TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 6, Pages 2187–2205
S 0002-9947(02)02964-1
Article electronically published on February 14, 2002

LOCAL SUBGROUPS AND THE STABLE CATEGORY

WAYNE W. WHEELER

Abstract. If G is a finite group and k is an algebraically closed field of characteristic $p > 0$, then this paper uses the local subgroup structure of G to define a category $\mathfrak{L}(G,k)$ that is equivalent to the stable category of all left kG-modules modulo projectives. A subcategory of $\mathfrak{L}(G,k)$ equivalent to the stable category of finitely generated kG-modules is also identified. The definition of $\mathfrak{L}(G,k)$ depends largely but not exclusively upon local data; one condition on the objects involves compatibility with respect to conjugations by arbitrary group elements rather than just elements of p-local subgroups.

1. Introduction

One of the main themes of modular representation theory, going back many decades to the fundamental work of R. Brauer, is the idea that the representations of a finite group are closely related to those of its local subgroups. To some extent this paper is intended to provide a general explanation for why this idea has proven to be so fruitful over such a long period. In particular, the work presented here shows that it is possible to use the local subgroups of a finite group to construct a category equivalent to the stable category.

Let G be a finite group, let k be an algebraically closed field of characteristic p, and let $\mathcal{P}(G)$ be the collection of all p-subgroups of G. The stable category kG-$\underline{\text{Mod}}$ is obtained by factoring out the projective modules from the category of all left kG-modules. Section 3 defines a category $\mathfrak{L}(G,k)$ in which the objects are essentially certain collections of modules. In particular, an object L in $\mathfrak{L}(G,k)$ determines a module $L(P)$ in $kN_G(P)$-$\underline{\text{Mod}}$ for each $P \in \mathcal{P}(G)$. Each module $L(P)$ must satisfy a condition on its variety, and the family of all modules determined by L must be compatible under conjugation and restriction. In the compatibility conditions conjugations by arbitrary group elements are allowed, so the definition of $\mathfrak{L}(G,k)$ does not depend solely on the local structure of G. The main result of this paper is that $\mathfrak{L}(G,k)$ is equivalent to kG-$\underline{\text{Mod}}$. If Q is a Sylow p-subgroup of G, let $\mathfrak{l}(G,k)$ be the full subcategory of $\mathfrak{L}(G,k)$ consisting of the objects L such that $L(Q)$ is stably isomorphic to a finitely generated module. Then $\mathfrak{l}(G,k)$ is equivalent to the full subcategory kG-$\underline{\text{mod}}$ of finitely generated modules in kG-$\underline{\text{Mod}}$.

The work presented here makes extensive use of Rickard's work on idempotent modules [3] as well as Benson, Carlson, and Rickard's theory of varieties for infinitely generated modules [1]. These topics are reviewed in Section 2. The third section defines the category $\mathfrak{L}(G,k)$ and a canonical functor $\mathfrak{F} : kG$-$\underline{\text{Mod}} \to \mathfrak{L}(G,k)$. It is possible to define the tensor product of a kG-module M and an object L of

Received by the editors January 2, 2001 and, in revised form, September 24, 2001.
2000 *Mathematics Subject Classification.* Primary 20C20.

$\mathfrak{L}(G,k)$, and the result is another object of $\mathfrak{L}(G,k)$. This idea is considered in Section 4. Finally, Section 5 is devoted to proving that the canonical functor $\mathfrak{F} : kG\text{-}\underline{\mathrm{Mod}} \to \mathfrak{L}(G,k)$ is an equivalence of categories.

2. Preliminary results

Throughout this paper G denotes a finite group, and k is an algebraically closed field of prime characteristic p. If $g \in G$ and H is a subgroup of G, let ${}^gH = gHg^{-1}$. For any kH-module M set $M{\uparrow}^G = kG \otimes_{kH} M$; if M is a kG-module, then $M{\downarrow}_H$ denotes the restriction of M to H. It will often be convenient simply to write $M{\downarrow}$ when the subgroup H can be inferred from the context. We write $\varepsilon : k_H^{\uparrow G} \to k$ for the augmentation map given by

$$\varepsilon\Big(\sum_{g \in G/H} g \otimes x_g\Big) = \sum_g x_g.$$

Let kG-Mod denote the category of all left kG-modules, and let kG-mod be the full subcategory of finitely generated kG-modules. If M and M' are kG-modules, let $\mathrm{PHom}_{kG}(M,M')$ denote the k-subspace of $\mathrm{Hom}_{kG}(M,M')$ consisting of those maps that factor through a projective kG-module. The stable category kG-$\underline{\mathrm{Mod}}$ has the same objects as kG-Mod, but the morphisms from M to M' in kG-$\underline{\mathrm{Mod}}$ are defined by setting

$$\underline{\mathrm{Hom}}_{kG}(M,M') = \mathrm{Hom}_{kG}(M,M')/\,\mathrm{PHom}_{kG}(M,M').$$

The full subcategory of kG-$\underline{\mathrm{Mod}}$ consisting of finitely generated kG-modules is denoted kG-$\underline{\mathrm{mod}}$. It is well known that the categories kG-$\underline{\mathrm{Mod}}$ and kG-$\underline{\mathrm{mod}}$ are triangulated (Theorem I.2.6 of [2]), and the translation functor is given by Ω^{-1}. For convenience we often identify Ω^{-1} with the isomorphic functor $\Omega^{-1}k \otimes -$.

If $\gamma : M \to M'$ is a kG-homomorphism, then we normally also write γ for the corresponding map in kG-$\underline{\mathrm{Mod}}$. In fact, we will generally only be concerned with maps in the stable category. In a few cases homomorphisms are defined in the module category, but even then it is always the image in kG-$\underline{\mathrm{Mod}}$ that is of interest.

Recall that if $\mathcal{T}$ is a triangulated category and $\mathcal{C}$ is a full triangulated subcategory of $\mathcal{T}$, then $\mathcal{C}$ is said to be a *thick* subcategory if it is closed under taking direct summands of objects. Now suppose that $\mathcal{C}$ is a thick subcategory of kG-$\underline{\mathrm{mod}}$. As in [3], we say that $\mathcal{C}$ is a *tensor-ideal* subcategory of kG-$\underline{\mathrm{mod}}$ if $M \otimes M'$ is in $\mathcal{C}$ whenever M is in $\mathcal{C}$ and M' is in kG-$\underline{\mathrm{mod}}$. Let $\mathcal{C}^{\oplus}$ denote the smallest full triangulated subcategory of kG-$\underline{\mathrm{Mod}}$ that contains $\mathcal{C}$ and is closed under arbitrary direct sums. A module M is said to be $\mathcal{C}$-*local* if $\underline{\mathrm{Hom}}_{kG}(C,M) = 0$ for all C in $\mathcal{C}$.

The following proposition summarizes the fundamental facts about idempotent modules that will be needed in the following sections. Proofs can be found in [3].

Proposition 2.1. *Let $\mathcal{C}$ be a tensor-ideal subcategory of kG-$\underline{\mathrm{mod}}$. For any object M in kG-$\underline{\mathrm{Mod}}$ there is a triangle*

$$e_{\mathcal{C}}(M) \longrightarrow M \longrightarrow f_{\mathcal{C}}(M) \longrightarrow \Omega^{-1}\big(e_{\mathcal{C}}(M)\big)$$

in kG-$\underline{\mathrm{Mod}}$ such that $e_{\mathcal{C}}(M)$ is in $\mathcal{C}^{\oplus}$ and $f_{\mathcal{C}}(M)$ is $\mathcal{C}$-local, and such a triangle is unique up to isomorphism. The morphism $e_{\mathcal{C}}(M) \to M$ is the universal map in kG-$\underline{\mathrm{Mod}}$ from an object of $\mathcal{C}^{\oplus}$ to M, and $M \to f_{\mathcal{C}}(M)$ is the universal map from M to a $\mathcal{C}$-local object. The modules $e_{\mathcal{C}}(k)$ and $f_{\mathcal{C}}(k)$ are idempotent in the *sense that* $e_{\mathcal{C}}(k) \otimes e_{\mathcal{C}}(k) \cong e_{\mathcal{C}}(k)$ *and* $f_{\mathcal{C}}(k) \otimes f_{\mathcal{C}}(k) \cong f_{\mathcal{C}}(k)$ *in* kG-$\underline{\mathrm{Mod}}$*. Moreover,* $e_{\mathcal{C}}(k) \otimes M \cong e_{\mathcal{C}}(M)$ *and* $f_{\mathcal{C}}(k) \otimes M \cong f_{\mathcal{C}}(M)$ *in* kG-$\underline{\mathrm{Mod}}$ *for any module* M*.*

In the following work it will often be necessary to consider the universal map $\eta : e_{\mathcal{C}}(k) \to k$ described in Proposition 2.1. By abuse of notation we usually use the same symbol η to denote this map for any thick subcategory $\mathcal{C}$ of kG-$\underline{\mathrm{mod}}$ and for any finite group G.

Benson, Carlson, and Rickard have used idempotent modules to develop a theory of varieties for arbitrary kG-modules, and we give a brief review of this theory. Although it is common to consider the maximal ideal spectrum of the cohomology ring $H^*(G,k)$, we will use the space $\operatorname{Proj} H^*(G,k)$ of all homogeneous prime ideals that do not contain the ideal $\bigoplus_{n=1}^{\infty} H^n(G,k)$. If I is a homogeneous ideal in $H^*(G,k)$, we set $\bar{V}_G(I) = \{\mathfrak{p} \in \operatorname{Proj} H^*(G,k) \mid I \subseteq \mathfrak{p}\}$. If M is a finitely generated kG-module, let $J(M)$ be the annihilator of $\operatorname{Ext}^*_{kG}(M,M)$ in $H^*(G,k)$, and let $\bar{V}_G(M) = \bar{V}_G\big(J(M)\big)$. Then $\bar{V}_G(M)$ is closed in $\operatorname{Proj} H^*(G,k)$, and $\bar{V}_G(k) = \operatorname{Proj} H^*(G,k)$.

If M is infinitely generated, however, then the definition of $\bar{V}_G(M)$ is more complicated. For $\mathfrak{p} \in \bar{V}_G(k)$ let $\mathcal{C}(\mathfrak{p})$ denote the full subcategory of kG-$\underline{\mathrm{mod}}$ consisting of all finitely generated kG-modules M' such that $\bar{V}_G(M') \subseteq \bar{V}_G(\mathfrak{p})$, and let $\mathcal{C}'(\mathfrak{p})$ denote the full subcategory of $\mathcal{C}(\mathfrak{p})$ consisting of all finitely generated modules M' such that $\mathfrak{p} \notin \bar{V}_G(M')$. Then $\mathcal{C}(\mathfrak{p})$ and $\mathcal{C}'(\mathfrak{p})$ are tensor-ideal subcategories of kG-$\underline{\mathrm{mod}}$. If M is an arbitrary kG-module, then $\bar{V}_G(M)$ can be defined as the collection of all primes $\mathfrak{p} \in \bar{V}_G(k)$ such that $f_{\mathcal{C}'(\mathfrak{p})} \otimes e_{\mathcal{C}(\mathfrak{p})} \otimes M$ is not projective.

Although the set $\bar{V}_G(M)$ is not necessarily closed in $\bar{V}_G(k)$ if M is not finitely generated, these sets do retain the most important properties of varieties for finitely generated modules. In particular, the following results hold for any kG-modules M and M':

(1) M is projective if and only if $\bar{V}_G(M) = \emptyset$;
(2) $\bar{V}_G(M \oplus M') = \bar{V}_G(M) \cup \bar{V}_G(M')$;
(3) $\bar{V}_G(M \otimes M') = \bar{V}_G(M) \cap \bar{V}_G(M')$.

The reader should be warned, however, that the definition of $\bar{V}_G(M)$ given here is not used in [1]. In that paper the authors define the so-called variety $\mathcal{V}_G(M)$ of a module M to be a collection of homogeneous irreducible subvarieties of the maximal ideal spectrum of $H^*(G,k)$. The results of [1] can be translated into the notation used here by observing that $V \in \mathcal{V}_G(M)$ if and only if the generic point of V lies in $\bar{V}_G(M)$.

