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THREE-DIVISIBLE FAMILIES OF SKEW LINES ON A SMOOTH PROJECTIVE QUINTIC

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Abstract. We give an example of a family of 15 skew lines on a quintic such that its class is divisible by 3. We study properties of the codes given by arrangements of disjoint lines on quintics.

1. Introduction

The aim of this note is to study sets of lines on quintic surfaces $Y \subset \mathbb{P}_3(\mathbb{C})$. A divisor Λ is said to be 3-*divisible* if its class in $\mathrm{NS}(Y)$ is divisible by 3. If

$$\Lambda = \sum_{1}^{p} L_i - \sum_{p+1}^{p+q} L_i,$$

with disjoint lines $L_i \subset Y$, is 3-divisible, we call it a (p,q)-*divisor*. In this note we give an example of a $(12,3)$-divisor (see Sect. 2). We prove bounds on the weight and the dimension of the code given by a (p,q)-divisor. We also collect restrictions on the numbers (p,q), see Sect. 4.

It is well-known that the maximal number of disjoint lines on a smooth quartic X_4 is 16 ([10]). If a family of skew lines on X_4 is even (i.e. divisible by 2), then it consists of eight or 16 lines ([10]). Every family of 16 skew lines on a quartic is a sum of even families of eight lines, whereas a family of eight lines is even under a condition that involves the existence of a configuration of rational or elliptic curves ([1]).

In the case of a smooth quintic surface Y, it is not known whether the bounds [9] (at most 30 skew lines), [13] (at most 147 lines) are sharp. One can see, however, that a family of 30 skew lines on Y, if there is any, defines a d-dimensional $\mathbb{F}_3$-code with $d \geq 4$. Moreover, by the Griesmer bound, such a Y must contain (p,q)-divisors with $p+q \leq 18$.

In this note we prove that the support of a (p,q)-divisor contains at least 15 lines, and that this bound is sharp. We give analogues of conditions [1] for a family of skew lines to be 3-divisible. We apply S.-L. Tan's generalization of Beauville's technique to prove that every (p,q)-divisor is a sum of (r,s)-divisors with $r, s \leq 15$. Finally, we show that a reduced 3-divisible family must consist of 15 lines, and compute invariants of the triple covers associated with (p,q)-divisors.

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2. Three-divisible divisors with at most 18 lines

Let Y be a smooth quintic surface in $\mathbb{P}_3(\mathbb{C})$. In this note we study properties of the divisors

$$\Lambda := \sum_1^p L_i - \sum_{p+1}^{p+q} L_i = 3\mathcal{L},$$

where the L_i are pairwise disjoint lines on Y and $\mathcal{L} \in \mathrm{Pic}(Y)$. The divisor Λ will be called a (p,q)-*divisor*. We assume $p \geq q$.

Let $\mathcal{K}_Y$ be the canonical divisor of Y. We have the equalities

$$\mathcal{L}^2 = -\frac{p+q}{3}, \quad (\mathcal{K}_Y - \mathcal{L})^2 = 5 - p + \frac{q}{3}.$$

Since $\deg(\mathcal{L}) = \frac{1}{3}(p-q)$, both p and q are divisible by 3. Moreover,

$$\mathrm{h}^0(\mathcal{L}) = 0 \text{ for every } p,\, q. \tag{2.1}$$

Indeed, suppose that $\mathcal{L}$ is effective. Then, since $\mathcal{L}.L_i = -1$ for $i \leq p$, we have $\mathcal{L} = L_1 + \ldots + L_p + D'$, with an effective D', so $\deg(\mathcal{L}) \geq p$. Contradiction.

By Riemann-Roch we have $\chi(\mathcal{L}) = 5 - \frac{p}{3}$, which implies that

$$\mathcal{K}_Y - \mathcal{L} = \mathcal{O}(1) - \mathcal{L} \text{ is effective for } p \leq 12, \tag{2.2}$$

$$\mathcal{O}(1) - \mathcal{L} \text{ is moving (i.e. } \mathrm{h}^0(\mathcal{O}(1) - \mathcal{L}) \geq 2) \text{ for } p \leq 9. \tag{2.3}$$

Lemma 2.1. *Let D be an effective divisor on Y with $D^2 < 4$.*

a) If $\deg(D) \leq 4$, then D is moving iff it is a plane quartic residual to a line $L_0 \subset Y$. It moves by the pencil of planes through L_0. In particular, $D^2 = 0$.

b) If $\deg(D) = 5$ and D is moving, then the unique fixed component of $|D|$ is a line L, and the generic $D_1 \in |D - L|$ is irreducible. Moreover, $L.D_1 < 2$.

