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NONLINEAR CAUCHY-RIEMANN OPERATORS IN $\mathbb{R}^n$

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ABSTRACT. This paper has arisen from an effort to provide a comprehensive and unifying development of the L^p-theory of quasiconformal mappings in $\mathbb{R}^n$. The governing equations for these mappings form nonlinear differential systems of the first order, analogous in many respects to the Cauchy-Riemann equations in the complex plane. This approach demands that one must work out certain variational integrals involving the Jacobian determinant. Guided by such integrals, we introduce two nonlinear differential operators, denoted by $\mathcal{D}^-$ and $\mathcal{D}^+$, which act on weakly differentiable deformations $f : \Omega \to \mathbb{R}^n$ of a domain $\Omega \subset \mathbb{R}^n$.

Solutions to the so-called Cauchy-Riemann equations $\mathcal{D}^- f = 0$ and $\mathcal{D}^+ f = 0$ are simply conformal deformations preserving and reversing orientation, respectively. These operators, though genuinely nonlinear, possess the important feature of being rank-one convex. Among the many desirable properties, we give the fundamental L^p-estimate

$$\|\mathcal{D}^+ f\|_p \le A_p(n)\|\mathcal{D}^- f\|_p.$$

In quest of the best constant $A_p(n)$, we are faced with fascinating problems regarding quasiconvexity of some related variational functionals. Applications to quasiconformal mappings are indicated.

1. INTRODUCTION

One of the first things we wish to discuss is how one can determine linear conformal transformations of $\mathbb{R}^n$. It is important to realize that such transformations in the plane can be described by a system of linear algebraic equations, while in higher dimensions the defining equations are inevitably nonlinear. In spite of this algebraic point of difference, conformal transformations, regardless of the dimension, can be given a uniform and esthetically pleasing description.

Throughout the entire paper we identify a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ with its $m \times n$ matrix $A = [a_{ij}]$, $1 \le i \le m$, $1 \le j \le n$, relative to the standard bases in $\mathbb{R}^m$ and $\mathbb{R}^n$. We supply the space $\mathbb{R}^{m\times n}$ of all such matrices with the inner product $\langle A|B\rangle = Tr(A^T B)$ and the induced Hilbert-Schmidt norm $\|A\| = \langle A|A\rangle^{\frac{1}{2}}$. Most of the time, however, we make use of the operator norm

$$|A| = \max\{|Ah|;\ |h| = 1\}. \tag{1.1}$$

We reserve the notation $\mathbb{R}^{n\times n}_+$ and $\mathbb{R}^{n\times n}_-$ for the sets of square matrices $A \in \mathbb{R}^{n\times n}$ with $\det A \ge 0$ and $\det A \le 0$, respectively.

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Let $\mathcal{O}(n) = \{A \in \mathbb{R}^{n\times n}; A^T A = I\}$ denote the orthogonal group of $\mathbb{R}^n$. The set of conformal matrices is then defined as

$$\mathcal{CO}(n) = \{\gamma A; A \in \mathcal{O}(n) \text{ and } \gamma \in \mathbb{R}\}. \tag{1.2}$$

It represents two classes of linear conformal mappings; the orientation-preserving and orientation-reversing. Accordingly, we denote

$$\mathcal{CO}_+(n) = \mathcal{CO}(n) \cap \mathbb{R}_+^{n\times n} \text{ and } \mathcal{CO}_-(n) = \mathcal{CO}(n) \cap \mathbb{R}_-^{n\times n}. \tag{1.3}$$

By virtue of Hadamard's inequality, we can characterize these classes by the following equations:

$$\mathcal{CO}_+(n): \qquad |A|^n = \det A \qquad \text{or, equivalently,} \qquad \|A\|^n = n^{\frac{n}{2}} \det A,$$

$$\mathcal{CO}_-(n): \qquad |A|^n = -\det A \qquad \text{or, equivalently,} \qquad \|A\|^n = -n^{\frac{n}{2}} \det A.$$

They indicate we should consider the nonlinear forms $|A|^n \pm \det A$ or $\|A\|^n \pm n^{\frac{n}{2}} \det A$, as a possibility of formulating Cauchy-Riemann equations in $\mathbb{R}^n$. However, for reasons to be discussed in the body of this paper, it will be advantageous to work with forms on $\mathbb{R}^{n\times n}$ whose degree of homogeneity equals exactly half of the dimension. Therefore, we look at the forms given by

$$2|A^+| := |A|^{\frac{n}{2}} + |A|^{-\frac{n}{2}} \det A, \quad 2|A^-| := |A|^{\frac{n}{2}} - |A|^{-\frac{n}{2}} \det A \tag{1.4}$$

One convincing argument for preferring (1.4) to the analogous forms with Hilbert-Schmidt norm comes from the following, rather deep, convexity result.

Theorem 1. *The matrix functions $\mathcal{H}^+, \mathcal{H}^- : \mathbb{R}^{n\times n} \to \mathbb{R}$ given by*

$$\mathcal{H}^\pm(A) = |A^\pm| = \frac{1}{2}(|A|^{\frac{n}{2}} \pm |A|^{-\frac{n}{2}} \det A), \qquad \textit{respectively} \tag{1.5}$$

are rank-one convex. That is to say, the function $t \mapsto \mathcal{H}^\pm(A + tX)$ of one real variable t is convex, whenever rank $X \leq 1$.

See also Theorem 5 for a treatment of slightly more general functions. Needless to say, rank-one convexity is a satisfactory substitute for linearity.

In what follows we shall apply (1.5) to the Jacobian matrices of mappings $f : \Omega \to \mathbb{R}^n$, where Ω is an open subset of $\mathbb{R}^n$. In this way we obtain nonlinear differential operators of the first order which, on account of Theorem 1, have a good chance of being lower semicontinuous.

There is another route to the forms $|A^\pm|$, directly from a study of linear mappings in the complex plane $\mathbb{C} = \{x = x + iy;\ x, y \in \mathbb{R}\}$. If $Az = az + b\bar{z}$, $a, b \in \mathbb{C}$, it is customary to view az and $b\bar{z}$ as the conformal and anticonformal part of A, respectively. Let us record for further use the following formulas:

$$|A| = |a| + |b|, \qquad \det A = |a|^2 - |b|^2, \qquad \|A\|^2 = |a|^2 + |b|^2. \tag{1.6}$$

If $A = [a_{ij}] \in \mathbb{R}^{2\times 2}$, the conformal and anticonformal part are represented by the matrices

$$A^\pm = \frac{1}{2}\begin{bmatrix} a_{11} \pm a_{22} & a_{12} \mp a_{21} \\ a_{21} \mp a_{12} & a_{22} \pm a_{11} \end{bmatrix} \in \mathcal{CO}_\pm(2). \tag{1.7}$$

In this way one obtains an orthogonal decomposition

$$\mathbb{R}^{2\times 2} = \mathcal{CO}_+(2) \oplus \mathcal{CO}_-(2). \tag{1.8}$$

It is worth noting that the classes

$$\mathcal{CO}_+(2) = \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}; \alpha, \beta \in \mathbb{R} \right\} \text{ and } \mathcal{CO}_-(2) = \left\{ \begin{bmatrix} \gamma & \delta \\ \delta & -\gamma \end{bmatrix}; \gamma, \delta \in \mathbb{R} \right\} \tag{1.9}$$

form 2-dimensional linear orthogonal subspaces of $\mathbb{R}^{2\times 2}$. Furthermore, formulas (1.4) actually give the operator norms of A^+ and A^-, respectively. Hence the notation. Similarly in higher dimensions, the expressions $|A^\pm|$ are nothing but the operator norms of the following matrices:

$$A^\pm = \frac{1}{2}(|A|^{\frac{n-2}{2}} A \pm |A|^{\frac{2-n}{2}} A^\sharp). \tag{1.10}$$

We refer to them as $\pm$ components of A. Here and subsequently, $A^\sharp$ denotes the cofactor matrix which is built up from all $(n-1)\times(n-1)$ subdeterminants of A, see (2.1) below. It is then immediate that

$$|A|^{\frac{n}{2}} = |A^+| + |A^-| \qquad \text{and} \qquad \det A = |A^+|^2 - |A^-|^2. \tag{1.11}$$

With the introduction of the $\pm$ components of a matrix, we can now define two nonlinear Cauchy-Riemann operators. They act on weakly differentiable mappings $f : \Omega \to \mathbb{R}^n$, where Ω is an open region in $\mathbb{R}^n$, according to the rules

$$\mathcal{D}^+ f = \frac{1}{2}(|Df|^{\frac{n-2}{2}} Df + |Df|^{\frac{2-n}{2}} D^\sharp f), \tag{1.12}$$

$$\mathcal{D}^- f = \frac{1}{2}(|Df|^{\frac{n-2}{2}} Df - |Df|^{\frac{2-n}{2}} D^\sharp f). \tag{1.13}$$

Here $Df = [\partial f^i/\partial x_j]$ stands for the differential of f and $D^\sharp f$ is the matrix of cofactors of Df, simply called codifferential of f. Under suitable regularity hypotheses on f the Cauchy-Riemann equations

$$\mathcal{D}^+ f = 0 \qquad \text{or} \qquad \mathcal{D}^- f = 0 \tag{1.14}$$

define two types of conformal deformations of Ω, reversing or preserving orientation. According to recent developments in the regularity theory of quasiconformal mappings [I3] [IM1] [IMNS] [M], [IM3] the natural setting for $\mathcal{D}^+$ and $\mathcal{D}^-$ is in the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, with $p \geq \frac{n}{2}$.

The Jacobian determinant $\mathcal{J}(x, f) = \det Df(x)$, being a null Lagrangian, plays a special role in the forthcoming integral inequalities. Let us begin with a consequence of Hadamard's inequality,

$$\left| \int_{\mathbb{R}^n} |Df(x)|^{p-n} \mathcal{J}(x, f) dx \right| \leq \lambda \int_{\mathbb{R}^n} |Df(x)|^p dx \tag{1.15}$$

which obviously holds when $\lambda = 1$. While it is certainly not apparent at this point, it is true that the constant λ can be strictly smaller than 1 for some exponents p. For instance, $\lambda = 0$ if $p = n$, due to obvious cancellations occurring in integration by parts of the Jacobian. Our primary estimate, serving as a starting point to many more integral inequalities, states:

Theorem 2. *There exists a smallest number $p_0(n) \in [\frac{n}{2}, n)$, called the critical point, such that for $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ with $p > p_0(n)$ we have*

$$\left| \int_{\mathbb{R}^n} |Df|^{p-n} \mathcal{J}(x, f) dx \right| \leq \lambda_p(n) \int_{\mathbb{R}^n} |Df|^p dx \tag{1.16}$$

where the least such constant, denoted by $\lambda_p(n)$*, must satisfy*

$$|1 - \frac{n}{p}| \leq \lambda_p(n) < 1. \tag{1.17}$$

To date, we have succeeded in computing the critical point in even dimensions:

Theorem 3. *Let* n *be an even integer. Then (1.16) holds for all* $p > \frac{n}{2}$ *and with some* $\lambda_p(n) < 1$*. Under the above notation this means that*

$$p_0(n) = l \quad \textit{for } n = 2l, \quad l = 1, 2, \ldots. \tag{1.18}$$

It occurs to us that this latter statement may also hold for $n = 3, 5, \ldots$. If so, we would have a powerful base for the L^p-theory of quasiconformal mappings, which still remains far from being conclusive in odd dimensions [I3], [IMNS], [M], [IM3].

L^p-estimates with large exponents for nonlinear partial differential forms can be traced back at least as far as [I2]. It is the case below the dimension that has driven us into truly new studies concerning nonlinear commutators of singular integrals [IS], [I4], [M]. But we shall not enter into these quite involved ideas. Instead, we shall adapt some results from [IL].

A fundamental character of inequalities like (1.16) is attested to by several applications to the theory of partial differential equations, as they supersede singular integrals for nonlinear problems. In this paper inequality (1.16) proves extremely useful in deriving a fundamental estimate for the Cauchy-Riemann operators.

Theorem 4. *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a mapping in the Sobolev class* $W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$*, with* $p > p_0(n)$*. Then*

$$\|\mathcal{D}^+ f\|_{\frac{2p}{n}} \leq \frac{1 + \lambda_p(n)}{1 - \lambda_p(n)} \|\mathcal{D}^- f\|_{\frac{2p}{n}}. \tag{1.19}$$

Thus the differential operators $\mathcal{D}^+$ and $\mathcal{D}^-$ capture all the geometric, algebraic and analytic spirit of the complex Cauchy-Riemann operators.

The category of mappings that one usually encounters in geometric function theory are the so-called quasiregular mappings [R], [R2], [V], [IM3]. We say that $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is K-quasiregular, $1 \leq K < \infty$, if

$$|Df(x)|^n \leq K \mathcal{J}(x, f) \qquad \text{a.e. in } \Omega. \tag{1.20}$$

The smallest number K here is referred to as the outer dilatation of f. In the complex plane the fundamental connection between such mappings and PDEs is via the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu(z) \frac{\partial f}{\partial z}, \qquad \text{where } |\mu(z)| \leq \frac{K-1}{K+1} < 1. \tag{1.21}$$

Here we have taken advantage of the complex Cauchy-Riemann operators

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}). \tag{1.22}$$

In studying the Beltrami equation a singular integral operator was introduced by Beurling, which is now known [AB] as the Beurling-Ahlfors transform

$$(S\omega)(z) = \frac{i}{2\pi} \int\int_{\mathbb{C}} \frac{\omega(\xi) d\xi \wedge d\bar{\xi}}{(z - \xi)^2}. \tag{1.23}$$

The characteristic property of this operator is that it converts the complex partial derivative $f_{\bar z}$ into f_z, whenever $f \in W^{1,p}(\mathbb{C})$ and $1 < p < \infty$. We write it symbolically as

$$\frac{\partial}{\partial z} = S \circ \frac{\partial}{\partial \bar z}. \tag{1.24}$$

One of the major advances in higher dimensional quasiconformal analysis was based on extension of the Beurling-Ahlfors transform to differential forms in $\mathbb{R}^n$ [DS], [IM1], [IM2], [I3], [M]. We will talk about it in considerable detail in Section 10. While it has not been possible to identify the critical point in odd dimensions, the Beurling-Ahlfors transform comes to the rescue in even dimensions as an effective tool in proving Theorem 3.

Note that inequality (1.19) in dimension 2 results in a bound of the norm of $S : L^p(\mathbb{C}) \to L^p(\mathbb{C})$, namely

$$\|S\omega\|_p \le A_p\|\omega\|_p, \quad \text{where } A_p = \frac{1+\lambda_p(2)}{1-\lambda_p(2)} \quad \text{and } 1 < p < 2. \tag{1.25}$$

The relevance of such bounds to elliptic PDEs and complex function theory has been evident to researchers for about fifty years, but no counterparts were available to treat n-dimensional problems. Very recently the Beurling-Ahlfors transform has been a source of new studies for some analysts, and much creative effort has gone into trying to find its p-norms [AIS], [BM-S], [BW], [I1], [IM2], [NV], [PV], [BM-H], [BL]. A long-standing conjecture asserts:

Conjecture 1. The p-norms of the Beurling-Ahlfors transform are given by

$$\|S\|_p = \max\{p-1,\ \frac{1}{p-1}\} = p^* - 1 \tag{1.26}$$

where $p^* = \max\{p, q\}$ and $\frac{1}{p} + \frac{1}{q} = 1$; confront with Conjecture 9.1.

