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VERLINDE BUNDLES AND GENERALIZED THETA LINEAR SERIES

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Abstract. In this paper we approach the study of generalized theta linear series on moduli of vector bundles on curves via vector bundle techniques on abelian varieties.

We study a naturally defined class of vector bundles on a Jacobian, called Verlinde bundles, in order to obtain information about duality between theta functions and effective global and normal generation on these moduli spaces.

Introduction

The purpose of this paper is to introduce and study some very interesting vector bundles on the Jacobian of a curve which are associated to generalized theta linear series on moduli spaces of vector bundles on that curve. By way of application, we establish effective bounds for the global and normal generation of such linear series, and we give new proofs of the known results and a new perspective on the familiar conjectures concerning duality for generalized theta functions. We also show that these bundles lead to new examples (in the spirit of [25]) of base points for the determinant linear series.

Let X be a smooth projective complex curve of genus $g \geq 2$, and let $U_X(r,0)$ be the moduli space of semistable vector bundles of rank r and degree 0 on X, and $SU_X(r)$ the moduli space of semistable rank r vector bundles with trivial determinant. We define the *Verlinde bundles* to be push-forwards of plurithetha line bundles on $U_X(r,0)$ to the Jacobian of X by the determinant map $\det : U_X(r,0) \to J(X)$:

$$E_{r,k} := \det_* \mathcal{O}(k\Theta_N),$$

where Θ_N is the generalized theta divisor associated to a line bundle $N \in \mathrm{Pic}^{g-1}(X)$ (see *1.1*). Their fibers are precisely the Verlinde vector spaces of level k theta functions $H^0(SU_X(r), \mathcal{L}^k)$, where $\mathcal{L}$ is the determinant line bundle on $SU_X(r)$. These bundles are the focus of the present paper. Our viewpoint is that they allow a translation of statements about spaces of generalized theta functions into global statements about vector bundles on Jacobians. This in turn opens the way towards a more geometric study of these spaces.

We study at some length the behavior of the Verlinde bundles under natural operations associated to vector bundles on abelian varieties, e.g. in the spirit of Mukai [17]. In particular, we focus on their Fourier-Mukai transforms $\widehat{E_{r,k}}$. An

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interesting picture is provided by the relationship between the bundles $E_{r,k}$ and $E_{k,r}$ via the Fourier transform. Identifying $J(X)$ with the dual $\mathrm{Pic}^0(J(X))$ in the canonical way, we prove:

Theorem. *There is a (non-canonical) isomorphism $r_J^* E_{r,k}^* \cong r_J^* \widehat{E_{k,r}}$, where r_J is the multiplication by r map on $J(X)$. In particular we have (cf.* [4]*)*

$$h^0(SU_X(k), \mathcal{L}^r) \cdot r^g = h^0(U_X(k,0), \mathcal{O}(r\Theta_N)) \cdot k^g.$$

The last statement is the duality result of [4] on dimensions of spaces of generalized theta functions. It is natural then, as in the case of theta functions, to try to understand stronger dualities between these bundles. The main result in this direction is:

Theorem. *For every $k \geq 2$, there is a canonical nontrivial map*

$$SD : E_{r,k}^* \longrightarrow \widehat{E_{k,r}}.$$

In the case $k = 1$, $E_{r,1}$ and $\widehat{E_{1,r}}$ are stable and SD is an isomorphism.

The last part follows once we prove a statement of independent interest concerning the stability of the Fourier-Mukai transform of an arbitrary polarization on an abelian variety. Note also that we are granting the Verlinde formula in the proof of the theorem.

We recall that in the context of nonabelian theta functions, one of the central problems is a conjectural duality between certain spaces of $GL(n)$ and $SL(n)$-theta functions, known as the *strange duality conjecture*. It asserts that there is a canonical isomorphism

$$H^0(SU_X(r), \mathcal{L}^k)^* \xrightarrow{\cong} H^0(U_X(k,0), \mathcal{O}(r\Theta_N)).$$

These two vector spaces can be seen as fibers of the vector bundles $E_{r,k}^*$ and $\widehat{E_{k,r}}$ respectively, and indeed the natural map SD in the theorem induces the strange duality map on the fibers. The conjecture is consequently equivalent to the global statement saying that SD is an isomorphism for every k. In particular, the theorem already implies the *strange duality at level* 1, initially proved in [2], saying that

$$H^0(SU_X(r), \mathcal{L})^* \xrightarrow{SD} H^0(J(X), \mathcal{O}(r\Theta_N))$$

is an isomorphism. Moreover, the first theorem above can also be seen as providing some global evidence for the general conjecture. One expects that an understanding of further properties characterizing the Verlinde bundles and the maps between them (e.g. as in Remark 3.13) could shed some new light on this whole circle of ideas.

Our main concrete application of the Verlinde bundles is to the study of effective global generation and normal generation for pluritheta line bundles on $U_X(r,0)$. The underlying idea is very simple: the determinant map makes $U_X(r,0)$ a fiber space over $J(X)$, and one expects that effective bounds on the base and the fibers should give bounds on the total space. For the fiberwise situation, i.e. the case of multiples of the determinant bundle on $SU_X(r)$, we make use of recent results proved in [24]. Thus the complications arise when one tries to understand the global or normal generation of $E_{r,k}$ on $J(X)$. We begin by proving that $E_{r,k}$ is globally generated if and only if $k \geq r+1$. The consequence of this fact for linear series is stated below in a form that allows algorithmic applications.

Theorem. *$\mathcal{O}(k\Theta_N)$ is globally generated on $U_X(r,0)$ as long as $k \geq r+1$ and $\mathcal{L}^k$ is globally generated on $SU_X(r)$. Moreover, $\mathcal{O}(r\Theta_N)$ is not globally generated.*

As mentioned above, this provides effective results when combined with the bound obtained in [24] §4 for $SU_X(r)$.

Corollary. *$\mathcal{O}(k\Theta_N)$ is globally generated on $U_X(r,0)$ if $k \geq \frac{(r+1)^2}{4}$.*

Note that the theorem suggests that the optimal result on $U_X(r,0)$ could be the global generation of $\mathcal{O}((r+1)\Theta_N)$, which goes beyond what we currently have on $SU_X(r)$. This is indeed true for moduli of rank 2 and rank 3 vector bundles (see 5.5) and accounts for the discrepancy between the way we are formulating the theorem and the statement of the corollary. For some general results and questions in the direction of optimal effective base point freeness the reader can consult [24], §4 and §5.

Moving to effective surjectivity of multiplication maps, we prove the following:

Theorem. (a) *The multiplication map*

$$H^0(E_{r,k}) \otimes H^0(E_{r,k}) \longrightarrow H^0(E_{r,k}^{\otimes 2})$$

is surjective for $k \geq 2r+1$.

(b) *Assume that $E_{r,k}$ on $J(X)$ and $\mathcal{L}^k$ on $SU_X(r)$ are globally generated. Then the multiplication map*

$$\mu_k : H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\mathcal{O}(k\Theta_N)) \longrightarrow H^0(\mathcal{O}(2k\Theta_N))$$

is surjective as long as the multiplication map $H^0(\mathcal{L}^k) \otimes H^0(\mathcal{L}^k) \to H^0(\mathcal{L}^{2k})$ on $SU_X(r)$ is surjective and $k \geq 2r+1$. In particular, for such k, $\mathcal{O}(k\Theta_N)$ is very ample.

To get an effective application in the case of rank 2 vector bundles one can use a theorem of Laszlo [14]. It is interesting to note that while the very ampleness of the determinant line bundle on $SU_X(2)$ has been the subject of various approaches (see e.g. [3], [6], [14]), no effective results seem to have been known for generalized theta bundles on $U_X(2,0)$.

Corollary. *For a generic curve X, the multiplication map*

$$H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\mathcal{O}(k\Theta_N)) \xrightarrow{\mu_k} H^0(\mathcal{O}(2k\Theta_N))$$

on $U_X(2,0)$ is surjective for $k \geq \max\{5, g-2\}$, and so $\mathcal{O}(k\Theta_N)$ is very ample for such k.

The proofs of both theorems make use of some very interesting recent results of Pareschi [22] (see *1.4*), drawing also on earlier work of Kempf, on cohomological criteria for global generation and normal generation of vector bundles on abelian varieties [1].

A somewhat surprising application of the Verlinde bundles is to the problem of finding base points for the linear system $|\mathcal{L}|$ on $SU_X(r)$ (see [1] §2 and [23] for a survey and results in this direction). We extend a method of Raynaud [25] to show that the Fourier-Mukai transforms $\widehat{E_{r,k}^*}$, restricted to certain embeddings of X in $J(X)$, provide new examples of base points for $|\mathcal{L}|$. Their ranks are given by the Verlinde formula at all levels $k \geq 1$.

[1] Pareschi uses these criteria in [22] to prove a conjecture of Lazarsfeld on syzygies of abelian varieties.

Theorem. *There exists an embedding $X \hookrightarrow J(X)$ such that $\widehat{E^*_{r,k}}_{|X}$ gives (via twisting by a line bundle) a base point for the linear system $|\mathcal{L}|$ on $SU_X(s_{k,r})$, where $s_{k,r} := h^0(SU_X(k), \mathcal{L}^r)$.*

This construction turns out to be a natural generalization of the examples given in [25], which in our language correspond to the level 1 Verlinde bundles.

Most of our results can be formulated in the setting of moduli of vector bundles of arbitrary rank and degree. To keep the arguments uniform and at a technical minimum we have chosen to explain all the details for the moduli spaces $U_X(r,0)$ and $SU_X(r)$. However, the last section contains complete statements of the analogous results that hold in the general case, with hints for the (minor) modifications needed in the proofs.

We briefly explain the structure of the paper, which is mainly ordered according to the increasing complexity of the methods involved. The first section provides the background material needed, with rather detailed statements included for the convenience of the reader. The definition and basic study of the Verlinde bundles occupy §2. The third section is an account of the Fourier duality picture for Verlinde bundles with applications to duality for generalized theta functions. In §4 we explain the construction of base points for the determinant linear series. The fifth section contains the main application to the effective global and normal generation problem for plurithetha linear series. Finally, in the last section we explain how the results can be extended to the setting of arbitrary degree vector bundles.

There are a few very interesting related topics that have been left out of this paper. The reader familiar with the theory of nonabelian theta functions will notice that there is a certain parallel between the particular form of the Verlinde bundles $E_{r,k}$ and (in a vague sense) the representation theory of the Verlinde vector spaces $H^0(SU_X(r), \mathcal{L}^k)$. In general one studies the representation theory of the Verlinde spaces in order to learn something about linear series and the geometry of the moduli spaces. Our hope is that a better understanding of the global properties of the $E_{r,k}$'s might, in the other direction, say something about the representations. From a different point of view, the theory developed here may be at least partially generalized to the setting of moduli of sheaves on smooth (irregular) surfaces. We are hoping to address these questions in future work.

1. Preliminaries and notations

1.1. Generalized theta divisors. Let $U_X(r,0)$ be the moduli space of equivalence classes of semistable vector bundles of rank r and degree 0 on X, $SU_X(r,L)$ the moduli space of vector bundles with fixed determinant $L \in \mathrm{Pic}^0(X)$, and $U^s_X(r,0)$ and $SU^s_X(r,L)$ the open subsets corresponding to stable bundles. We recall the construction and some basic facts about generalized theta divisors on these moduli spaces, drawing especially on [5]. Analogous constructions work for any degree d, as we will recall in the last section.