Proof. a) An irreducible curve $\subset Y$ of degree ≤ 3 has arithmetic genus ≤ 1 and negative self-intersection on Y, a surface of general type. Hence if D is moving, then $\deg(D) = 4$, and the generic $C \in |D|$ is irreducible. If the generic $C \in |D|$ is singular, we find a base point at which all $C \in |D|$ have multiplicity ≥ 2, which yields $D^2 \geq 2^2$, and contradicts our assumption. By adjunction and Castelnuovo's inequality, the generic $C \in |D|$ is planar. Let L_0 be the line in $|\mathcal{O}(1) - C|$. Then $L_0.C = 4$, which implies $|D| = |\mathcal{O}(1) - L_0|$.

b) If the generic $C \in |D|$ is irreducible, we can find a smooth quintic in $|D|$. By Castelnuovo's inequality the only smooth irreducible quintic $D \subset Y$ with non-negative self-intersection is the planar one. Hence $|D| = |D_1| + L$, where L is a line. By Bertini ([6, Lemma on p.536]), components of the generic $C \in |D_1|$ have non-negative self-intersection, and C is irreducible.

If $L.D_1 \geq 2$, we have $D_1^2 < 7 - 2D_1.L < 4$, so a) implies that D_1 is planar. Thus $D_1 \in |\mathcal{O}(1) - L|$, and $D^2 = 5$. Contradiction. □

Theorem 2.2. *If a divisor of the type (p,q) is 3-divisible, then either $p \geq 15$, or $p = 12$ and $q \geq 3$, or $p = q = 9$.*

Proof. Assume $0 < p \leq 9$. Then (see (2.3)) $D := \mathcal{O}(1) - \mathcal{L}$ is a moving divisor of degree $5 + \frac{1}{3}(q-p) \leq 5$ with $D^2 < 4$.

Suppose $\deg(D) = 4$. Lemma 2.1.a implies $\mathcal{O}(1) - \mathcal{L} = \mathcal{O}(1) - L_0$, where $L_0 \subset Y$ is a line. Hence $\mathcal{L}$ is effective, which contradicts (2.1).

Assume $0 < p = q \leq 6$ (i.e. $\deg(D) = 5$). Let L be the fixed component of $|D|$. Then $(\mathcal{L} + D_1).L = (\mathcal{K}_Y - L).L = 4$. Since $3\mathcal{L}.L \leq 6$, we get $D_1.L \geq 2$. The latter cannot happen by Lemma 2.1.b.

For $(p, q) = (12, 0)$, we have $\deg(D) = 1$, so D is a line with $D^2 = -7$. □

Example 2.3. Consider the surface $S := \mathrm{V}(X_1^4 X_2 + X_2^4 X_1 + X_3^4 X_4 + X_4^4 X_3)$. We introduce the following notation:

ω (resp. ε) stands for a primitive root of unity of degree 3 (resp. 5),

$A_{i,j}$ denotes the line $\mathrm{V}(X_i, X_j)$, where $1 \leq i \leq 2 < j \leq 4$,

$C_{l,s}$ is the line that passes through $(0 : 1 : 0 : -\eta^4)$, $(-\eta : 0 : 1 : 0)$, where $\eta := \omega^l \varepsilon^s$ and $0 \leq l \leq 2$, $0 \leq s \leq 4$,

$D_{l,s}$ is the line that passes through $(1 : 0 : 0 : -\eta^4)$, $(0 : -\eta : 1 : 0)$,

Q_s, where $0 \leq s \leq 4$, denotes the quadric $\mathrm{V}(X_1 X_2 - \varepsilon^{2s} X_3 X_4)$.

