implies the structure completely. This algebra is again special biserial of infinite type. Hence, the principal block of \(H_q(5)\) is tame for \(l = 2\).

We remark that both \(H_q(4)\) and \(H_q(5)\) when \(l = 2\) are not quite of “dihedral type” as defined in [Erd], since they have tubes of rank two. These algebras are of domestic representation type.

REFERENCES


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