Let N be a line bundle of degree $g-1$ on X. Then $\chi(E \otimes N) = 0$ for all $E \in U_X(r,0)$. Consider the subset of $U^s_X(r,0)$

$$\Theta^s_N := \{E \in U^s_X(r,0) |\ h^0(E \otimes N) \neq 0\}$$

and the analogous set in $SU^s_X(r,L)$. One can prove (see [5] (7.4.2)) that Θ^s_N is a hypersurface in $U^s_X(r,0)$ (resp. $SU^s_X(r,L)$). Denote by Θ_N the closure of Θ^s_N in $U_X(r,0)$ and $SU_X(r,L)$. As we vary N, these hypersurfaces are called

generalized theta divisors. It is proved in [5], Theorem A, that $U_X(r,0)$ and $SU_X(r,L)$ are locally factorial, and so the generalized theta divisors determine line bundles $\mathcal{O}(\Theta_N)$ on these moduli spaces. We have the following important facts:

Theorem 1.1 ([5, Theorem B])**.** *The line bundle $\mathcal{O}(\Theta_N)$ on $SU_X(r,L)$ does not depend on the choice of N. The Picard group of $SU_X(r,L)$ is isomorphic to $\mathbf{Z}$, generated by $\mathcal{O}(\Theta_N)$.*

The line bundle in the theorem above, independent of the choice of N, is denoted by $\mathcal{L}$ and is called the *determinant bundle*.

Theorem 1.2 ([5, Theorem C])**.** *The inclusions* $\mathrm{Pic}(J(X)) \subset \mathrm{Pic}(U_X(r,0))$ *(given by the determinant morphism) and* $\mathbf{Z}\cdot\mathcal{O}(\Theta_N) \subset \mathrm{Pic}(U_X(r,0))$ *induce an isomorphism*

$$\mathrm{Pic}(U_X(r,0)) \cong \mathrm{Pic}(J(X)) \oplus \mathbf{Z}.$$

More generally, for any vector bundle F of rank k and degree $k(g-1)$, we can define

$$\Theta_F^s := \{E \in U_X^s(r,0) |\ h^0(E\otimes F) \neq 0\}$$

and denote by Θ_F the closure of Θ_F^s in $U_X(r,0)$. It is clear that, for generic F at least, Θ_F is strictly contained in $U_X(r,0)$ (in which case it is again a divisor). It is useful to know what is the dependence of $\mathcal{O}(\Theta_F)$ on F, and in this direction we have:

Proposition 1.3 ([5] (7.4.3) and [4] Prop.3)**.** *Let F and G be two vector bundles of slope $g-1$ on X. If* $\mathrm{rk}(F) = m\cdot\mathrm{rk}(G)$*, then*

$$\mathcal{O}(\Theta_F) \cong \mathcal{O}(\Theta_G)^{\otimes m} \otimes \det{}^*(\det F \otimes (\det \mathrm{G})^{\otimes -m}),$$

where we use the natural identification of $\mathrm{Pic}^0(X)$ *with* $\mathrm{Pic}^0(J(X))$*. In particular, if F has rank k and N is a line bundle of degree $g-1$, we get*

$$\mathcal{O}(\Theta_F) \cong \mathcal{O}(\Theta_N)^{\otimes k} \otimes \det{}^*(\det F \otimes N^{\otimes -k}).$$

1.2. A convention on theta divisors. When looking at generalized theta divisors Θ_N and the corresponding linear series, it will be convenient to consider the line bundle N to be a *theta characteristic*, i.e. satisfying $N^{\otimes 2} \cong \omega_X$. The assumption brings some simplifications to most of the arguments, but on the other hand this case implies all the results for arbitrary N. This is true since for any N and M in $\mathrm{Pic}^{g-1}(X)$, if $\xi := N\otimes M^{-1}$, twisting by ξ gives an automorphism

$$U_X(r,0) \xrightarrow{\otimes\xi} U_X(r,0)$$

by which $\mathcal{O}(\Theta_M)$ corresponds to $\mathcal{O}(\Theta_N)$. As an example, we will freely use isomorphisms of the type $r_J^*\mathcal{O}_J(\Theta_N) \cong \mathcal{O}_J(r^2\Theta_N)$, where r_J is multiplication by r on $J(X)$, which in general would work only up to numerical equivalence. One can easily see that in each particular proof the arguments could be worked out in the general situation with little extra effort (the main point is that the cohomological arguments work even if we use numerical equivalence instead of linear equivalence).

It is worth mentioning another convention about the notation that we will be using. The divisors Θ_N make sense of course on both $U_X(r,0)$ for $r \geq 2$ and $J(X) = U_X(1,0)$, and in some proofs both versions will be used. We will denote the associated line bundle simply by $\mathcal{O}(\Theta_N)$ if Θ_N lives on $U_X(r,0)$, and by $\mathcal{O}_J(\Theta_N)$ if it lives on the Jacobian.

1.3. The Fourier-Mukai transform on an abelian variety. Here we give a brief overview of some basic facts on the Fourier-Mukai transform on an abelian variety, following the original paper of Mukai [17]. Let X be an abelian variety of dimension g, $\widehat{X}$ its dual and $\mathcal{P}$ the Poincaré line bundle on $X \times \widehat{X}$, normalized so that $\mathcal{P}_{|X\times\{0\}}$ and $\mathcal{P}_{|\{0\}\times\widehat{X}}$ are trivial. To any coherent sheaf $\mathcal{F}$ on X we can associate the sheaf $p_{2*}({p_1}^*\mathcal{F}\otimes\mathcal{P})$ on $\widehat{X}$ via the natural diagram

$$\begin{tikzcd} & X\times\widehat{X} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \widehat{X} \end{tikzcd}$$

This correspondence gives a functor

$$\mathcal{S} : \mathrm{Coh(X)} \to \mathrm{Coh}(\widehat{X}).$$

If we denote by $D(X)$ and $D(\widehat{X})$ the derived categories of $\mathrm{Coh}(X)$ and $\mathrm{Coh}(\widehat{X})$, then the derived functor $\mathbf{R}\mathcal{S} : D(X) \to D(\widehat{X})$ is defined (and called the Fourier functor), and one can consider $\mathbf{R}\widehat{\mathcal{S}} : D(\widehat{X}) \to D(X)$ in a similar way. Mukai's main theorem is the following:

Theorem 1.4 ([17, 2.2])**.** *The Fourier functor establishes an equivalence of categories between $D(X)$ and $D(\widehat{X})$. More precisely, there are isomorphisms of functors*

$$\mathbf{R}\mathcal{S}\circ\mathbf{R}\widehat{\mathcal{S}} \cong (-1_{\widehat{X}})^*[-g],$$

$$\mathbf{R}\widehat{\mathcal{S}}\circ\mathbf{R}\mathcal{S} \cong (-1_X)^*[-g].$$

In this paper we will essentially have to deal with the simple situation when by applying the Fourier functor we get back another vector bundle, i.e. a complex with only one nonzero (and locally free) term. This is packaged in the following definition (see [17] (2.3)):

Definition 1.5. A coherent sheaf $\mathcal{F}$ on X satisfies I.T. (index theorem) with *index* j if $H^i(\mathcal{F}\otimes\alpha) = 0$ for all $\alpha \in \mathrm{Pic}^0(X)$ and all $i \neq j$. In this situation we have $R^i\mathcal{S}(\mathcal{F}) = 0$ for all $i \neq j$, and by the base change theorem $R^j\mathcal{S}(\mathcal{F})$ is locally free. We denote $R^j\mathcal{S}(\mathcal{F})$ by $\widehat{\mathcal{F}}$ and call it the *Fourier transform* of $\mathcal{F}$. Note that then $\mathbf{R}\mathcal{S}(\mathcal{F}) \cong \widehat{F}[-j]$.

For later reference we list some basic properties of the Fourier transform that will be used repeatedly throughout the paper:

Proposition 1.6. *Let X be an abelian variety and $\widehat{X}$ its dual, and let $\mathbf{R}\mathcal{S}$ and $\mathbf{R}\widehat{\mathcal{S}}$ be the corresponding Fourier functors. Then the following are true:*

(1) [17, (3.4)]. *Let Y be an abelian variety, $f : Y \to X$ an isogeny and $\hat{f} : \widehat{X} \to \widehat{Y}$ the dual isogeny. Then there are isomorphisms of functors*

$$f^*\circ\mathbf{R}\widehat{\mathcal{S}}_X \cong \mathbf{R}\widehat{\mathcal{S}}_Y\circ\hat{f}_*,$$

$$f_*\circ\mathbf{R}\widehat{\mathcal{S}}_Y \cong \mathbf{R}\widehat{\mathcal{S}}_X\circ\hat{f}^*.$$

(2) [17, (3.7)]. *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X, and define their Pontrjagin product by $\mathcal{F} * \mathcal{G} := \mu_*({p_1}^*\mathcal{F} \otimes {p_2}^*\mathcal{G})$, where $\mu : X \times X \to X$ is the multiplication on X. Then we have the following isomorphisms:*

$$\mathbf{R}\mathcal{S}(\mathcal{F} * \mathcal{G}) \cong \mathbf{R}\mathcal{S}(\mathcal{F}) \otimes \mathbf{R}\mathcal{S}(\mathcal{G}),$$

$$\mathbf{R}\mathcal{S}(\mathcal{F} \otimes \mathcal{G}) \cong \mathbf{R}\mathcal{S}(\mathcal{F}) * \mathbf{R}\mathcal{S}(\mathcal{G})[g],$$

where the operations on the right hand side should be thought of in the derived category.

(3) [17, (3.11)]. *Let L be a nondegenerate line bundle on X of index i, i.e. $h^i(L) \neq 0$ and $h^j(L) = 0$ for all $j \neq i$. Then by* [20] *§16, I.T. holds for L and there is an isomorphism*

$$\phi_L^* \widehat{L} \cong H^i(L) \otimes L^{-1} \ (\cong \bigoplus_{|\chi(L)|} L^{-1}),$$

with ϕ_L the isogeny canonically defined by L.

(4) [17, (3.1)]. *Let $\mathcal{F}$ be a coherent sheaf on X, $x \in \widehat{X}$ and $P_x \in \mathrm{Pic}^0(X)$ the corresponding line bundle. Then we have an isomorphism*

$$\mathbf{R}\mathcal{S}(\mathcal{F} \otimes P_x) \cong t_x^* \mathbf{R}\mathcal{S}(\mathcal{F}),$$

where t_x is translation by x.

(5) [17, (2.4), (2.5), (2.8)]. *Assume that E satisfies I.T. with index i. Then $\widehat{E}$ satisfies I.T. with index $g - i$, and in this case*

$$\chi(E) = (-1)^i \cdot \mathrm{rk}(\widehat{E}).$$

Moreover, there are isomorphisms $\mathrm{Ext}^k(E, E) \cong \mathrm{Ext}^k(\widehat{E}, \widehat{E})$ for all k, and in particular E is simple if and only if $\widehat{E}$ is simple.

1.4. Global generation and normal generation of vector bundles on abelian varieties. For the reader's convenience, in this subsection we give a brief account of some very recent results and techniques in the study of vector bundles on abelian varieties – following work of Pareschi [22] – in a form convenient for our purposes. The underlying theme is to give useful criteria for the global generation and surjectivity of multiplication maps of such vector bundles.

Let X be an abelian variety and E a vector bundle on X. Building on earlier work of Kempf [12], Pareschi proves the following cohomological criterion for global generation:

Theorem 1.7 ([22, (2.1)]). *Assume that E satisfies the following vanishing property:*

$$h^i(E \otimes \alpha) = 0, \ \forall \alpha \in \mathrm{Pic}^0(X) \ \textit{and} \ \forall i > 0.$$

Then for any ample line bundle L on X, $E \otimes L$ is globally generated.

In another direction, in order to attack questions about multiplication maps of the form

$$H^0(E) \otimes H^0(F) \longrightarrow H^0(E \otimes F) \tag{1}$$

for E and F vector bundles on X, the right notion turns out to be that of skew Pontrjagin product:

Definition 1.8. Let E and F be coherent sheaves on the abelian variety X. Then the *skew Pontrjagin product* of E and F is defined by

$$E\hat{*}F := p_{1*}((p_1+p_2)^*E \otimes p_2^*F)$$

where $p_1, p_2 : X \times X \to X$ are the two projections.