On account of (1.25), see also Section 14, a search for the best constant $\lambda = \lambda_p(n)$ in (1.16) sounds quite appealing. The more general conjecture which we are facing is:

Conjecture 2. For each $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$, we have

$$\left|\int |Df(x)|^{p-n}\mathcal{J}(x,f)dx\right| \le |1-\frac{n}{p}|\int |Df(x)|^p dx. \tag{1.27}$$

Obviously, this estimate has significant meaning only for $p > \frac{n}{2}$. Now the natural direction to go is to examine carefully the integral functionals

$$\mathcal{J}[f] = \mathcal{J}^{\pm}_{\lambda,p}[f] = \int_\Omega \left[\lambda|Df(x)|^p \pm |Df(x)|^{p-n}\mathcal{J}(x,f)\right] dx \tag{1.28}$$

for $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, where $p \ge \frac{n}{2}$ and $|1-\frac{n}{p}| \le \lambda \le 1$. Our main result, which takes the great part of this work, is concerned with rank-one convexity of the integrand in (1.28).

Theorem 5. *The matrix functions $\mathcal{F} : \mathbb{R}^{n\times n} \to \mathbb{R}$, defined by*

$$\mathcal{F}(A) = \mathcal{F}^{\pm}_{\lambda,p}(A) = \lambda|A|^p \pm |A|^{p-n}\det A, \tag{1.29}$$

are rank-one convex for all $\lambda \ge |1-\frac{n}{p}|$ and $p \ge \frac{n}{2}$. Moreover, $|1-\frac{n}{p}|$ is the smallest possible value of λ for which rank-one convexity of $\mathcal{F}$ holds.

In the light of this result it may very well be that the above variational functionals are quasiconvex. That is to say,

$$\mathcal{J}[f] \geq \mathcal{J}[f_0] \tag{1.30}$$

whenever f_0 is a linear transformation and $f - f_0 \in W_0^{1,p}(\Omega, \mathbb{R}^n)$. We put off discussing this question, along with some new perspectives on rank-one convexity, to Section 12, where we extend ourselves to differential forms. In the case of $n = 2$ and $\lambda = |1 - \frac{2}{p}|$ it is rewarding to express $\mathcal{J}[f]$ in terms of the complex derivatives of f. One obtains the integral functional

$$\mathcal{B}_p[f] = \int\int_\Omega [(p^* - 1)|f_{\bar{z}}| - |f_z|]\,[|f_{\bar{z}}| + |f_z|]^{p-1}\,. \tag{1.31}$$

The nonlinear expression

$$B(x, y) = [(p^* - 1)x - y][x + y]^{p-1} \tag{1.32}$$

has already emerged in Burkholder's theory of stochastic integrals and martingale inequalities [B1], [B2]. Hence our new notation. We shall benefit from convexity properties of Burkholder's forms. Related to these forms are some functionals introduced and studied by V. Šverák [S1], [S3]. The interested reader is also referred to an expository article by A. Baernstein and S. Montgomery-Smith [BM-S] for more discussion and results.

Coming to an end, we should point out that there are more results and problems along the lines of this introduction which we discuss in the forthcoming sections. Applications to quasiconformal mappings are given in Section 14. Although our approach has not yet been able to establish sharp L^p-estimates for quasiconformal mappings, it gives us at least new insights gleaned from the results we have obtained. In a way, this is our final goal, which culminates in a precise version of the Caccioppoli estimate.

Theorem 6. *Let $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a K-quasiregular mapping, where the Sobolev exponent and the dilatation are constrained by the condition*

$$\lambda_p(n)K < 1, \qquad p > p_0(n). \tag{1.33}$$

Then, for each test function $\eta \in C_0^\infty(\Omega, \mathbb{R}^n)$,

$$\int_\Omega |\eta|^p |Df|^p \leq C_p(n, K) \int_\Omega |f|^p |\nabla\eta|^p. \tag{1.34}$$

Caccioppoli's inequality, as evidenced by current literature [I3], [IM1], [IMNS], [M], [R], attains the status of a primary tool in multidimensional quasiconformal analysis.

2. Conformal matrices

Consider the determinant function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ and its differential $\det' : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. For $A \in \mathbb{R}^{n\times n}$ the entries of the matrix $\det'(A)$ are cofactors of A. We denote it simply by $A^\sharp$, and recall that

$$A^\sharp_{ij} = [(-1)^{i+j} \det M_{ij}] \tag{2.1}$$

where M_{ij} stands for the $(n-1)\times(n-1)$ submatrix obtained from A by deleting its i^{th} row and j^{th} column. We have two Laplace expansions of the determinant:

$$\text{down the } j^{\text{th}} \text{ column: } \det A = \sum_{i=1}^{n} a_{ij} A^{\sharp}_{ij},$$

$$\text{across the } i^{\text{th}} \text{ row: } \det A = \sum_{j=1}^{n} a_{ij} A^{\sharp}_{ij}.$$

These are particular cases of the component version of the familiar Cramer's equations

$$A^T A^{\sharp} = A^{\sharp} A^T = (\det A) I. \tag{2.2}$$

It is easily seen from these equations that A is conformal if and only if

$$A^{\sharp} = \alpha A \tag{2.3}$$

where the scalar factor α is given by $\pm\alpha = |A|^{n-2} = n^{\frac{2-n}{2}} \|A\|^{n-2}$ for $A \in \mathcal{CO}_{\pm}(n)$, respectively. Hence,

$$\begin{aligned} \mathcal{CO}_+(n) &= \{A; |A|^{n-2}A - A^{\sharp} = 0\} = \{A; \|A\|^{n-2}A - n^{\frac{n-2}{2}} A^{\sharp} = 0\}, \\ \mathcal{CO}_-(n) &= \{A; |A|^{n-2}A + A^{\sharp} = 0\} = \{A; \|A\|^{n-2}A + n^{\frac{n-2}{2}} A^{\sharp} = 0\}. \end{aligned} \tag{2.4}$$

Another route to these equations is by looking at the potential wells (minima) of the matrix functions

$$\Phi_{\pm}(A) := \|A\|^n \pm n^{\frac{n}{2}} \det A.$$

The conformal matrices, being stationary points of $\Phi_{\pm}$, must satisfy the corresponding Lagrange-Euler system

$$\Phi'_{\pm}(A) = n(\|A\|^{n-2}A \pm n^{\frac{n-2}{2}} A^{\sharp}) = 0. \tag{2.5}$$

Similar reasoning applies to the equations

$$|A|^{n-2}A \pm A^{\sharp} = 0. \tag{2.6}$$

For reasons already explained in the Introduction we prefer these latter equations. We divide equations (2.6) by $|A|^{\frac{n-2}{2}}$ to make them homogeneous of degree $\frac{n}{2}$.

Having performed these preliminary steps, we can now introduce the $\pm$ components of an arbitrary matrix $A \in \mathbb{R}^{n\times n}$:

$$A^+ = \frac{1}{2}(|A|^{\frac{n-2}{2}} A + |A|^{\frac{2-n}{2}} A^{\sharp}), \tag{2.7}$$

$$A^- = \frac{1}{2}(|A|^{\frac{n-2}{2}} A - |A|^{\frac{2-n}{2}} A^{\sharp}). \tag{2.8}$$

Another justification for preferring these latter expressions to (2.6) can be traced back to an example in [IM1]. In this paper we have shown that the critical Sobolev exponent for the regularity theory of conformal deformations equals half of the dimension.

Next, we note the commutation rules

$$(OA)^{\pm} = OA^{\pm}, \qquad (AO)^{\pm} = A^{\pm}O, \qquad \text{with } O \in \mathcal{O}_+(n). \tag{2.9}$$

The resemblance with the planar case is clearly visible in the following list of identities:

Lemma 2.1. *For all* $A \in \mathbb{R}^{n\times n}$ *we have*

$$|A|^{\frac{n-2}{2}} A = A^+ + A^-, \tag{2.10}$$

$$|A^+| = \frac{1}{2}(|A|^{\frac{n}{2}} + |A|^{-\frac{n}{2}} \det A), \tag{2.11}$$

$$|A^-| = \frac{1}{2}(|A|^{\frac{n}{2}} - |A|^{-\frac{n}{2}} \det A), \tag{2.12}$$

$$|A|^{\frac{n}{2}} = |A^+| + |A^-|, \tag{2.13}$$

$$\det A = |A^+|^2 - |A^-|^2. \tag{2.14}$$

Proof. We only give the proof of formulas (2.11) and (2.12); the others are quite straightforward. Here we may assume that $A \neq 0$. With the aid of two orthogonal transformations we can diagonalize A as

$$A = P\Lambda Q, \qquad P, Q \in \mathcal{O}_+(n),$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}, \qquad \begin{array}{l} \lambda_n = |A| > 0, \\ \lambda_1 \dots \lambda_n = \det A. \end{array}$$

On account of the commutation rules (2.9) we are reduced to showing that

$$\left| |\Lambda|^{\frac{n-2}{2}} \Lambda \pm |\Lambda|^{\frac{2-n}{2}} \Lambda^\sharp \right| = |\Lambda|^{\frac{n}{2}} \pm |\Lambda|^{-\frac{n}{2}} \det \Lambda. \tag{2.15}$$

Note that $\Lambda^\sharp$ is also diagonal and its ii-entry equals $\lambda_1 \dots \widehat{\lambda_i} \dots \lambda_n$, where $\widehat{\lambda_i}$ means that the factor λ_i is omitted. The task is now to show that the expressions

$$|\lambda_n^{\frac{n-2}{2}} \lambda_i \pm \lambda_n^{\frac{2-n}{2}} \lambda_1 \dots \widehat{\lambda_i} \dots \lambda_n|, \qquad i = 1, 2, \dots, n,$$

assume the largest value for $i = n$. Squaring, we obtain

$$\lambda_n^{n-2} \lambda_i^2 + \lambda_n^{2-n} \lambda_1^2 \dots \widehat{\lambda_i^2} \dots \lambda_n^2 \pm 2\lambda_1 \dots \lambda_n.$$

The lemma will be proved once we show that

$$\lambda_n^{n-2} \lambda_i^2 + \lambda_n^{2-n} \lambda_1^2 \dots \widehat{\lambda_i^2} \dots \lambda_n^2 \leq \lambda_n^n + \lambda_n^{2-n} \lambda_1^2 \dots \lambda_{n-1}^2 \tag{2.16}$$

or, equivalently

$$\lambda_n^{2-n} \lambda_1^2 \dots \widehat{\lambda_i^2} \dots \lambda_{n-1}^2 (\lambda_n^2 - \lambda_i^2) \leq \lambda_n^{n-2} (\lambda_n^2 - \lambda_i^2).$$

This latter inequality is true since $|\lambda_k^2| \leq \lambda_n^2$ for all $k = 1, 2, \dots, n$, finishing the proof. □

For use in Section 14, we introduce the so-called Beltrami's matrix of $A \in \mathbb{R}^{n\times n}$, where we assume that $\det A > 0$:

$$\mu(A) = (A^-)(A^+)^{-1}. \tag{2.17}$$

As the matrix $2A^T A^+ = |A|^{\frac{n-2}{2}} A^T A + |A|^{\frac{2-n}{2}} (\det A) I$ is positive definite, A^+ is invertible.

Lemma 2.2. *For matrices with positive determinant we have*

$$|\mu(A)| = \frac{|A^-|}{|A^+|} < 1. \tag{2.18}$$

Proof. We first diagonalize A and compute the i-th singular value of μ. In the notation of the previous proof this reads as

$$\mu_i = \left| \frac{\lambda_n^{\frac{n-2}{2}} \lambda_i - \lambda_n^{\frac{2-n}{2}} \lambda_1 \dots \widehat{\lambda_i} \dots \lambda_n}{\lambda_n^{\frac{n-2}{2}} \lambda_i + \lambda_n^{\frac{2-n}{2}} \lambda_1 \dots \widehat{\lambda_i} \dots \lambda_n} \right|$$

or, equivalently,

$$|\mu_i|^2 = \frac{\lambda_n^{n-2}\lambda_i + \lambda_n^{2-n}\lambda_1^2 \dots \widehat{\lambda_i^2} \dots \lambda_n^2 - 2\lambda_1 \dots \lambda_n}{\lambda_n^{n-2}\lambda_i + \lambda_n^{2-n}\lambda_1^2 \dots \widehat{\lambda_i^2} \dots \lambda_n^2 + 2\lambda_1 \dots \lambda_n}.$$

Applying (2.16), we see that the largest of all $|\mu_i|$ happens when $i = n$, establishing inequality (2.18). □

3. On Multiple Singular Values

In the following pages we study singular values of the linear transformations $A : \mathbb{R}^n \to \mathbb{R}^m$ with $1 \leq m \leq n$. The positive square roots of the eigenvalues of the matrix $A^T A \in \mathbb{R}^{n\times n}$ are the singular values of A, denoted by $\sigma_i = \sigma_i(A)$, $i = 1, \dots, n$, where $0 \leq \sigma_1 \leq \cdots \leq \sigma_n = |A|$.

Note that the special orthogonal group $SO(n)$ is a connected Lie group of dimension $\binom{n}{2} = \frac{n(n-1)}{2}$, while the full orthogonal group $\mathcal{O}(n)$ has precisely two components: $\mathcal{O}_+(n)$ and $\mathcal{O}_-(n)$ with determinant ± 1, respectively. More generally, for $1 \leq m \leq n$ we say that a linear transformation $A : \mathbb{R}^m \to \mathbb{R}^n$ is orthogonal if $A^T A = I \in \mathbb{R}^{m\times m}$. The set of such matrices, denoted by $\mathcal{O}(n, m)$, is known as Stiefel's manifold. It is a smooth compact manifold of dimension

$$\dim \mathcal{O}(n, m) = \frac{m(2n - m - 1)}{2}. \tag{3.1}$$

In particular, $\mathcal{O}(n, 1) = S^{n-1}$ and $\mathcal{O}(n, n) = \mathcal{O}(n)$ We recall that each square matrix $A \in \mathbb{R}^{n\times n}$ can be diagonalized as

$$A = \mathcal{Q} O \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} O^T \tag{3.2}$$

where $\sigma_1, \dots, \sigma_n$ are the singular values of A and O, $\mathcal{Q} \in \mathcal{O}(n)$. Such a decomposition is not unique when A has multiple singular values. Our aim here is to examine the algebraic subvariety $\mathbb{R}_k^{n\times n} \subset \mathbb{R}^{n\times n}$ which consists of matrices having at least one singular value of multiplicity k, $k = 1, 2, \dots, n$. For the purpose of computing its dimension we can certainly restrict ourselves to the matrices $A \in \mathbb{R}_k^{n\times n}$ such that $0 \leq \sigma_1 \leq \cdots \leq \sigma_{n-k+1} = \cdots = \sigma_n$; the other possible cases are handled in much the same way by permuting the indices. Thus each such matrix can be written as

$$\begin{aligned} A &= \sigma_n \mathcal{Q} + \mathcal{Q} O \begin{bmatrix} \sigma_1 - \sigma_n & \dots & 0 \\ 0 & \sigma_{n-k} - \sigma_n & \\ 0 & & 0 \end{bmatrix} O^T \\ &= \sigma_n \mathcal{Q} + \mathcal{Q} P \begin{pmatrix} \sigma_1 - \sigma_n & & 0 \\ & \ddots & \\ 0 & & \sigma_{n-k} - \sigma_n \end{pmatrix} P^T \end{aligned}$$

where $P \in \mathcal{O}(n, n-k)$ is obtained from O by deleting its last k columns. It then follows that every such matrix $A \in \mathbb{R}_k^{n\times n}$ can be parametrized by:

the numbers $\sigma_1, \dots, \sigma_{n-k}, \sigma_n \in \mathbb{R}_+$,
the orthogonal matrices $\mathcal{Q} \in \mathcal{O}(n)$,
the Stiefel matrices $P \in \mathcal{O}(n, n-k)$.