Observe that $Q_s \cap S$ consists of the lines $A_{i,j}$, $C_{l,s}$, $D_{l,s}$, where $l = 0, 1, 2$, $i = 1, 2$, $j = 3, 4$. Moreover, $C_{l,s}$, $A_{1,4}$, $A_{2,3}$ belong to one ruling of Q_s, whereas $D_{l,s}$, $A_{1,3}$, $A_{2,4}$ belong to the other one. Fix an $s_0 \in \{0, \ldots, 4\}$ and define

$$\Lambda := \sum_{\substack{j=0,\ldots,2 \\ s \neq s_0}} C_{j,s} - \sum_{j=0,\ldots,2} D_{j,s_0}.$$

Observe that $C_{l_1,s_1} \cap D_{l_2,s_2} \neq \emptyset$ iff $s_1 = s_2$, which implies that $\mathrm{supp}(\Lambda)$ consists of 15 disjoint lines. Furthermore, $A_{1,3}$, $A_{2,4}$ meet the twelve lines $C_{l,s} \subset \mathrm{supp}(\Lambda)$, and do not meet the lines D_{j,s_0}.

One can prove that S contains precisely 55 lines, and that the maximal number of pairwise disjoint lines on S is 19 (e.g. the family

$$C_{i,j}, \mathrm{V}(X_1, X_3 + \omega X_4), \mathrm{V}(X_2, X_3 + X_4), \mathrm{V}(X_3, X_1 + \omega X_2), \mathrm{V}(X_4, X_1 + X_2)$$

consists of 19 skew lines). Moreover, the only divisors supported by at least 15 skew lines which meet all of those 55 lines with multiplicities divisible by 3 are the ten divisors defined as Λ. The fact that Λ is a $(12, 3)$-divisor results from Lemma 2.4.

Lemma 2.4. *Let $L_1, \ldots, L_{15}$ be a family of skew lines on a smooth projective quintic Y. Then the following conditions are equivalent:*

(1) $3 \mid \sum_1^{12} L_i - \sum_{13}^{15} L_i$ *in* $\mathrm{H}^2(Y, \mathbb{Z})$,

(2) there exist disjoint lines $M_1, M_2 \subset Y$ such that for $k = 1, 2$

$$M_k \cap L_i \neq \emptyset \text{ for } i \leq 12, \text{ and } M_k \cap L_i = \emptyset \text{ for } i = 13, 14, 15.$$

Moreover, in this case $M_1 + M_2 \in |\mathcal{K}_Y - \frac{1}{3}(\sum_1^{12} L_i - \sum_{13}^{15} L_i)|$.

Proof. "(1) ⇒ (2)": Put $D := \mathcal{K}_Y - \mathcal{L}$, where $3\mathcal{L} = \sum_1^{12} L_i - \sum_{13}^{15} L_i$. Since $D^2 = -6$, the system $|D|$ contains no irreducible conic. As $\chi(\mathcal{L}) = 1$, the system $|\mathcal{K}_Y - \mathcal{L}|$ must contain a sum of disjoint lines $M_1 + M_2$. Then $D.M_k = -3 = 1 - \mathcal{L}.M_k$. Thus $3\mathcal{L}.M_k = 12$, and $M_k.L_i = 1$ iff $i \leq 12$.

"(2) ⇒ (1)": Put $F := 3\mathcal{K}_Y - 3(M_1 + M_2) - (\sum_1^{12} L_i - \sum_{13}^{15} L_i)$. Then

$$F^2 = 0 \text{ and } (\mathcal{K}_Y.F) = 0$$

imply $F = r\mathcal{K}_Y$, where $r \in \mathbb{Q}$, by the Hodge index theorem (recall that $\mathrm{H}^2(Y, \mathbb{Z})$ is torsion-free). But $0 = F^2 = 5r^2$ shows that $F = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$. □

Let $L_1, \ldots, L_{18}$ be a family of skew lines on a smooth quintic Y.

Lemma 2.5. *The following conditions are equivalent:*
(1) $3 \mid \sum_1^{12} L_i - \sum_{13}^{18} L_i$ *in* $\mathrm{H}^2(Y, \mathbb{Z})$,
(2) there exists a smooth rational cubic $C \subset Y$ *such that*

$$C.L_i = 2 \ \textit{for} \ i \leq 12, \ \textit{and} \ C \cap L_i = \emptyset \ \textit{for} \ i > 12.$$

Moreover, in this case $C \in |\mathcal{K}_Y - \frac{1}{3}(\sum_1^{12} L_i - \sum_{13}^{18} L_i)|$.