The following is a simple but essential result relating the skew Pontrjagin product to the surjectivity of the multiplication map (1). It is a restatement of [22] (1.1) in a form convenient to us, and we reproduce Pareschi's argument for the sake of completeness.

Proposition 1.9. *Assume that $E\hat{*}F$ is globally generated and that $h^i(t_x^*E \otimes F) = 0$ for all $x \in X$ and all $i > 0$, where t_x is the translation by x. Then for all $x \in X$ the multiplication map*

$$H^0(t_x^*E) \otimes H^0(F) \longrightarrow H^0(t_x^*E \otimes F)$$

is surjective, and in particular (1) is surjective.

Proof. The fact that $h^i(t_x^*E \otimes F) = 0$ for all $i > 0$ and all $x \in X$ implies by base change that $E\hat{*}F$ is locally free with fiber

$$E\hat{*}F(x) \cong H^0(t_x^*E \otimes F).$$

We also have a natural isomorphism

$$\varphi : H^0(E\hat{*}F) \xrightarrow{\sim} H^0(E) \otimes H^0(F),$$

obtained as follows. On one hand, by Leray we naturally have

$$H^0(p_{1*}((p_1+p_2)^*E \otimes p_2^*F)) \cong H^0((p_1+p_2)^*E \otimes p_2^*F),$$

and on the other hand the automorphism (p_1+p_2, p_2) of $X \times X$ induces an isomorphism

$$H^0((p_1+p_2)^*E \otimes p_2^*F) \cong H^0(E) \otimes H^0(F),$$

so φ is obtained by composition. If we identify $H^0(E) \otimes H^0(F)$ with both $H^0(E\hat{*}F)$ (via φ) and $H^0(t_x^*E) \otimes H^0(F)$ (via $t_x^* \times id$), then it is easily seen that the multiplication map

$$H^0(t_x^*E) \otimes H^0(F) \longrightarrow H^0(t_x^*E \otimes F)$$

coincides with the evaluation map

$$H^0(E\hat{*}F) \xrightarrow{ev_x} E\hat{*}F(x),$$

and this proves the assertion. □

Remark 1.10. There is a clear relationship between the skew Pontrjagin product and the usual Pontrjagin product as defined in 1.6(2). In fact (see [22] (1.2)),

$$E\hat{*}F \cong E * (-1_X)^*F.$$

If F is symmetric the two notions coincide, and one can hope to apply results like 1.6(2). This whole circle of ideas will be used in Section 5 below.

Finally we extract another result on multiplication maps that will be useful in the sequel. It is a particular case of [22] (3.8), implicit in Kempf's work [12] on syzygies of abelian varieties.

Proposition 1.11. *Let E be a vector bundle on X, L an ample line bundle and $m \geq 2$ an integer. Assume that*

$$h^i(E \otimes L^{\otimes k} \otimes \alpha) = 0, \ \forall i > 0, \ \forall k \geq -2 \text{ and } \forall \alpha \in \mathrm{Pic}^0(X).$$

Then the multiplication map

$$H^0(L^{\otimes m}) \otimes H^0(E) \longrightarrow H^0(L^{\otimes m} \otimes E)$$

is surjective.

2. The Verlinde bundles $E_{r,k}$

In the present section we define the main objects of this paper and study their first properties. These are vector bundles on the Jacobian of X which are naturally associated to generalized theta line bundles on the moduli spaces $U_X(r,0)$ and play a key role in what follows.

Definition 2.1. For all positive integers r and k and every $N \in \mathrm{Pic}^{g-1}(X)$, the (r,k)-*Verlinde* bundle on $J(X)$ is the vector bundle

$$E_{r,k} \ (= E^N_{r,k}) := \det{}_* \mathcal{O}(k\Theta_N),$$

where $\det : U_X(r,0) \to J(X)$ is the determinant map. The dependence of the definition on the choice of N is implicitly assumed, but not emphasized by the notation. Recall that by the convention of subsection 1.2 we will (and it is enough to) assume that N is a theta characteristic. Also, most of the time the rank r is fixed and we refer to $E_{r,k}$ as the *level* k Verlinde bundle.

Remark 2.2. The fibers of the determinant map are the moduli spaces of vector bundles of fixed determinant $SU_X(r,L)$ with $L \in \mathrm{Pic}^0(X)$. The restriction of $\mathcal{O}(k\Theta_N)$ to such a fiber is exactly $\mathcal{L}^k$, where $\mathcal{L}$ is the determinant bundle. Since on $SU_X(r)$ the dualizing sheaf is isomorphic to $\mathcal{L}^{-2r}$, by the rational singularities version of the Kodaira vanishing theorem these have no higher cohomology, and it is clear that $E_{r,k}$ is a vector bundle of rank $s_{r,k} := h^0(SU_X(r), \mathcal{L}^k)$. The fiber of $E_{r,k}$ at a point L is naturally isomorphic to the Verlinde vector space $H^0(SU_X(r,L), \mathcal{L}^k)$, and this justifies the terminology "Verlinde bundle" that we are using.

Much of the study of the vector bundles $E_{r,k}$ is governed by the fact that they decompose very nicely when pulled back via multiplication by r. To see this, first recall from [4], §2 and §4, that there is a cartesian diagram

$$\begin{tikzcd} SU_X(r) \times J(X) \arrow[r, "\tau"] \arrow[d, "p_2"'] & U_X(r,0) \arrow[d, "\det"] \\ J(X) \arrow[r, "r_J"] & J(X) \end{tikzcd}$$

where τ is the tensor product of vector bundles, p_2 is the projection on the second factor and r_J is multiplication by r. The top and bottom maps are étale covers of degree r^{2g}, and one finds in [4] the formula

$$\tau^* \mathcal{O}(\Theta_N) \cong \mathcal{L} \boxtimes \mathcal{O}_J(r\Theta_N). \tag{2}$$

Using the notation $V_{r,k} := H^0(SU_X(r), \mathcal{L}^k)$, we have the following simple but very important fact:

Lemma 2.3. $r_J^* E_{r,k} \cong V_{r,k} \otimes \mathcal{O}_J(kr\Theta_N)$.

Proof. By the push-pull formula (see [8], III.9.3) and (2) we obtain

$$\begin{aligned} r_J^* E_{r,k} &\cong r_J^* \det{}_* \mathcal{O}(k\Theta_N) \cong p_{2*}\tau^*\mathcal{O}(k\Theta_N) \\ &\cong p_{2*}(\mathcal{L}^k \boxtimes \mathcal{O}_J(kr\Theta_N)) \cong V_{r,k} \otimes \mathcal{O}_J(kr\Theta_N). \end{aligned}$$

□

An immediate consequence of this property is the following (recall that a vector bundle is called *polystable* if it decomposes as a direct sum of stable bundles of the same slope):

Corollary 2.4. *$E_{r,k}$ is an ample vector bundle, polystable with respect to any polarization on $J(X)$.*

Proof. Both properties can be checked up to finite covers (see e.g. [11] §3.2), and they are obvious for $r_J^* E_{r,k}$. □

For future reference it is also necessary to study how the bundles $E_{r,k}$ behave under the Fourier-Mukai transform (recall the definitions from subsection 1.3).

Lemma 2.5. *$E_{r,k}$ satisfies I.T. with index 0, and so $\widehat{E_{r,k}}$ is a vector bundle of rank $h^0(U_X(r,0), \mathcal{O}(k\Theta_N))$ satisfying I.T. with index g.*

Proof. Using the usual identification between $\mathrm{Pic}^0(J(X))$ and $\mathrm{Pic}^0(X)$, we have to show that $H^i(E_{r,k} \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(X)$ and all $i > 0$. But $H^i(E_{r,k} \otimes P)$ is a direct summand in $H^i(r_{J*} r_J^*(E_{r,k} \otimes P))$, and so it is enough to have the vanishing of $H^i(r_J^*(E_{r,k} \otimes P))$. This is obvious by the formula in 2.3. By 1.6(5) we have

$$rk(\widehat{E_{r,k}}) = h^0(E_{r,k}) = h^0(U_X(r,0), \mathcal{O}(k\Theta_N)).$$

The last statement follows also from 1.6(5). □

By Serre duality, one obtains in a similar way the statement:

Lemma 2.6. *$E_{r,k}^*$ satisfies I.T. with index g, and so $\widehat{E_{r,k}^*}$ is a vector bundle of rank $h^0(U_X(r,0), \mathcal{O}(k\Theta_N))$ satisfying I.T. with index 0.*

The Verlinde bundles are particularly easy to compute when k is a multiple of the rank r, but note that for general k the decomposition of $E_{r,k}$ into stable factors is not so easy to describe.

Proposition 2.7. *For all $m \geq 1$*

$$E_{r,mr} \cong \bigoplus_{s_{r,mr}} \mathcal{O}_J(m\Theta_N) \ (\cong V_{r,mr} \otimes \mathcal{O}_J(m\Theta_N)).$$

Proof. By 2.3 we have

$$r_J^* E_{r,mr} \cong V_{r,mr} \otimes \mathcal{O}_J(mr^2\Theta_N).$$

Since N is a theta characteristic, it follows that Θ_N is symmetric, and so $r_J^*\mathcal{O}(m\Theta_N) \cong \mathcal{O}_J(mr^2\Theta_N)$. We get

$$r_J^* E_{r,mr} \cong r_J^*(V_{r,mr} \otimes \mathcal{O}(m\Theta_N)).$$

Note now that the diagram giving 2.3 is equivariant with respect to the X_r, the group of r-torsion points of $J(X)$, if we let X_r act on $SU_X(r) \times J(X)$ by $(\xi, (F, L)) \to (F, L \otimes \xi)$, by twisting on $U_X(r,0)$, by translation on the left $J(X)$ and trivially on the right $J(X)$. Also the action of X_r on $\tau^*\mathcal{O}(mr\Theta_N) \cong \mathcal{L}^{mr} \boxtimes \mathcal{O}_J(mr^2\Theta_N)$ is on

$\mathcal{O}_J(mr^2\Theta_N)$ the same as the natural (pullback) action. By chasing the diagram we see then that the vector bundle isomorphism above is equivariant with respect to the natural X_r action on both sides. Since we have an an induced isomorphism

$$r_{J*}r_J^*E_{r,mr} \cong r_{J*}r_J^*(V_{r,mr} \otimes \mathcal{O}(m\Theta_N))$$

and $E_{r,mr}$ and $V_{r,mr} \otimes \mathcal{O}_J(m\Theta_N)$ are both eigenbundles with respect to the trivial character, the lemma follows. □

The rest of the section will be devoted to a further study of these bundles in the case $k = 1$, where more tools are available. The main result is that $E_{r,1}$ is a simple vector bundle, and this fact will be exploited in the next section. We show this after proving a very simple lemma, to the effect that twisting by r-torsion line bundles does not change $E_{r,1}$.