Hence
(3.3)

$$\dim \mathbb{R}_k^{n\times n} \le n - k + 1 + \frac{n(n-1)}{2} + \frac{(n-k)(k+n-1)}{2} = n^2 + 1 - \frac{k(k+1)}{2}.$$

As a matter of fact, elementary examples reveal that the dimension of $\mathbb{R}_k^{n\times n}$ is precisely equal to

$$\dim \mathbb{R}_k^{n\times n} = n^2 + 1 - \frac{k(k+1)}{2}. \tag{3.4}$$

But we shall only need inequality (3.3). Applying it for $k = 2$, we find that the variety of matrices having at least one double singular value has dimension bounded by

$$\dim \mathbb{R}_2^{n\times n} \le n^2 - 2. \tag{3.5}$$

Note that the space $\mathbb{R}^{n\times n}$ inherits naturally the Lebesgue measure from $\mathbb{R}^{n^2}$. We are now in a position to prove the following:

Proposition 3.1. *Given a nonzero matrix $X \in \mathbb{R}^{n\times n}$, for almost every $A \in \mathbb{R}^{n\times n}$ the straight line*

$$\mathcal{L}_A = \{A + tX; -\infty < t < \infty\} \subset \mathbb{R}^{n\times n}$$

consists of matrices with distinct singular values.

Proof. Let $\mathcal{A} \subset \mathbb{R}^{n\times n}$ denote the set of all matrices $A \in \mathbb{R}^{n\times n}$ for which the line $\mathcal{L}_A$ intersects $\mathbb{R}_2^{n\times n}$. Consider the orthogonal projection

$$\pi : \mathbb{R}^{n\times n} \to \mathcal{X}$$

where $\mathcal{X}$ consists of matrices orthogonal to X. Thus $\mathcal{X}$ is an (n^2-1)-dimensional subspace of $\mathbb{R}^{n\times n}$. Now, if the set $\mathcal{A} \subset \mathbb{R}^{n\times n}$ had positive Lebesgue measure, then its projection $\pi(\mathcal{A}) \subset \mathcal{X}$ would have positive (n^2-1)-dimensional Lebesgue measure on $\mathcal{X}$, by Fubini's theorem. In particular, its Hausdorff dimension would equal (n^2-1). On the other hand, since π is a Lipschitz map and $\pi(\mathcal{A}) \subset \pi(\mathbb{R}_2^{n\times n})$,

$$\dim \pi(\mathcal{A}) \le \dim \pi(\mathbb{R}_2^{n\times n}) \le \dim(\mathbb{R}_2^{n\times n}) \le n^2 - 2$$

This contradiction shows that the set $\mathcal{A} \subset \mathbb{R}^{n\times n}$ has Lebesgue measure zero, as desired. □

4. Second Variation of the Norm Function

Let us begin with a diagonal matrix

$$A = \text{diag}\,[a_1, \dots, a_n] = \begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix}$$

and two arbitrary $n \times n$ matrices $X = [x_{ij}]$, $Y = [y_{ij}]$. We are interested in finding the eigenvalues of the matrix $A + tX + t^2Y$, at least for a small parameter $t \in \mathbb{R}$. In order to get an asymptotic result for the k-th eigenvalue, it is necessary to assume that a_k differs from the other diagonal entries. With this assumption we can use the Implicit Function Theorem to solve the algebraic equation

$$\det(A + tX + t^2Y - \lambda I) = 0$$

for $\lambda = \lambda(t)$ to be an analytic function near $t = 0$, such that $\lambda(0) = a_k$.

Lemma 4.1. *The Maclaurin expansion of $\lambda_k = \lambda(t)$ takes the form*

$$\lambda_k(t) = a_k + tx_{kk} + t^2(y_{kk} + \sum_{i\neq k} \frac{x_{ik}x_{ki}}{a_k - a_i}) + \text{ higher powers of } t. \tag{4.1}$$

Proof. We begin by identifying the lowest three terms of the polynomial

$$t \mapsto \det(\Lambda + tX + t^2Y),$$

with arbitrary diagonal matrix $\Lambda = \operatorname{diag}[\lambda_1, \dots, \lambda_n]$. It is a simple matter to see that

$$\begin{aligned} \det(\Lambda + tX + t^2Y) &= \lambda_1\lambda_2\dots\lambda_n + t\sum_{i=1}^{n} \lambda_1 \dots \widehat{\lambda_i} \dots \lambda_n x_{ii} \\ &+ t^2\Big[\sum_{i=1}^{n} \lambda_1 \dots \widehat{\lambda_i} \dots \lambda_n y_{ii} \\ &+ \sum_{1\le i\le j\le n} \lambda_1 \dots \widehat{\lambda_i} \dots \widehat{\lambda_j} \dots \lambda_n (x_{ii}x_{jj} - x_{ij}x_{ji})\Big] + \dots . \end{aligned} \tag{4.2}$$

Throughout these pages the notation $\widehat{\lambda_i}$ means that the factor λ_i is omitted in the product. We shall make use of formula (4.2) only when one of the λ's is equal to zero, say $\lambda_k = 0$. In this case we have the slightly simpler formula

$$\begin{aligned} \det(\Lambda + tX + t^2Y) &= t\lambda_1 \dots \widehat{\lambda_k} \dots \lambda_n x_{kk} + t^2\big[\lambda_1 \dots \widehat{\lambda_k} \dots \lambda_n y_{kk} \\ &+ \sum_{i\neq k} \lambda_1 \dots \widehat{\lambda_i} \dots \widehat{\lambda_k} \dots \lambda_n (x_{ii}x_{kk} - x_{ik}x_{ki})\big] + \dots . \end{aligned}$$

We can now derive formula (4.1). To this effect, let us write $\lambda_k(t) = a_k + t\alpha_k + t^2\beta_k + \dots$. According to this latter expansion the characteristic equation for $\lambda_k(t)$ takes the form

$$\begin{aligned} &\det\big[A - a_kI + t(X - \alpha_kI) + t^2(Y - \beta_kI)\big] \\ &\quad = t(a_1 - a_k)\dots(\widehat{a_k - a_k})\dots(a_n - a_k)(x_{kk} - \alpha_k) \\ &\quad + t^2\big\{(a_1 - a_k)\dots(\widehat{a_k - a_k})\dots(a_n - a_k)(y_{kk} - \beta_k) \\ &\quad + \sum_{i\neq k}(a_1 - a_k)\dots(\widehat{a_i - a_k})\dots(\widehat{a_k - a_k})\dots(a_n - a_k) \\ &\quad \times [(x_{ii} - \alpha_k)(x_{kk} - \alpha_k) - x_{ik}x_{ki}]\big\} + \dots = 0. \end{aligned}$$

First we find that $\alpha_k = x_{kk}$, and then it is immediate that

$$\beta_k = y_{kk} + \sum_{i\neq k} \frac{x_{ik}x_{ki}}{a_k - a_i}$$

as desired. □

We are now able to compute the second order Maclaurin polynomial for the norm function.

Proposition 4.1. *Suppose* $A = \mathrm{diag}\,[a_1, \ldots, a_n]$, *where* $|a_1| \le \cdots \le |a_{n-1}| < a_n$. *Then for each* $X \in \mathbb{R}^{n\times n}$ *we have*

$$|A + tX| = a_n + tx_{nn} + \frac{1}{2}t^2 \sum_{i=1}^{n-1} \frac{a_n(x_{in}^2 + x_{ni}^2) + 2a_i x_{in} x_{ni}}{a_n^2 - a_i^2} + \ldots. \tag{4.3}$$

Proof. The square of the norm is simply the largest eigenvalue of the matrix

$$\begin{aligned}(A + tX)^T(A + tX) &= A^T A + t(X^T A + A^T X) + t^2 X^T X \\ &= \mathrm{diag}\,[a_1^2, \ldots, a_n^2] + t[a_i x_{ij} + a_j x_{ji}] + t^2 \left[\sum_{l=1}^{n} x_{li} x_{lj}\right].\end{aligned}$$

By virtue of Lemma 4.1 we find that

$$|A + tX|^2 = a_n^2 + 2ta_n x_{nn} + t^2 \left[\sum_{i=1}^{n} x_{in}^2 + \sum_{i=1}^{n-1} \frac{(a_i x_{in} + a_n x_{ni})^2}{a_n^2 - a_i^2}\right] + \ldots. \tag{4.4}$$

Taking the square root yields (4.3). □

5. Some Rank-One Convex Functionals

The main purpose of this section is to prove Theorem 5. We begin by analyzing the borderline case of $\lambda = |1 - \frac{n}{p}|$. Therefore, we first look at the matrix function $\mathcal{F} = \mathcal{F}_p^{\pm} : \mathbb{R}^{n\times n} \to \mathbb{R}$, defined by

$$\mathcal{F}(A) = \mathcal{F}_p^{\pm}(A) = \left|1 - \frac{n}{p}\right| |A|^p \pm |A|^{p-n} \det A \tag{5.1}$$

where the exponent p ranges over the interval $\left[\frac{n}{2}, \infty\right)$. Here and in the sequel, when no confusion can arise, we abbreviate this notation by dropping both the subscript p and the $\pm$ sign.

Proposition 5.1. *The functions* $\mathcal{F}_p^{\pm}$ *are rank-one convex, whenever* $p \ge \frac{n}{2}$. *Furthermore, the factor* $|1 - \frac{n}{p}|$ *is the smallest possible for this statement to hold.*

Proof. First, and throughout this proof, we fix a rank-one matrix $X \in \mathbb{R}^{n\times n}$. Our task is to show that for each $A \in \mathbb{R}^{n\times n}$ the function $\varphi(t) = \mathcal{F}(A + tX)$ is convex on the real line $\mathbb{R}$. As a matter of fact we only need to check this property for matrices A in some dense subset of $\mathbb{R}^{n\times n}$, because convexity is preserved after passing to the limit.

We choose A's to have the property that the straight line $\{A + tX, -\infty < t < \infty\}$ consists of matrices with distinct singular values. By virtue of Proposition 3.1 such matrices form a dense subset of $\mathbb{R}^{n\times n}$. With this assumption φ becomes an analytic function on the real line. Our claim is that $\varphi''(t) \ge 0$ for all $t \in \mathbb{R}$. It suffices to prove it for $t = 0$; the general case will follow from this particular one via the identity $\varphi''(t_0) = \frac{d^2}{ds^2}\mathcal{F}(A_0 + sX)_{|s=0} \ge 0$, where $A_0 = A + t_0 X$ and $s = t - t_0$. For computational simplicity, we also wish to arrange that A be a diagonal matrix. Well, one can always diagonalize A with the aid of two orthogonal transformations. Accordingly, we can write $A = PA'Q$, where $P, Q \in \mathcal{O}_+(n)$ and $A' = \mathrm{diag}\,[a_1, \ldots, a_n]$. The numbers $0 \le |a_1| < |a_2| < \cdots < |a_{n-1}| < a_n$ are just singular values of A. In order to justify the reduction to the diagonal case we make the following observations:

For arbitrary $P, Q \in \mathcal{O}_+(n)$ and $Y \in \mathbb{R}^{n\times n}$ we have the identities

$$\mathcal{F}^{\pm}(PY) = \mathcal{F}^{\pm}(Y) = \mathcal{F}^{\pm}(YQ).$$

Here the $\pm$ signs in the left and right hand side correspond to those standing in the middle.

The singular values of $A' + tX' = P^T(A+tX)Q^T$ are the same as those of $A+tX$, and the new matrix $X' = P^T X Q^T$ here has still rank 1. In other words, the line through A' in the direction of X' consists of matrices with distinct singular values.

Summarizing, it involves no loss of generality in assuming that A is diagonal. In addition to A being diagonal, we may also normalize A by requiring that $|A| = 1$, as the function $\mathcal{F}$ is homogeneous. More specifically,

$$A = \operatorname{diag}\,[a_1, \dots, a_{n-1}, 1] \quad \text{where } 0 \le |a_1| \le \cdots \le |a_{n-1}| < 1. \tag{5.2}$$

Concerning the rank-one matrix $X \in \mathbb{R}^{n\times n}$, we can write it as the tensor product of two vectors

$$X = [x_{ij}] = [p_i q_j], \qquad i, j = 1, \dots, n, \tag{5.3}$$

where $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ are some vectors in $\mathbb{R}^n$.

Having disposed of these preliminary reductions, we can now proceed to show that

$$\varphi''(0) \ge 0 \tag{5.4}$$

where $\varphi(t) = |1 - \frac{n}{p}||A + tX|^p \pm |A+tX|^{p-n}\det(A+tX)$ and the matrices A and X are specified by (5.2) and (5.3).