Now let V be a closed subset of $\bar{V}_G(k)$, and let $\mathcal{C}(V)$ denote the full subcategory of kG-$\underline{\mathrm{mod}}$ consisting of all finitely generated kG-modules M such that $\bar{V}_G(M) \subseteq V$. For simplicity we write e_V for $e_{\mathcal{C}(V)}$ and f_V for $f_{\mathcal{C}(V)}$. If H is a subgroup of G, let $\operatorname{res}^*_{G,H} : \bar{V}_H(k) \to \bar{V}_G(k)$ denote the map induced by the restriction homomorphism $\operatorname{res}_{G,H} : H^*(G,k) \to H^*(H,k)$.

Proposition 2.2. *Let* V *and* W *be closed subsets of* $\bar{V}_G(k)$*, and let* H *be a subgroup of* G*. Then*

(1) $e_V \otimes e_W \cong e_{V\cap W}$ *and* $f_V \otimes f_W \cong f_{V\cup W}$ *in* kG-$\underline{\mathrm{Mod}}$*;*
(2) $e_V{\downarrow_H} \cong e_{(\operatorname{res}^*_{G,H})^{-1}(V)}$ *and* $f_V{\downarrow_H} \cong f_{(\operatorname{res}^*_{G,H})^{-1}(V)}$*;*
(3) $\bar{V}_G(e_V) = V$ *and* $\bar{V}_G(f_V) = \bar{V}_G(k) - V$*.*

Proof. The first two statements are proven in [3]; the third statement is Proposition 3.1 of [4]. ∎

Proposition 2.3. *Suppose that H is a subgroup of G.*

(1) *If M is a kG-module, then $\bar{V}_H(M{\downarrow_H}) = (\mathrm{res}^*_{G,H})^{-1}\big(\bar{V}_G(M)\big)$.*
(2) *If M is a kH-module, then $\bar{V}_G(M{\uparrow^G}) = \mathrm{res}^*_{G,H}\big(\bar{V}_H(M)\big)$.*

Proof. See Propositions 4.1 and 4.2 of [4]. □

Now suppose that P is an arbitrary p-subgroup of G, and define

$$V_{G,P} = \mathrm{res}^*_{G,P}\big(\bar{V}_P(k)\big).$$

We shall write $e_{G,P}$ for $e_{V_{G,P}}$ and $f_{G,P}$ for $f_{V_{G,P}}$. If E is an elementary abelian p-subgroup of G, set

$$V^-_{G,E} = \bigcup_{E_0<E} V_{G,E_0}$$

and $V^+_{G,E} = V_{G,E} - V^-_{G,E}$. Then the module $\lambda_{G,E} = f_{V^-_{G,E}} \otimes e_{V_{G,E}}$ satisfies $\bar{V}_G(\lambda_{G,E}) = V^+_{G,E}$.

Proposition 2.4. *Let E be an elementary abelian p-subgroup of G, and set $N = N_G(E)$. Then $\lambda_{N,E}^{\uparrow G}$ is stably isomorphic to $\lambda_{G,E}$. Moreover, the map $1 \otimes \varepsilon\eta{\uparrow^G}$: $\lambda_{G,E} \otimes e_{N,E}^{\uparrow G} \to \lambda_{G,E}$ is a stable isomorphism.*

Proof. The first statement is Proposition 4.5 of [4]. The proof of that proposition also shows that the second statement is true, although it is not explicitly stated in [4]. □

Lemma 2.5. *Let V, W, and X be closed subsets of $\bar{V}_G(k)$, and assume that $W \cap X \subseteq V$. Then there are canonical stable isomorphisms $(f_V \otimes e_W) \oplus (f_V \otimes e_X) \to f_V \otimes e_{W\cup X}$ and $f_V \otimes e_W \to f_{V\cup X} \otimes e_W$.*

Proof. See Lemma 3.2 of [4]. □

Let r be the p-rank of G. For $0 \le s \le r$ let

$$V_s = \bigcup_E V_{G,E},$$

where the union is taken over all elementary abelian p-subgroups E of G with $\operatorname{rank} E \le s$. Set $e_s = e_{V_s}$ and $f_s = f_{V_s}$.

The proof of the following result is similar to that of Proposition 4.3 of [4].

Proposition 2.6. *Let $0 \le s \le r-1$, and let $E_1, \ldots, E_n$ be a set of representatives for the conjugacy classes of elementary abelian p-subgroups of rank $s+1$ in G. Then*

$$f_s \otimes e_{s+1} \cong \bigoplus_{i=1}^n \lambda_{G,E_i}.$$

Proof. For $1 \le i \le n$ set $V^{(i)}_{s+1} = V_{G,E_i}$, and let $V^{(n+1)}_{s+1}$ be the union of all components of V_{s+1} of dimension at most s. Then $V_{s+1} = V^{(1)}_{s+1} \cup \cdots \cup V^{(n+1)}_{s+1}$, and we prove by induction on t that $f_s \otimes e_{V^{(1)}_{s+1}\cup\cdots\cup V^{(t)}_{s+1}} \cong \bigoplus_{i=1}^t f_s \otimes e_{V^{(i)}_{s+1}}$ for $1 \le t \le n+1$. There is nothing to prove if $t = 1$, so assume that $1 < t \le n+1$. Because $(V^{(1)}_{s+1} \cup \cdots \cup V^{(t-1)}_{s+1}) \cap V^{(t)}_{s+1} \subseteq V_s$, Lemma 2.5 and the inductive assumption give

$$f_s \otimes e_{V^{(1)}_{s+1}\cup\cdots\cup V^{(t)}_{s+1}} \cong (f_s \otimes e_{V^{(1)}_{s+1}\cup\cdots\cup V^{(t-1)}_{s+1}}) \oplus (f_s \otimes e_{V^{(t)}_{s+1}}) \cong \bigoplus_{i=1}^t f_s \otimes e_{V^{(i)}_{s+1}},$$

as desired. Moreover, $V_{s+1}^{(n+1)} \subseteq V_s$, so that $f_s \otimes e_{V_{s+1}^{(n+1)}} \cong 0$ in kG-$\underline{\mathrm{Mod}}$. Hence

$$f_s \otimes e_{s+1} \cong \bigoplus_{i=1}^{n+1} f_s \otimes e_{V_{s+1}^{(i)}} \cong \bigoplus_{i=1}^{n} f_s \otimes e_{V_{G,E_i}}.$$

Fix i with $1 \leq i \leq n$, and let V_i' be the union of all components of V_s that are not contained in V_{G,E_i}^-. Then $V_{G,E_i} \cap V_i' \subseteq V_{G,E_i}^-$, so Lemma 2.5 implies that

$$f_s \otimes e_{V_{G,E_i}} = f_{V_{G,E_i}^- \cup V_i'} \otimes e_{V_{G,E_i}} \cong f_{V_{G,E_i}^-} \otimes e_{V_{G,E_i}} = \lambda_{G,E_i}.$$

Thus $f_s \otimes e_{s+1} \cong \bigoplus_{i=1}^n \lambda_{G,E_i}$, as desired. □

Proposition 2.7. *Let $0 \leq s \leq r-1$, and let $E_1, \ldots, E_n$ be a set of representatives for the conjugacy classes of elementary abelian p-subgroups of rank $s+1$ in G. Set $N_j = N_G(E_j)$ for all j, and let $\eta_j : e_{N_j,E_j} \to k$ be the universal map described in Proposition 2.1. Then for $1 \leq j \leq n$ there are maps $\alpha_j : e_{N_j,E_j}^{\uparrow G} \to e_{s+1}$ such that $\eta\alpha_j = \varepsilon\eta_j{\uparrow}^G$ and $1 \otimes \left(\bigoplus_{j=1}^n \alpha_j\right) : f_s \otimes \left(\bigoplus_{j=1}^n e_{N_j,E_j}^{\uparrow G}\right) \to f_s \otimes e_{s+1}$ is a stable isomorphism.*

Proof. For each j there is a commutative diagram

$$\begin{array}{ccc} e_{s+1} \otimes e_{N_j,E_j}^{\uparrow G} & \xrightarrow{1\otimes\varepsilon\eta_j{\uparrow}^G} & e_{s+1} \otimes k \\ {\scriptstyle \eta\otimes 1}\downarrow & & \downarrow{\scriptstyle \eta\otimes 1} \\ k \otimes e_{N_j,E_j}^{\uparrow G} & \xrightarrow[1\otimes\varepsilon\eta_j{\uparrow}^G]{} & k \otimes k, \end{array}$$

and $\bar{V}_G(e_{N_j,E_j}^{\uparrow G}) = V_{G,E_j} \subseteq V_{s+1}$, so that $\eta \otimes 1 : e_{s+1} \otimes e_{N_j,E_j}^{\uparrow G} \to e_{N_j,E_j}^{\uparrow G}$ is a stable isomorphism for all j. Set $\alpha_j = (1 \otimes \varepsilon\eta_j{\uparrow}^G)(\eta \otimes 1)^{-1}$. Then the above diagram shows that $\eta\alpha_j = \varepsilon\eta_j{\uparrow}^G$ for all j.

To prove that $1 \otimes \left(\bigoplus_{j=1}^n \alpha_j\right) : f_s \otimes \left(\bigoplus_{j=1}^n e_{N_j,E_j}^{\uparrow G}\right) \to f_s \otimes e_{s+1}$ is a stable isomorphism, it suffices to prove that $1 \otimes 1 \otimes \left(\bigoplus_{j=1}^n \varepsilon\eta_j{\uparrow}^G\right) : f_s \otimes e_{s+1} \otimes \left(\bigoplus_{j=1}^n e_{N_j,E_j}^{\uparrow G}\right) \to f_s \otimes e_{s+1}$ is a stable isomorphism. Now $f_s \otimes e_{s+1} \cong \bigoplus_{i=1}^n \lambda_{G,E_i}$, and $\bar{V}_G(\lambda_{G,E_i} \otimes e_{N_j,E_j}^{\uparrow G}) = V_{G,E_i}^+ \cap V_{G,E_j} = \emptyset$ if $i \neq j$. Thus it is only necessary to show that $\bigoplus_{j=1}^n (1 \otimes \varepsilon\eta_j{\uparrow}^G) : \bigoplus_{j=1}^n (\lambda_{G,E_j} \otimes e_{N_j,E_j}^{\uparrow G}) \to \bigoplus_{j=1}^n \lambda_{G,E_j}$ is a stable isomorphism. But this is an immediate consequence of Proposition 2.4. □

3. The category of local modules

The main purpose of this section is to define the category $\mathfrak{L}(G,k)$ of G-local modules and a canonical functor $\mathfrak{F} : kG\text{-}\underline{\mathrm{Mod}} \to \mathfrak{L}(G,k)$ that will be studied in the following sections. It will be useful to begin by fixing some notation. Let $\mathcal{P}(G)$ be the collection of all p-subgroups of G. If $P \in \mathcal{P}(G)$, then throughout the remainder of the paper we usually write N for $N_G(P)$. Similar notation will be used for normalizers of other p-subgroups. For example, if $P_0, P_1 \in \mathcal{P}(G)$, then we write N_0 for $N_G(P_0)$ and N_1 for $N_G(P_1)$.