Lemma 2.8. *$E_{r,1} \otimes P_\xi \cong E_{r,1}$ for any r-torsion line bundle P_ξ on $J(X)$ corresponding to an r-torsion $\xi \in \mathrm{Pic}^0(X)$ by the usual identification.*

Proof. By definition and the projection formula one has

$$E_{r,1} \otimes P_\xi \cong \det{}_*(\mathcal{O}(\Theta_N) \otimes \det{}^* P_\xi) \cong \det{}_*\mathcal{O}(\Theta_{N\otimes\xi}),$$

where the last isomorphism is an application of 1.3. Now the following commutative diagram:

$$\begin{array}{ccc} U_X(r,0) & \xrightarrow{\otimes\xi} & U_X(r,0) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} \\ J(X) & \xrightarrow{id} & J(X) \end{array}$$

shows that $\det_*\mathcal{O}(\Theta_N) \cong \det_*\mathcal{O}(\Theta_{N\otimes\xi})$, which is exactly the statement of the lemma. □

Proposition 2.9. *$E_{r,1}$ is a simple vector bundle.*

Proof. This follows from a direct computation of the number of endomorphisms of $E_{r,1}$. By Lemma 2.3 and the Verlinde formula at level 1, $r_J^*E_{r,1} \cong \bigoplus_{r^g} \mathcal{O}_J(r\Theta_N)$. Then

$$h^0(r_{J*}r_J^*(E_{r,1}^* \otimes E_{r,1})) = h^0(r_J^*(E_{r,1}^* \otimes E_{r,1}))$$

$$= h^0((\bigoplus_{r^g} \mathcal{O}_J(-r\Theta_N)) \otimes (\bigoplus_{r^g} \mathcal{O}_J(r\Theta_N))) = r^{2g}.$$

On the other hand, since r_J is a Galois cover with Galois group X_r, we have the formula $r_{J*}\mathcal{O}_J \cong \bigoplus_{\xi\in X_r} P_\xi$. Combined with lemma 2.8, this gives

$$r_{J*}r_J^*(E_{r,1}^* \otimes E_{r,1}) \cong \bigoplus_{\xi\in X_r} E_{r,1}^* \otimes E_{r,1} \otimes P_\xi \cong \bigoplus_{r^{2g}} E_{r,1}^* \otimes E_{r,1}.$$

The two relations imply that $h^0(E_{r,1}^* \otimes E_{r,1}) = 1$, so $E_{r,1}$ is simple. □

Remark 2.10. Using an argument similar to the one given above, plus the Verlinde formula, it is not hard to see that the bundles $E_{r,k}$ are not simple for $k \geq 2$. In the case when k is a multiple of r they even decompose as direct sums of line bundles, as we have already seen in 2.7. This shows why some special results that we will obtain in the case $k = 1$ do not admit straightforward extensions to higher k's.

3. Stability of Fourier transforms and duality for generalized theta functions

One of the main features of the vector bundles $E_{r,1}$, already observed in the previous section, is that they are simple. We will see below that in fact they are even stable (with respect to any polarization on $J(X)$). This fact, combined with the fact that $\widehat{\mathcal{O}_J(r\Theta_N)}$ is also stable (see Proposition 3.1 below), gives simple proofs of some results of Beauville-Narasimhan-Ramanan [2] and Donagi-Tu [4]. Note that for the proofs of these applications it is enough to use the simpleness of the vector bundles mentioned above.

Proposition 3.1. *$\widehat{E_{1,r}} = \widehat{\mathcal{O}_J(r\Theta_N)}$ is stable with respect to any polarization on $J(X)$.*

This is a consequence of a more general fact of independent interest, saying that this is indeed true for an arbitrary nondegenerate line bundle on an abelian variety. Recall that a line bundle A on the abelian variety X is called *nondegenerate* if $\chi(A) \neq 0$. By [20] §16 this implies that there is a unique i (the *index* of A) such that $H^i(A) \neq 0$.

Proposition 3.2. *Let A be a nondegenerate line bundle on an abelian variety X. The Fourier-Mukai transform $\widehat{A}$ is stable with respect to any polarization on X.*

Proof. Let's begin by fixing a polarization on $\widehat{X}$, so that stability will be understood with respect to this polarization. Consider the isogeny defined by A,

$$\begin{array}{rccc} \phi_A : & X & \longrightarrow & \mathrm{Pic}^0(X) \cong \widehat{X}, \\ & x & \rightsquigarrow & t_x^* A \otimes A^{-1}. \end{array}$$

If i is the index of A, it follows from 1.6(3) that $\phi_A^* \widehat{A} \cong V \otimes A^{-1}$, where $V := H^i(A)$. As we already mentioned in the proof of 2.4, by [11] §3.2 this already implies that $\widehat{A}$ is polystable. On the other hand, by 1.6(5) the Fourier transform of any line bundle is simple, so $\widehat{A}$ must be stable. □

Remark 3.3. It is worth noting that one can avoid the use of [11] §3.2 quoted above and use only the easier fact that semistability is preserved by finite covers. More precisely, $\widehat{A}$ has to be semistable, but assume that it is not stable. Then we can choose a maximal destabilizing subbundle $F \subsetneq \widehat{A}$, which must obviously be semistable and satisfy $\mu(F) = \mu(\widehat{A})$. Again $\phi_A^* F$ must be semistable, with respect to the pull-back polarization. But $\phi_A^* F \subsetneq \phi_A^* \widehat{A} \cong V \otimes A^{-1}$ and by semistability this implies that $\phi_A^* F \cong V' \otimes A^{-1}$ with $V' \subsetneq V$. This situation is overruled by the presence of the action of Mumford's theta-group $\mathcal{G}(A)$. Recall from [19] that A is endowed with a natural $\mathcal{G}(A)$-linearization of weight 1. On the other hand, $\phi_A^* F$ has a natural $K(A)$-linearization which can be seen as a $\mathcal{G}(A)$-linearization of weight 0, where $K(A)$ is the kernel of the isogeny ϕ_A above. By tensoring we obtain a weight 1 $\mathcal{G}(A)$-linearization on $V' \otimes \mathcal{O}_X \cong \phi_A^* F \otimes A$ and thus an induced weight 1 representation on V'. It is known though from [19] that V is the unique irreducible representation of $\mathcal{G}(A)$ up to isomorphism, so we get a contradiction.

We now return to the study of the relationship between the bundles $E_{r,k}$ and their Fourier transforms. First note that throughout the rest of the paper we will use the fact that $J(X)$ is canonically isomorphic to its dual $\mathrm{Pic}^0(J(X))$, and consequently we will use the same notation for both. Thus all the Fourier transforms should be

thought of as coming from the dual via this identification, although since there is no danger of confusion this will not be visible in the notation. The fiber of $E_{r,k}$ over a point $\xi \in J(X)$ is $H^0(SU_X(r,\xi),\mathcal{L}^k)$, while the fiber of $\widehat{E_{k,r}}$ over the same point is canonically isomorphic to $H^0(J(X), E_{k,r} \otimes P_\xi)$. By 1.3 the latter is isomorphic to $H^0(U_X(k,0), \mathcal{O}(r\Theta_{N\otimes\eta}))$, where $\eta^{\otimes r} \cong \xi$ (it is easy to see that this does not depend on the choice of η). The strange duality conjecture (see [1] §8 and [4] §5) says that there is a canonical isomorphism

$$H^0(SU_X(r,\xi),\mathcal{L}^k)^* \cong H^0(U_X(k,0), \mathcal{O}(r\Theta_{N\otimes\eta}))$$

which will be described more precisely later in this section. This suggests then that one should somehow relate the vector bundles $E_{r,k}$ and $\widehat{E_{k,r}}$ via the diagram

$$\begin{tikzcd}
U_X(r,0) \arrow[dd, "\det"'] & & U_X(k,0) \arrow[dd, "\det"] \\
& J(X)\times J(X) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
J(X) & & J(X)
\end{tikzcd}$$

The first proposition treats the case $k=1$ and establishes the fact that the dual of $E_{r,1}$ is nothing else but the Fourier transform of $\mathcal{O}_J(r\Theta_N)$.

Proposition 3.4. $E^*_{r,1} \cong \widehat{E_{1,r}}$.

Proof. By the duality theorem 1.4, it is enough to show that $\widehat{E^*_{r,1}} \cong (-1_J)^* E_{1,r}$. But $E_{1,r}$ is just $\mathcal{O}_J(r\Theta_N)$, which is symmetric since N is a theta characteristic. So what we have to prove is

$$\widehat{E^*_{r,1}} \cong \mathcal{O}_J(r\Theta_N).$$

Since r_J is Galois with Galois group X_r, we have (see e.g [21] (2.1))

$$r_J^* r_{J*} \widehat{E^*_{r,1}} \cong \bigoplus_{\xi\in X_r} t_\xi^* \widehat{E^*_{r,1}}.$$

But translates commute with tensor products via the Fourier transform (cf. 1.6(4)), and so

$$t_\xi^* \widehat{E^*_{r,1}} \cong \widehat{E^*_{r,1} \otimes P_\xi} \cong E^*_{r,1},$$

where as usual P_ξ is the line bundle in $\mathrm{Pic}^0(J(X))$ that corresponds to ξ, and the last isomorphism follows from 2.8. Thus we get

$$r_J^* r_{J*} \widehat{E^*_{r,1}} \cong \bigoplus_{r^{2g}} \widehat{E^*_{r,1}},$$

and the idea is to compute this bundle in a different way, by using the behavior of the Fourier transform under isogenies. More precisely, by applying 1.6(1) we get the isomorphisms

$$r_J^* r_{J*} \widehat{E^*_{r,1}} \cong r_J^* \widehat{r_J^* E^*_{r,1}} \cong r_J^* (V^*_{r,1} \otimes \mathcal{O}_J(-r\Theta_N))\widehat{\ }$$

$$\cong \bigoplus_{r^g} r_J^* \mathcal{O}_J \widehat{(-r\Theta_N)} \cong \bigoplus_{r^{2g}} \mathcal{O}_J(r\Theta_N).$$

The second isomorphism follows from 2.3, while the fourth follows from 1.6(3) and the Verlinde formula at level 1. The outcome is the isomorphism

$$\bigoplus_{r^{2g}} \widehat{E^*_{r,1}} \cong \bigoplus_{r^{2g}} \mathcal{O}_J(r\Theta_N).$$

Now $\widehat{E^*_{r,1}}$ is simple since $E_{r,1}$ is simple, so the previous isomorphism implies the stronger fact that

$$\widehat{E^*_{r,1}} \cong \mathcal{O}_J(r\Theta_N).$$

□

Example 3.5. The relationship between the Chern character of a sheaf and that of its Fourier transform established in [18] (1.18) allows us to easily compute the first Chern class of $E_{r,1}$ as a consequence of the previous proposition. More precisely, if θ is the class of a theta divisor on $J(X)$, we have

$$c_1(\widehat{\mathcal{O}_J(r\Theta_N)}) = \mathrm{ch}_1(\widehat{\mathcal{O}_J(r\Theta_N)}) = (-1)\cdot PD_{2g-2}(\mathrm{ch}_{g-1}(\mathcal{O}_J(r\Theta_N)))$$

$$= (-1)\cdot PD_{2g-2}(r^{g-1}\cdot\theta^{g-1}/(g-1)!) = -r^{g-1}\cdot\theta,$$

where $PD_{2g-2} : H^{2g-2}(J(X),\mathbf{Z}) \to H^2(J(X),\mathbf{Z})$ is the Poincaré duality. We get

$$c_1(E_{r,1}) = r^{g-1}\cdot\theta.$$

We are able to prove the analogous fact for higher k's only modulo multiplication by r.

Proposition 3.6. $r_J^* E^*_{r,k} \cong r_J^* \widehat{E_{k,r}}$.

Proof. We know that $k_J^* E_{k,r} \cong \bigoplus_{s_{k,r}} \mathcal{O}_J(kr\Theta_N)$, so by 1.6(1) we obtain

$$k_{J*}\widehat{E_{k,r}} \cong \bigoplus_{s_{k,r}} \widehat{\mathcal{O}_J(kr\Theta_N)}.$$

As in the previous proposition, since the Galois group of k_J is X_k, we have

$$k_J^* k_{J*}\widehat{E_{k,r}} \cong \bigoplus_{\xi\in X_k} t_\xi^*\widehat{E_{k,r}} \quad \text{and so} \quad (kr)_J^* k_{J*}\widehat{E_{k,r}} \cong \bigoplus_{\xi\in X_k} r_J^* t_\xi^*\widehat{E_{k,r}}.$$

Moreover, the isomorphisms above show as before that $\widehat{E_{k,r}}$ has to be semistable with respect to an arbitrary polarization. On the other hand, by 1.6(3) we have

$$(kr)_J^* k_{J*}\widehat{E_{k,r}} \cong \bigoplus_{s_{k,r}} (kr)_J^*\widehat{\mathcal{O}_J(kr\Theta_N)} \cong \bigoplus_{k^g r^g s_{k,r}} \mathcal{O}_J(-kr\Theta_N).$$

This gives us the isomorphism

$$\bigoplus_{\xi\in X_k} r_J^* t_\xi^*\widehat{E_{k,r}} \cong \bigoplus_{k^g r^g s_{k,r}} \mathcal{O}_J(-kr\Theta_N),$$

which in particular implies (recall semistability) that

$$r_J^*\widehat{E_{k,r}} \cong \bigoplus_{s_{r,k}} \mathcal{O}_J(-kr\Theta_N) \cong r_J^* E^*_{r,k}.$$

The important fact that the last index of summation is $s_{r,k}$ follows from the well-known "symmetry" of the Verlinde formula, which is

$$r^g s_{k,r} = k^g s_{r,k}.$$

□

The propositions above allow us to give alternative proofs of some results in [2] and [4] concerning duality between spaces of generalized theta functions. We first show how one can recapture a theorem of Donagi-Tu in the present context. The full version of the theorem (i.e. for arbitrary degree) can be obtained by the same method (cf. Section 6).