Proof. "(1) $\Rightarrow$ (2)": Define D and $\mathcal{L}$ as in the proof of Lemma 2.4. By the equality (2.2) the system $|D|$ contains a (possibly reducible) cubic C. Since $\mathrm{p_a}(C) = 0$ and $C.L_i = 2$ (resp. $C.L_i = 0$) for $i \leq 12$ (resp. $i > 12$), it suffices to prove that C is irreducible.

Suppose that the contrary holds. Since $D^2 = -5$, we must have $C = C_2 + L$, where C_2 (resp. L) is a (possibly singular) conic (resp. a line) and $L.C_2 = 1$. Thus $\mathcal{L}.L = (\mathcal{K}_Y - C_2 - L).L = 3$. Hence $C_2.3\mathcal{L} = 15$, and there exist two lines L_i, L_j such that $L_i.C_2 = L_j.C_2 = 2$. Contradiction.

"(2) $\Rightarrow$ (1)": Apply the Hodge index theorem. □

Lemma 2.6. *The following conditions are equivalent:*
(1) $3 \mid \sum_1^{9} L_i - \sum_{10}^{18} L_i$ *in* $\mathrm{H}^2(Y, \mathbb{Z})$,
(2) there exist disjoint lines $M_1, M_2 \subset Y$ *such that*

$$M_1 \cap L_i \neq \emptyset \ \textit{for} \ i \leq 9, \ \textit{and} \ M_1 \cap L_i = \emptyset \, \textit{for} \, i > 9,$$

$$M_2 \cap L_i \neq \emptyset \ \textit{for} \ i > 9, \ \textit{and} \ M_2 \cap L_i = \emptyset \, \textit{for} \, i \leq 9.$$

Moreover, in this case $\sum_1^{9} L_i - \sum_{10}^{18} L_i = 3(M_2 - M_1)$ *in* $\mathrm{H}^2(Y, \mathbb{Z})$.

Proof. "(1) $\Rightarrow$ (2)": Put $D_1 := \mathcal{K}_Y - \mathcal{L}$, $D_2 := \mathcal{K}_Y + \mathcal{L}$, where $3\mathcal{L} = \sum_1^9 L_i - \sum_{10}^{18} L_i$. By Lemma 2.1.b $|D_k| = |C_k| + M_k$, where M_k is a line. Lemma 2.1.a implies that C_k is planar. Hence $C_1.L_i \leq 1$. Since $D_1.L_i = 2$ (resp. $D_1.L_i = 0$) for $i \leq 9$ (resp. $i > 9$), we get $M_1.L_i = 1$ (resp. $M_1.L_i = 0$) when $i \leq 9$ (resp. $i > 9$). The proof for M_2 is analogous.

As $(C_1 + M_1).M_2 = 1 - \mathcal{L}.M_2 = 4$ and C_1 is planar, we have $M_1.M_2 = 0$.

"(2) $\Rightarrow$ (1)": Define $F := 3(M_2 - M_1) - (\sum_1^9 L_i - \sum_{10}^{18} L_i)$ and apply the Hodge index theorem. □

3. Covers and codes given by (p, q)-divisors

Let us consider the mapping $\varphi : \mathbb{F}_3^{p+q} \rightarrow \mathrm{Pic}(Y) \otimes \mathbb{F}_3$ given by the formula

$$\varphi : \sum_1^{p+q} \mu_j e_j \mapsto \sum_1^{p+q} \mu_j L_j. \tag{3.1}$$

Observe that every vector (word) in $\ker(\varphi)$ corresponds to an effective 3-divisible $\Lambda' \subset \Lambda$ (we write $\Lambda' \subset \Lambda$ instead of $\mathrm{supp}(\Lambda') \subset \mathrm{supp}(\Lambda)$).

In this section we prove a lower bound on $\dim_{\mathbb{F}_3}(\ker \varphi)$.