For simplicity of notation, let $N(t) = |A + tX|$ and $D(t) = \det(A + tX)$. Hence φ takes the form

$$\varphi(t) = \left|1 - \frac{n}{p}\right| N^p(t) \pm N^{p-n}(t) D(t). \tag{5.5}$$

By Proposition 4.1 we find at once that $N(0) = 1$, $N'(0) = p_n q_n$ and

$$N''(0) = \sum_{i=1}^{n-1} \frac{p_n^2 q_i^2 + p_i^2 q_n^2 + 2a_i p_i q_i p_n q_n}{1 - a_i^2} \tag{5.6}$$

Analogous formulas for the determinant function can be obtained from (4.2) when applied to $Y = 0$ and $X = [p_i q_i]$. Accordingly, $D(0) = a_1 \dots a_{n-1}$, $D'(0) = a_1 \dots a_{n-1} p_n q_n + \sum_{i=1}^{n-1} a_1 \dots \widehat{a_i} \dots a_{n-1} p_i q_i$, and $D''(0) = 0$. We are now able to compute the second order derivatives at zero of the two terms in formula (5.5):

$$\begin{aligned} \left|1 - \frac{n}{p}\right| (N^p)'' &= |p - n| \big[(p-1)(N')^2 + N''\big] \\ &= |p-n| \big[(p-1) p_n^2 q_n^2 + N''\big], \end{aligned} \tag{5.7}$$

$$\begin{aligned} (N^{p-n} D)'' &= (p-n)\big[2D'N' + (p-n-1)(N')^2 D + DN''\big] \\ &= (p-n)\big[(p-n+1) a_1 \dots a_{n-1} p_n^2 q_n^2 \\ &\quad + a_1 \dots a_{n-1} N'' + 2 p_n q_n \sum_{i=1}^{n-1} a_1 \dots \widehat{a_i} \dots a_{n-1} p_i q_i\big]. \end{aligned} \tag{5.8}$$

We shall have established (5.4) if we prove that

$$|(N^{p-n}D)''| \leq |1 - \frac{n}{p}|(N^p)''. \tag{5.9}$$

The proof falls naturally into two parts. The first is quite straightforward:

$$\left|(p-n+1)a_1 \dots a_{n-1} p_n^2 q_n^2\right| \leq (p-1)p_n^2 q_n^2$$

in view of $p \geq \frac{n}{2}$ and $|a_1 \dots a_{n-1}| < 1$. The second is

$$\left|a_1 \dots a_{n-1} N'' + 2p_n q_n \sum_{i=1}^{n-1} a_1 \dots \widehat{a_i} \dots a_{n-1} p_i q_i\right| \leq N''. \tag{5.10}$$

Taking into account formula (5.6), we are reduced to showing that

$$\left|a_1 \dots a_{n-1} \frac{p_n^2 q_i^2 + p_i^2 q_n^2 + 2a_i p_i q_i p_n q_n}{1 - a_i^2} + 2a_1 \dots \widehat{a_i} \dots a_{n-1} p_i q_i p_n q_n\right| \leq \frac{p_n^2 q_i^2 + p_i^2 q_n^2 + 2a_i p_i q_i p_n q_n}{1 - a_i^2}.$$

We want this to hold for all $i = 1, 2, \dots, n-1$. Observe the common factor $|a_1 \dots \widehat{a_i} \dots a_{n-1}|$ in the left hand side. Since this factor is less than 1, we are left with the task of verifying the following inequality:

$$\left|a_i(p_n^2 q_i^2 + p_i^2 q_n^2) + 2p_i q_i p_n q_n\right| \leq p_n^2 q_i^2 + p_i^2 q_n^2 + 2a_i p_i q_i p_n q_n.$$

After squaring both sides, this inequality becomes evident:

$$(1 - a_i^2)(p_n^2 q_i^2 - p_i^2 q_n^2)^2 \geq 0$$

and (5.4) follows. Therefore, the function $t \longmapsto \mathcal{F}_p^{\pm}(A + tX)$ is convex as claimed.

The proof above actually shows that the factor $|1 - \frac{n}{p}|$ in the definition of $\mathcal{F}_p^{\pm}$ cannot be replaced by any smaller one. To see this, we must generalize the function φ at (5.5) by introducing a new parameter:

$$\varphi_\lambda(t) = \lambda|A + tX|^p \pm |A + tX|^{p-n} \det(A + tX)$$

with some positive λ in place of $|1 - \frac{n}{p}|$. With the notation of our previous proof we now look more closely at $\varphi_\lambda''(0)$. By letting $q_1 = \dots = q_{n-1} = 0$, $q_n = 1$ and $p_1 = 1$, $p_2 = \dots = p_n = 0$ we obtain

$$N(0) = 1, \qquad N'(0) = 0, \qquad N''(0) = \frac{1}{1 - a_1^2},$$
$$D(0) = a_1 \dots a_{n-1}, \qquad D'(0) = 0, \qquad D''(0) = 0.$$

In this particular case, the formula for $\varphi_\lambda''(0)$ is remarkably simple:

$$\varphi_\lambda''(0) = [p\lambda \pm (p-n)a_1 \dots a_{n-1}]N''(0). \tag{5.11}$$

Here we point out that $a_1, \dots, a_{n-1}$ can be arbitrary real numbers ranging in the open interval $(-1, 1)$. On the other hand, convexity of φ_λ demands that $\varphi_\lambda''(0) \geq 0$, hence $\lambda \geq |1 - \frac{n}{p}|$. This completes the proof of Proposition 5.1. □

Now Theorem 5 is straightforward.

Proof of Theorem 5. For $\lambda \geq |1-\frac{n}{p}|$ one can decompose

$$\mathcal{F}^{\pm}_{\lambda,p}(A) = \mathcal{F}^{\pm}_{p}(A) + \left(\lambda - \left|1-\frac{n}{p}\right|\right)|A|^p.$$

The first term is rank-one convex, while the second one is even convex. This implies rank-one convexity of $\mathcal{F}^{\pm}_{\lambda,p}$. □

Remark 5.1. It is of interest to know whether the so-called conformal stored energy integrands

$$2\mathcal{H}^{\pm}_p(A) = |A|^p \pm |A|^{p-n}\det A \tag{5.12}$$

are rank-one convex. This holds exactly for $p \geq \frac{n}{2}$. The affirmative answer is contained in Theorem 5 applied to $\lambda = 1$. That the rank-one convexity of $\mathcal{H}^{\pm}_p$ fails for $p < \frac{n}{2}$ can be seen from the example just described, again by letting $\lambda = 1$ in (5.11). One question still unanswered is whether the conformal stored energies are polyconvex. With the aid of the conformal and anticonformal part of a matrix $A \in \mathbb{R}^{n\times n}$, as defined by (2.7) and (2.8), we can construct more rank-one convex functions. For instance,

$$\mathcal{W}^{\pm}_{\alpha,\beta}(A) = |A^{\pm}|^{\alpha}|A|^{\beta} = [\mathcal{H}^{\pm}_p(A)]^{\alpha} \tag{5.13}$$

with $\alpha \geq 1$, $\beta \geq 0$ and $p = \frac{n}{2} + \frac{\beta}{\alpha}$. Since $\mathcal{H}^{\pm}_p$ is rank-one convex, so is the function $\mathcal{W}^{\pm}_{\alpha,\beta}$.

Another interesting inference from Proposition 5.1 arises by passing to the limit as $p \to n^+$. Note that

$$\lim_{p\to n^+} \frac{n}{p-n}[\mathcal{F}^{\pm}_p(A) \mp \det A] = |A|^n \pm (\det A)\log|A|^n.$$

Corollary 5.1. *The functions $\mathcal{L}^{\pm} : \mathbb{R}^{n\times n} \to \mathbb{R}$, defined by*

$$\mathcal{L}^{\pm}(A) = |A|^n \pm (\det A)\log|A|^n \tag{5.14}$$

are rank-one convex.

6. The Key Inequality for Jacobians

In this section we prove Theorem 2. Variational integrals whose absolute minima are conformal deformations have already been studied in [IL], [MSY], [Y1], [Y2] and [YZ]. Of particular relevance to us will be the functional

$$I_{p,n}[f] = \int_{\mathbb{R}^n} \|Df(x)\|^{p-n}\left[\|Df(x)\|^n - n^{\frac{n}{2}}J(x,f)\right]dx. \tag{6.1}$$

The heart of the matter is that the integrand here is neither convex nor coercive. However, after integration, we obtain a functional which is coercive. This simply means that there is $\delta < 1$ such that

$$I_{p,n}[f] \geq (1-\delta)\int_{\mathbb{R}^n} \|Df(x)\|^p\,dx \tag{6.2}$$

for all mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ in the Sobolev space $W^{1,p}(\mathbb{R}^n,\mathbb{R}^n)$. This estimate is worthless if $\delta \geq 1$. In [IL], see Corollary 9.1, we proved that $0 \leq \delta = \delta(n,p) < 1$ for all $p \geq n-\epsilon$. Unfortunately, the arguments therein give no information about how large the number $\epsilon = \epsilon(n) \in (0,\frac{n}{2})$ can be. That is why we will not be able to determine explicitly the critical point $p_0(n)$ for inequality (1.16), though we will succeed in proving that $p_0(n) < n$.

Proof of Theorem 2. Using the above notation, we claim that inequality (1.16) holds with the constant $1-(1-\delta)n^{-\frac{p}{2}}<1$ in place of $\lambda_p(n)$, for all $p \geq n-\epsilon$. To this end, we introduce the temporary notation

$$I(t) = t^{\frac{p}{n}-1}[t - n^{\frac{n}{2}}J(x,f)]$$

for the integrand of $I_{p,n}$, where $t = \|Df(x)\|^n$. In view of Hadamard's inequality we only need to examine $I(t)$ for $t \geq n^{\frac{n}{2}}J(x,f)$. In this range, we easily conclude that $I(t)$ is increasing. Next, we recall an inequality between the Hilbert-Schmidt norm and the operator norm. It asserts that $\|Df(x)\| \leq \sqrt{n}\,|Df(x)|$. From monotonicity of $I(t)$ it follows that

$$\begin{aligned} n^{\frac{p}{2}}\int |Df|^{p-n}\big[|Df|^n - J(x,f)\big] &= \int I(n^{\frac{n}{2}}|Df|) \geq \int I(\|Df\|^n) \\ &= I_{p,n}[f] \geq (1-\delta)\int \|Df\|^p \geq (1-\delta)\int |Df|^p. \end{aligned}$$

Hence, we see at once that

$$\int_{\mathbb{R}^n} |Df(x)|^{p-n}J(x,f)\,dx \leq \lambda \int_{\mathbb{R}^n} |Df(x)|^p\,dx \tag{6.3}$$

with $\lambda = 1-(1-\delta)n^{-\frac{p}{2}} < 1$, as claimed. The only point remaining concerns the sign of the integral in the left hand side. We can change this sign, if necessary, by applying inequality (6.3) to the so-called conjugate mapping

$$\bar{f} = (f^1, f^2, \ldots, f^{n-1}, -f^n). \tag{6.4}$$

Notice that the norm of $D\bar{f}(x)$ is the same as that of f, while $J(x,\bar{f}) = -J(x,f)$. This makes it legitimate to take the absolute value in the left hand side of (6.3), establishing Theorem 2. □

Let us remark finally that the latter trick of reversing orientation via conjugate mapping will prove useful many times in the sequel.

We close this section with some possible conclusions and additional comments about Conjecture 2. By analogy to Corollary 5.1 it is tempting to divide both sides of (1.27) by $|p-n|$ and then pass to the limit as p approaches n. This procedure is perfectly valid at least for mappings f of class $C_0^\infty(\Omega,\mathbb{R}^n)$, which involves evaluation of the limit in the left hand side by using L'Hôpital's Rule. To see this we point out that the integral of the Jacobian vanishes. We then conjecture the rather surprising inequality

$$\left|\int_\Omega J(x,f)\log|Df(x)|^n\,dx\right| \leq \int_\Omega |Df(x)|^n\,dx. \tag{6.5}$$

Although the integral in the left hand side may not converge for arbitrary f in $W_0^{1,n}(\Omega,\mathbb{R}^n)$, we were able to give meaning to this integral. One alternative for defining the left hand side of (6.5) with $f \in W_0^{1,n}(\Omega,\mathbb{R}^n)$ is

$$\int_\Omega J(x,f)\log|Df(x)|^n\,dx = \frac{-id}{dt}\int_\Omega |Df|^{int}J(x,f)\,dx \tag{6.6}$$

where the derivative is computed at $t=0$. For yet other methods we refer the reader to [IV]. Inequality (6.5) with an additional constant factor in the right hand side was already known in [GI]. That this constant cannot be smaller than 1 becomes clear from the next section.

7. A Quest for Extremals

Conjecture 2 was inspired by the observation that the matrix functions

$$\mathcal{F}_p^{\pm}(A) = \left|1 - \frac{n}{p}\right| |A|^p \pm |A|^{p-n} \det A \tag{7.1}$$

are rank-one convex for all $p \geq \frac{n}{2}$. Questing for further evidence takes us to a study of the variational functionals

$$\mathcal{J}_p^{\pm}[f] = \int_\Omega \left[\left|1 - \frac{n}{p}\right| |Df(x)|^p \pm |Df(x)|^{p-n} J(x,f)\right] dx \tag{7.2}$$

defined on mappings $f = (f^1, \ldots, f^n) : \Omega \to \mathbb{R}^n$ in the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$. It is our belief that $\mathcal{J}_p^{\pm}$ are quasiconvex, which simply means that

$$\mathcal{J}_p^{\pm}[f] \geq \mathcal{J}_p^{\pm}[f_0] \tag{7.3}$$

whenever f_0 is an affine mapping and $f - f_0 \in W_0^{1,p}(\Omega, \mathbb{R}^n)$. Inequality (1.27) is none other than the case $f_0 = 0$. The aim of this section is to demonstrate that equality in (7.3) may actually occur for a considerable number of mappings. All examples we are given here turn out to solve the Euler-Lagrange equations for $\mathcal{J}_p^{\pm}$, reinforcing Conjecture 2. Similar examples in dimension 2 have been obtained independently by A. Baernstein and S. Montgomery-Smith [BM-S]. Observe that on the supposition of (7.3) we would also have $\mathcal{J}_{\lambda,p}^{\pm}[f] \geq \mathcal{J}_{\lambda,p}^{\pm}[f_0]$, see (1.28) for the definition of $\mathcal{J}_{\lambda,p}^{\pm}$. However, if λ is strictly greater than the critical number $|1 - \frac{n}{p}|$, equality would occur only for $f = f_0$.

We begin with radially symmetric mappings

Proposition 7.1. *Let ρ be an arbitrary nonnegative and absolutely continuous function on each interval $[a,b]$, $0 < a < b < \infty$, and suppose that*

$$\rho'(t)[2\rho(t) + t\rho'(t)] \leq 0 \qquad a.e., \tag{7.4}$$

$$\int_0^\infty \rho^p(t) t^{n-1}\, dt < \infty, \qquad \text{where } p \geq \frac{n}{2}. \tag{7.5}$$

Consider the radial mapping $f(x) = x\rho(|x|)$. Then both f and its conjugate $\bar{f} = (f^1, \ldots, f^{n-1}, -f^n)$ belong to $W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$ and satisfy the equations

$$\mathcal{J}_p^{-}[f] = \mathcal{J}_p^{+}[\bar{f}] = 0 \qquad \text{if } p \geq n, \tag{7.6}$$

$$\mathcal{J}_p^{+}[f] = \mathcal{J}_p^{-}[\bar{f}] = 0 \qquad \text{if } \frac{n}{2} \leq p \leq n. \tag{7.7}$$

Let us point out that condition (7.4) holds if $\rho(t)$ decreases slowly enough so that $t^2\rho(t)$ is increasing.