Theorem 3.7 ([4, Theorem 1])**.** *For any $L \in \mathrm{Pic}^0(X)$ and any $N \in \mathrm{Pic}^{g-1}(X)$, we have*

$$h^0(SU_X(k,L), \mathcal{L}^r) \cdot r^g = h^0(U_X(k,0), \mathcal{O}(r\Theta_N)) \cdot k^g.$$

Proof. We will actually prove the following equality:

$$h^0(SU_X(r,L), \mathcal{L}^k) = h^0(U_X(k,0), \mathcal{O}(r\Theta_N)).$$

The statement will then follow from the same symmetry of the Verlinde formula $r^g s_{k,r} = k^g s_{r,k}$ mentioned in the proof of 3.6. To this end we can use Proposition 3.6 to obtain $\mathrm{rk}(E^*_{r,k}) = \mathrm{rk}(\widehat{E_{k,r}})$. But on one hand

$$\mathrm{rk}(E^*_{r,k}) = h^0(SU_X(r,L), \mathcal{L}^k),$$

while on the other hand, by 1.6(5),

$$\mathrm{rk}(\widehat{E_{k,r}}) = h^0(J(X), E_{k,r}) = h^0(U_X(k,0), \mathcal{O}(r\Theta_N)),$$

as required. □

It is worth mentioning that since we are assuming the Verlinde formula all throughout, an important particular case of the theorem above is

Corollary 3.8. [2, Theorem 2]. $h^0(U_X(r,0), \mathcal{O}(\Theta_N)) = 1$.

As it is very well known, this fact is essential in setting up the strange duality. As an application in this direction, the results above also lead to a simple proof of the strange duality at level 1, which was first proved in [2], Theorem 3. It is important, though, to emphasize again that here, unlike in the quoted paper, the Verlinde formula is granted. So the purpose of the next application is to show how, with the knowledge of the Verlinde numbers, the strange duality at level 1 can simply be seen as the solution of a stability problem for vector bundles on $J(X)$. This may also provide a global method for understanding the conjecture for higher levels. A few facts in this direction will be mentioned at the end of this section (cf. Remark 3.13).

Before turning to the proof, we need the following general result, which is a globalization of §3 in [4].

Proposition 3.9. *Consider the tensor product map*

$$\tau : U_X(r,0) \times U_X(k,0) \longrightarrow U_X(kr,0)$$

and the map

$$\phi := \det \times \det : U_X(r,0) \times U_X(k,0) \longrightarrow J(X) \times J(X).$$

Then

$$\tau^*\mathcal{O}(\Theta_N) \cong p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P},$$

where $\mathcal{P}$ is a Poincaré line bundle on $J(X)\times J(X)$, normalized so that $\mathcal{P}_{|\{0\}\times J(X)} \cong \mathcal{O}_J$.

Proof. We will compare the restrictions of the two line bundles to fibers of the projections. First fix $F \in U_X(k,0)$. We have

$$\tau^*\mathcal{O}(\Theta_N)|_{U_X(r,0)\times\{F\}} \cong \tau_F^*\mathcal{O}(\Theta_N) \cong \mathcal{O}(\Theta_{F\otimes N}),$$

where τ_F is the map given by twisting with F. But by 1.3 one has

$$\mathcal{O}(\Theta_{F\otimes N}) \cong \mathcal{O}(k\Theta_N) \otimes \det{}^*(\mathrm{detF})(\cong \mathcal{O}(k\Theta_N) \otimes \det{}^*(\mathcal{P}_{|J(X)\times\{\det F\}})).$$

On the other hand, obviously

$$p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P}|_{U_X(r,0)\times\{F\}} \cong \mathcal{O}(k\Theta_N) \otimes \det{}^*(\det F).$$

Let's now fix $E \in U_X(r,0)$ such that $\det E \cong \mathcal{O}_X$. Using the same 1.3, we get

$$\tau^*\mathcal{O}(\Theta_N)|_{\{E\}\times U_X(k,0)} \cong \mathcal{O}(\Theta_{E\otimes N}) \cong \mathcal{O}(r\Theta_N),$$

and we also have

$$p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P}|_{\{E\}\times U_X(k,0)}$$

$$\cong \mathcal{O}(r\Theta_N) \otimes \det{}^*(\mathcal{P}_{|\{\mathcal{O}_X\}\times J(X)}) \cong \mathcal{O}(r\Theta_N).$$

The desired isomorphism follows now from the see-saw principle (see e.g. [20], I.5.6). □

Theorem 3.7 tells us that there is essentially a unique nonzero section

$$s \in H^0(U_X(kr,0), \mathcal{O}(\Theta_N)),$$

which induces via τ a nonzero section

$$t \in H^0(U_X(r,0)\times U_X(k,0), \tau^*\mathcal{O}(\Theta_N)$$

$$\cong H^0(U_X(r,0)\times U_X(k,0), p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P}).$$

But notice that from the projection formula we get

$$\phi_*(p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P})$$

$$\cong \phi_*(p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N)) \otimes \mathcal{P} \cong p_1^*E_{r,k} \otimes p_2^*E_{k,r} \otimes \mathcal{P},$$

so t induces a section (denoted also by t):

$$0 \neq t \in H^0(J(X)\times J(X), p_1^*E_{r,k} \otimes p_2^*E_{k,r} \otimes \mathcal{P}) \cong H^0(E_{r,k} \otimes \widehat{E_{k,r}}).$$

This is nothing else but a globalization of the section defining the strange duality morphism, as explained in [4] §5 (simply because for any $\xi \in \mathrm{Pic}^0(X)$ the restriction of τ in 3.9 to $SU_X(r,\xi)\times U_X(k,0)$ is again the tensor product map

$$\tau : SU_X(r,\xi)\times U_X(k,0) \longrightarrow U_X(kr,0)$$

and $\tau^*\mathcal{O}(\Theta_N) \cong \mathcal{L}^k \boxtimes \mathcal{O}(r\Theta_N))$. In other words, t corresponds to a morphism of vector bundles

$$SD : E_{r,k}^* \longrightarrow \widehat{E_{k,r}},$$

which fiberwise is exactly the strange duality morphism

$$H^0(SU_X(r,\xi),\mathcal{L}^k)^* \xrightarrow{SD} H^0(U_X(k,0),\mathcal{O}(r\Theta_{N\otimes\eta})),$$

where $\eta^{\otimes r}\cong\xi$. Since this global morphism collects together all the strange duality morphisms as we vary ξ, the strange duality conjecture is equivalent to SD being an isomorphism.

Conjecture 3.10. *$SD : E^*_{r,k} \longrightarrow \widehat{E_{k,r}}$ is an isomorphism of vector bundles.*

In this context Proposition 3.6 can be seen as a weak form of "global" evidence for the conjecture in the case $k \geq 2$. For $k = 1$ it can now be easily proved.

Theorem 3.11. *$SD : E^*_{r,1} \longrightarrow \widehat{E_{1,r}}$ is an isomorphism.*

Corollary 3.12 (cf. [2, Theorem 3])**.** *The level 1 strange duality morphism*

$$H^0(SU_X(r),\mathcal{L})^* \xrightarrow{SD} H^0(J(X),\mathcal{O}(r\Theta_N))$$

is an isomorphism.

Proof. (of 3.11) All the ingredients necessary for proving this have been discussed above: by 3.1 and 3.4, the bundles $E^*_{r,1}$ and $\widehat{E_{1,r}}$ are isomorphic and stable. This means that SD is essentially the unique nonzero morphism between them, and it must be an isomorphism. □

Remark 3.13. One step towards 3.10 is a better understanding of the properties of the kernel F of SD. A couple of interesting remarks in this direction can already be made. Since $E_{r,k}$ and $\widehat{E_{k,r}}$ are polystable of the same slope, the same will be true about F. On the other hand, some simple calculus involving 1.3 and 1.6(4) shows that F gets multiplied by a line bundle in $\mathrm{Pic}^0(J(X))$ when we translate it, so in the language of Mukai (e.g. [18] §3) it is a semi-homogeneous vector bundle (although clearly not homogeneous, i.e. not fixed by all translations).

4. A generalization of Raynaud's examples

In this section we would like to discuss a generalization of the examples of base points of the theta linear system $|\mathcal{L}|$ on $SU_X(m)$ constructed by Raynaud in [25]. For a survey of this circle of ideas the reader can consult [1], §2. Let us recall here (see [23] §2) only that for a semistable vector bundle E of rank m to induce a base point of $|\mathcal{L}|$ it is sufficient that it satisfies the property

$$0 \leq \mu(E) \leq g-1 \text{ and } h^0(E\otimes L) \neq 0 \text{ for } L \in \mathrm{Pic}^0(X) \text{ generic.}$$

The examples of Raynaud are essentially the restrictions of $\widehat{E^*_{1,r}} = \widehat{\mathcal{O}_J(-r\Theta_N)}$ to some embedding of the curve X in the Jacobian. We will generalize this by considering the Fourier transform of higher level Verlinde bundles $\widehat{E^*_{k,r}}$ with $k \geq 2$.

For simplicity, let's fix k and r and denote $F := \widehat{E^*_{k,r}}$. This is a vector bundle by lemma 2.6. Consider also an arbitrary embedding

$$j : X \to J(X)$$

and denote by E the restriction $F_{|X}$.

Proposition 4.1. *E is a semistable vector bundle.*

Proof. This follows basically from the proof of Proposition 3.6. One can see in a completely analogous way that

$$r_J^* F \cong r_J^* E_{r,k} \cong \bigoplus_{s_{r,k}} \mathcal{O}_J(kr\Theta_N).$$

Now if we consider Y to be the preimage of X by r_J, this shows that $r_J^* F_{|Y}$ is semistable, and so $F_{|X}$ is semistable. □

Proposition 4.2. *There is an embedding of X in $J(X)$ such that $E = F_{|X}$ satisfies $H^0(E \otimes L) \neq 0$ for $L \in \mathrm{Pic}^0(X)$ generic.*

Proof. The proof goes as in [25] (3.1), and we repeat it here for convenience. Choose U a nonempty open subset of $J(X)$ on which $(-1_J)^* E_{k,r}$ is trivial. By Mukai's duality theorem [17] (2.2) we know that $\widehat{F} \cong (-1_J)^* E_{k,r}^*$, so there exists a nonzero section $s \in \Gamma({p_2}^{-1}(U), p_1^* F \otimes \mathcal{P})$. Choose now an $x_0 \in J(X)$ such that $s_{|\{x_0\} \times U} \neq 0$ and consider an embedding of X in $J(X)$ passing through x_0. The image of s in $\Gamma(X \times U, p_1^* F \otimes \mathcal{P})$ is nonzero, and this implies that $H^0(E \otimes L) \neq 0$ for L generic. □

We are only left with computing the invariants of E.