Suppose that for every $P \in \mathcal{P}(G)$ we have a module $L(P)$ in kN-$\underline{\mathrm{Mod}}$ such that $\bar{V}_N\big(L(P)\big) \subseteq V_{N,P}$. Assume in addition that whenever $P_1 \subseteq P_2$ in $\mathcal{P}(G)$, there is a homomorphism $\phi_{P_1,P_2} : L(P_1)\downarrow_{N_1\cap N_2} \to L(P_2)\downarrow_{N_1\cap N_2}$ such that the diagram

$$\begin{array}{ccc} L(P_1)\downarrow_{N_1\cap N_2} & \xrightarrow{\phi_{P_1,P_2}} & L(P_2)\downarrow_{N_1\cap N_2} \\ {\scriptstyle \psi_{P_1,P_2}}\downarrow & & \| \\ e_{N_1\cap N_2,P_1}\otimes L(P_2)\downarrow_{N_1\cap N_2} & \xrightarrow[\eta\otimes 1]{} & L(P_2)\downarrow_{N_1\cap N_2} \end{array}$$

commutes for some stable isomorphism ψ_{P_1,P_2}. Assume that $\phi_{P,P} = 1_{L(P)}$ for all $P \in \mathcal{P}(G)$ and that $\phi_{P_2,P_3} \circ \phi_{P_1,P_2} = \phi_{P_1,P_3}$ in $k[N_1 \cap N_2 \cap N_3]$-Mod whenever $P_1 \subseteq P_2 \subseteq P_3$ in $\mathcal{P}(G)$. For each $g \in G$ assume that there is a stable isomorphism $c_g(P) : g \otimes L(P) \to L({}^gP)$ such that the diagram

$$\begin{array}{ccc} g\otimes h\otimes L(P) & \xrightarrow{\cong} & gh\otimes L(P) \\ {\scriptstyle g\otimes c_h(P)}\downarrow & & \downarrow{\scriptstyle c_{gh}(P)} \\ g\otimes L({}^hP) & \xrightarrow[c_g({}^hP)]{} & L({}^{gh}P) \end{array}$$

commutes for all $g, h \in G$ and all $P \in \mathcal{P}(G)$; assume in addition that if $g \in N$, then $c_g(P) : g \otimes L(P) \to L(P)$ is the map given by $g \otimes x \mapsto gx$. Finally, suppose that if $P_1 \subseteq P_2$ and $g \in G$, then there is a commutative diagram

$$\begin{array}{ccc} \big(g\otimes L(P_1)\big)\downarrow_{{}^g(N_1\cap N_2)} & \xrightarrow{c_g(P_1)} & L({}^gP_1)\downarrow_{{}^g(N_1\cap N_2)} \\ {\scriptstyle g\otimes\phi_{P_1,P_2}}\downarrow & & \downarrow{\scriptstyle \phi_{{}^gP_1,{}^gP_2}} \\ \big(g\otimes L(P_2)\big)\downarrow_{{}^g(N_1\cap N_2)} & \xrightarrow[c_g(P_2)]{} & L({}^gP_2)\downarrow_{{}^g(N_1\cap N_2)}. \end{array}$$

Then we say that (L, ϕ, c) is a *G-local module* (over k). We usually abbreviate the notation by writing L for the G-local module (L, ϕ, c). For simplicity we sometimes also write ϕ instead of ϕ_{P_1,P_2} when the subgroups P_1 and P_2 can be determined from the context.

If $L = (L, \phi, c)$ and $L' = (L', \phi', c')$ are G-local modules, let $\xi(P) : L(P) \to L'(P)$ be a map in kN-Mod for all $P \in \mathcal{P}(G)$. Assume that there is a commutative diagram

$$\begin{array}{ccc} g\otimes L(P) & \xrightarrow{c_g(P)} & L({}^gP) \\ {\scriptstyle g\otimes\xi(P)}\downarrow & & \downarrow{\scriptstyle \xi({}^gP)} \\ g\otimes L'(P) & \xrightarrow[c'_g(P)]{} & L'({}^gP) \end{array}$$

for all $P \in \mathcal{P}(G)$ and $g \in G$; assume in addition that if $P_1, P_2 \in \mathcal{P}(G)$ and $P_1 \subseteq P_2$, then there is a commutative diagram

$$\begin{array}{ccc} L(P_1)\downarrow_{N_1\cap N_2} & \xrightarrow{\phi_{P_1,P_2}} & L(P_2)\downarrow_{N_1\cap N_2} \\ {\scriptstyle \xi(P_1)}\downarrow & & \downarrow{\scriptstyle \xi(P_2)} \\ L'(P_1)\downarrow_{N_1\cap N_2} & \xrightarrow[\phi'_{P_1,P_2}]{} & L'(P_2)\downarrow_{N_1\cap N_2}. \end{array}$$

Under these circumstances we say that the sequence of maps $\xi = \{\xi(P)\}_{P\in\mathcal{P}(G)}$ is a *G-local homomorphism.*

Let $\mathfrak{L}(G,k)$ denote the category in which the objects are the G-local modules and the morphisms are the G-local homomorphisms. Our next objective is to define a canonical functor $\mathfrak{F} : kG\text{-}\underline{\mathrm{Mod}} \to \mathfrak{L}(G,k)$. For any kG-module M set $(\mathfrak{F}M)(P) = e_{N,P}\otimes M{\downarrow}_N$ for all $P\in\mathcal{P}(G)$. The map $\phi_{P_1,P_2} : (\mathfrak{F}M)(P_1)\to(\mathfrak{F}M)(P_2)$ is defined to be the composition

$$e_{N_1,P_1}{\downarrow}_{N_1\cap N_2}\otimes M{\downarrow}_{N_1\cap N_2} \xrightarrow{(1\otimes\eta\otimes 1)^{-1}} e_{N_1,P_1}{\downarrow}\otimes e_{N_2,P_2}{\downarrow}\otimes M{\downarrow}$$
$$\xrightarrow{\eta\otimes 1\otimes 1} e_{N_2,P_2}{\downarrow}_{N_1\cap N_2}\otimes M{\downarrow}_{N_1\cap N_2}$$

whenever $P_1\subseteq P_2$. For each $P\in\mathcal{P}(G)$ and each $g\in G$ there is a stable isomorphism

$$g\otimes e_{N,P} \xrightarrow{\left((g\otimes 1)\otimes\eta\right)^{-1}} (g\otimes e_{N,P})\otimes e_{^gN,{}^gP} \xrightarrow{(g\otimes\eta)\otimes 1} e_{^gN,{}^gP}.$$

Combining this map with the isomorphism $g\otimes M{\downarrow}_N\to M{\downarrow}_{^gN}$ given by $g\otimes m\mapsto gm$, we obtain a stable isomorphism $c_g(P)$ by taking the composition

$$g\otimes(e_{N,P}\otimes M{\downarrow}_N)\cong(g\otimes e_{N,P})\otimes(g\otimes M{\downarrow}_N)\cong e_{^gN,{}^gP}\otimes M{\downarrow}_{^gN}.$$

If $\gamma : M\to M'$ is a map in kG-$\underline{\mathrm{Mod}}$, then $\mathfrak{F}\gamma$ is the G-local homomorphism satisfying $(\mathfrak{F}\gamma)(P) = 1\otimes\gamma : e_{N,P}\otimes M{\downarrow}_N\to e_{N,P}\otimes M'{\downarrow}_N$ for all $P\in\mathcal{P}(G)$.

We have now defined a canonical functor $\mathfrak{F} : kG\text{-}\underline{\mathrm{Mod}}\to\mathfrak{L}(G,k)$. It will sometimes be useful to consider the analogous functor $kH\text{-}\underline{\mathrm{Mod}}\to\mathfrak{L}(H,k)$ for some subgroup H of G. By abuse of notation we use the same symbol $\mathfrak{F}$ to denote this functor for any subgroup of G.

The following proposition characterizes the isomorphisms in $\mathfrak{L}(G,k)$. The proof is straightforward and is left to the reader.

Proposition 3.1. *Let $\xi : L\to L'$ be a G-local homomorphism. Then ξ is an isomorphism in $\mathfrak{L}(G,k)$ if and only if $\xi(P) : L(P)\to L'(P)$ is an isomorphism for all $P\in\mathcal{P}(G)$.*

4. Tensor products

If one thinks of a G-local module essentially as a kG-module, then one would expect to be able to define the tensor product of two G-local modules. Such a definition is indeed possible, but for our purposes it will be more useful to consider the tensor product of a kG-module and a G-local module. This construction has the advantage of being slightly easier to define and to use. The main result of this section is the existence of a certain natural isomorphism relating tensor products and the functor $\mathfrak{F}$. This isomorphism is needed in the next section to show that $\mathfrak{F}$ is an equivalence.

Let M be a kG-module, and let $L = (L,\phi,c)$ be a G-local module. For any $P\in\mathcal{P}(G)$ set $(M\otimes L)(P) = M{\downarrow}_N\otimes L(P)$. If $P_1,P_2\in\mathcal{P}(G)$ with $P_1\subseteq P_2$, then there is a stable isomorphism $1\otimes\phi_{P_1,P_2} : M{\downarrow}_{N_1\cap N_2}\otimes L(P_1){\downarrow}_{N_1\cap N_2}\to M{\downarrow}_{N_1\cap N_2}\otimes L(P_2){\downarrow}_{N_1\cap N_2}$. For any $g\in G$ and $P\in\mathcal{P}(G)$ let $m_g(P) : g\otimes M{\downarrow}_N\to M{\downarrow}_{^gN}$ be the isomorphism given by $g\otimes x\mapsto gx$. Then there is a stable isomorphism given by the composition

$$g\otimes\big(M{\downarrow}_N\otimes L(P)\big)\cong(g\otimes M{\downarrow}_N)\otimes\big(g\otimes L(P)\big)\xrightarrow{m_g(P)\otimes c_g(P)} M{\downarrow}_{^gN}\otimes L(^gP).$$

With these definitions it is easy to see that $M \otimes L = (M \otimes L, 1 \otimes \phi, m \otimes c)$ is again a G-local module. In order to provide a connection between this definition and the functor $\mathfrak{F}$, we begin with the following lemma.

Lemma 4.1. *Let L be a G-local module, and let $P_1, P_2 \in \mathcal{P}(G)$. Then there is a stable isomorphism $\theta_L(P_1, P_2)$ of $k[N_1 \cap N_2]$-modules given by the composition*

$$e_{N_1,P_1}{\downarrow}_{N_1\cap N_2} \otimes L(P_2){\downarrow} \xrightarrow[\cong]{(1\otimes\eta\otimes\phi)^{-1}} e_{N_1,P_1}{\downarrow} \otimes e_{N_2,P_2}{\downarrow} \otimes L(P_1 \cap P_2){\downarrow} \xrightarrow[\cong]{\eta\otimes 1\otimes\phi} e_{N_2,P_2}{\downarrow} \otimes L(P_1){\downarrow}.$$

Moreover, the isomorphism $\theta_L(P_1, P_2)$ is natural in L. For each $g \in G$ there is a commutative diagram in $k[{}^g(N_1 \cap N_2)]$-$\underline{\mathrm{Mod}}$ of the form

$$\begin{array}{ccccc} g \otimes \big(e_{N_1,P_1}{\downarrow} \otimes L(P_2){\downarrow}\big) & \cong & e_{{}^gN_1,{}^gP_1}{\downarrow} \otimes \big(g \otimes L(P_2)\big){\downarrow} & \xrightarrow{1\otimes c_g(P_2)} & e_{{}^gN_1,{}^gP_1}{\downarrow} \otimes L({}^gP_2){\downarrow} \\ \Big\downarrow{\scriptstyle g\otimes\theta_L(P_1,P_2)} & & & & \Big\downarrow{\scriptstyle \theta_L({}^gP_1,{}^gP_2)} \\ g \otimes \big(e_{N_2,P_2}{\downarrow} \otimes L(P_1){\downarrow}\big) & \cong & e_{{}^gN_2,{}^gP_2}{\downarrow} \otimes \big(g \otimes L(P_1)\big){\downarrow} & \xrightarrow[1\otimes c_g(P_1)]{} & e_{{}^gN_2,{}^gP_2}{\downarrow} \otimes L({}^gP_1){\downarrow}. \end{array}$$