Proof. We first compute the differential and the Jacobian determinant of f:

$$Df(x) = \rho(|x|)I + \frac{x \otimes x}{|x|}\rho'(|x|), \tag{7.8}$$

$$J(x,f) = \rho^n(|x|) + |x|\rho'(|x|)\rho^{n-1}(|x|) = -J(x,\bar{f}). \tag{7.9}$$

In order to compute the operator norm of $Df(x)$ we look at the expression

$$|Df(x)h|^2 = \rho^2|h|^2 + \rho'(2\rho + |x|\rho')\frac{\langle x,h\rangle^2}{|x|}$$

where h is an arbitrary unit vector in $\mathbb{R}^n$. By (7.4) we see that the second term is nonpositive and the supremum is attained when h is orthogonal to x. Hence

$$|Df(x)| = \rho(|x|) = |D\bar{f}(x)|. \tag{7.10}$$

We can now proceed to examine the integrands $\mathcal{F}_p^{\pm}$. For $p \geq n$, we have

$$\begin{aligned}\mathcal{F}_p^-(Df) = \mathcal{F}_p^+(D\bar{f}) &= (1-\frac{n}{p})\rho^p - \rho^{p-n}(\rho^n + |x|\rho'\rho^{n-1})\\ &= -(\frac{n}{p}\rho^p + |x|\rho'\rho^{n-1}).\end{aligned}$$

Similar considerations apply to the case $\frac{n}{2} \leq p \leq n$ and yield

$$\mathcal{F}_p^+(Df) = \mathcal{F}_p^-(D\bar{f}) = \frac{n}{p}\rho^p + |x|\rho'\rho^{p-1}.$$

It remains to compute the integral of $\mathcal{F}_p^{\pm}(Df)$. By using polar coordinates we find that

$$\begin{aligned}\int_{\mathbb{R}^n}&\left[n\rho^p(|x|) + p|x|\rho'(|x|)\rho^{n-1}(|x|)\right]dx\\ &= \omega_{n-1}\int_0^\infty [nt^{n-1}\rho^p(t) + pt^n\rho'(t)\rho^{p-1}(t)]\,dt\\ &= \omega_{n-1}\int_0^\infty [t^n\rho^p(t)]'\,dt = 0.\end{aligned}$$

To justify this latter step one has to take into account condition (7.5), the detailed verification being left to the reader. □

Remark 7.1. The integrands $\mathcal{F} = \mathcal{F}_p^{\pm} : \mathbb{R}^{n\times n} \to \mathbb{R}$ are C^∞-smooth only on a dense open subset of $\mathbb{R}^{n\times n}$. This is exactly the set of matrices $A \in \mathbb{R}^{n\times n}$ for which $|A|^2$ is a simple eigenvalue of A^TA, see Section 3. Rank-one convexity at such a point $A \in \mathbb{R}^{n\times n}$ is equivalent to the Legendre-Hadamard condition

$$\sum_{1\leq i,j,k,l\leq n} \frac{\partial^2\mathcal{F}(A)}{\partial a_{ij}\partial a_{kl}}\xi^i\xi^k\varsigma^j\varsigma^l \geq 0 \tag{7.11}$$

for $\xi, \varsigma \in \mathbb{R}^n$. This also means that the Euler-Lagrange system

$$\operatorname{div}\,\mathcal{F}'(Df) = \sum_{j=1}^n \frac{\partial}{\partial x_j}\left[\frac{\partial\mathcal{F}(Df)}{\partial f_j^i}\right] = \sum_{1\leq j,k,l\leq n} \frac{\partial^2\mathcal{F}(Df)}{\partial f_j^i\partial f_l^k}\frac{\partial^2 f^k}{\partial x_j\partial x_l} = 0 \tag{7.12}$$

for $i = 1, 2, \ldots, n$, is elliptic, consult for instance L. Nirenberg [N]. Although we do not pursue the matter here, the mappings in Proposition 7.1 are among the stationary solutions to this system.

Examples showing that equality may occur in (7.3) can now be constructed by modifying the radial mappings in Proposition 7.1. With the aid of the same function $\rho = \rho(|x|)$ we define a mapping on the ball $\mathbb{B} = \{x;\ |x-a| \leq r\}$ by

$$f(x) = \frac{\rho(|x-a|)}{\rho(r)}(x-a) + a. \tag{7.13}$$

We see at once that $f \in W^{1,p}(\mathbb{B}, \mathbb{R}^n)$ and $f(x) = f_0(x) = x$ on $\partial\mathbb{B}$. Moreover,

$$\begin{cases}\mathcal{J}_p^-[f] = \mathcal{J}_p^-[f_0] & \text{if } p \geq n,\\ \mathcal{J}_p^+[f] = \mathcal{J}_p^+[f_0] & \text{if } \frac{n}{2} \leq p \leq n.\end{cases} \tag{7.14}$$

The dual identities are also available for the conjugate mapping:

$$\begin{cases} \mathcal{J}_p^+[\bar{f}] = \mathcal{J}_p^+[\bar{f}_0] & \text{if } p \geq n, \\ \mathcal{J}_p^-[\bar{f}] = \mathcal{J}_p^-[\bar{f}_0] & \text{if } \frac{n}{2} \leq p \leq n. \end{cases} \tag{7.15}$$

As before, these are also stationary solutions of the Euler-Lagrange equations. More examples can be made up of a finite, or even infinite, number of radial solutions. Let $\mathbb{B}_j = \{x; |x - a_j| < r_j\} \subset \Omega$, $j = 1, 2, \dots$, be mutually disjoint balls, where Ω is a given bounded open set in $\mathbb{R}^n$. We define

$$f(x) = \begin{cases} \frac{\rho_j(|x-a_j|)}{\rho_j(r_j)}(x - a_j) + a_j & \text{if } x \in \mathbb{B}_j, \\ x & \text{if } x \in \Omega \setminus \bigcup \mathbb{B}_j, \end{cases}$$

where the ρ_j satisfy the hypotheses of Proposition 7.1. In addition it is required that

$$\sum |\rho_j(r_j)|^{-p} \int_0^{r_j} |\rho_j(t)|^p t^{n-1}\, dt < \infty$$

For instance, when $1 \leq \rho_j(t) \leq M$, for all $j = 1, 2, \dots$ and $0 < t < \infty$, this sum is dominated by $M^p \operatorname{vol}(\Omega)$. A little calculation reveals that f belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$. It is evident that f makes equality hold in (7.3), where $f_0(x) \equiv x$.

8. Burkholder's Inequality Revisited

Recall the constant $p^* - 1 = \max\{p - 1, (p-1)^{-1}\}$, $1 < p < \infty$. An elementary verification shows that

$$p^* - 1 \geq \max\{p^{\frac{1}{p-1}} - 1, (p^{\frac{1}{p-1}} - 1)^{-1}\} := p^{\#} - 1. \tag{8.1}$$

The following results are a useful addition to Burkholder's inequality [B1], [B2].

Lemma 8.1. *For all nonnegative numbers x and y we have*

$$p(Mx - y)(x + y)^{p-1} \leq (1 + M)^{p-1} M^{1-p} (M^p x^p - y^p) \tag{8.2}$$

provided $M \geq p^{\#} - 1$, and

$$(Mx - y)(x + y)^{p-1} \geq Mx^p - y^p \tag{8.3}$$

provided $M \geq p^* - 1$.

The above lower bounds for M are sharp.

Proof. First we are concerned with inequality (8.3). The condition $M \geq p^* - 1$ is forced by testing this inequality near the points $(0, 1)$ and $(1, 0)$. The details are left to the reader. Now consider the following function of one real variable $x \geq 0$:

$$\varphi(x) = [pM + (M - p + 1)x](1 + x)^{p-2} - pM.$$

Its derivative is easily seen to be nonnegative. Indeed, we have

$$\varphi'(x) = (p - 1)(1 + x)^{p-3}[(M - p + 1)x - 1 + (p - 1)M] \geq 0$$

which is clear from $M \geq p^* - 1$. Hence we find that $\varphi(x) \geq \varphi(0) = 0$. Next, consider the function

$$\psi(x) = (Mx - 1)(x + 1)^{p-1} - (Mx^p - 1).$$

Since $\psi'(x) = x^{p-1}\varphi(\frac{1}{x}) \geq 0$, we must have $\psi(x) \geq \psi(0) = 0$. Finally, we obtain the desired estimate

$$(Mx - y)(x + y)^{p-1} - (Mx^p - y^p) = y^p \psi\Big(\frac{x}{y}\Big) \geq 0.$$

The proof of inequality (8.2) is adapted from [B1], with slight changes. First, there is no loss of generality in assuming that $x + y = 1$. The task is to examine the function

$$\gamma(x) = M^p x^p - (1 - x)^p - pM^{p-1}(1 + M)^{1-p}(Mx + x - 1).$$

We shall have established inequality (8.2) if we prove that $\gamma(x) \geq 0$ for all $0 \leq x \leq 1$. The case $p = 2$ is obvious:

$$\gamma(x) = \frac{M - 1}{M + 1}(Mx + x - 1)^2 \geq 0.$$

From now on p will be different from 2, and therefore, $M > 1$. We look at the relative minima of γ in the open interval $0 < x < 1$. For this purpose, we first compute

$$\gamma'(x) = pM^p x^{p-1} + p(1 - x)^{p-1} - pM^{p-1}(1 + M)^{2-p}$$

and

$$\gamma''(x) = p(p - 1)[M^p x^{p-2} - (1 - x)^{p-2}].$$

This latter formula shows that γ has exactly one inflection point. Observe that between two relative minima there must be at least two inflection points. Consequently, γ admits at most one relative minimum. As a matter of fact, the relative minimum takes place at the point $x_0 = (1 + M)^{-1}$. Indeed, we have $\gamma'(x_0) = 0$ and $\gamma''(x_0) = p(p - 1)M^{p-2}(1 + M)^{3-p}(M - 1) > 0$. Since $\gamma(x_0) = 0$, it remains to check at the end-points, which will determine the lower bound for the constant M. We must have

$$\gamma(0) = pM^{p-1}(1 + M)^{1-p} - 1 \geq 0$$

and

$$\gamma(1) = M^p - pM^p(1 + M)^{1-p} \geq 0.$$

These two conditions exactly mean that $M \geq p^{\#} - 1$, as is easy to see. □

9. Basic Estimate for the Cauchy-Riemann Operators

In this section we derive inequality (1.19) from Theorem 2. We wish to express this inequality in terms of the Cauchy-Riemann operators $\mathcal{D}^+ f$ and $\mathcal{D}^- f$. Identities (1.11) yield

$$|Df|^{\frac{n}{2}} = |\mathcal{D}^+ f| + |\mathcal{D}^- f| \qquad \text{and} \qquad J(x, f) = |\mathcal{D}^+ f|^2 - |\mathcal{D}^- f|^2. \tag{9.1}$$

On substituting into (1.16), we obtain

$$\int \left(\frac{1 + \lambda_p}{1 - \lambda_p} |\mathcal{D}^- f| - |\mathcal{D}^+ f| \right) (|\mathcal{D}^+ f| + |\mathcal{D}^- f|)^{\frac{2p}{n} - 1} \geq 0. \tag{9.2}$$

Now we want to apply inequality (8.2) with $M = \frac{1+\lambda_p}{1-\lambda_p}$ and the exponent $\frac{2p}{n} > 1$ in place of p. It is important to pay attention to the lower bound for M. On account of (1.17) we find that

$$M \geq \frac{1+|1-\frac{n}{p}|}{1-|1-\frac{n}{p}|} = \max\left\{\frac{2p-n}{n}, \frac{n}{2p-n}\right\}$$
$$= \left(\frac{2p}{n}\right)^* - 1 \geq \left(\frac{2p}{n}\right)^\# - 1$$

as needed. Accordingly, we obtain

$$\int \left[\left(\frac{1+\lambda_p}{1-\lambda_p}\right)^{\frac{2p}{n}} |\mathcal{D}^- f|^{\frac{2p}{n}} - |\mathcal{D}^+ f|^{\frac{2p}{n}}\right] \geq 0$$

which is none other than inequality (1.19), completing the proof of Theorem 4.

It is natural to try to relate this result to Conjecture 1. Before embarking on this discussion, however, we rephrase our inequality as

$$\|\mathcal{D}^+ f\|_p \leq A_p(n) \|\mathcal{D}^- f\|_p \tag{9.4}$$

for $p > \frac{2p_0(n)}{n}$, where

$$A_p(n) = \frac{1+\lambda_{\frac{np}{2}}(n)}{1-\lambda_{\frac{np}{2}}(n)}. \tag{9.5}$$

One can interchange $\mathcal{D}^+$ and $\mathcal{D}^-$ in (9.4) by simply applying it to the conjugate mapping $\bar{f}$. Recall that in even dimensions the critical point $p_0(n)$ equals $\frac{n}{2}$. Hence (9.4) holds for all $p > 1$. By virtue of Conjecture 2 this may also be true in odd dimensions. Another inference from Conjecture 2 would be that

$$A_p(n) = \frac{1+|1-\frac{2}{p}|}{1-|1-\frac{2}{p}|} = p^* - 1. \tag{9.6}$$

In this way we arrive at the n-dimensional counterpart of Conjecture 1, which precisely parallels the L^p-inequality for the Cauchy-Riemann operators in the complex plane.

Conjecture 9.1. For each dimension $n \geq 2$ and $p > 1$ we have

$$\|\mathcal{D}^+ f\|_p \leq (p^* - 1) \|\mathcal{D}^- f\|_p \tag{9.7}$$

whenever $f \in W_0^{1,\frac{np}{2}}(\Omega, \mathbb{R}^n)$. The constant $p^* - 1$ is sharp.

We end this section by showing that (9.7) fails if the constant $p^* - 1$ is replaced by a smaller one.