Proposition 4.3. *The rank and the slope of E are given by*

$$\mathrm{rk}(E) = s_{r,k} = h^0(SU_X(r), \mathcal{L}^k) \quad \textit{and} \quad \mu(E) = gk/r.$$

Proof. By 1.6(5) we have

$$\mathrm{rk}(F) = \mathrm{rk}(\widehat{E_{k,r}^*}) = h^0(E_{k,r}) = h^0(U_X(k,0), \mathcal{O}(r\Theta_N)).$$

But from the proof of 3.7 we know that $h^0(U_X(k,0), \mathcal{O}(r\Theta_N)) = h^0(SU_X(r,0), \mathcal{L}^k)$, and so $\mathrm{rk}(F) = s_{r,k}$. To compute the slope of E, first notice that by the proof of Proposition 4.1 we know that

$$r^{2g} \cdot \mu(E) = \deg(\mathcal{O}_J(kr\Theta_N)_{|Y}).$$

But $r_J^* \mathcal{O}_J(kr\Theta_N) \equiv \mathcal{O}_J(kr^3\Theta_N)$, so

$$r^2 \cdot \deg(\mathcal{O}_J(kr\Theta_N)_{|Y}) = \deg(r_J^* \mathcal{O}_J(kr\Theta_N)_{|Y}) = r^{2g} \cdot \deg(\mathcal{O}_J(kr\Theta_N)_{|X}) = r^{2g+1} kg.$$

Combining the two equalities, we get

$$\mu(E) = \deg(\mathcal{O}_J(kr\Theta_N)_{|Y})/r^{2g} = gk/r.$$

□

In conclusion, for each r and k we obtain a semistable vector bundle E on X of rank equal to the Verlinde number $s_{r,k}$ and of slope gk/r, satisfying the property that $H^0(E \otimes L) \neq 0$ for L generic in $\mathrm{Pic}^0(X)$. So as long as $k < r$ and r divides gk we obtain new examples of base points for $|\mathcal{L}|$ on the moduli spaces $SU_X(s_{r,k})$, as explained above. Raynaud's examples correspond to the case $k = 1$. See also [23] for a study of a different kind of examples of such base points and for bounds on the dimension of the base locus of $|\mathcal{L}|$.

5. Linear series on $U_X(r,0)$

The main application of the Verlinde vector bundles concerns the global generation and normal generation of line bundles on $U_X(r,0)$. The specific goal is to give effective bounds for multiples of the generalized theta line bundles that satisfy the properties mentioned above. In this direction analogous results for $SU_X(r)$ will be used. The starting point is the following general result:

Proposition 5.1. *Let $f : X \to Y$ be a flat morphism of projective schemes, with reduced fibers. Let L be a line bundle on X and denote $E := f_*L$. Assume that if X_y denotes the fiber of f over $y \in Y$ the following conditions hold:*

(i) $h^1(L) = 0$.

(ii) $h^i(L_{|X_y}) = 0, \forall y \in Y, \forall i > 0$.

Then L is globally generated as long as $L_{|X_y}$ is globally generated for all $y \in Y$ and E is globally generated.

Proof. Start with $x \in X$ and consider $y = f(x)$. By (i) we have the exact sequence on X:

$$0 \longrightarrow H^0(L \otimes \mathcal{I}_{X_y}) \longrightarrow H^0(L) \longrightarrow H^0(L_{|X_y}) \longrightarrow H^1(L \otimes \mathcal{I}_{X_y}) \longrightarrow 0.$$

The global generation of $L_{|X_y}$ implies that there exists a section $s \in H^0(L_{|X_y})$ such that $s(x) \neq 0$. We would like to lift s to some $\bar{s}$, so it is enough to prove that $H^1(L \otimes \mathcal{I}_{X_y}) = 0$. The fibers of f are reduced, so $\mathcal{I}_{X_y} \cong f^*\mathcal{I}_{\{y\}}$. Condition (ii) implies, by the base change theorem, that $R^i f_*L = 0$ for all $i > 0$, and so by the projection formula we also get $R^i f_*(L \otimes \mathcal{I}_{X_y}) = 0$ for all $i > 0$. The Leray spectral sequence then gives $H^i(E) \cong H^i(L)$ and $H^i(E \otimes \mathcal{I}_{\{y\}}) \cong H^i(L \otimes \mathcal{I}_{Xy})$ for all $i > 0$. The first isomorphism implies that there is an exact sequence

$$0 \longrightarrow H^0(E \otimes \mathcal{I}_{\{y\}}) \longrightarrow H^0(E) \xrightarrow{ev_y} H^0(E_y) \longrightarrow H^1(E \otimes \mathcal{I}_{\{y\}}) \longrightarrow 0.$$

But E is globally generated, which means that ev_y is surjective. This implies the vanishing of $H^1(E \otimes \mathcal{I}_{\{y\}})$, which by the second isomorphism is equivalent to $H^1(L \otimes \mathcal{I}_{X_y}) = 0$. □

The idea is to apply this result to the situation when the map is $\det : U_X(r,0) \to J(X)$, $L = \mathcal{O}(k\Theta_N)$ and $E = E_{r,k}$. Modulo detecting for what k global generation is attained, the conditions of the proposition are satisfied. The fiberwise global generation problem (i.e. the $SU_X(r)$ case) has been given some effective solutions in the literature. The most recent is the author's result in [24], improving earlier bounds of Le Potier [16] and Hein [9]. We show in [24] §4 that $\mathcal{L}^k$ on $SU_X(r)$ is globally generated if $k \geq \frac{(r+1)^2}{4}$. We now turn to the effective statement for $E_{r,k}$, and in this direction we make essential use of Pareschi's cohomological criterion described in Section 1.

Proposition 5.2. *$E_{r,k}$ is globally generated if and only if $k \geq r+1$.*

Proof. The trick is to write $E_{r,k}$ as $E_{r,k} \otimes \mathcal{O}_J(-\Theta_N) \otimes \mathcal{O}_J(\Theta_N)$. Denote $E_{r,k} \otimes \mathcal{O}_J(-\Theta_N)$ by F. Pareschi's criterion 1.7 says in our case that $E_{r,k}$ is globally generated as long as the condition

$$h^i(F \otimes \alpha) = 0, \ \forall \alpha \in \mathrm{Pic}^0(X), \ \forall i > 0,$$

is satisfied (in fact under this assumption $F \otimes A$ will be globally generated for every ample line bundle A). Arguing as usual, $h^i(F \otimes \alpha) = 0$ is implied by $h^i(r_J^*(F \otimes \alpha)) = 0$. We have

$$r_J^* F \cong r_J^* E_{r,k} \otimes r_J^* \mathcal{O}_J(-\Theta_N) \cong \mathcal{O}_J((kr - r^2)\Theta_N),$$

and this easily gives the desired vanishing for $k \geq r+1$. On the other hand, from 2.7 we know that $E_{r,r} \cong \bigoplus_{s_{r,r}} \mathcal{O}_J(\Theta_N)$, which is clearly not globally generated. This shows that the bound is optimal. □

Combining all these, we obtain effective bounds for global generation on $U_X(r,0)$ in terms of the analogous bounds on $SU_X(r)$. We prefer to state the general result in a non-effective form though, in order to emphasize that it applies algorithmically (but see the corollaries for effective statements):

Theorem 5.3. *$\mathcal{O}(k\Theta_N)$ is globally generated on $U_X(r,0)$ as long as $k \geq r+1$ and $\mathcal{L}^k$ is globally generated on $SU_X(r)$. Moreover, $\mathcal{O}(r\Theta_N)$ is not globally generated.*

Proof. The first part follows by puting together 5.2 and 5.1 in our particular setting. To prove that $\mathcal{O}(r\Theta_N)$ is not globally generated, let us begin by assuming the contrary. Then the restriction $\mathcal{L}^r$ of $\mathcal{O}(r\Theta_N)$ to any of the fibers $SU_X(r,L)$ is also globally generated.

Choose in particular a line bundle L in the support of Θ_N on $J(X)$. Restriction to the fiber gives the following long exact sequence on cohomology:

$$0 \longrightarrow H^0(\mathcal{O}(r\Theta_N) \otimes \mathcal{I}_{SU_X(r,L)}) \longrightarrow H^0(\mathcal{O}(r\Theta_N))$$

$$\xrightarrow{\alpha} H^0(\mathcal{L}^r) \longrightarrow H^1(\mathcal{O}(r\Theta_N) \otimes \mathcal{I}_{SU_X(r,L)}) \longrightarrow 0.$$

As in the proof of 5.1, this sequence can be written in terms of the cohomology of $E_{r,r}$:

$$0 \longrightarrow H^0(E_{r,r} \otimes \mathcal{I}_{\{L\}}) \longrightarrow H^0(\mathcal{O}(r\Theta_N)) \xrightarrow{\alpha} H^0(\mathcal{L}^r) \longrightarrow H^1(E_{r,r} \otimes \mathcal{I}_{\{L\}}) \longrightarrow 0.$$

The assumption on $\mathcal{O}(r\Theta_N)$ ensures the fact that the map α in the sequence above is nonzero, and as a result

$$h^1(E_{r,r} \otimes \mathcal{I}_{\{L\}}) < h^0(\mathcal{L}^r) = s_{r,r}.$$

On the other hand we use again the fact 2.7 that $E_{r,r}$ is isomorphic to $\bigoplus_{s_{r,r}} \mathcal{O}_J(\Theta_N)$. The additional hypothesis that $L \in \Theta_N$ then says that

$$h^1(E_{r,r} \otimes \mathcal{I}_{\{L\}}) = s_{r,r},$$

which is a contradiction. □

As suggested above, by combining this with the effective result on $SU_X(r)$ given in [24] §4, we get

Corollary 5.4 (cf. [24] (5.3))**.** *$\mathcal{O}(k\Theta_N)$ is globally generated on $U_X(r,0)$ for*

$$k \geq \frac{(r+1)^2}{4}.$$

It is important, as noted in the introduction, to emphasize the fact that the $SU_X(r)$ bound may still allow for improvement. Thus the content and formulation of the theorem certainly go beyond this corollary. For moduli spaces of vector bundles of rank 2 and 3 though, in view of the second part of 5.3, we actually have optimal results (see also [24] (5.4)).

Corollary 5.5. *(i) $\mathcal{O}(3\Theta_N)$ is globally generated on $U_X(2,0)$.*
(ii) $\mathcal{O}(4\Theta_N)$ is globally generated on $U_X(3,0)$.

These are natural extensions of the fact that $\mathcal{O}(2\Theta_N)$ is globally generated on $J(X) \cong U_X(1,0)$ (see e.g. [7], Theorem 2, p. 317). In [24] §5 we also state some questions and conjectures about optimal bounds in general.

Remark 5.6. A similar technique can be applied to study the base point freeness of more general linear series on $U_X(r,0)$. This is done in [24] (5.9).

Remark 5.7. A result analogous to 5.1, combined with a more careful study of the cohomological properties of $E_{r,k}$, gives information about effective separation of points by the linear series $|k\Theta_N|$. We will not insist on this aspect in the present article.

In the same spirit of studying properties of linear series on $U_X(r,0)$ via vector bundle techniques, one can look at multiplication maps on spaces of sections and normal generation. The Verlinde bundles are again an essential tool. The underlying theme is the study of surjectivity of the multiplication map

$$H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\mathcal{O}(k\Theta_N)) \xrightarrow{\mu_k} H^0(\mathcal{O}(2k\Theta_N)).$$

In this respect we have to start by assuming that k is already chosen such that $\mathcal{L}^k$ and $E_{r,k}$ are globaly generated (in particular, $k \geq r+1$). As proved in Theorem 5.3, this also induces the global generation of $\mathcal{O}(k\Theta_N)$. The method will be to look at the kernels of various multiplication maps–in the spirit of [15] for example–and study their cohomology vanishing properties. Let M_k on $U_X(r,0)$ and $M_{r,k}$ on $J(X)$ be the vector bundles defined by the sequences

$$0 \longrightarrow M_k \longrightarrow H^0(\mathcal{O}(k\Theta_N)) \otimes \mathcal{O} \longrightarrow \mathcal{O}(k\Theta_N) \longrightarrow 0 \tag{3}$$

and

$$0 \longrightarrow M_{r,k} \longrightarrow H^0(E_{r,k}) \otimes \mathcal{O}_J \longrightarrow E_{r,k} \longrightarrow 0. \tag{4}$$

By twisting (3) with $\mathcal{O}(k\Theta_N)$ and taking cohomology, it is clear that the surjectivity of μ_k is equivalent to $H^1(M_k \otimes \mathcal{O}(k\Theta_N)) = 0$.