Furthermore, if $P_1 \subseteq P_2$ and $P \in \mathcal{P}(G)$, then there is a commutative diagram in $k[N_1 \cap N_2 \cap N]$-$\underline{\mathrm{Mod}}$ given by

$$\begin{array}{ccccc} e_{N_1,P_1}{\downarrow} \otimes e_{N_2,P_2}{\downarrow} \otimes L(P){\downarrow} & & \xrightarrow{\cong} & & e_{N_2,P_2}{\downarrow} \otimes e_{N_1,P_1}{\downarrow} \otimes L(P){\downarrow} \\ \Big\downarrow{\scriptstyle 1\otimes\theta_L(P_2,P)} & & & & \Big\downarrow{\scriptstyle 1\otimes\theta_L(P_1,P)} \\ e_{N_1,P_1}{\downarrow} \otimes e_{N,P}{\downarrow} \otimes L(P_2){\downarrow} & & & & e_{N_2,P_2}{\downarrow} \otimes e_{N,P}{\downarrow} \otimes L(P_1){\downarrow} \\ & \searrow{\scriptstyle \eta\otimes1\otimes1} & & \swarrow{\scriptstyle \eta\otimes1\otimes\phi} & \\ & & e_{N,P}{\downarrow} \otimes L(P_2){\downarrow}. & & \end{array}$$

Proof. Set $P_3 = P_1 \cap P_2$. Because $N_1 \cap N_2 \subseteq N_3$, we can restrict the maps ϕ_{P_3,P_i} to $N_1 \cap N_2$ for $i = 1, 2$, and we claim that the resulting maps $1 \otimes \eta \otimes \phi_{P_3,P_2}{\downarrow}$ and $\eta \otimes 1 \otimes \phi_{P_3,P_1}{\downarrow}$ occurring in the definition of $\theta_L(P_1, P_2)$ are isomorphisms. There is a triangle of $k[N_1 \cap N_2]$-modules of the form

$$e_{N_2\cap N_3,P_3}{\downarrow} \otimes L(P_2){\downarrow} \xrightarrow{\eta\otimes1} L(P_2){\downarrow} \to f_{N_2\cap N_3,P_3}{\downarrow} \otimes L(P_2){\downarrow} \to \Omega^{-1}k \otimes e_{N_2\cap N_3,P_3}{\downarrow} \otimes L(P_2){\downarrow}.$$

Because $\bar{V}_{N_1\cap N_2}\big(e_{N_1,P_1}{\downarrow} \otimes f_{N_2\cap N_3,P_3}{\downarrow} \otimes L(P_2){\downarrow}\big) = \emptyset$, the map $\eta\otimes1$ occurring in this triangle becomes an isomorphism upon tensoring with $e_{N_1,P_1}{\downarrow}$. The commutativity of the diagram

$$\begin{array}{ccc} e_{N_1,P_1}{\downarrow}_{N_1\cap N_2} \otimes e_{N_2,P_2}{\downarrow}_{N_1\cap N_2} \otimes L(P_3){\downarrow}_{N_1\cap N_2} & \xrightarrow{1\otimes\eta\otimes\phi_{P_3,P_2}{\downarrow}} & e_{N_1,P_1}{\downarrow}_{N_1\cap N_2} \otimes L(P_2){\downarrow}_{N_1\cap N_2} \\ \Big\downarrow{\scriptstyle 1\otimes1\otimes\psi_{P_3,P_2}}\ \cong & & \Big\| \\ e_{N_1,P_1}{\downarrow} \otimes e_{N_2,P_2}{\downarrow} \otimes e_{N_2\cap N_3,P_3}{\downarrow} \otimes L(P_2){\downarrow} & \xrightarrow{1\otimes\eta\otimes\eta\otimes1} & e_{N_1,P_1}{\downarrow}_{N_1\cap N_2} \otimes L(P_2){\downarrow}_{N_1\cap N_2} \\ \Big\downarrow{\scriptstyle 1\otimes\eta\otimes1\otimes1}\ \cong & & \Big\| \\ e_{N_1,P_1}{\downarrow}_{N_1\cap N_2} \otimes e_{N_2\cap N_3,P_3}{\downarrow}_{N_1\cap N_2} \otimes L(P_2){\downarrow}_{N_1\cap N_2} & \xrightarrow[1\otimes\eta\otimes1]{\cong} & e_{N_1,P_1}{\downarrow}_{N_1\cap N_2} \otimes L(P_2){\downarrow}_{N_1\cap N_2} \end{array}$$

shows that $1\otimes\eta\otimes\phi_{P_3,P_2}{\downarrow}$ is an isomorphism. Similarly, $\eta\otimes1\otimes\phi_{P_3,P_1}{\downarrow}$ is an isomorphism, and so is $\theta_L(P_1, P_2)$. It is straightforward to check that this isomorphism is natural in L.

Now suppose in addition that $g \in G$. Observe that there is a commutative diagram in $k[{}^g(N_1 \cap N_2)]$-$\underline{\mathrm{Mod}}$ of the form

$$\begin{array}{ccc}
g \otimes \big(e_{N_1,P_1}{\downarrow} \otimes e_{N_2,P_2}{\downarrow} \otimes L(P_3){\downarrow}\big) & \xrightarrow{g\otimes(1\otimes\eta\otimes\phi)} & g \otimes \big(e_{N_1,P_1}{\downarrow} \otimes L(P_2){\downarrow}\big) \\
\cong\downarrow & & \downarrow\cong \\
e_{{}^gN_1,{}^gP_1}{\downarrow} \otimes e_{{}^gN_2,{}^gP_2}{\downarrow} \otimes \big(g \otimes L(P_3)\big){\downarrow} & \xrightarrow{1\otimes\eta\otimes(g\otimes\phi)} & e_{{}^gN_1,{}^gP_1}{\downarrow} \otimes \big(g \otimes L(P_2)\big){\downarrow} \\
{\scriptstyle 1\otimes1\otimes c_g(P_3)}\downarrow & & \downarrow{\scriptstyle 1\otimes c_g(P_2)} \\
e_{{}^gN_1,{}^gP_1}{\downarrow} \otimes e_{{}^gN_2,{}^gP_2}{\downarrow} \otimes L({}^gP_3){\downarrow} & \xrightarrow[1\otimes\eta\otimes\phi]{} & e_{{}^gN_1,{}^gP_1}{\downarrow} \otimes L({}^gP_2){\downarrow}.
\end{array}$$

Combining this diagram with the analogous diagram in which P_1 and P_2 are interchanged, we conclude that the first diagram given in the statement of the lemma commutes.

Finally, if $P_1 \subseteq P_2$ and $P \in \mathcal{P}(G)$, then it is straightforward to verify that the last diagram given in the statement of the lemma commutes. □

Proposition 4.2. *Let L be a G-local module, and let $P \in \mathcal{P}(G)$. Then there is an isomorphism $\mathfrak{F}\big(L(P){\uparrow}^G\big) \cong e_{N,P}^{\uparrow G} \otimes L$, and this isomorphism is natural in L.*

Proof. Let $P_0 \in \mathcal{P}(G)$. By Lemma 4.1 there are isomorphisms

$$\begin{aligned}
e_{N_0,P_0} \otimes L(P){\uparrow}^G{\downarrow}_{N_0} &\cong \bigoplus_{g\in N_0\backslash G/N} e_{N_0,P_0} \otimes \big(g \otimes L(P)\big){\downarrow}_{{}^gN\cap N_0}{\uparrow}^{N_0} \\
&\cong \bigoplus_g \big(e_{N_0,P_0}{\downarrow}_{{}^gN\cap N_0} \otimes L({}^gP){\downarrow}_{{}^gN\cap N_0}\big){\uparrow}^{N_0} \\
&\cong \bigoplus_g \big(e_{{}^gN,{}^gP}{\downarrow}_{{}^gN\cap N_0} \otimes L(P_0){\downarrow}_{{}^gN\cap N_0}\big){\uparrow}^{N_0} \\
&\cong \bigoplus_g (g \otimes e_{N,P}){\downarrow}_{{}^gN\cap N_0}{\uparrow}^{N_0} \otimes L(P_0) \\
&\cong e_{N,P}^{\uparrow G}{\downarrow}_{N_0} \otimes L(P_0),
\end{aligned}$$

and it is straightforward to check that these isomorphisms are natural in L. Let $\Phi(P_0) : e_{N_0,P_0} \otimes L(P){\uparrow}^G{\downarrow}_{N_0} \to e_{N,P}^{\uparrow G}{\downarrow}_{N_0} \otimes L(P_0)$ be the above isomorphism. Then it is only necessary to show that $\Phi : \mathfrak{F}\big(L(P){\uparrow}^G\big) \to e_{N,P}^{\uparrow G} \otimes L$ is a G-local homomorphism.

Suppose that $P_1, P_2 \in \mathcal{P}(G)$ with $P_1 \subseteq P_2$. We wish to show that there is a commutative diagram

$$\begin{array}{ccccc}
e_{N_1,P_1}{\downarrow}_{N_1\cap N_2} \otimes L(P){\uparrow}^G{\downarrow}_{N_1\cap N_2} & \xleftarrow[\cong]{1\otimes\eta\otimes1} & e_{N_1,P_1}{\downarrow} \otimes e_{N_2,P_2}{\downarrow} \otimes L(P){\uparrow}^G{\downarrow} & \xrightarrow{\eta\otimes1\otimes1} & e_{N_2,P_2}{\downarrow} \otimes L(P){\uparrow}^G{\downarrow} \\
{\scriptstyle \Phi(P_1)}\downarrow & & & & \downarrow{\scriptstyle \Phi(P_2)} \\
e_{N,P}^{\uparrow G}{\downarrow}_{N_1\cap N_2} \otimes L(P_1){\downarrow}_{N_1\cap N_2} & & \xrightarrow[1\otimes\phi_{P_1,P_2}]{} & & e_{N,P}^{\uparrow G}{\downarrow} \otimes L(P_2){\downarrow}.
\end{array}$$

If $g \in G$, then Lemma 4.1 shows that in $k[{}^gN \cap N_1 \cap N_2]$-Mod there is a commutative diagram given by

$$\begin{array}{ccc}
e_{N_1,P_1}\!\downarrow \otimes e_{N_2,P_2}\!\downarrow \otimes L({}^gP)\!\downarrow & \xrightarrow{\cong} & e_{N_2,P_2}\!\downarrow \otimes e_{N_1,P_1}\!\downarrow \otimes L({}^gP)\!\downarrow \\
{\scriptstyle 1\otimes\theta_L(P_2,{}^gP)}\downarrow & & \downarrow{\scriptstyle 1\otimes\theta_L(P_1,{}^gP)} \\
e_{N_1,P_1}\!\downarrow \otimes e_{{}^gN,{}^gP}\!\downarrow \otimes L(P_2)\!\downarrow & & e_{N_2,P_2}\!\downarrow \otimes e_{{}^gN,{}^gP}\!\downarrow \otimes L(P_1)\!\downarrow \\
& \searrow^{\eta\otimes 1\otimes 1} \quad {}^{\eta\otimes 1\otimes\phi}\swarrow & \\
& e_{{}^gN,{}^gP}\!\downarrow \otimes L(P_2)\!\downarrow. &
\end{array}$$

After inducing from ${}^gN \cap N_1 \cap N_2$ to $N_1 \cap N_2$ and taking the direct sum over $g \in (N_1 \cap N_2)\backslash G/N$, some tedious verifications show that there is a commutative diagram

$$\begin{array}{ccc}
e_{N_2,P_2}\!\downarrow \otimes e_{N_1,P_1}\!\downarrow \otimes L(P)\!\uparrow^G\downarrow_{N_1\cap N_2} & \xrightarrow{\eta\otimes\Phi(P_1)} & e_{N,P}^{\uparrow G}\!\downarrow_{N_1\cap N_2} \otimes L(P_1)\!\downarrow_{N_1\cap N_2} \\
\cong\downarrow & & \downarrow{\scriptstyle 1\otimes\phi_{P_1,P_2}} \\
e_{N_1,P_1}\!\downarrow \otimes e_{N_2,P_2}\!\downarrow \otimes L(P)\!\uparrow^G\downarrow_{N_1\cap N_2} & \xrightarrow[\eta\otimes\Phi(P_2)]{} & e_{N,P}^{\uparrow G}\!\downarrow_{N_1\cap N_2} \otimes L(P_2)\!\downarrow_{N_1\cap N_2}.
\end{array}$$

It is easy to see that this is equivalent to the commutativity of the desired diagram.