Example 9.1. For each $p > 1$ there exist mappings $f = f_R \in W^{1,\frac{np}{2}}(\mathbb{R}^n, \mathbb{R}^n)$, $R > 1$, such that

$$\lim_{R\to\infty} \frac{\|\mathcal{D}^+ f_R\|_p}{\|\mathcal{D}^- f_R\|_p} = p^* - 1. \tag{9.8}$$

Proof. We only outline the essential parts of our calculation. Given any $p > 1$ and $R > 1$, we define

$$f(x) = \begin{cases} x & \text{if } |x| \le 1, \\ x|x|^{-\frac{2}{p}} & \text{if } 1 \le |x| \le R, \\ x|x|^{-2}R^{2-\frac{2}{p}} & \text{if } |x| \ge R. \end{cases}$$

Elementary computation shows that

$$|\mathcal{D}^+ f|^p = \begin{cases} 1 & \text{if } |x| \le 1, \\ (\frac{p-1}{p})^p |x|^{-n} & \text{if } 1 \le |x| \le R, \\ 0 & \text{if } |x| \ge R, \end{cases}$$

$$|\mathcal{D}^- f|^p = \begin{cases} 0 & \text{if } |x| \le 1, \\ (\frac{1}{p})^p |x|^{-n} & \text{if } 1 \le |x| \le R, \\ R^{np-n}|x|^{-np-p} & \text{if } |x| \ge R. \end{cases}$$

Integrating in polar coordinates, we arrive at the formula

$$\frac{\|\mathcal{D}^+ f\|_p^p}{\|\mathcal{D}^- f\|_p^p} = \frac{(\frac{p-1}{p})^p \log R + \frac{1}{n}}{(\frac{1}{p})^p \log R + \frac{1}{(np-n+p)R^p}}.$$

Finally, letting R go to ∞, we conclude that

$$\lim_{R\to\infty} \frac{\|\mathcal{D}^+ f_R\|_p}{\|\mathcal{D}^- f_R\|_p} = p - 1.$$

This solves the problem when $p \ge 2$. The case $1 < p \le 2$ is handled by applying this result to the conjugate mapping. Indeed, $|\mathcal{D}^\pm \bar{f}| = |\mathcal{D}^\mp f|$ and hence

$$\lim_{R\to\infty} \frac{\|\mathcal{D}^+ \bar{f}_R\|_p}{\|\mathcal{D}^- \bar{f}_R\|_p} = \frac{1}{p-1}.$$

This completes the proof of (9.8). □

10. The Beurling-Ahlfors Transform on Differential Forms

We denote by $\bigwedge = \bigwedge(\mathbb{R}^n) = \bigoplus_{l=0}^n \bigwedge^l(\mathbb{R}^n)$ the exterior algebra of $\mathbb{R}^n$. The space $\bigwedge^l = \bigwedge^l(\mathbb{R}^n)$, $l = 1, 2, \ldots, n$, is spanned by the l-covectors $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_l}$ corresponding to all l-tuples $I = (i_1, \ldots, i_l)$ with $1 \le i_1 < \cdots < i_l \le n$. As a convention we put $\bigwedge^0(\mathbb{R}^n) = \mathbb{R}$. Then $\bigwedge(\mathbb{R}^n)$ is a graded algebra with respect to the wedge product. The inner product in $\bigwedge(\mathbb{R}^n)$, denoted by $\langle \quad | \quad \rangle$, is defined by declaring the covectors

$$1, dx_1, \ldots, dx_n, dx_1 \wedge dx_2, \ldots, dx_{n-1} \wedge dx_n, \ldots, dx_1 \wedge \cdots \wedge dx_n$$

to form an orthonormal basis. Then the Hodge star operator $\star : \bigwedge^l(\mathbb{R}^n) \to \bigwedge^{n-l}(\mathbb{R}^n)$ is determined uniquely from the rule

$$\alpha \wedge \star\beta = \langle \alpha|\beta \rangle dx_1 \wedge \cdots \wedge dx_n$$

for all $\alpha, \beta \in \bigwedge^l(\mathbb{R}^n)$.

Let Ω be an open subset of $\mathbb{R}^n$. A differential form of degree l on Ω is simply a function, or Schwartz distribution, with values in $\bigwedge^l(\mathbb{R}^n)$. We denote the space of l-forms on Ω by $\mathcal{D}'(\Omega, \bigwedge^l)$, $l = 0, 1, \ldots, n$. Continuing in this fashion, we denote by

$L^p(\Omega, \bigwedge^l)$ and $W^{1,p}(\Omega, \bigwedge^l)$ the Lebesgue and Sobolev spaces of l-forms, respectively. The primary differential operators on forms are the exterior differential

$$d : W^{1,p}\big(\Omega, \textstyle\bigwedge^l\big) \to L^p\big(\Omega, \textstyle\bigwedge^{l+1}\big)$$

and its formal adjoint

$$d^* = (-1)^{nl-1} \star d\star : W^{1,p}\big(\Omega, \textstyle\bigwedge^{l+1}\big) \to L^p\big(\Omega, \textstyle\bigwedge^l\big).$$

Then the Hodge-de Rham decomposition asserts that each form $\omega \in L^p(\mathbb{R}^n, \bigwedge^l)$, $1 < p < \infty$, can be written as

$$\omega = d\alpha + d^*\beta \tag{10.1}$$

with $\alpha \in W^{1,p}(\mathbb{R}^n, \bigwedge^{l-1})$ and $\beta \in W^{1,p}(\mathbb{R}^n, \bigwedge^{l+1})$. The exact and coexact components of ω can be expressed uniquely in terms of ω by using Riesz transforms in $\mathbb{R}^n$. With the aid of this decomposition the Beurling-Ahlfors transform

$$S : L^p\big(\mathbb{R}^n, \textstyle\bigwedge^l\big) \to L^p\big(\mathbb{R}^n, \textstyle\bigwedge^l\big) \tag{10.2}$$

is defined by

$$S\omega = d\alpha - d^*\beta. \tag{10.3}$$

Thus S is a singular integral operator acting as the identity on exact forms and as minus the identity on coexact forms. In particular, S is an isometry in $L^2(\mathbb{R}^n, \bigwedge^l)$. In recent years this operator has attained a profound interest for many analysts in the theory of quasiconformal mappings and the governing PDEs. One fundamental question still unanswered concerns the sharp constant in the inequality

$$\|S\omega\|_p \le A_p(n)\|\omega\|_p, \qquad 1 < p < \infty. \tag{10.4}$$

It is not by accident that we use the same notation for $A_p(n)$ as in (9.4). While we have not been able to provide a definite answer to this question, there are strong arguments to expect that $A_p(n)$ is dimension free and equals $p^* - 1$ [IM2], [BL], [BW], [BM-H], [NV]. We will encounter related questions in Section 12, reinforcing this conjecture.

11. Computing the Critical Point in Even Dimension

Let $f = (f^1, \dots, f^n) : \mathbb{R}^n \to \mathbb{R}^n$, $n = 2l$, be an arbitrary mapping of Sobolev class $W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$, where $p > l$. With every l-tuple $I = (i_1, \dots, i_l)$, $1 \le i_1 < \cdots < i_l \le n$, we associate two differential l-forms: the exact form

$$d\alpha = \pm df^{i_1} \wedge \cdots \wedge df^{i_l} = \pm d(f^{i_1} df^{i_2} \wedge \cdots \wedge df^{i_l}) \tag{11.1}$$

and the coexact form

$$d^*\beta = \star df^{i'_1} \wedge \cdots \wedge df^{i'_l}. \tag{11.2}$$

Here and subsequently, $I' = (i'_1, \dots, i'_l)$ stands for the so-called complementary l-tuple obtained from $(1, 2, \dots, 2l)$ by deleting the indices in I. The $\pm$ sign in the definition of $d\alpha$ must be chosen to ensure that

$$\langle d\alpha | d^*\beta \rangle dx = df^1 \wedge \cdots \wedge df^n = \mathcal{J}(x, f)dx. \tag{11.3}$$

Next define

$$\omega_I^+ = d\alpha + d^*\beta \quad \text{ and } \quad \omega_I^- = d\alpha - d^*\beta = S(\omega_I^+). \tag{11.4}$$

Inequality (10.4) yields

$$\|\omega_I^{\mp}\|_{2r} \le A_{2r}(n)\|\omega_I^{\pm}\|_{2r}, \qquad r = \frac{p}{n} > \frac{1}{2}. \tag{11.5}$$

Permuting indices, we can certainly claim that

$$\int_{\mathbb{R}^n} \Big(\sum_I |\omega_I^{\mp}|^2\Big)^r \le M_r(n) \int_{\mathbb{R}^n} \Big(\sum_I |\omega_I^{\pm}|^2\Big)^r \tag{11.6}$$

summations being taken over all l-tuples $I = (i_1, \dots, i_l)$, $i \le i_1 < \cdots < i_l \le n$. Although it is not obvious at this point, it is true that (11.6) holds with the constant $M_r(n) = [A_{2r}(n)]^2$. We shall not pursue a proof of this observation here, since the forthcoming arguments also work with an arbitrary constant $M_r(n)$. Next we express the sums in (11.6) by means of minors of the differential matrix $Df \in \mathbb{R}^{n\times n}$. In view of (11.3) and by elementary exterior algebra we obtain

$$\begin{aligned}\sum_I |\omega_I^{\pm}|^2 &= \sum_{1\le i_1<\cdots<i_l\le n} (|df^{i_1}\wedge\cdots\wedge df^{i_l}|^2 + |df^{i'_1}\wedge\cdots\wedge df^{i'_l}|^2) \\ &\pm \sum_{1\le i_1<\cdots<i_l\le n} \mathcal{J}(x,f) = 2\big[\|D^\natural f\|^2 \pm \tbinom{n}{l}\mathcal{J}(x,f)\big]\end{aligned}$$

where the symbol $\|D^\natural f\|^2$ stands for the sum of squares of all possible $l \times l$-subdeterminants

$$\|D^\natural f\|^2 = \sum_{\substack{1\le i_1<\cdots<i_l\le n \\ 1\le j_1<\cdots<j_l\le n}} \left|\frac{\partial(f^{i_1},\dots,f^{i_l})}{\partial(x_{j_1},\dots,x_{j_l})}\right|^2. \tag{11.7}$$

It then follows from (11.6) that

(11.8)
$$\int_{\mathbb{R}^n}\Big[\|D^\natural f(x)\|^2 \mp \tbinom{n}{l}\mathcal{J}(x,f)\Big]^r dx \le M_r(n)\int_{\mathbb{R}^n}\Big[\|D^\natural f(x)\|^2 \pm \tbinom{n}{l}\mathcal{J}(x,f)\Big]^r dx.$$

To go any further we need a lemma.

Lemma 11.1. *Given $M > 0$ and $r \ge \frac{1}{2}$, there exist $0 < \lambda < 1$ and $C > 0$ such that*

$$M(t+s)^r - (t-s)^r \le Ct^{r-1}(\lambda t + s) \tag{11.9}$$

for all positive t and $-t \le s \le t$.

Proof. In view of homogeneity we can assume that $t = 1$. The function

$$F(s) = M(1+s)^r - (1-s)^r, \qquad -1 \le s \le 1,$$

is easily seen to be increasing from -2^r to 2^rM. Fix a number $-1 < a < 0$ such that $F(a) < 0$. We then claim that (11.9) holds with the following constants:

$$\lambda = \frac{2^rM + aF(a)}{2^rM - F(a)} \qquad \text{and} \qquad C = \frac{2^rM - F(a)}{1+a}.$$

It is evident that $0 < aF(a) < -F(a)$; thus $0 < \lambda < 1$ and $C > 0$. Furthermore,

$$\begin{aligned}&\text{if } -1 \le s \le a, \quad \text{then } F(s) \le F(a) = C(\lambda - 1) \le C(\lambda + s),\\ &\text{if } \quad a < s \le 1, \quad \text{then } F(s) \le 2^rM = C(\lambda + a) \le C(\lambda + s).\end{aligned}$$

In either case we arrive at (11.9) for $t = 1$, as desired. □

Having proved this inequality, we can now apply (11.8), where we denote $s = \binom{n}{l}\mathcal{J}(x,f)$ and $t = \|D^\natural f(x)\|^2$. Hadamard's inequality for $l \times l$ minors [IL] shows that $-t \le s \le t$. On substituting these quantities into (11.9), upon integration and in view of (11.8) we obtain

$$\int_{\mathbb{R}^n} \|D^\natural f(x)\|^{2r-2}\Big[\lambda\|D^\natural f(x)\|^2 \pm \tbinom{n}{l}\mathcal{J}(x,f)\Big]dx \ge 0. \tag{11.10}$$

The resulting estimate deserves a statement in its own right

Corollary 11.1. *Let $n = 2l$, $l = 1, 2, \ldots,$ and let $f \in W^{1,sl}(\mathbb{R}^n, \mathbb{R}^n)$ with $s > 1$. Then there exists a constant $0 < \lambda_s(n) < 1$ such that*

$$\binom{n}{l}\left|\int_{\mathbb{R}^n} \|D^\natural f(x)\|^{s-2}\mathcal{J}(x,f)dx\right| \le \lambda_s(n)\int_{\mathbb{R}^n}\|D^\natural f(x)\|^s dx. \tag{11.11}$$

We conjecture that the best constant here equals $|1 - \frac{2}{s}|$. If so, $\lambda_s(n)$ would be dimension free.

Our goal now is to replace $\|D^\natural f\|^2$ in (11.10) by $|Df|^n$. Recalling that $r = \frac{p}{n}$, we find the pointwise inequality

$$|Df|^{p-n}[\lambda\,|Df|^n \pm \mathcal{J}(x,f)] \ge \tbinom{n}{l}^{-r}\|D^\natural f\|^{2r-2}[\lambda\,\|D^\natural f\|^2 \pm \tbinom{n}{l}\mathcal{J}(x,f)] \tag{11.12}$$

whenever $1 > \lambda \ge |1 - \frac{n}{p}|$. For the proof, the left hand side is regarded as an increasing function with respect to $t = |Df|^n \ge |\mathcal{J}(x,f)|$. By virtue of the Hadamard type inequality $|Df|^n \ge \binom{n}{l}^{-1}\|D^\natural f\|^2$, see for instance [IL], we obtain (11.12). Integrating over $\mathbb{R}^n$ and on account of (11.10), we conclude with the estimate

$$\int_{\mathbb{R}^n} |Df(x)|^{p-n}[\lambda\,|Df(x)|^n \pm \mathcal{J}(x,f)]dx \ge 0$$

for some $1 > \lambda = \lambda(n,p) \ge |1 - \frac{n}{p}|$ and all $p > \frac{n}{2}$. This exactly means that the critical point equals half of the dimension.

12. New Perspectives on Rank-One Convexity

Convexity hypotheses for variational functionals are traditionally investigated in the context of integral functionals for vector fields $f = (f^1, \ldots, f^m) : \Omega \to \mathbb{R}^m$, where Ω is an open subset of $\mathbb{R}^n$. Here we wish to discuss variational integrals defined on m-tuples of differential forms

$$\Phi = (\varphi^1, \ldots, \varphi^m) : \Omega \to \textstyle\bigwedge^{l_1 - 1} \times \cdots \times \bigwedge^{l_m - 1}. \tag{12.1}$$

We shall try to understand the relationship between what is known and what we can get from these generalizations. The integrals we have in mind take the form

$$\mathcal{E}[\Phi] = \int_\Omega E(d\Phi) \tag{12.2}$$

where $d\Phi = (d\varphi^1, \ldots, d\varphi^m)$ and $E : \bigwedge^{l_1} \times \cdots \times \bigwedge^{l_m} \to \mathbb{R}$.