On the other hand, the global generation of $\mathcal{O}(k\Theta_N)$ implies that the natural map $\det^* E_{r,k} \to \mathcal{O}(k\Theta_N)$ is surjective, so we can consider the vector bundle K defined by the following sequence:

$$0 \longrightarrow K \longrightarrow \det^* E_{r,k} \longrightarrow \mathcal{O}(k\Theta_N) \longrightarrow 0. \tag{5}$$

Remark 5.8. Fixing $L \in \mathrm{Pic}^0(X)$, we can also look at the evaluation sequence for $\mathcal{L}^k$ on $SU_X(r,L)$:

$$0 \longrightarrow M_{\mathcal{L}^k} \longrightarrow H^0(\mathcal{L}^k) \otimes \mathcal{O}_{SU_X} \longrightarrow \mathcal{L}^k \longrightarrow 0.$$

The sequence (5) should be interpreted as globalizing this picture. It induces the above sequence when restricted to the fiber of the determinant map over each L.

The study of vanishing for $M_{r,k}$ and K will be the key to obtaining the required vanishing for M_k. This is reflected in the top exact sequence in the following commutative diagram, obtained from (3),(4) and (5) as an application of the snake lemma:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \det^* M_{r,k} \arrow[r] \arrow[d] & M_k \arrow[r] \arrow[d] & K \arrow[r] & 0 \\
 & \det^* H^0(E_{r,k}) \otimes \mathcal{O} \arrow[r, "\cong"] \arrow[d] & H^0(k\Theta_N) \otimes \mathcal{O} \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] & \det^* E_{r,k} \arrow[r] \arrow[d] & \mathcal{O}(k\Theta_N) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We again state our result in part (b) of the following theorem in a form that allows algorithmic applications. The main ingredient is an effective normal generation bound for $E_{r,k}$, which is the content of part (a).

Theorem 5.9. (a) *The multiplication map*

$$H^0(E_{r,k}) \otimes H^0(E_{r,k}) \longrightarrow H^0(E_{r,k}^{\otimes 2})$$

is surjective for $k \geq 2r+1$.

(b) *Under the global generation assumptions formulated above, the multiplication map*

$$\mu_k : H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\mathcal{O}(k\Theta_N)) \longrightarrow H^0(\mathcal{O}(2k\Theta_N))$$

is surjective as long as the multiplication map $H^0(\mathcal{L}^k) \otimes H^0(\mathcal{L}^k) \to H^0(\mathcal{L}^{2k})$ *on* $SU_X(r)$ *is surjective and* $k \geq 2r+1$.

Proof. (a) This is not hard to deal with when k is a multiple of r, since we know from 2.7 that $E_{r,k}$ decomposes in a particularly nice way. To tackle the general case though, we have to appeal to 1.9. Concretely, we have to see precisely when the skew Pontrjagin product

$$E_{r,k} \widehat{*} E_{r,k} \cong E_{r,k} * (-1_J)^* E_{r,k}$$

is globally generated, and since by the initial choice of a theta characteristic $E_{r,k}$ is symmetric, this is the same as the global generation of the usual Pontrjagin product $E_{r,k} * E_{r,k}$. As an aside, recall from 1.9 that this would imply the surjectivity of all the multiplication maps

$$H^0(t_x^* E_{r,k}) \otimes H^0(E_{r,k}) \longrightarrow H^0(t_x^* E_{r,k} \otimes E_{r,k})$$

for all $x \in J(X)$.

The global generation of this Pontrjagin product is in turn another application of the general cohomological criterion 1.7 for vector bundles on abelian varieties. We first prove that $E_{r,k} * E_{r,k}$ also has a nice decomposition when pulled back by

an isogeny, namely this time by multiplication by $2r$. Denote by F the Fourier transform $\widehat{E_{r,k}}$, so that $\widehat{F} \cong (-1_J)^* E_{r,k} \cong E_{r,k}$. Then we have the following isomorphisms:

$$E_{r,k} * E_{r,k} \cong \widehat{F} * \widehat{F} \cong \widehat{F \otimes F},$$

where the second one is obtained by the correspondence between the Potrjagin product and the tensor product via the Fourier-Mukai transform, as in 1.6(2). Next, as in the previous sections, we look at the behavior of our bundle when pulled back via certain isogenies (cf. 1.6(1))

$$k_{J*}(E_{r,k} * E_{r,k}) \cong k_J^* \widehat{F \otimes k_J^* F}. \tag{6}$$

In 3.6 we proved that

$$k_J^* F \cong k_J^* \widehat{E_{r,k}} \cong k_J^* E_{k,r}^* \cong \bigoplus_{s_{k,r}} \mathcal{O}_J(-kr\Theta_N),$$

and by plugging this into (6) we obtain

$$k_{J*}(E_{r,k} * E_{r,k}) \cong (\bigoplus_{s_{k,r}} \mathcal{O}_J(-kr\Theta_N) \otimes \bigoplus_{s_{k,r}} \mathcal{O}_J(-kr\Theta_N))\widehat{\ }$$

$$\cong \bigoplus_{s_{k,r}^2} \widehat{\mathcal{O}_J(-2kr\Theta_N)}.$$

Finally we apply $(2r)_J^* \circ k_J^*$ to both sides of the isomorphism above and use the behavior of the Fourier transform of a line bundle when pulled back via the isogeny that it determines (see 1.6(3)). Since $E_{r,k} * E_{r,k}$ is a direct summand in $k_J^* k_{J*}(E_{r,k} * E_{r,k})$, we obtain the desired decomposition

$$(2r)_J^*(E_{r,k} * E_{r,k}) \cong \bigoplus \mathcal{O}_J(2kr\Theta_N). \tag{7}$$

This allows us to apply a trick analogous to the one used in the proof of 5.2. Namely, (7) implies that

$$(2r)_J^*(E_{r,k} * E_{r,k} \otimes \mathcal{O}_J(-\Theta_N)) \cong \bigoplus \mathcal{O}_J((2kr - 4r^2)\Theta_N).$$

Thus, if we denote by $U_{r,k}$ the vector bundle $E_{r,k} * E_{r,k} \otimes \mathcal{O}_J(-\Theta_N)$, we clearly have

$$h^i(U_{r,k} \otimes \alpha) = 0, \ \forall \alpha \in \mathrm{Pic}^0(X), \ \forall i > 0 \text{ and } \forall k \geq 2r+1.$$

Pareschi's criterion 1.7 immediately gives then that $E_{r,k} * E_{r,k}$ is globally generated for $k \geq 2r+1$, since

$$E_{r,k} * E_{r,k} \cong U_{r,k} \otimes \mathcal{O}_J(\Theta_N).$$

(b) We will show the vanishing of $H^1(M_k \otimes \mathcal{O}(k\Theta_N))$. By the top sequence in the diagram preceding the theorem, it is enough to prove that

$$H^1(K \otimes \mathcal{O}(k\Theta_N)) = 0 \ \text{ and } \ H^1(\det{}^* M_{r,k} \otimes \mathcal{O}(k\Theta_N)) = 0.$$

First we prove the vanishing of $H^1(K \otimes \mathcal{O}(k\Theta_N))$. The key point is to identify the pull-back of K by the étale cover τ in the diagram

$$\begin{array}{ccc} SU_X(r) \times J(X) & \xrightarrow{\tau} & U_X(r,0) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \det} \\ J(X) & \xrightarrow{r_J} & J(X) \end{array}$$

described in section 3. In the pull-back sequence

$$0 \longrightarrow \tau^* K \longrightarrow \tau^* \det{}^* E_{r,k} \longrightarrow \tau^* \mathcal{O}(k\Theta_N) \longrightarrow 0$$

we can identify $\tau^* \det{}^* E_{r,k}$ with $p_2^* r_J{}^* E_{r,k}$ and $\tau^* \mathcal{O}(k\Theta_N)$ with $\mathcal{L}^k \boxtimes \mathcal{O}_J(kr\Theta_N)$. In other words we have the exact sequence

$$0 \longrightarrow \tau^* K \longrightarrow H^0(\mathcal{L}^k) \otimes \mathcal{O}_J(kr\Theta_N) \longrightarrow \mathcal{L}^k \boxtimes \mathcal{O}_J(kr\Theta_N) \longrightarrow 0,$$

which shows that the following isomorphism holds (cf. 5.8):

$$\tau^* K \cong M_{\mathcal{L}^k} \boxtimes \mathcal{O}_J(kr\Theta_N).$$

Finally we obtain the isomorphism

$$\tau^*(K \otimes \mathcal{O}(k\Theta_N)) \cong (M_{\mathcal{L}^k} \otimes \mathcal{L}^k) \boxtimes \mathcal{O}_J(2kr\Theta_N).$$

Certainly by the argument mentioned earlier the surjectivity of the multiplication map $H^0(\mathcal{L}^k) \otimes H^0(\mathcal{L}^k) \to H^0(\mathcal{L}^{2k})$ is also equivalent to $H^1(M_{\mathcal{L}^k} \otimes \mathcal{L}^k) = 0$. The required vanishing is then an easy application of the Künneth formula.

The next step is to prove the vanishing of $H^1(\det{}^* M_{r,k} \otimes \mathcal{O}(k\Theta_N))$. From the projection formula we know that

$$R^i \det{}_*(\det{}^* M_{r,k} \otimes \mathcal{O}(k\Theta_N)) \cong M_{r,k} \otimes R^i \det{}_* \mathcal{O}(k\Theta_N) = 0 \ \text{ for all } i > 0,$$

since obviously $R^i \det{}_* \mathcal{O}(k\Theta_N) = 0$ for all $i > 0$. The Leray spectral sequence reduces then our problem to proving the vanishing $H^1(M_{r,k} \otimes E_{r,k}) = 0$, which is basically equivalent to the surjectivity of the multiplication map

$$H^0(E_{r,k}) \otimes H^0(E_{r,k}) \longrightarrow H^0(E_{r,k}^{\otimes 2}).$$

This is the content of part (a). □

Corollary 5.10. *For k as in 5.9, $\mathcal{O}(k\Theta_N)$ is very ample.*

Proof. Since Θ_N is ample, by a standard argument the assertion is true if the multiplication maps

$$H^0(\mathcal{O}(k\Theta_N) \otimes H^0(\mathcal{O}(kl\Theta_N)) \longrightarrow H^0(\mathcal{O}(k(l+1)\Theta_N))$$

are surjective for $l \geq 1$. For $l = 1$ this is proved in the theorem, and the case $l \geq 2$ is similar but easier. □

Remark 5.11. The case of line bundles in the theorem above (i.e. $r = 1$) is the statement for Jacobians of a well known theorem of Koizumi (see [13] and [26]). Applied to that particular case, the method of proof in Step 2 above is of course implicit in Pareschi's paper [22].

For an effective bound implied by the previous theorem we have to restrict ourselves to the case of rank 2 vector bundles, since to the best of our knowledge nothing is known about multiplication maps on $SU_X(r)$ for $r \geq 3$.

Corollary 5.12. *For a generic curve X the multiplication map*

$$H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\mathcal{O}(k\Theta_N)) \xrightarrow{\mu_k} H^0(\mathcal{O}(2k\Theta_N))$$

on $U_X(2,0)$ is surjective for $k \geq \max\{5, g-2\}$, and so $\mathcal{O}(k\Theta_N)$ is very ample for such k.