Finally, suppose that $g \in G$ and that $P_0 \in \mathcal{P}(G)$. We must show that there is a commutative diagram in $k[{}^gN_0]$-Mod of the form

$$\begin{array}{ccccc}
g \otimes \left(e_{N_0,P_0} \otimes L(P)\!\uparrow^G\downarrow\right) & \cong & e_{{}^gN_0,{}^gP_0} \otimes \left(g \otimes L(P)\!\uparrow^G\downarrow\right) & \longrightarrow & e_{{}^gN_0,{}^gP_0} \otimes L(P)\!\uparrow^G\downarrow \\
{\scriptstyle g\otimes\Phi(P_0)}\downarrow & & & & \downarrow{\scriptstyle \Phi({}^gP_0)} \\
g \otimes \left(e_{N,P}^{\uparrow G}\!\downarrow_{N_0} \otimes L(P_0)\right) & \cong & e_{N,P}^{\uparrow G}\!\downarrow_{{}^gN_0} \otimes \left(g \otimes L(P_0)\right) & \longrightarrow & e_{N,P}^{\uparrow G}\!\downarrow_{{}^gN_0} \otimes L({}^gP_0).
\end{array}$$

Let T be a set of representatives for the (N_0, N) double cosets in G. It follows from Lemma 4.1 that there is a commutative diagram

$$\begin{array}{ccc}
g \otimes \left(e_{N_0,P_0} \otimes L(P)\!\uparrow^G\downarrow_{N_0}\right) & \xrightarrow{\cong} & e_{{}^gN_0,{}^gP_0} \otimes L(P)\!\uparrow^G\downarrow_{{}^gN_0} \\
\downarrow & & \downarrow \\
\bigoplus_{h\in T}\left(g \otimes \left(e_{N_0,P_0}\!\downarrow \otimes (h \otimes L(P))\!\downarrow_{{}^hN\cap N_0}\uparrow^{N_0}\right)\right) & \xrightarrow{\cong} & \bigoplus_h e_{{}^gN_0,{}^gP_0} \otimes \left(gh \otimes L(P)\right)\!\downarrow_{{}^{gh}N\cap{}^gN_0}\uparrow^{{}^gN_0} \\
\downarrow & & \downarrow \\
\bigoplus_{h\in T}\left(g \otimes (e_{N_0,P_0}\!\downarrow \otimes L({}^hP)\!\downarrow_{{}^hN\cap N_0})\!\uparrow^{N_0}\right) & \xrightarrow{\cong} & \bigoplus_{h\in T}\left(e_{{}^gN_0,{}^gP_0}\!\downarrow \otimes L({}^{gh}P)\!\downarrow\right)\!\uparrow^{{}^gN_0} \\
\downarrow & & \downarrow \\
\bigoplus_{h\in T}\left(g \otimes (e_{{}^hN,{}^hP}\!\downarrow \otimes L(P_0)\!\downarrow)\!\uparrow^{N_0}\right) & \xrightarrow{\cong} & \bigoplus_{h\in T}\left(e_{{}^{gh}N,{}^{gh}P}\!\downarrow \otimes L({}^gP_0)\!\downarrow\right)\!\uparrow^{{}^gN_0} \\
\downarrow & & \downarrow \\
g \otimes \left(e_{N,P}^{\uparrow G}\!\downarrow_{N_0} \otimes L(P_0)\right) & \xrightarrow{\cong} & e_{N,P}^{\uparrow G}\!\downarrow_{{}^gN_0} \otimes L({}^gP_0),
\end{array}$$

and this completes the proof. $\square$

It is possible to define not only the tensor product but also operations such as induction and restriction of G-local modules. In the case of restriction, for example, let L be a G-local module, and let $H \subseteq G$. Define an H-local module $L{\downarrow_H}$ by setting

$$L{\downarrow_H}(P) = L(P){\downarrow_{H\cap N}}$$

for every p-subgroup $P \subseteq H$. Note that $\bar{V}_N\big(L(P)\big) \subseteq V_{N,P}$, so

$$\bar{V}_{H\cap N}\big(L{\downarrow_H}(P)\big) = (\mathrm{res}^*_{N,H\cap N})^{-1}\big(\bar{V}_N\big(L(P)\big)\big) \subseteq (\mathrm{res}^*_{N,H\cap N})^{-1}(V_{N,P}) = V_{H\cap N,P}.$$

If P_1 and P_2 are p-subgroups of H with $P_1 \subseteq P_2$, then the map $\phi_{P_1,P_2} : L(P_1){\downarrow_{N_1\cap N_2}} \to L(P_2){\downarrow_{N_1\cap N_2}}$ gives a commutative diagram

$$\begin{array}{ccc} L(P_1){\downarrow_{N_1\cap N_2\cap H}} & \xrightarrow{\phi_{P_1,P_2}\downarrow} & L(P_2){\downarrow_{N_1\cap N_2\cap H}} \\ {\scriptstyle \psi_{P_1,P_2}\downarrow}\big\downarrow & & \big\| \\ e_{N_1\cap N_2,P_1}{\downarrow} \otimes L(P_2){\downarrow} & \xrightarrow[\eta\otimes 1]{} & L(P_2){\downarrow_{N_1\cap N_2\cap H}}. \end{array}$$

Moreover, $(\mathrm{res}^*_{N_1\cap N_2,N_1\cap N_2\cap H})^{-1}(V_{N_1\cap N_2,P_1}) = V_{N_1\cap N_2\cap H,P_1}$ so that

$$e_{N_1\cap N_2,P_1}{\downarrow_{N_1\cap N_2\cap H}} \cong e_{N_1\cap N_2\cap H,P_1}.$$

Finally, if $h \in H$, then $c_h(P) : h \otimes L(P) \to L({}^hP)$ is a stable isomorphism, so $c_h(P){\downarrow_{{}^hN\cap H}} : h \otimes L{\downarrow_H}(P) \to L{\downarrow_H}({}^hP)$ is also a stable isomorphism. It is easy to check that the resulting object $L{\downarrow_H} = (L{\downarrow_H}, \phi{\downarrow_H}, c{\downarrow_H})$ is an H-local module. If $\xi : L \to L'$ is a G-local homomorphism, set $\xi{\downarrow_H}(P) = \xi(P){\downarrow_{N\cap H}} : L(P){\downarrow_{N\cap H}} \to L'(P){\downarrow_{N\cap H}}$ for every p-subgroup P of H. Then $\xi{\downarrow_H} : L{\downarrow_H} \to L'{\downarrow_H}$ is an H-local homomorphism, so there is a restriction functor $\mathrm{res}_{G,H} : \mathfrak{L}(G,k) \to \mathfrak{L}(H,k)$.

Using arguments similar to those given in the proof of Proposition 4.2, one can show that for any $P \in \mathcal{P}(G)$ there is a natural isomorphism $\mathfrak{F}\big(L(P)\big) \cong e_{N,P} \otimes L{\downarrow_N}$ in $\mathfrak{L}(N,k)$. It is much easier, however, to deduce the existence of this isomorphism from Theorem 5.9, so we postpone any further discussion of the restriction functor to the next section.

5. An equivalence of categories

The current section is devoted to showing that the canonical functor $\mathfrak{F} : kG\text{-}\underline{\mathrm{Mod}} \to \mathfrak{L}(G,k)$ is an equivalence of categories. Constructing an explicit adjoint seems to be rather difficult, so we prove that $\mathfrak{F}$ is an equivalence by showing that it is essentially surjective, full, and faithful. As a first step toward proving faithfulness, we begin with a result showing that any G-local homomorphism is uniquely determined by its value on a Sylow p-subgroup.

Proposition 5.1. *Let L and L' be G-local modules, and let $\xi : L \to L'$ be a G-local homomorphism such that $\xi(P) = 0$ for some Sylow p-subgroup P of G. Then $\xi = 0$.*

Proof. Let Q be a p-subgroup of G, and let R be a Sylow p-subgroup of $N_G(Q)$. Then there is an element $g \in G$ with ${}^gR \subseteq P$. The commutative diagram

$$\begin{array}{ccc} g \otimes L(Q) & \xrightarrow{c_g(Q)} & L({}^gQ) \\ {\scriptstyle g\otimes\xi(Q)}\big\downarrow & & \big\downarrow{\scriptstyle \xi({}^gQ)} \\ g \otimes L'(Q) & \xrightarrow[c'_g(Q)]{} & L'({}^gQ) \end{array}$$

shows that $\xi(Q) = 0$ if and only if $\xi({}^gQ) = 0$. In order to show that $\xi(Q) = 0$, then, we may replace Q by ${}^gQ$ and may therefore assume that $g = 1$ and that $R \subseteq P$.

The commutative diagram

$$\begin{array}{ccc}
L(Q){\downarrow}_{N_G(Q)\cap N_G(P)} & \xrightarrow{\phi_{Q,P}} & L(P){\downarrow}_{N_G(Q)\cap N_G(P)} \\
{\scriptstyle \xi(Q)\downarrow}\big\downarrow & & \big\downarrow{\scriptstyle \xi(P)\downarrow=0} \\
L'(Q){\downarrow}_{N_G(Q)\cap N_G(P)} & \xrightarrow{\phi'_{Q,P}} & L'(P){\downarrow}_{N_G(Q)\cap N_G(P)} \\
{\scriptstyle \psi_{Q,P}}\big\downarrow & & \big\| \\
e_{N_G(Q)\cap N_G(P),Q} \otimes L'(P){\downarrow} & \xrightarrow[\eta\otimes 1]{} & L'(P){\downarrow}_{N_G(Q)\cap N_G(P)}
\end{array}$$

shows that $(\eta\otimes 1)\circ\psi_{Q,P}\circ\xi(Q){\downarrow} = 0$. But $\bar{V}_{N_G(Q)\cap N_G(P)}\big(L(Q){\downarrow}\big) \subseteq V_{N_G(Q)\cap N_G(P),Q}$, so the map $(\eta\otimes 1)\circ\psi_{Q,P}\circ\xi(Q){\downarrow}$ factors uniquely through $\eta\otimes 1$, and hence $\psi_{Q,P}\circ \xi(Q){\downarrow} = 0$. Because $\psi_{Q,P}$ is a stable isomorphism, it follows that $\xi(Q){\downarrow}_{N_G(Q)\cap N_G(P)} = 0$ and hence $\xi(Q){\downarrow}_R = 0$. Thus

$$\xi(Q) = \frac{1}{|N_G(Q):R|}\,\mathrm{Tr}_R^{N_G(Q)}\big(\xi(Q){\downarrow}_R\big) = 0,$$

as desired. □

Let P be a Sylow p-subgroup of G. Set $N = N_G(P)$, and let $g \in G$. If Q is a Sylow p-subgroup of ${}^gN \cap N$, then $Q \subseteq {}^gP \cap P \subseteq {}^gN \cap N$, and hence $Q = {}^gP \cap P$. It follows that the map $\eta : e_{{}^gN\cap N, {}^gP\cap P} \to k$ is a stable isomorphism, and hence so are

$$\phi_{{}^gP\cap P,{}^gP} : L({}^gP\cap P){\downarrow}_{{}^gN\cap N} \to L({}^gP){\downarrow}_{{}^gN\cap N}$$

and

$$\phi_{{}^gP\cap P,P} : L({}^gP\cap P){\downarrow}_{{}^gN\cap N} \to L(P){\downarrow}_{{}^gN\cap N}.$$

Definition 5.2. Let L and L' be G-local modules, let P be a Sylow p-subgroup of G, and set $N = N_G(P)$. Suppose that $\gamma : L(P) \to L'(P)$ is a kN-homomorphism. We say that γ is *G-stable* if there is a commutative diagram in $k[{}^gN\cap N]$-$\underline{\mathrm{Mod}}$ of the form

$$\begin{array}{ccccccc}
\big(g\otimes L(P)\big){\downarrow} & \xrightarrow{c_g(P)} & L({}^gP){\downarrow} & \xrightarrow{\phi^{-1}} & L({}^gP\cap P){\downarrow} & \xrightarrow{\phi} & L(P){\downarrow} \\
{\scriptstyle g\otimes\gamma}\big\downarrow & & & & & & \big\downarrow{\scriptstyle \gamma} \\
\big(g\otimes L'(P)\big){\downarrow} & \xrightarrow[c'_g(P)]{} & L'({}^gP){\downarrow} & \xrightarrow[(\phi')^{-1}]{} & L'({}^gP\cap P){\downarrow} & \xrightarrow[\phi']{} & L'(P){\downarrow}
\end{array}$$

for all $g \in G$.