One special feature of this setting is that the partial differentiation occurs only via the exterior derivative operator. Therefore, the natural spaces of differential forms in which to look for the minima of (12.2) are the Sobolev classes

$$\mathcal{W}^{d,p}(\Omega, \textstyle\bigwedge^{l-1}) = \{\omega \in \mathcal{D}'(\Omega, \bigwedge^{l-1}); d\omega \in L^p(\Omega, \bigwedge^l)\}, \quad \text{where } 1 \le p \le \infty. \tag{12.3}$$

We say that $\omega \in \mathcal{W}^{d,p}(\Omega, \bigwedge^{l-1})$ has vanishing tangential part on $\partial\Omega$ if its zero extension to $\mathbb{R}^n$ belongs to $\mathcal{W}^{d,p}(\mathbb{R}^n, \bigwedge^{l-1})$. The space of such forms will be denoted by $\mathcal{W}_T^{d,p}(\Omega, \bigwedge^{l-1})$. Note that $C_0^\infty(\Omega, \bigwedge^{l-1})$ is dense in $\mathcal{W}_T^{d,p}(\Omega, \bigwedge^{l-1})$ for $1 \le p < \infty$. That is, for each $\omega \in \mathcal{W}_T^{d,p}(\Omega, \bigwedge^{l-1})$ one can find a sequence $\omega_j \in C_0^\infty(\Omega, \bigwedge^{l-1})$ such that $d\omega_j \to d\omega$ in $L^p(\Omega, \bigwedge^l)$. One type of boundary constraints for the energy functional $\mathcal{E}[\Phi]$ is to prescribe the tangential part of $\Phi = (\varphi^1, \dots, \varphi^m)$ on $\partial\Omega$ in the distributional sense, called the Hodge boundary data.

The first task which we are facing is to solve the so-called interface problem for differential forms. A linear function $\rho : \mathbb{R}^n \to \mathbb{R}$, with $d\rho \neq 0$, defines a plane $\{x; \rho(x) = 0\}$ which forms the common boundary between two parts of $\mathbb{R}^n$:

$$G_+ = \{x \in \mathbb{R}^n; \rho(x) \ge 0\} \quad \text{and} \quad G_- = \{x \in \mathbb{R}^n; \rho(x) \le 0\}.$$

Suppose we are given two differential forms

$$\omega^+ : G_+ \to \textstyle\bigwedge^{l-1}(\mathbb{R}^n) \quad \text{and} \quad \omega^- : G_- \to \textstyle\bigwedge^{l-1}(\mathbb{R}^n)$$

whose coefficients are linear functions. Set

$$\omega = \begin{cases} \omega^+ & \text{in } G_+, \\ \omega^- & \text{in } G_-. \end{cases} \tag{12.4}$$

It is of interest to know whether ω belongs to $\mathcal{W}^{d,\infty}(\mathbb{R}^n, \bigwedge^{l-1})$. If so, we would have

$$d\omega = \begin{cases} \xi & \text{in } G_+, \\ \zeta & \text{in } G_-, \end{cases} \tag{12.5}$$

where $\xi = d\omega^+ \in \bigwedge^l(\mathbb{R}^n)$ and $\zeta = d\omega^- \in \bigwedge^l(\mathbb{R}^n)$. Furthermore, one can show that

$$\xi - \zeta = \mu \wedge d\rho \tag{12.6}$$

for some covector $\mu \in \bigwedge^{l-1}(\mathbb{R}^n)$. We shall refer to (12.6) as *Hadamard's jump condition* for forms. It turns out that this condition is also sufficient for covectors $\xi, \zeta \in \bigwedge^l(\mathbb{R}^n)$ to be written in the way described above, the verification being left to the reader.

Next consider two m-tuples

$$\mathcal{A} = (\alpha^1, \dots, \alpha^m) \quad \text{and} \quad \mathcal{B} = (\beta^1, \dots, \beta^m) \tag{12.7}$$

of covectors $\alpha^i, \beta^i \in \bigwedge^{l_i}$, $i = 1, 2, \dots, m$. Just as for the case of $m \times n$ matrices [B], [M1], [D], we say that $\mathcal{A}$ and $\mathcal{B}$ are rank-one connected if

$$\text{rank } (\mathcal{A} - \mathcal{B}) \le 1. \tag{12.8}$$

This means that there exists $\varphi \in \bigwedge^1(\mathbb{R}^n)$ such that

$$\mathcal{A} - \mathcal{B} = \mathcal{M} \wedge \varphi \tag{12.9}$$

where $\mathcal{M} = (\mu^1, \dots, \mu^m)$ with some $\mu^i \in \bigwedge^{l_i - 1}(\mathbb{R}^n)$. Equation (12.9) is understood coordinatewise, that is, $\alpha^i - \beta^i = \mu^i \wedge \varphi$ for all indices $i = 1, 2, \dots, m$.

Now, let us briefly look at the basis for calling the functional (12.2) quasiconvex, polyconvex and rank-one convex. Throughout we assume that the integrand $E : \bigwedge^{l_1} \times \cdots \times \bigwedge^{l_m} \to \mathbb{R}$ is at least continuous.

Definition 12.1. The integrand $E : \bigwedge^{l_1} \times \cdots \times \bigwedge^{l_m} \to \mathbb{R}$ is said to be quasiconvex if for every $\xi = (\xi^1, \ldots, \xi^m) \in \bigwedge^{l_1} \times \cdots \times \bigwedge^{l_m}$

$$\int_{\mathbb{R}^n} [E(\xi + d\Psi) - E(\xi)] \geq 0, \tag{12.10}$$

whenever $\Psi = (\psi^1, \ldots, \psi^m) \in C_0^\infty(\mathbb{R}^n, \bigwedge^{l_1 - 1} \times \cdots \times \bigwedge^{l_m - 1})$.

One algebraic inference from this definition is that E is convex in every direction $\mathcal{X} = (\chi^1, \ldots, \chi^m) \in \bigwedge^{l_1} \times \cdots \times \bigwedge^{l_m}$ with rank $\mathcal{X} \leq 1$. We say that E is rank-one convex. The proof that this is so is similar to that for the vectorial case [D], [M1], and is left for the interested reader, see also [IVV].

The term null Lagrangian pertains to the integrands of the form

$$E_0(\xi^1, \ldots, \xi^m) = \sum_{k=1}^{m} \sum_{1 \leq i_1, \ldots, i_k \leq m} \langle \alpha_{i_1 \ldots i_k} | \xi^{i_1} \wedge \cdots \wedge \xi^{i_k} \rangle \tag{12.11}$$

where $\alpha_{i_1 \ldots i_k} \in \bigwedge^{l_{i_1} + \cdots + l_{i_k}}(\mathbb{R}^n)$.

We say that a function $E : \bigwedge^{l_1} \times \cdots \times \bigwedge^{l_m} \to \mathbb{R}$ is polyconvex if it is a convex function of null Lagrangians. In particular, for polyconvex functions we have

$$E(\xi) - E(\zeta) \geq \sum_{k=1}^{m} \sum_{1 \leq i_1, \ldots, i_k \leq m} \langle \alpha_{i_1 \ldots i_k}(\zeta) | \xi^{i_1} \wedge \cdots \wedge \xi^{i_k} - \zeta^{i_1} \wedge \cdots \wedge \zeta^{i_k} \rangle \tag{12.12}$$

for all $\xi, \zeta \in \bigwedge^{l_1} \times \cdots \times \bigwedge^{l_m}$. Here we emphasize that the coefficients $\alpha_{i_1 \ldots i_k}$ are functions only of the variable ζ. Because of some cancellations occurring upon integration of wedge products of exact differential forms, it follows that polyconvex functions are quasiconvex. Moreover, one can deduce from (12.12) that E is actually convex in all directions $\mathcal{X} = (\chi^1, \ldots, \chi^m)$ such that $\chi^i \wedge \chi^j = 0$ for $i, j = 1, \ldots, m$. In general, this range of directions is much wider than that of rank-one convex functions, except in the case of vector fields when $l_1 = \ldots = l_m = 1$. To settle matters finally, we introduce what is perhaps the most restrictive and yet realistic variation of the notion of rank-one convexity.

Definition 12.2. A continuous function $E : \bigwedge^{l_1} \times \cdots \times \bigwedge^{l_m} \to \mathbb{R}$ is said to be convex in singular directions if the function $t \mapsto E(\xi + t\mathcal{X})$ is convex whenever $\det \mathcal{X} := \chi^1 \wedge \cdots \wedge \chi^m = 0$.

Note that for $E : \mathbb{R}^{n \times n} \to \mathbb{R}$ this means that E is convex in the directions of matrices with zero determinant. In case $m = 2$ and $l_1 = l_2 = 1$ the above definition coincides with that of rank-one convexity, regardless of the dimension $n \geq 2$. It is perhaps worth mentioning here that in dimension 2 we actually have a slightly stronger result than Proposition 5.1, namely

Proposition 12.1. *Let the matrix functions $\mathcal{F}_p^\pm : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be given by*

$$\mathcal{F}_p^\pm(A) = |1 - \tfrac{2}{p}|\, |A|^p \pm |A|^{p-2} \det A, \qquad 1 < p < \infty. \tag{12.13}$$

Then $\mathcal{F}_p^+$ ($\mathcal{F}_p^-$, respectively) is convex in the direction of any matrix with nonnegative (nonpositive) determinant.

To see this we recall Burkholder's form

$$B_p(X, Y) = [(p^* - 1)\,\|X\| - \|Y\|]\,[\|X\| + \|Y\|]^{p-1} \tag{12.14}$$

where X and Y are elements of some Hilbert space. It was proved by Burkholder [B1], [B2] that the function

$$t \mapsto B_p(X + tA, Y + tB) \tag{12.15}$$

of one real variable is convex, whenever $\|A\| \geq \|B\|$. This, in view of formula (1.31), proves the proposition.

Continuing in this fashion, we encounter a new framework of the L^p-bounds for the Beurling-Ahlfors transform $S : L^p(\mathbb{R}^n, \bigwedge^l) \to L^p(\mathbb{R}^n, \bigwedge^l)$. For this purpose, we apply inequality (8.2) to obtain

$$(p^* - 1)|S\omega|^p - |\omega|^p \geq p(1 - \frac{1}{p^*})^{p-1}[(p^* - 1)|S\omega| - |\omega|]\,[|S\omega| + |\omega|]^{p-1}. \tag{12.16}$$

Now, the sharp constant $A_p(n)$ in (10.4) would equal $p^* - 1$ on the supposition that

$$\int_{\mathbb{R}^n} [(p^* - 1)|S\omega| - |\omega|]\,[|S\omega| + |\omega|]^{p-1} \geq 0. \tag{12.17}$$

The left hand side can be viewed as a functional defined on a pair of differential forms,

$$d\Phi = (d\varphi^1, d\varphi^2) \in L^p(\mathbb{R}^n, \textstyle\bigwedge^l) \times L^p(\mathbb{R}^n, \bigwedge^{n-l}). \tag{12.18}$$

We want to argue by analogy with Burkholder's functional (1.31). Therefore, it is worthwhile to introduce the following notation:

$$\partial\Phi = \frac{1}{2}(d\varphi^1 + \star d\varphi^2) \qquad \text{and} \qquad \bar{\partial}\Phi = \frac{1}{2}(d\varphi^1 - \star d\varphi^2). \tag{12.19}$$

On substituting $\omega = \partial\Phi$ and $S\omega = \bar{\partial}\Phi$ we are reduced to an energy functional of the form

$$\mathcal{E}[\Phi] = \int [(p^* - 1)|\bar{\partial}\Phi| - |\partial\Phi|]\,[|\partial\Phi| + |\bar{\partial}\Phi|]^{p-1}. \tag{12.20}$$

Theorem 12.1. *The integrand of* (12.20) *is convex in singular directions.*

Proof. We set $\xi^+ = \frac{1}{2}(\xi^1 + \star\xi^2)$ and $\xi^- = \frac{1}{2}(\xi^1 - \star\xi^2)$ for $\xi = (\xi^1, \xi^2) \in \bigwedge^l \times \bigwedge^{n-l}$. Then the integrand takes the form

$$E(\xi) = [(p^* - 1)|\xi^-| - |\xi^+|]\,[|\xi^-| + |\xi^+|]^{p-1}.$$

The direction of $\mathcal{X} = (\chi^1, \chi^2) \in \bigwedge^l \times \bigwedge^{n-l}$ is singular whenever $\chi^1 \wedge \chi^2 = 0$. We want to examine convexity of the function

$$E(\xi + t\mathcal{X}) = [(p^* - 1)|\xi^- + t\mathcal{X}^-| - |\xi^+ + t\mathcal{X}^+|]\,[|\xi^- + t\mathcal{X}^-| + |\xi^+ + t\mathcal{X}^+|]^{p-1}.$$

Note that this function is of the form (12.15). In view of Burkholder's result the proof is completed by showing that $|\mathcal{X}^-| = |\mathcal{X}^+|$. We have

$$|\mathcal{X}^-|^2 - |\mathcal{X}^+|^2 = \frac{1}{4}|\chi^1 - \star\chi^2|^2 - \frac{1}{4}|\chi^1 + \star\chi^2|^2 = -\langle \chi^1, \star\chi^2 \rangle.$$

On the other hand, $\langle \chi^1, \star\chi^2 \rangle dx = \chi^1 \wedge \star\star\chi^2 = (-1)^{nl-l}\chi^1 \wedge \chi^2 = 0$, as desired. □

There are enough arguments to expect that functions which are convex in singular directions are quasiconvex, though the converse need not be true. If — and this is probably the case — the functional (12.20) is quasiconvex, then, combining (12.17) with (12.16), we would find that the norm of the Beurling-Ahlfors transform $S : L^p(\mathbb{R}^n, \bigwedge^l) \to L^p(\mathbb{R}^n, \bigwedge^l)$ equals $p^* - 1$.

13. The Operators ∂^+ and ∂^-

It is rewarding to make some explicit calculation for the Beurling-Ahlfors transform $S : L^p(\mathbb{R}^n, \bigwedge^l) \to L^p(\mathbb{R}^n, \bigwedge^l)$ in the special case of dimension $n = 2l$, $l = 1, \dots$. Let us first recall some exterior algebra. We will denote by $e_1, e_2, \dots, e_n$ the standard ordered basis of $\mathbb{R}^n$. Then the standard orthonormal basis (ordered lexicographically) of $\bigwedge^l = \bigwedge^l(\mathbb{R}^n)$ consists of the exterior products:

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_l}, \quad I = (i_1, \dots, i_l), \quad 1 \le i_1 < \ldots < i_l \le n.$$

With this notation, the Hodge star operator $\star : \bigwedge^l(\mathbb{R}^n) \to \bigwedge^{n-l}(\mathbb{R}^n)$ can be defined by the formula

$$\star e_K = (-1)^{|K| + nl + \frac{1}{2}(l^2 - l)} e_{K'} \tag{13.1}$$

for $K = (k_1, \dots, k_l)$, $1 \le k_1 < \cdots < k_l \le n$, $|K| = k_1 + \cdots + k_l$. Here, as before, K' stands for the complementary $(n-l)$-tuple obtained from the sequence $(1, 2, \dots, n)$ by deleting the indices of K.