Proof. This follows by a theorem of Laszlo [14], which says that on a generic curve the multiplication map

$$S^k H^0(\mathcal{L}^2) \longrightarrow H^0(\mathcal{L}^{2k})$$

on $SU_X(2)$ is surjective for $k \geq g-2$. See also [1] §4 for a survey of results in this direction. □

Remark 5.13. A more refined study along these lines gives analogous results in the extended setting of higher syzygies and N_p properties. We hope to come back to this somewhere else.

We would like to end this section with another application to multiplication maps. Although probably not of the same significance as the previous results, it still brings some new insight through the use of methods characteristic to abelian varieties. Recall from 1.2 that the Picard group of $U_X(r,0)$ is generated by $\mathcal{O}(\Theta_N)$ and the preimages of line bundles on $J(X)$. We want to study "mixed" multiplication maps of the form

$$H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\det{}^*\mathcal{O}_J(m\Theta_N)) \xrightarrow{\alpha} H^0(\mathcal{O}(k\Theta_N) \otimes \det{}^*\mathcal{O}_J(m\Theta_N)). \tag{8}$$

Proposition 5.14. *The multiplication map α in (8) is surjective if $m \geq 2$ and $k \geq 2r+1$.*

Proof. By repeated use of the projection formula, from the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\det{}^*\mathcal{O}_J(m\Theta_N)) & \xrightarrow{\alpha} & H^0(\mathcal{O}(k\Theta_N) \otimes \det{}^*\mathcal{O}_J(m\Theta_N)) \\ \cong \downarrow & & \downarrow \cong \\ H^0(E_{r,k}) \otimes H^0(\mathcal{O}_J(m\Theta_N)) & \xrightarrow{\beta} & H^0(E_{r,k} \otimes \mathcal{O}_J(m\Theta_N)) \end{array}$$

we see that it is enough to prove the surjectivity of the multiplication map β on $J(X)$.

This is an application of the cohomological criterion 1.11 going back to Kempf [12]. What we need to check is that

$$h^i(E_{r,k} \otimes \mathcal{O}_J(l\Theta_N) \otimes \alpha) = 0, \ \forall i > 0, \ \forall l \geq -2 \text{ and } \forall \alpha \in \mathrm{Pic}^0(J(X)).$$

It is again enough to prove this after pulling back by multiplication by r. Now

$$r_J^*(E_{r,k} \otimes \mathcal{O}_J(l\Theta_N) \otimes \alpha) \cong \bigoplus_{s_{r,k}} \mathcal{O}_J((kr + lr^2)\Theta_N) \otimes r_J^*\alpha.$$

Hence the required vanishings are obvious as long as $l \geq -2$ and $k \geq 2r+1$. □

6. Variants for arbitrary degree

As it is natural to expect, some of the facts discussed in the previous sections for moduli spaces of vector bundles of degree 0 can be extended to arbitrary degree. On the other hand, as the reader might have already observed, there are results that do not admit (at least straightforwardly) such extensions. In this last section we would like to emphasize what can and what cannot be generalized using the present techniques.

Fix arbitrary positive integers r and d. Then we can look at the moduli space $U_X(r,d)$ of semistable vector bundles of rank r and degree d. For $A \in \mathrm{Pic}^d(X)$, denote also by $SU_X(r,A)$ the moduli space of rank r bundles with fixed determinant A. We will write $SU_X(r,d)$ when it is not important what determinant is involved. On these moduli spaces one can construct generalized theta divisors as in the degree 0 case. More precisely, denote

$$h = \gcd(r,d),\ r_1 = r/h \text{ and } d_1 = d/h.$$

Then for any vector bundle F of rank r_1 and degree d_1, we can consider Θ_F to be the closure in $U_X(r,d)$ of the locus

$$\Theta_F^s = \{E|\ h^0(E \otimes F) \neq 0\} \subset U_X^s(r,d).$$

This does not always have to be a proper subset, but it is so for generic F (see [10]) and in that case Θ_F is a divisor. We can of course do the same thing with vector bundles of rank kr_1 and degree kd_1 for any $k \geq 1$, and a formula analogous to 1.3 holds. On $SU_X(r,d)$ there are similar divisors Θ_F, and they all determine the same determinant line bundle. As before, this generates $\mathrm{Pic}(SU_X(r,d))$ and is denoted by $\mathcal{L}$.

To study the linear series determined by these divisors we define Verlinde type bundles as in Section 2. They will now depend on two parameters (again not emphasized by the notation). Namely, fix $F \in U_X(r_1, r_1(g-1) - d_1)$ and $L \in \mathrm{Pic}^d(X)$. Consider the composition

$$\pi_L : U_X(r,d) \xrightarrow{\det} \mathrm{Pic}^d(X) \xrightarrow{\otimes L^{-1}} J(X),$$

and define

$$E_{r,d,k}\ (= E_{r,d,k}^{F,L}) := \pi_{L*}\mathcal{O}(k\Theta_F).$$

This is a vector bundle on $J(X)$ of rank $s_{r,d,k} := h^0(SU_X(r,d), \mathcal{L}^k)$. There is again a fiber diagram

$$\begin{array}{ccc} SU_X(r,A) \times J(X) & \xrightarrow{\tau} & U_X(r,d) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi_L} \\ J(X) & \xrightarrow{r_J} & J(X) \end{array}$$

where τ is given by tensor product and the top and bottom maps are Galois with Galois group X_r. By [4] §3 one has the formula

$$\tau^*\mathcal{O}(\Theta_F) \cong \mathcal{L} \boxtimes \mathcal{O}_J(krr_1\Theta_N),$$

where $N \in \mathrm{Pic}^{g-1}(X)$ is a line bundle such that $N^{\otimes r} \cong L \otimes (\det F)^{\otimes h}$. As in 2.3 we obtain the decomposition:

$$r_J^* E_{r,d,k} \cong \bigoplus_{s_{r,d,k}} \mathcal{O}_J(krr_1\Theta_N).$$

The basic duality setup via Fourier-Mukai transform presented in Section 3 can be extended with a little extra care to this general setting. The purpose is to relate linear series on the complementary moduli spaces

$$SU_X(r,d) \quad \text{and} \quad U_X(kr_1, kr_1(g-1) - kd_1)$$

(cf. [4] §5), and this is realized via a diagram of the form

$$\begin{tikzcd}
U_X(r,d) \arrow[dd, "\pi_L"'] & & U_X(kr_1, kr_1(g-1) - kd_1) \arrow[dd, "\pi_M"] \\
& J(X) \times J(X) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
J(X) & & J(X)
\end{tikzcd}$$

where $L \in \mathrm{Pic}^d(X)$, $M \in \mathrm{Pic}^{kr_1(g-1)-kd_1}(X)$, and π_L and π_M are defined as above. One can also choose a vector bundle $G \in U_X(r_1, d_1)$ and consider the Verlinde bundle $E_{kr_1, kr_1(g-1)-kd_1, h}$ associated to G and M. As before, there is an obvious tensor product map

$$U_X(r,d) \times U_X(kr_1, kr_1(g-1) - kd_1) \xrightarrow{\tau} U_X(krr_1, krr_1(g-1)).$$

With the extra (harmless) assumption on our choices that $L \cong (\det G)^{\otimes h}$ and $M \cong (\det F)^{\otimes k}$ we can show exactly as in 3.9 that

$$\tau^* \mathcal{O}(\Theta) \cong \mathcal{O}(k\Theta_F) \boxtimes \mathcal{O}(h\Theta_G) \otimes (\pi_L \times \pi_M)^* \mathcal{P},$$

where $\mathcal{P}$ is a normalized Poincaré line bundle on $J(X) \times J(X)$. Note that this is slightly different from 3.9 in the sense that we are twisting up to slope $g-1$, and on $U_X(krr_1, krr_1(g-1))$, Θ represents the canonical theta divisor. The two formulations are of course equivalent.

The unique nonzero section of $\mathcal{O}(\Theta)$ induces then a nonzero map

$$SD : E_{r,d,k}^* \longrightarrow (E_{kr_1, kr_1(g-1)-kd_1, h})\widehat{\ },$$

and the global formulation of the full strange duality conjecture is:

Conjecture 6.1. *(cf.* [4] *§5) SD is an isomorphism.*

On the positive side, the properties of the kernel of this map described in 3.13 still hold. On the other hand, the method of proof of Theorem 3.11 cannot be used for arbitrary degree even in the level 1 situation. The point is that these new Verlinde type bundles may always fail to be simple. Probably the most suggestive example is the case of r and d coprime, when the other extreme is attained for any k.

Example 6.2. If $\gcd(r,d) = 1$, then $E_{r,d,k}$ decomposes as a direct sum of line bundles for any k. More precisely,

$$E_{r,d,k} = \bigoplus_{s_{r,d,k}} \mathcal{O}(k\Theta_N),\ \forall k \geq 1.$$

This can be seen by imitating the proof of 2.7.

On a more modest note, the main result of [4] can be naturally integrated into these global arguments on $J(X)$. It is obtained by calculus with Fourier transforms in the spirit of Section 3, and we do not repeat the argument here.

Proposition 6.3 ([4], Theorem 1)**.** $h^0(U_X(r,d), \mathcal{O}(k\Theta_F)) = \frac{k^g}{h^g} \cdot s_{r,d,k}$.

Turning to effective global generation and normal generation on $U_X(r,d)$, the picture described in Section 5 completely extends, with the appropriate modifications, to the general case. All the effective bounds turn out to depend on the number $h = \gcd(r,d)$. The global generation result analogous to 5.3 is formulated as follows:

Theorem 6.4. *$\mathcal{O}(k\Theta_F)$ is globally generated on $U_X(r,d)$ as long as $k \geq h+1$ and $\mathcal{L}^k$ is globally generated on $SU_X(r,d)$. Moreover, $\mathcal{O}(k\Theta_F)$ is not globally generated for $k \leq h$.*

In [24] §4 it is proved that $\mathcal{L}^k$ is globally generated on $SU_X(r,d)$ for $k \geq \max\{\frac{(r+1)^2}{4r}h,\ \frac{r^2}{4s}h\}$, where s is an invariant of the moduli space that we will not define here, but satisfying $s \geq h$, so that in particular $k \geq \frac{(r+1)^2}{4}$ always works. This implies, then,

Corollary 6.5. *$\mathcal{O}(k\Theta_F)$ is globally generated on $U_X(r,d)$ for*

$$k \geq \max\{\frac{(r+1)^2}{4r}h, \frac{r^2}{4s}h\}.$$

This again produces optimal results in the case of rank 2 and rank 3 vector bundles (see [24] (5.7)):

Corollary 6.6. *$\mathcal{O}(2\Theta_F)$ is globally generated on $U_X(2,1)$ and $U_X(3,\pm 1)$.*

In the same vein, the normal generation result 5.9 can be generalized to

Theorem 6.7. *The multiplication map*

$$\mu_k : H^0(\mathcal{O}(k\Theta_F)) \otimes H^0(\mathcal{O}(k\Theta_F)) \longrightarrow H^0(\mathcal{O}(2k\Theta_F))$$

on $U_X(r,d)$ is surjective as long as the multiplication map $H^0(\mathcal{L}^k) \otimes H^0(\mathcal{L}^k) \to H^0(\mathcal{L}^{2k})$ on $SU_X(r,d)$ is surjective and $k \geq 2h+1$. For such k, $\mathcal{O}(k\Theta_F)$ is very ample.

Corollary 6.8. *For X generic, μ_k is surjective on $U_X(2,1)$ if $k \geq \max\{3, \frac{g-2}{2}\}$.*

Proof. This is a consequence of 6.7 and [14], where it is proved that on $SU_X(2,1)$

$$S^k H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L}^k)$$

is surjective for $k \geq g-2$. □

It is certainly not hard to formulate further results corresponding to 5.14. We leave this to the interested reader.

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