The following result is easy to verify, and the proof is left to the reader.

Proposition 5.3. *Let L and L' be G-local modules, let P be a Sylow p-subgroup of G, and set $N = N_G(P)$. Suppose that $\gamma : L(P) \to L'(P)$ is a kN-homomorphism. If there is a G-local homomorphism $\xi : L \to L'$ with $\xi(P) = \gamma$, then γ is G-stable.*

Let L and L' be G-local modules, and let P be a Sylow p-subgroup of G. Set $N = N_G(P)$. We will define a homomorphism

$$T = T_P^{L,L'} : \underline{\mathrm{Hom}}_{kN}\big(L(P), L'(P)\big) \to \mathrm{Hom}_{\mathfrak{L}(G,k)}(L, L').$$

Because P is a Sylow p-subgroup of G, we have $e_{N,P} \cong k_N$. The maps $\iota : k_G \to k_N^{\uparrow G}$ and $\varepsilon : k_N^{\uparrow G} \to k_G$ given by $\iota(x) = \sum_{g \in G/N} g \otimes x$ and $\varepsilon\big(\sum_{g\in G/N} g \otimes x_g\big) = \sum_g x_g$ satisfy $\varepsilon\iota = 1_k$ because $|G : N| \equiv 1 \pmod p$. If $\gamma : L(P) \to L'(P)$ is a kN-homomorphism, let $T\gamma : L \to L'$ be the G-local homomorphism given by the composition

$$\begin{aligned} L \xrightarrow{\iota\otimes 1} k_N^{\uparrow G} \otimes L \cong e_{N,P}^{\uparrow G} \otimes L \cong \mathfrak{F}\big(L(P){\uparrow}^G\big) \xrightarrow{\mathfrak{F}(\gamma\uparrow^G)} \mathfrak{F}\big(L'(P){\uparrow}^G\big) \\ \cong e_{N,P}^{\uparrow G} \otimes L' \cong k_N^{\uparrow G} \otimes L' \xrightarrow{\varepsilon\otimes 1} L'. \end{aligned}$$

Proposition 5.4. *Let L and L' be G-local modules, and let P be a Sylow p-subgroup of G. Set $N = N_G(P)$, and let $\gamma : L(P) \to L'(P)$ be a kN-homomorphism.*

1. *If γ is G-stable, then $T\gamma : L \to L'$ is a G-local homomorphism with $(T\gamma)(P) = \gamma$.*
2. *If $L = \mathfrak{F}M$ and $L' = \mathfrak{F}M'$ for some kG-modules M and M', then $T\gamma = \mathfrak{F}(\mathrm{Tr}_N^G\,\gamma)$.*

Proof. To compute $(T\gamma)(P)$, we begin by observing that the definition of the isomorphism in Proposition 4.2 shows that the composition of the isomorphisms

$$\begin{aligned} L(P){\uparrow}^G{\downarrow}_N \cong e_{N,P} \otimes L(P){\uparrow}^G{\downarrow}_N = \mathfrak{F}\big(L(P){\uparrow}^G\big)(P) \\ \cong e_{N,P}^{\uparrow G}{\downarrow}_N \otimes L(P) \cong k_N^{\uparrow G}{\downarrow}_N \otimes L(P) \end{aligned}$$

is given by

$$\begin{aligned} L(P){\uparrow}^G{\downarrow}_N &\cong \bigoplus_{g\in N\backslash G/N} \big(g \otimes L(P)\big){\downarrow}_{{}^gN\cap N}{\uparrow}^N \\ &\cong \bigoplus_g L({}^gP){\downarrow}_{{}^gN\cap N}{\uparrow}^N \\ &\cong \bigoplus_g L({}^gP\cap P){\downarrow}_{{}^gN\cap N}{\uparrow}^N \\ &\cong \bigoplus_g L(P){\downarrow}_{{}^gN\cap N}{\uparrow}^N \\ &\cong \bigoplus_g k_{{}^gN\cap N}^{\uparrow G} \otimes L(P) \\ &\cong k_N^{\uparrow G}{\downarrow}_N \otimes L(P). \end{aligned}$$

If γ is G-stable, then there is a commutative diagram

$$\begin{array}{ccccccc} L(P) & \cong & k\otimes L(P) & \xrightarrow{\iota\otimes 1} & k_N^{\uparrow G}{\downarrow}_N \otimes L(P) & \cong & L(P){\uparrow}^G{\downarrow}_N \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle 1\otimes\gamma} & & \downarrow{\scriptstyle 1\otimes\gamma} & & \downarrow{\scriptstyle\gamma\uparrow^G\downarrow_N} \\ L'(P) & \cong & k\otimes L'(P) & \xleftarrow[\varepsilon\otimes 1]{} & k_N^{\uparrow G}{\downarrow}_N \otimes L'(P) & \cong & L'(P){\uparrow}^G{\downarrow}_N. \end{array}$$

Hence $(T\gamma)(P) = \gamma$, and (1) holds.

Now suppose that $L = \mathfrak{F}M$ and $L' = \mathfrak{F}M'$. Identifying $e_{N,P}$ with k_N, we may assume that $L(P) = M{\downarrow}_N$ and $L'(P) = M'{\downarrow}_N$. Then the isomorphism

$$k_N^{\uparrow G}{\downarrow}_N \otimes L(P) = k_N^{\uparrow G}{\downarrow}_N \otimes M{\downarrow}_N \cong M{\downarrow}_N{\uparrow}^G{\downarrow}_N = L(P){\uparrow}^G{\downarrow}_N$$

is given by $(g \otimes 1) \otimes m \mapsto g \otimes g^{-1}m$, and similarly for L' instead of L. It follows that the composition

$$\begin{aligned} M{\downarrow_N} &\xrightarrow{\iota\otimes 1} k_N^{\uparrow G}{\downarrow_N} \otimes M{\downarrow_N} \cong M{\downarrow_N}{\uparrow^G}{\downarrow_N} \\ &\xrightarrow{\gamma{\uparrow^G}{\downarrow_N}} M'{\downarrow_N}{\uparrow^G}{\downarrow_N} \cong k_N^{\uparrow G}{\downarrow_N} \otimes M'{\downarrow_N} \xrightarrow{\varepsilon\otimes 1} M'{\downarrow_N} \end{aligned}$$

is equal to $\operatorname{Tr}_N^G \gamma$. Thus $T\gamma : \mathfrak{F}M \to \mathfrak{F}M'$ and $\mathfrak{F}(\operatorname{Tr}_N^G \gamma) : \mathfrak{F}M \to \mathfrak{F}M'$ are G-local homomorphisms with $(T\gamma)(P) = \mathfrak{F}(\operatorname{Tr}_N^G \gamma)(P)$. Proposition 5.1 now implies that $T\gamma = \mathfrak{F}(\operatorname{Tr}_N^G \gamma)$, as desired. ☐

Corollary 5.5. *The functor $\mathfrak{F} : kG\text{-}\underline{\mathrm{Mod}} \to \mathfrak{L}(G,k)$ is full.*

Proof. Let M and M' be kG-modules, and let $\xi : \mathfrak{F}M \to \mathfrak{F}M'$ be a G-local homomorphism. Let P be a Sylow p-subgroup of G, and set $N = N_G(P)$. Then $\xi(P)$ is G-stable, so $T\xi(P) : \mathfrak{F}M \to \mathfrak{F}M'$ satisfies $\big(T\xi(P)\big)(P) = \xi(P)$. Hence $\xi = T\xi(P) = \mathfrak{F}\big(\operatorname{Tr}_N^G \xi(P)\big)$, and $\mathfrak{F}$ is full. ☐

As another consequence of Proposition 5.4, we present a result that strengthens the statement of Proposition 3.1.

Corollary 5.6. *Let $\xi : L \to L'$ be a G-local homomorphism, and let P be a Sylow p-subgroup of G. Then ξ is an isomorphism if and only if $\xi(P)$ is an isomorphism.*

Proof. It is clear that if ξ is an isomorphism, then $\xi(P)$ is an isomorphism. To prove the converse, suppose that $\xi(P)$ is an isomorphism. Because $\xi(P)$ is G-stable, the map $\xi(P)^{-1} : L'(P) \to L(P)$ is also G-stable. By Proposition 5.4 there is a G-local homomorphism $\xi' : L' \to L$ with $\xi'(P) = \xi(P)^{-1}$. Then $(\xi'\xi)(P) = 1_{L(P)}$ and $(\xi\xi')(P) = 1_{L'(P)}$, so $\xi'\xi = 1_L$ and $\xi\xi' = 1_{L'}$ by Proposition 5.1. Thus ξ is an isomorphism. ☐

The following technical lemma is needed to show that the functor $\mathfrak{F}$ is essentially surjective.

Lemma 5.7. *Let $\xi_1 : L_1 \to L_1'$ and $\xi_2 : L_2 \to L_2'$ be G-local homomorphisms, and let P be a Sylow p-subgroup of G. Set $N = N_G(P)$. Suppose that $\gamma : L_1(P) \to L_2(P)$ and $\gamma' : L_1'(P) \to L_2'(P)$ are kN-homomorphisms such that there is a commutative diagram*

$$\begin{array}{ccc} L_1(P) & \xrightarrow{\quad\xi_1(P)\quad} & L_1'(P) \\ {\scriptstyle\gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle\gamma'} \\ L_2(P) & \xrightarrow[\quad\xi_2(P)\quad]{} & L_2'(P). \end{array}$$

Then $(T\gamma') \circ \xi_1 = \xi_2 \circ (T\gamma)$.

Proof. The result follows immediately from the commutativity of the diagram

$$\begin{array}{ccccccccc} L_1 & \xrightarrow{\iota\otimes 1} & k_N^{\uparrow G} \otimes L_1 & \cong & \mathfrak{F}\big(L_1(P){\uparrow^G}\big) & \xrightarrow{\mathfrak{F}(\gamma{\uparrow^G})} & \mathfrak{F}\big(L_2(P){\uparrow^G}\big) & \cong & k_N^{\uparrow G} \otimes L_2 & \xrightarrow{\varepsilon\otimes 1} & L_2 \\ {\scriptstyle\xi_1}\Big\downarrow & & \Big\downarrow{\scriptstyle 1\otimes\xi_1} & & \Big\downarrow{\scriptstyle\mathfrak{F}(\xi_1(P){\uparrow^G})} & & \Big\downarrow{\scriptstyle\mathfrak{F}(\xi_2(P){\uparrow^G})} & & \Big\downarrow{\scriptstyle 1\otimes\xi_2} & & \Big\downarrow{\scriptstyle\xi_2} \\ L_1' & \xrightarrow[\iota\otimes 1]{} & k_N^{\uparrow G} \otimes L_1' & \cong & \mathfrak{F}\big(L_1'(P){\uparrow^G}\big) & \xrightarrow[\mathfrak{F}(\gamma'{\uparrow^G})]{} & \mathfrak{F}\big(L_2'(P){\uparrow^G}\big) & \cong & k_N^{\uparrow G} \otimes L_2' & \xrightarrow[\varepsilon\otimes 1]{} & L_2'. \end{array}$$

☐

Let r denote the p-rank of G, and suppose that s is an integer with $0 \le s \le r$. Let $kG\text{-}\underline{\mathrm{Mod}}(s)$ denote the full subcategory of $kG\text{-}\underline{\mathrm{Mod}}$ consisting of all modules M with $e_s \otimes M \cong M$, and let $\mathfrak{L}_s(G,k)$ be the full subcategory of $\mathfrak{L}(G,k)$ consisting of all objects L such that $e_s \otimes L \cong L$. It is easy to check that if M is any kG-module, then M is in $kG\text{-}\underline{\mathrm{Mod}}(s)$ if and only if $\mathfrak{F}M$ is in $\mathfrak{L}_s(G,k)$. In particular, the restriction of $\mathfrak{F}$ to $kG\text{-}\underline{\mathrm{Mod}}(s)$ defines a functor $\mathfrak{F}_s : kG\text{-}\underline{\mathrm{Mod}}(s) \to \mathfrak{L}_s(G,k)$.