Let $A = (A^i_j) \in \mathbb{R}^{n \times n}$ be the matrix of a linear map. Thus

$$Ae_k = \sum_{i=1}^{n} A^i_k e_i, \qquad \text{for } k = 1, 2, \dots, n. \tag{13.2}$$

The l-th exterior power of A is the linear map $A_\# : \bigwedge^l(\mathbb{R}^n) \to \bigwedge^l(\mathbb{R}^n)$ defined by $A_\#(e_{k_1} \wedge \cdots \wedge e_{k_l}) = (Ae_{k_1}) \wedge \cdots \wedge (Ae_{k_l})$. Hence

$$A_\# e_K = \sum_I A^I_K e_I, \qquad K = (k_1, \dots, k_l), \tag{13.3}$$

where the summation is over all ordered l-types $I = (i_1, \dots, i_l)$ and A^I_K is the $l \times l$ subdeterminant of the matrix A given by

$$A^I_K = \det \begin{bmatrix} A^{i_1}_{k_1} & \dots & A^{i_1}_{k_l} \\ \vdots & & \vdots \\ A^{i_1}_{k_l} & \dots & A^{i_l}_{k_l} \end{bmatrix}. \tag{13.4}$$

The exterior powers are not the only linear transformations of the space $\bigwedge^l(\mathbb{R}^n)$. Set $N = \binom{n}{l}$ and let $\mathbb{R}^{N \times N}$ denote the space of all linear mappings from $\bigwedge^l(\mathbb{R}^n)$ into itself. In the style of previous conventions, we will identify an element $\mathcal{A} \in \mathbb{R}^{N \times N}$ with its $N \times N$ matrix:

$$\mathcal{A} = (\mathcal{A}^I_K) : \textstyle\bigwedge^l(\mathbb{R}^n) \to \bigwedge^l(\mathbb{R}^n) \tag{13.5}$$

where

$$I = (i_1, \dots, i_l), \quad 1 \le i_1 < \cdots < i_l \le n,$$
$$K = (k_1, \dots, k_l), \quad 1 \le k_1 < \cdots < k_l \le n.$$

From now on, we assume that $n = 2l$, so $N = \binom{n}{l} = 2\binom{2l-1}{l}$. Thus the Hodge star operator $\star \in \mathbb{R}^{N \times N}$ is represented by the matrix whose (I, K) entry equals $(-1)^{|K| + \frac{1}{2}(l^2 - l)} \delta^I_{K'}$. This is an orthogonal matrix since

$$\star^T \star = \mathbb{I} \in \mathbb{R}^{N \times N} \tag{13.6}$$

as is easy to check. With every $\mathcal{A} \in \mathbb{R}^{N\times N}$ we associate the so-called complementary matrix $\mathcal{A}' \in \mathbb{R}^{N\times N}$ given by

$$(\mathcal{A}')^I_J = (-1)^{|I|+|J|}\mathcal{A}^{I'}_{J'}. \tag{13.7}$$

It is immediate that $\mathcal{A}' = \star^T \mathcal{A}\star$. The following interplay between the operators $\star$, $'$, $\#$ and transpose is satisfying for the sequel:

$$\begin{cases} (AB)_\# & = A_\# B_\#, \qquad & (\mathcal{A}\mathcal{B})' = \mathcal{A}'\mathcal{B}', \\ (A^T)_\# & = (A_\#)^T := A^T_\#, \qquad & (\mathcal{A}^T)' = (\mathcal{A}')^T, \\ (A^{-1})_\# & = (A_\#)^{-1} := A^{-1}_\#, \qquad & \star\mathcal{A}' = \mathcal{A}\star, \\ A^T_\# \star A_\# & = (\det A)\star, \qquad & A^T_\#\mathcal{A}' = (\det A)\mathbb{I}. \end{cases} \tag{13.8}$$

On the analogy of (1.7), we have an orthogonal decomposition of a matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$ as

$$\mathcal{A} = \mathcal{A}^+ + \mathcal{A}^- \tag{13.9}$$

where

$$\mathcal{A}^+ = \frac{1}{2}(\mathcal{A} + \mathcal{A}') \qquad \text{and} \qquad \mathcal{A}^- = \frac{1}{2}(\mathcal{A} - \mathcal{A}'). \tag{13.10}$$

The $\pm$ components are possessed of many nice properties, including

$$\|\mathcal{A}^+\|^2 + \|\mathcal{A}^-\|^2 = \|\mathcal{A}\|^2. \tag{13.11}$$

If $\mathcal{A} = A_\#$ for some $A \in \mathbb{R}^{n\times n}$, then

$$\|\mathcal{A}^+\|^2 - \|\mathcal{A}^-\|^2 = N \det A. \tag{13.12}$$

Moreover, nonsingular matrices $A \in \mathcal{CO}_\pm(n)$ can be described by the equations

$$(A_\#)^\pm = 0, \qquad \text{respectively.} \tag{13.13}$$

It is therefore of interest to look at the pair of differential operators defined on weakly differentiable mappings $f : \Omega \to \mathbb{R}^n$ by the rule

$$\partial^+ f = (Df)^+_\# \qquad \text{and} \qquad \partial^- f = (Df)^-_\#. \tag{13.14}$$

Thus $\partial^+, \partial^- : \mathcal{W}^{1,lp}(\Omega, \mathbb{R}^n) \to L^p(\Omega, \mathbb{R}^{N\times N})$, for all exponents $1 \le p \le \infty$. Note that these operators depend linearly on the $l\times l$ minors of the differential matrix. We then gain some advantage over the operators $\mathcal{D}^+$ and $\mathcal{D}^-$ from the weak continuity properties of the minors [B], [R1], [M1], [M2], see also [I5] for a recent account.

Three results merit mentioning here

Theorem 13.1. *If the mappings $f_j : \Omega \to \mathbb{R}^n$ converge to $f : \Omega \to \mathbb{R}^n$ weakly in $\mathcal{W}^{1,l}(\Omega, \mathbb{R}^n)$, then $\partial^\pm f_j \to \partial^\pm f$ in the sense of Schwartz distributions. That is,*

$$\lim_{j\to\infty} \int \varphi \partial^\pm f_j = \int \varphi \partial^\pm f$$

for every test function $\varphi \in C^\infty_0(\Omega)$.

Theorem 13.2. *For each $f \in \mathcal{W}^{1,lp}(\mathbb{R}^{2l}, \mathbb{R}^{2l})$ and $1 < p < \infty$, we have*

$$\|\partial^- f\|_p \le A_p(n)\|\partial^+ f\|_p. \tag{13.15}$$

The proof is straightforward by reducing (13.15) to the L^p-inequality for the Beurling-Ahlfors transform $S : L^p(\mathbb{R}^n, \bigwedge^l) \to L^p(\mathbb{R}^n, \bigwedge^l)$. Once again we conjecture that the sharp constant at (13.15) equals $A_p(n) = p^* - 1$. As before, the success of this conjecture will depend on proving that the matrix function

$$\mathcal{K}_p(A) = [(p^* - 1)\|A^+_\#\| - \|A^-_\#\|]\,[\|A^+_\#\| + \|A^-_\#\|]^{p-1} \tag{13.16}$$

is quasiconvex. The following result supports this conjecture

Theorem 13.3. *The function $\mathcal{K}_p : \mathbb{R}^{n\times n} \to \mathbb{R}$ is rank-one convex for all $1 < p < \infty$.*

Proof. We only give the main idea of the calculation. Fix two matrices $A, X \in \mathbb{R}^{n\times n}$, where rank $X \le 1$. The theorem will be established once we prove that the function

$$\begin{aligned}&\mathcal{K}_p(A + tX)\\ &= [(p^* - 1)\|(A + tX)^+_\#\| - \|(A + tX)^-_\#\|]\,[\|(A + tX)^+_\#\| + \|(A + tX)^-_\#\|]^{p-1}\end{aligned}$$

is convex for $t \in \mathbb{R}$. Note that $(A+tX)_\#$ is linear with respect to t, as rank $X \le 1$. Thus, we can write

$$(A + tX)_\# = A_\# + t\mathcal{R}$$

where $\mathcal{R} \in \mathbb{R}^{N\times N}$ depends on X (linearly) and A (polynomially of degree $l - 1$). Hence $(A + tX)^\pm_\# = A^\pm_\# + t\mathcal{R}^\pm$, respectively. This observation brings us again to Burkholder's form (12.14). We only need to show that $\|\mathcal{R}^+\| = \|\mathcal{R}^-\|$, or equivalently

$$\|(A + tX)^+_\# - A^+_\#\|^2 = \|(A + tX)^-_\# - A^-_\#\|^2. \tag{13.17}$$

With the aid of identity (13.12) and orthogonality properties of the + and − components we are reduced to showing that

$$N \det(A + tX) + N \det A = 2\langle (A + tX)_\# | A^+_\# - A^-_\# \rangle.$$

We lose no generality in assuming that A is invertible, since such matrices are dense in $\mathbb{R}^{n\times n}$. Using the identity $A^+_\# - A^-_\# = A'_\#$ and (13.8), we arrive at the equation

$$N \det A[(I + tA^{-1}X) + \det I] = 2\langle (I + tA^{-1}X)_\# | A^T_\# A'_\# \rangle$$

where $A^T_\# A'_\# = (\det A)\mathbb{I} \in \mathbb{R}^{N\times N}$. We still have rank $A^{-1}X \le 1$. Hence the left and the right hand side of this equation are linear functions in t, and are reduced to

$$N[2 + t \text{ Trace } (A^{-1}X)] = 2 \text{ Trace } (I + tA^{-1}X)_\#.$$

With the notation $Y = A^{-1}X$, we have

$$N[2 + t(Y^1_1 + Y^2_2 + \ldots Y^n_n)] = 2 \sum_{1\le i_1<\cdots<i_l\le n} [1 + t(Y^{i_1}_{i_1} + Y^{i_2}_{i_2} + \cdots + Y^{i_l}_{i_l})],$$

which is obviously true. □

14. Applications to Quasiconformal Mappings

In this last section we give a brief account of how our estimates apply to quasiconformal mappings, though we do not discuss far-reaching implications of these results. The interested reader is referred to [IM1], [I3], [IMNS], [M], [IM3] for a fuller treatment of this subject.

Recall that $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is K-quasiregular ($1 \le K < \infty$) if

$$|Df(x)|^n \le K J(x,f) \qquad \text{a.e. in } \Omega. \tag{14.1}$$

It is advantageous to express the distortion condition (14.1) in terms of the Cauchy-Riemann operators. We obtain the differential inequality

$$|\mathcal{D}^- f(x)| \le \frac{K-1}{K+1} |\mathcal{D}^+ f(x)|. \tag{14.2}$$

In association with this inequality one can introduce the so-called Beltrami equation

$$\mathcal{D}^- f(x) = \mu(x) \mathcal{D}^+ f(x) \tag{14.3}$$

where $\mu : \Omega \to \mathbb{R}^{n\times n}$ is a measurable function defined by

$$\mu(x) = \begin{cases} \mu(Df(x)) & \text{if } J(x,f) > 0, \\ 0 & \text{if } J(x,f) = 0. \end{cases} \tag{14.4}$$

Here in the right hand side, μ stands for the Beltrami matrix of the differential, see formula (2.17).

That f solves equation (14.3) is of course a tautology. Nevertheless, we are hoping to investigate quasiregular mappings by looking at all possible solutions to this PDE. It is important to realize that in view of Lemma 2.2 we have the following uniform bound:

$$|\mu(x)| \le \frac{K-1}{K+1} < 1 \qquad \text{a.e. in } \Omega. \tag{14.5}$$

Just as in the familiar case of the complex Beltrami equation, the L^p-estimate of the Cauchy-Riemann operators (stated in Theorem 4) is the key to the regularity theory of quasiconformal mappings. Of course, estimate (1.16) works equally well. In order to make use of (1.16) it will be necessary to multiply the mapping $f : \Omega \to \mathbb{R}^n$ by a nonnegative test function $\varphi \in C_0(\Omega)$. This gives us a new mapping $\varphi f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$, for which we can write

$$\mathcal{J}(\varphi f) = \int_{\mathbb{R}^n} \mathcal{F}(\varphi Df + f \otimes \nabla\varphi) \ge 0, \tag{14.6}$$

where

$$\mathcal{F}(A) = \lambda_p(n)|A|^p - |A|^{p-n} \det A \qquad \text{for } A \in \mathbb{R}^{n\times n} \tag{14.7}$$

with some $1 > \lambda_p(n) \ge |1 - \frac{n}{p}|$ and $p > p_0(n)$. The rest of our calculation is routine, though quite involved. We only outline the main steps. An elementary analysis shows that $|\mathcal{F}(X+Y) - \mathcal{F}(Y)| \le C_p(n)|X|(|X|+|Y|)^{p-1}$, for arbitrary matrices X and Y. Hence, by Young's inequality,

$$\mathcal{F}(X+Y) \le \mathcal{F}(Y) + \epsilon|Y|^p + C_p(n,\epsilon)|X|^p \tag{14.8}$$

where $\epsilon > 0$ can be arbitrarily small. Combining (14.6) and (14.7) yields

$$\int \varphi^p |Df|^{p-n} J(x,f)dx - \lambda_p(n) \int |\varphi Df|^p \le \epsilon \int |\varphi Df|^p + C_p(n,\epsilon) \int |f \otimes \nabla\varphi|^p.$$

It is at this point that we apply the distortion condition (14.1) to obtain

$$(1 - \lambda_p(n)K - \epsilon K) \int |\varphi Df|^p \leq C_p(n,\epsilon)K \int |f \otimes \nabla\varphi|^p.$$

From now on we impose the following condition:

$$\lambda_p(n)K < 1. \tag{14.9}$$

This requirement is certainly fulfilled by the exponents p which are close to n. Indeed, the function $p \to \lambda_p(n)$ is continuous and equals zero at $p = n$.

As a final step we choose ϵ small enough to conclude with the inequality

$$\int_\Omega |\varphi Df|^p \leq C_p(n,K) \int_\Omega |f \otimes \nabla\varphi|^p \tag{14.10}$$

which is slightly stronger than (1.34). The proof of Theorem 6 is completed by noting that $|f \otimes \nabla\varphi| \leq |f|\,|\nabla\eta|$ for $\varphi = |\eta|$.

Caccioppoli inequalities are of far-reaching importance for the development of geometric function theory. We have already explored them to obtain results concerning removable singularities, distortion of Hausdorff dimension, and the regularity theory of very weak solutions of the governing PDEs [IM1], [I3], [IMNS], [PV], [IS], [IM3].

It is worthwhile recalling Conjecture 2, which says that $\lambda_p(n) = |1 - \frac{n}{p}|$ for all $p > \frac{n}{2}$. If true, condition (14.9) would mean that

$$\frac{nK}{K+1} < p < \frac{nK}{K-1}. \tag{14.11}$$

One might expect, therefore, that the range of Sobolev exponents for K-quasiregular mappings in $\mathbb{R}^n$ is precisely the interval (14.11). This is known in dimension 2 due to the work of K. Astala [A1], see also [A2], [A3], [AIS] and [PV] for further developments. Caccioppoli's inequality for the full range of exponents, as given by (14.11), remains one of the outstanding problems in multidimensional quasiconformal analysis.

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