Put $\mathcal{L}' := \mathcal{L} + \sum_{p+1}^{p+q} L_i$. Riemann-Roch yields that

$$\chi(-\mathcal{L}') = 5 - \frac{q}{3}, \ \text{and} \ \chi(-2\mathcal{L}') = 5 - \frac{p}{3} - 2q.$$

The bundle $\mathcal{L}'$ defines a $3:1$ cyclic cover $Z \rightarrow Y$ branched over the lines L_i (see [3, I.§17]) . Its structure sheaf $\mathcal{O}_Z$ is the $\mathcal{O}_Y$-algebra

$$\mathcal{O}_Y \oplus \mathcal{O}_Y(-\mathcal{L}') \oplus \mathcal{O}_Y(-2\mathcal{L}'). \tag{3.2}$$

The surface Z is smooth over the lines L_i, where $i \leq p$. It is singular (with local equation $z^3 = x^2$) over the lines L_i for $i > p$. Let $\nu : X \to Z$ be the normalization of Z, and let $\tilde{L}_i \subset X$ be the (reduced) rational curve lying over L_i. Then X is smooth and we have the exact normalization sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_X \to \bigoplus_{p+1}^{p+q} \mathrm{I}_{\tilde{L}_i}/\mathrm{I}^2_{\tilde{L}_i} \to 0, \tag{3.3}$$

where $\mathrm{I}_{\tilde{L}_i}/\mathrm{I}^2_{\tilde{L}_i} = \mathcal{O}_{\tilde{L}_i}(1)$ and $\chi(\mathrm{I}_{\tilde{L}_i}/\mathrm{I}^2_{\tilde{L}_i}) = 2$. Hence $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Z) + 2q$. By (3.2) we have $\chi(\mathcal{O}_X) = 15 - \frac{p+q}{3}$.

Lemma 3.1.

$$\mathrm{h}^1(\mathcal{O}_X) \geq \frac{p}{3} - 5.$$

Proof. The sequence (3.3) implies that $\mathrm{h}^1(\mathcal{O}_X) = \mathrm{h}^1(\mathcal{O}_Z) - 2q$. But

$$\mathrm{h}^1(\mathcal{O}_Z) = \mathrm{h}^1(\mathcal{O}_Y) + \mathrm{h}^1(-\mathcal{L}') + \mathrm{h}^1(-2\mathcal{L}') \geq -\chi(-2\mathcal{L}') = \frac{p}{3} + 2q - 5.$$

□

Observe that Z is given by the Galois triple cover data $(\mathcal{L}, \mathcal{L}^2, 1, c)$ (see [8, Remark 7.6]), where $1 \in \mathcal{O}_Y$ and $\mathrm{V}(c) = \sum_1^p L_i + 2\sum_{p+1}^{p+q} L_i$ (i.e. $\sum_1^p L_i + 2\sum_{p+1}^{p+q} L_i$ is the divisor of zeroes of $c \in \mathrm{H}^0(\mathcal{L}^3)$). By [8, Prop. 7.5] X is given by $(\mathcal{L}, \tilde{M}, \tilde{b}, \tilde{c})$, where $\tilde{M} \in \mathrm{Pic}(Y)$, $\mathrm{V}(\tilde{b}) = \sum_{p+1}^{p+q} L_i$, and $\mathrm{V}(\tilde{c}) = \sum_1^p L_i$.

We will need the following version of [7, Lemma 3.2.1]:

Lemma 3.2.

$$\dim_{\mathbb{F}_3}(\ker\varphi) \geq 1 + h^1(\mathcal{O}_X).$$

Proof ([7]). Let $\mathrm{Tor}_3(X)$ denote the group of 3-torsions in $\mathrm{Pic}(X)$ and let $\mathrm{Tor}_3(X)^G$ stand for its subgroup of divisors invariant under the action of the cyclic group G of the automorphisms of the cover. One can imitate the proof of [4, Lemma 2] (see [7, Lemma 1.2.1] for details) to construct an isomorphism

$$\mathrm{Tor}_3(X)^G \simeq \ker\varphi / \mathbb{F}_3 e,$$

where $e = \sum_1^p e_i + \sum_{p+1}^{p+q} 2e_i \in \mathbb{F}_3^{p+q}$. This yields the equality

$$\dim_{\mathbb{F}_3}(\ker\varphi) = 1 + \dim_{\mathbb{F}_3}(\mathrm{Tor}_3(X)^G).$$

[7, Cor. 1.2.3] gives the inequality $\dim_{\mathbb{F}_3}(\mathrm{Tor}_3(X)^G) \geq \frac{1}{2}\dim_{\mathbb{F}_3}(\mathrm{Tor}_3(X))$.