Lemma 5.8. *Let $0 \le s \le r-1$, and assume that the functor $\mathfrak{F}_s : kG\text{-}\underline{\mathrm{Mod}}(s) \to \mathfrak{L}_s(G,k)$ is essentially surjective. Then $\mathfrak{F}_{s+1}$ is also essentially surjective.*

Proof. Let L be an object of $\mathfrak{L}_{s+1}(G,k)$. Then $e_s \otimes L$ is an object of $\mathfrak{L}_s(G,k)$, so there is a module M in $kG\text{-}\underline{\mathrm{Mod}}(s)$ such that $\mathfrak{F}M \cong e_s \otimes L$. Let $\xi : \mathfrak{F}M \to L$ denote the composition

$$\mathfrak{F}M \cong e_s \otimes L \xrightarrow{\eta\otimes 1} L.$$

Then $1 \otimes \xi : e_s \otimes \mathfrak{F}M \to e_s \otimes L$ is an isomorphism.

Let $E_1, \dots, E_n$ be representatives for the conjugacy classes of elementary abelian p-subgroups of rank $s+1$ in G. Fix i with $1 \le i \le n$, and consider the map $\beta_{E_i} = \beta_s \otimes 1$ occurring in the triangle

$$e_s{\downarrow_{N_i}} \otimes L(E_i) \longrightarrow L(E_i) \longrightarrow f_s{\downarrow_{N_i}} \otimes L(E_i) \xrightarrow{\beta_s\otimes 1} \Omega^{-1}k \otimes e_s{\downarrow_{N_i}} \otimes L(E_i).$$

Using the isomorphisms

$$\begin{aligned} &\underline{\mathrm{Hom}}_{kN_i}\big(f_s{\downarrow_{N_i}} \otimes L(E_i), \Omega^{-1}k \otimes e_s{\downarrow_{N_i}} \otimes L(E_i)\big)\\ &\quad\cong \underline{\mathrm{Hom}}_{kN_i}\big(f_s{\downarrow_{N_i}} \otimes L(E_i), \Omega^{-1}k \otimes (\mathfrak{F}M)(E_i)\big)\\ &\quad= \underline{\mathrm{Hom}}_{kN_i}\big(f_s{\downarrow} \otimes L(E_i), \Omega^{-1}k \otimes e_{N_i,E_i} \otimes M{\downarrow}\big)\\ &\quad\cong \underline{\mathrm{Hom}}_{kN_i}\big(f_s{\downarrow_{N_i}} \otimes L(E_i), \Omega^{-1}k \otimes M{\downarrow_{N_i}}\big)\\ &\quad\cong \underline{\mathrm{Hom}}_{kG}\big(f_s \otimes L(E_i){\uparrow^G}, \Omega^{-1}k \otimes M\big), \end{aligned}$$

we obtain a map $\beta'_{E_i} : f_s \otimes L(E_i){\uparrow^G} \to \Omega^{-1}k \otimes M$ corresponding to β_{E_i}. Set $M_{s+1} = \bigoplus_{i=1}^n L(E_i){\uparrow^G}$. Taking the direct sum of the maps β'_{E_i} for $1 \le i \le n$ gives a map $\beta : f_s \otimes M_{s+1} \to \Omega^{-1}k \otimes M$. Let M' be the kG-module defined by the triangle

$$M \xrightarrow{\mu} M' \longrightarrow f_s \otimes M_{s+1} \xrightarrow{\beta} \Omega^{-1}k \otimes M.$$

Then $\bar{V}_G(M') \subseteq \bar{V}_G(M) \cup \bar{V}_G(f_s \otimes M_{s+1}) \subseteq V_{s+1}$, so M' is an object of $kG\text{-}\underline{\mathrm{Mod}}(s+1)$. We will show that $\mathfrak{F}M' \cong L$.

Observe that for $1 \le i \le n$ the map $\beta'_{E_i} : f_s \otimes L(E_i){\uparrow^G} \to \Omega^{-1}k \otimes M$ factors as

$$\begin{aligned} &f_s \otimes L(E_i){\uparrow^G} \xrightarrow{\beta_s\otimes 1} \Omega^{-1}k \otimes e_s \otimes L(E_i){\uparrow^G} \cong \Omega^{-1}k \otimes \big(e_s{\downarrow_{N_i}} \otimes L(E_i)\big){\uparrow^G}\\ &\quad\cong \Omega^{-1}k \otimes (e_{N_i,E_i} \otimes M{\downarrow_{N_i}}){\uparrow^G} \cong \Omega^{-1}k \otimes e_{N_i,E_i}^{\uparrow G} \otimes M \xrightarrow{1\otimes\varepsilon\eta_i{\uparrow^G}\otimes 1} \Omega^{-1}k \otimes M. \end{aligned}$$

Let P be a Sylow p-subgroup of G, and set $N = N_G(P)$. Then the map $(\mathfrak{F}\beta'_{E_i})(P)$ in $kN\text{-}\underline{\mathrm{Mod}}$ is given by the composition

$$\begin{aligned} e_{N,P} \otimes f_s{\downarrow_N} \otimes L(E_i){\uparrow^G}{\downarrow_N} &\to e_{N,P} \otimes \Omega^{-1}k \otimes e_{N_i,E_i}^{\uparrow G}{\downarrow} \otimes M{\downarrow}\\ &\xrightarrow{1\otimes 1\otimes\varepsilon\eta_i{\uparrow^G}\otimes 1} e_{N,P} \otimes \Omega^{-1}k \otimes M{\downarrow_N}. \end{aligned}$$

After a permutation of the tensor factors, the first map in this composition is just the composition around the diagram

$$\begin{array}{ccc}
f_s{\downarrow_N} \otimes e_{N,P} \otimes L(E_i){\uparrow^G}{\downarrow_N} & \cong & f_s{\downarrow_N} \otimes e^{\uparrow G}_{N_i,E_i}{\downarrow_N} \otimes L(P) \\
\Big\downarrow{\scriptstyle \beta_s\otimes 1\otimes 1} & & \Big\downarrow{\scriptstyle \beta_s\otimes 1\otimes 1} \\
\Omega^{-1}k \otimes e_s{\downarrow_N} \otimes e_{N,P} \otimes L(E_i){\uparrow^G}{\downarrow_N} & \cong & \Omega^{-1}k \otimes e_s{\downarrow_N} \otimes e^{\uparrow G}_{N_i,E_i}{\downarrow_N} \otimes L(P) \\
\Big\downarrow{\scriptstyle \left(1\otimes 1\otimes 1\otimes \xi(E_i)\uparrow\downarrow\right)^{-1}} & & \Big\downarrow{\scriptstyle \left(1\otimes 1\otimes 1\otimes \xi(P)\right)^{-1}} \\
\Omega^{-1}k \otimes e_s{\downarrow_N} \otimes e_{N,P} \otimes (e_{N_i,E_i} \otimes M{\downarrow_{N_i}}){\uparrow^G}{\downarrow_N} & \cong & \Omega^{-1}k \otimes e_s{\downarrow_N} \otimes e^{\uparrow G}_{N_i,E_i}{\downarrow_N} \otimes e_{N,P} \otimes M{\downarrow_N} \\
\Big\downarrow{\scriptstyle 1\otimes\eta\otimes 1\otimes 1 \;\cong} & & \Big\downarrow{\scriptstyle \cong\; 1\otimes\eta\otimes 1\otimes 1\otimes 1} \\
\Omega^{-1}k \otimes e_{N,P} \otimes (e_{N_i,E_i} \otimes M{\downarrow_{N_i}}){\uparrow^G}{\downarrow_N} & \cong & \Omega^{-1}k \otimes e^{\uparrow G}_{N_i,E_i}{\downarrow_N} \otimes e_{N,P} \otimes M{\downarrow_N},
\end{array}$$

in which the middle square commutes by Proposition 4.2. It follows, therefore, that the triangle in which the third map is

$$e_{N,P} \otimes f_s{\downarrow_N} \otimes M_{s+1}{\downarrow_N} \xrightarrow{1\otimes\beta} e_{N,P} \otimes \Omega^{-1}k \otimes M{\downarrow_N} \cong \Omega^{-1}k \otimes e_{N,P} \otimes M{\downarrow_N}$$

is isomorphic to the triangle in which the third map is given by the composition

$$\begin{aligned}
f_s{\downarrow_N} \otimes \Big(\bigoplus_{i=1}^{n} e^{\uparrow G}_{N_i,E_i}{\downarrow_N}\Big) \otimes L(P) &\xrightarrow{\beta_s\otimes 1\otimes 1} \Omega^{-1}k \otimes e_s{\downarrow_N} \otimes \Big(\bigoplus_{i=1}^{n} e^{\uparrow G}_{N_i,E_i}{\downarrow_N}\Big) \otimes L(P) \\
&\xrightarrow{\left(1\otimes 1\otimes 1\otimes\xi(P)\right)^{-1}} \Omega^{-1}k \otimes e_s{\downarrow_N} \otimes \Big(\bigoplus_{i=1}^{n} e^{\uparrow G}_{N_i,E_i}{\downarrow_N}\Big) \otimes e_{N,P} \otimes M{\downarrow_N} \\
&\xrightarrow{1\otimes 1\otimes(\bigoplus \varepsilon\eta_i\uparrow\downarrow)\otimes 1\otimes 1} \Omega^{-1}k \otimes e_s{\downarrow_N} \otimes e_{N,P} \otimes M{\downarrow_N}.
\end{aligned}$$

Let us call this composition $\tilde{\beta}$. For $1 \le i \le n$ let $\alpha_i : e^{\uparrow G}_{N_i,E_i} \to e_{s+1}$ be the map defined in Proposition 2.7, and let $\alpha : \bigoplus_{i=1}^{n} e^{\uparrow G}_{N_i,E_i} \to e_{s+1}$ be the direct sum of the α_i. Then $\eta\alpha = \bigoplus_{i=1}^{n} \varepsilon\eta_i{\uparrow^G}$, and one can check that there is a commutative diagram

$$\begin{array}{ccc}
f_s{\downarrow_N} \otimes \left(\bigoplus_{i=1}^{n} e^{\uparrow G}_{N_i,E_i}{\downarrow_N}\right) \otimes L(P) & \xrightarrow{\tilde{\beta}} & \Omega^{-1}k \otimes e_s{\downarrow_N} \otimes e_{N,P} \otimes M{\downarrow_N} \\
\Big\downarrow{\scriptstyle 1\otimes\alpha\otimes 1} & & \Big\downarrow{\scriptstyle 1\otimes 1\otimes\xi(P)} \\
f_s{\downarrow_N} \otimes e_{s+1}{\downarrow_N} \otimes L(P) & \xrightarrow[\beta_s\otimes\eta\otimes 1]{} & \Omega^{-1}k \otimes e_s{\downarrow_N} \otimes L(P).
\end{array}$$

Now $1 \otimes \alpha : f_s \otimes \left(\bigoplus_{i=1}^{n} e^{\uparrow G}_{N_i,E_i}\right) \to f_s \otimes e_{s+1}$ is a stable isomorphism by Proposition 2.7, and $1 \otimes \xi(P) : e_s{\downarrow_N} \otimes e_{N,P} \otimes M{\downarrow_N} \to e_s{\downarrow_N} \otimes L(P)$ is also an isomorphism.