(To prove that inequality, define $\beta : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ by $\beta(\eta) = \sigma^*\eta - \eta$, where $\sigma \neq \mathrm{id}$, $\sigma \in G$. We have the exact sequence

$$0 \longrightarrow \mathrm{Tor}_3(X)^G \longrightarrow \mathrm{Tor}_3(X) \xrightarrow{\beta} \mathrm{Im}(\beta) \longrightarrow 0.$$

But the sequence $0 \longrightarrow \mathrm{Tor}_3(X)^G \cap \mathrm{Im}(\beta) \longrightarrow \mathrm{Im}(\beta) \xrightarrow{\beta} \mathrm{Im}(\beta^2)$ is exact and $\mathrm{Im}(\beta^2) \subset \pi^*\mathrm{Tor}_3(Y) = 0$, where $\pi : X \to Z \to Y$ is the cover.)

The inequality $\dim_{\mathbb{F}_3}(\mathrm{Tor}_3(X)) \geq 2 \cdot \mathrm{h}^1(\mathcal{O}_X)$ concludes the proof. □

Corollary 3.3. *For every (p,q)-divisor, the following inequality holds:*

$$\dim_{\mathbb{F}_3}(\ker\varphi) \geq \frac{p}{3} - 4.$$

Theorem 3.4. *Every (p,q)-divisor Λ, where $p > 15$ or $q > 15$, is a sum of (r_i, s_i)-divisors Λ_i, with both $r_i, s_i \leq 15$.*

4. Reduced (p,q)-divisors

For $v \in \mathbb{F}_3^n$, one defines its *weight* $\mathrm{w}(v)$ as the number of its non-zero coordinates. An $[n,d,r]$-*code* is a d-dimensional subspace of $\mathbb{F}_3^n$ such that r is the minimum of weights of its elements. We have the Griesmer bound ([11])

$$n \geq \sum_{i=0}^{i=d-1} \lceil \tfrac{r}{3^i} \rceil . \tag{4.1}$$

Lemma 4.1. *There are no $(p,0)$-divisors for $p = 18, \ldots, 24$.*

Proof. Fix a $(p,0)$-divisor Λ, where $18 \leq p \leq 24$, and consider the map φ (see (3.1)). By Cor. 3.3, $\ker(\varphi)$ is a $[p,d,r]$-code with $d \geq 2$.

Suppose that $\Lambda_1 \subset \Lambda$, where Λ_1 is a $(15-s,s)$-divisor and $s \leq 3$. Then $\Lambda - \Lambda_1$ is a $(p-15,s)$-divisor. Thm. 2.2 yields that $p \geq 27$. Thus $r \geq 18$.

By (4.1) Λ is a $(24,0)$-divisor, and it must contain an $(18-s,s)$-divisor Λ_2 with $s \leq 9$. Then $\Lambda - \Lambda_2$ is a $(6,s)$-divisor with $s \leq 9$. Contradiction. □

Remark 4.2. Let Λ be an $(18,3)$-divisor. One can see that it contains no $(15-s,s)$-divisors. Hence, by (4.1), there are no $(18,3)$-divisors.

Lemma 4.3. *There are no $(27,0)$-divisors.*

Proof. Let $3|\Lambda$, where $\Lambda := \sum_1^{27} L_i$. Thm 2.2 implies that Λ can contain only $(9,9)$, $(12,12)$, $(12,3)$ and $(3,12)$ divisors. By Cor. 3.3 and (4.1) it contains a $(12,3)$-divisor. We can assume that $\sum_1^{12} L_i - \sum_{24}^{27} L_i = 3\mathcal{L}_1$ in $\mathrm{Pic}(Y)$. Then

$$\Lambda = 3\mathcal{L}_1 + 3\mathcal{L}_2 + 3\sum_{24}^{27} L_i,$$

where $3\mathcal{L}_2 = \sum_{13}^{24} L_i - \sum_{24}^{27} L_i$. Since $(\mathcal{K}_Y - \mathcal{L}_1).(\mathcal{K}_Y - \mathcal{L}_2) = -2$, by Lemma 2.4, there exist lines M_1, M_2, M_3 such that $M_k + M_3 \in |\mathcal{K}_Y - \mathcal{L}_k|$ for $k = 1,2$. M_3 meets L_i iff $i \leq 24$, so $(M_1 + M_3).\mathcal{L}_2 = 4$ gives $M_1.L_i = 0$ for $i > 12$.

Claim 1. $\{v \in \ker(\varphi) \,|\, w(v) \neq 18\} = \mathrm{span}\{3\mathcal{L}_1,\, 3\mathcal{L}_2\}$.

Proof. Suppose $3\mathcal{L}_3 \subset \Lambda$, where $3\mathcal{L}_3$ is a $(12,3)$-divisor in which precisely $\hat{p}$ (resp. $\hat{q}$) lines L_{25}, L_{26}, L_{27} appear with multiplicity 1 (resp. -1). Observe that $3|(\hat{p} - \hat{q}) = 9 - 3\mathcal{L}_3.M_3$. Since

$$((\mathcal{K}_Y - \mathcal{L}_1) + (\mathcal{K}_Y - \mathcal{L}_2)).(\mathcal{K}_Y - \mathcal{L}_3) = -5 + (\hat{p} - \hat{q}) < 0,$$

we can assume that $(\mathcal{K}_Y - \mathcal{L}_1).(\mathcal{K}_Y - \mathcal{L}_3) < 0$. If $C \in |\mathcal{K}_Y - \mathcal{L}_3|$, then C contains either M_1 or M_3, so $\hat{p} = 0$.

In the first case, we have $3\mathcal{L}_3 = \sum_1^{12} L_i - \sum_{k=1}^{3} L_{j_k}$, so either $\mathcal{L}_3 = \mathcal{L}_1$ (i.e. $\hat{q} = 3$) or all $j_k \leq 24$ ($\hat{q} = 0$). If the latter holds, then $(\mathcal{K}_Y - \mathcal{L}_1).(\mathcal{K}_Y - \mathcal{L}_3) = -5$, which contradicts Lemma 2.4.

Suppose that $M_3 \subset C$. We have $3\mathcal{L}_3 = \sum_{k=1}^{12} L_{j_k} - \sum_{25}^{27} L_i$ with $j_k \leq 24$. Then we can assume $j_k \leq 12$ for $k \leq 6$, whence $\mathcal{L}_1.\mathcal{L}_3 \leq -3$, so $(\mathcal{K}_Y - \mathcal{L}_1).(\mathcal{K}_Y - \mathcal{L}_3) < -3$. Lemma 2.4 implies $\mathcal{L}_1 = \mathcal{L}_3$.

If $\Lambda_1 \subset \Lambda$ is a $(12,12)$-divisor, then $\Lambda - \Lambda_1$ is a $(3,12)$-divisor. Thus $\pm(3\mathcal{L}_1 - 3\mathcal{L}_2)$ are the only $(12,12)$-divisors $\subset \Lambda$. □Claim 1

Claim 2. *In every* $(9,9)$*-divisor precisely one of the lines* L_{25}*,* L_{26}*,* L_{27} *appears with multiplicity* 1*, and precisely one with multiplicity* -1*.*

Proof. Let $\sum_{k=1}^{9} L_{l_k} - \sum_{k=10}^{18} L_{l_k} = 3(N_2 - N_1)$, where l_k is a permutation and N_1, N_2 are lines. Put $\mathcal{L}_3 := N_2 - N_1$, $3\mathcal{L}_4 := \sum_{k=19}^{27} L_{l_k} - \sum_{k=1}^{9} L_{l_k}$, where $\mathcal{L}_4 \in \mathrm{Pic}(Y)$. Then $\mathcal{L}_3.\mathcal{L}_4 = 3$. Thus, by Lemma 2.6, there exists a line $N_3 \subset Y$ such that $N_3.(N_1 + N_2) = 0$ and $N_3.L_{l_k} = 1$ iff $k > 18$. Moreover, $N_1.L_{l_k} = 1$ (resp. $N_2.L_{l_k} = 1$) iff $k < 9$ (resp. $9 < k \leq 18$).