It follows that there are isomorphisms of triangles

$$\begin{array}{ccccccc}
e_{N,P}\otimes M{\downarrow} & \xrightarrow{1\otimes\mu} & e_{N,P}\otimes M'{\downarrow} & \longrightarrow & f_s{\downarrow}\otimes e_{N,P}\otimes M_{s+1}{\downarrow} & \longrightarrow & \Omega^{-1}k\otimes e_{N,P}\otimes M{\downarrow} \\
\cong\Big\downarrow{\scriptstyle(\eta\otimes 1\otimes 1)^{-1}} & & \Big\| & & \cong\Big\downarrow{\scriptstyle\gamma} & & {\scriptstyle(1\otimes\eta\otimes 1\otimes 1)^{-1}}\Big\downarrow \\
e_s{\downarrow}\otimes e_{N,P}\otimes M{\downarrow} & \xrightarrow{\eta\otimes 1\otimes\mu} & e_{N,P}\otimes M'{\downarrow} & \longrightarrow & f_s{\downarrow}\otimes\big(\bigoplus_i e^{\uparrow G}_{N_i,E_i}{\downarrow}\big)\otimes L(P) & \xrightarrow{\tilde\beta} & \Omega^{-1}k\otimes e_s{\downarrow}\otimes e_{N,P}\otimes M{\downarrow} \\
\Big\downarrow{\scriptstyle 1\otimes\xi(P)} & & {\scriptstyle\zeta}\Big\downarrow & & \Big\downarrow{\scriptstyle 1\otimes\alpha\otimes 1} & & {\scriptstyle 1\otimes 1\otimes\xi(P)}\Big\downarrow \\
e_s{\downarrow}\otimes L(P) & \xrightarrow[\eta\otimes 1]{} & L(P) & \longrightarrow & f_s{\downarrow}\otimes e_{s+1}{\downarrow}\otimes L(P) & \xrightarrow[\beta_s\otimes\eta\otimes 1]{} & \Omega^{-1}k\otimes e_s{\downarrow}\otimes L(P)
\end{array}$$

for some $\zeta : e_{N,P}\otimes M'{\downarrow}_N \to L(P)$.

The first and last triangles in this commutative diagram correspond to G-local homomorphisms

$$\mathfrak{F}M \xrightarrow{\mathfrak{F}\mu} \mathfrak{F}M' \longrightarrow f_s\otimes\mathfrak{F}M_{s+1} \longrightarrow \Omega^{-1}k\otimes\mathfrak{F}M$$

and

$$e_s\otimes L \xrightarrow{\eta\otimes 1} L \longrightarrow f_s\otimes e_{s+1}\otimes L \longrightarrow \Omega^{-1}k\otimes e_s\otimes L.$$

Because there are G-local homomorphisms

$$\mathfrak{F}M \xrightarrow{(\eta\otimes 1)^{-1}} e_s\otimes\mathfrak{F}M \xrightarrow{1\otimes\xi} e_s\otimes L$$

and

$$\begin{aligned} f_s\otimes\mathfrak{F}M_{s+1} &\cong f_s\otimes\bigoplus_{i=1}^n \mathfrak{F}\big(L(E_i){\uparrow^G}\big) \cong f_s\otimes\big(\bigoplus_{i=1}^n e^{\uparrow G}_{N_i,E_i}\big)\otimes L \\ &\xrightarrow{1\otimes\alpha\otimes 1} f_s\otimes e_{s+1}\otimes L, \end{aligned}$$

Proposition 5.3 shows that the compositions

$$e_{N,P}\otimes M{\downarrow}_N \xrightarrow{(\eta\otimes 1\otimes 1)^{-1}} e_s{\downarrow}_N\otimes e_{N,P}\otimes M{\downarrow}_N \xrightarrow{1\otimes\xi(P)} e_s{\downarrow}_N\otimes L(P)$$

and

$$\begin{aligned} f_s{\downarrow}_N\otimes e_{N,P}\otimes M_{s+1}{\downarrow}_N &\underset{\cong}{\xrightarrow{\gamma}} f_s{\downarrow}\otimes\big(\bigoplus_{i=1}^n e^{\uparrow G}_{N_i,E_i}{\downarrow}\big)\otimes L(P) \\ &\xrightarrow{1\otimes\alpha\otimes 1} f_s{\downarrow}_N\otimes e_{s+1}{\downarrow}_N\otimes L(P) \end{aligned}$$

are both G-stable. Define

$$\sigma = (1\otimes\xi(P))\circ(\eta\otimes 1\otimes 1)^{-1} : e_{N,P}\otimes M{\downarrow}_N \to e_s{\downarrow}_N\otimes L(P).$$

Then Proposition 5.4 and Lemma 5.7 imply that there is a commutative diagram of triangles

$$\begin{array}{ccccccc}
e_{N,P}\otimes M{\downarrow}_N & \xrightarrow{1\otimes\mu} & e_{N,P}\otimes M'{\downarrow}_N & \longrightarrow & f_s{\downarrow}_N\otimes e_{N,P}\otimes M_{s+1}{\downarrow}_N & \longrightarrow & \Omega^{-1}k\otimes e_{N,P}\otimes M{\downarrow}_N \\
{\scriptstyle\sigma}\Big\downarrow & & {\scriptstyle(T\zeta)(P)}\Big\downarrow & & \Big\downarrow{\scriptstyle(1\otimes\alpha\otimes 1)\gamma} & & \Big\downarrow{\scriptstyle 1\otimes\sigma} \\
e_s{\downarrow}_N\otimes L(P) & \xrightarrow[\eta\otimes 1]{} & L(P) & \longrightarrow & f_s{\downarrow}_N\otimes e_{s+1}{\downarrow}_N\otimes L(P) & \longrightarrow & \Omega^{-1}k\otimes e_s{\downarrow}_N\otimes L(P).
\end{array}$$

Hence $(T\zeta)(P) : e_{N,P}\otimes M'{\downarrow}_N \to L(P)$ is an isomorphism. Corollary 5.6 implies that $T\zeta : \mathfrak{F}M' \to L$ is an isomorphism in $\mathfrak{L}_{s+1}(G,k)$, and this completes the proof. □

We can now prove the main result of the paper.

Theorem 5.9. *The functor* $\mathfrak{F} : kG\text{-}\underline{\mathrm{Mod}} \to \mathfrak{L}(G,k)$ *is an equivalence of categories.*

Proof. The functor $\mathfrak{F}_0 : kG\text{-}\underline{\mathrm{Mod}}(0) \to \mathfrak{L}_0(G,k)$ is trivially an equivalence because all objects of both $kG\text{-}\underline{\mathrm{Mod}}(0)$ and $\mathfrak{L}_0(G,k)$ are isomorphic to zero. For $1 \le s \le r$ Lemma 5.8 implies by induction that $\mathfrak{F}_s$ is essentially surjective. In particular, $\mathfrak{F} = \mathfrak{F}_r$ is essentially surjective.

Now suppose that $\gamma : M \to M'$ is a kG-homomorphism such that $\mathfrak{F}\gamma = 0$. Let P be a Sylow p-subgroup of G, and set $N = N_G(P)$. Identifying $e_{N,P}$ with k_N, we see that $0 = (\mathfrak{F}\gamma)(P) = \gamma{\downarrow_N}$ and hence $\gamma = 0$. Thus $\mathfrak{F}$ is faithful. Since $\mathfrak{F}$ is full by Corollary 5.5, it follows that $\mathfrak{F}$ is an equivalence, as desired. □

Assume that P is a Sylow p-subgroup of G, and set $N = N_G(P)$. Let $\mathfrak{l}(G,k)$ be the full subcategory of $\mathfrak{L}(G,k)$ consisting of the objects L such that $L(P)$ is stably isomorphic to a finitely generated kN-module. If M is a finitely generated kG-module, then $\mathfrak{F}M$ is an object of $\mathfrak{l}(G,k)$, so the restriction of $\mathfrak{F}$ defines a functor $\mathfrak{f} : kG\text{-}\underline{\mathrm{mod}} \to \mathfrak{l}(G,k)$.

Corollary 5.10. *The functor* $\mathfrak{f} : kG\text{-}\underline{\mathrm{mod}} \to \mathfrak{l}(G,k)$ *is an equivalence of categories.*

Proof. Theorem 5.9 implies that $\mathfrak{f}$ is full and faithful, so it is only necessary to show that $\mathfrak{f}$ is essentially surjective. Let L be an object of $\mathfrak{l}(G,k)$. Then there is a kG-module M with $\mathfrak{F}M \cong L$ in $\mathfrak{L}(G,k)$, and we may assume without loss of generality that M has no projective summands. The definition of $\mathfrak{l}(G,k)$ shows that $M{\downarrow_N} \cong M_0 \oplus Q$ in kN-Mod for some finitely generated module M_0 and some projective module Q. But M is a summand of $M{\downarrow_N}{\uparrow^G} \cong M_0{\uparrow^G} \oplus Q{\uparrow^G}$, and M has no projective summands. Thus M is a summand of $M_0{\uparrow^G}$, so that M is finitely generated. It follows that $\mathfrak{f}M \cong L$ in $\mathfrak{l}(G,k)$, as desired. □

We now prove the result on restrictions of G-local modules stated in Section 4.

Corollary 5.11. *Let L be an object of $\mathfrak{L}(G,k)$, and let $P \in \mathcal{P}(G)$. Then there is an isomorphism $\mathfrak{F}\big(L(P)\big) \cong e_{N,P} \otimes L{\downarrow_N}$ in $\mathfrak{L}(N,k)$, and the isomorphism is natural in L.*

Proof. Let M be a kG-module such that there is an isomorphism $\xi : L \to \mathfrak{F}M$ in $\mathfrak{L}(G,k)$. Then we obtain isomorphisms

$$\begin{aligned}\mathfrak{F}\big(L(P)\big) \xrightarrow{\mathfrak{F}(\xi(P))} \mathfrak{F}(e_{N,P} \otimes M{\downarrow_N}) &\cong e_{N,P} \otimes \mathfrak{F}(M{\downarrow_N}) \\ &\cong e_{N,P} \otimes (\mathfrak{F}M){\downarrow_N} \cong e_{N,P} \otimes L{\downarrow_N}\end{aligned}$$

in $\mathfrak{L}(N,k)$. It is easy to check that the resulting isomorphism $\mathfrak{F}\big(L(P)\big) \cong e_{N,P} \otimes L{\downarrow_N}$ is natural. □

If B is a block of kG, then it seems reasonable to expect that there is a category that is equivalent to the stable category $B\text{-}\underline{\mathrm{Mod}}$ and has a definition similar to that of $\mathfrak{L}(G,k)$. Unfortunately, the definition of $\mathfrak{L}(G,k)$ does not seem to be compatible with the block structure of kG. It is not clear, therefore, whether an appropriate construction can be obtained simply by modifying the techniques used here; significantly different ideas may be necessary.

References

[1] D. J. Benson, J. F. Carlson, and J. Rickard, *Complexity and varieties for infinitely generated modules, II,* Math. Proc. Cambridge Philos. Soc. 120 (1996), 597–615. MR **97f:**20008

[2] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, Cambridge Univ. Press, Cambridge, 1988. MR **89e:**16035

[3] J. Rickard, *Idempotent modules in the stable category,* J. London Math. Soc. (2) 56 (1997), 149–170. MR **998d:**20058

[4] W. W. Wheeler, *Quillen stratification for the stable module category,* Quart. J. Math. Oxford (2) 50 (1999), 355–369. MR **2000m:**20085

Center for Communications Research, 4320 Westerra Court, San Diego, California 92121

E-mail address: wheeler@member.ams.org