Let p_j (resp. q_j) be the number of the lines L_i, where $i \leq 12$ (resp. $i > 24$), that meet N_j. Assume that $q_1 \geq q_3 \geq q_2$. Suppose that $(q_1, q_2, q_3) \neq (1,1,1)$. Then $q_1 \geq 2$ and $q_2 = 0$. Observe that $3 \,|\, (p_1 - q_1) = 3\mathcal{L}_1.N_1$.

If $p_1 > 0$, then $p_1 \geq 2$, so at least two lines L_i meet N_1, M_1, M_3. Hence $N_1.(M_1 + M_3) = 0$, and $(\mathcal{K}_Y - \mathcal{L}_1).\mathcal{L}_3 = (M_1 + M_3).(N_2 - N_1) \geq 0$. Since

$$((\mathcal{K}_Y - \mathcal{L}_1) + (\mathcal{K}_Y - \mathcal{L}_2)).\mathcal{L}_3 = -(q_1 - q_2) \leq -2,$$

we have $(M_2 + M_3).(N_2 - N_1) \leq -2$, which contradicts $M_3.N_1 = 0$. Therefore, $p_1 = 0$ and $q_1 = 3$.

Repeating the same reasoning for L_i, where $12 < i \leq 24$, and $3\mathcal{L}_2$, one gets $N_1.L_i = 0$ for $i \leq 24$. Contradiction. $\square_{\text{Claim 2}}$

Fix a $(9,9)$-divisor $3\mathcal{L}_3 \subset \Lambda$. Let $3\mathcal{L}_4 \subset \Lambda$ be a $(9,9)$-divisor. One can see that in the word given by one of the divisors $\pm 3\mathcal{L}_3 \pm 3\mathcal{L}_4$ all the lines L_{25}, L_{26}, L_{27} appear either with multiplicity 1 or with multiplicity 0, so (by Claim 2) that word is not given by a $(9,9)$-divisor, and (by Claim 1) belongs to $\mathrm{span}\{3\mathcal{L}_1, 3\mathcal{L}_2\}$. Therefore, $3\mathcal{L}_4 \in \mathrm{span}\{3\mathcal{L}_1, 3\mathcal{L}_2, 3\mathcal{L}_3\}$, so $\dim(\ker(\varphi)) = 3$, which contradicts Cor. 3.3. $\square$

Remark 4.4. Similar reasoning yields that there are no $(21,3)$-divisors.

Theorem 4.5. *If a divisor of the type* $(p,0)$ *is divisible by* 3*, then* $p = 15$*.*

Proof. We are to prove that there are no $(30,0)$-divisors. Let $3 \,|\, \Lambda := \sum_1^{30} L_i$.

Thm 2.2 implies that Λ contains no (p,q)-divisors with $18 \leq p < 30$. If $\Lambda_1 \subset \Lambda$ is a $(12,6)$-divisor, then $\Lambda_2 := \Lambda - \Lambda_1$ is a $(12,6)$-divisor, and $(\mathcal{K}_Y - \frac{1}{3}\Lambda_1).(\mathcal{K}_Y - \frac{1}{3}\Lambda_2) = -1$, which contradicts Lemma 2.5.

Every (p,q)-divisor $\Lambda_1 \subset \Lambda$ defines the 3-divisible divisors $\Lambda \pm \Lambda_1$. Thus $\ker(\varphi)$ has the folowing word distribution (we omit $(0,0)$,$\pm(30,0)$):

$\pm(15,0)$	$\pm(12,3)$	$(9,9)$	$\pm(15,3)$	$\pm(12,9)$	$\pm(15,12)$	$\pm(15,15)$
$4\mathrm{d}_1$	$2\mathrm{d}_2$	$2\mathrm{d}_3$	$2\mathrm{d}_2$	$4\mathrm{d}_3$	$2\mathrm{d}_2$	$2\mathrm{d}_1$

where $d_1 \leq 1$ (because $(15,0) - (15,0) = (15 - s, 15 - s)$, and Thm 2.2 implies $s \leq 6$. Then $(15,0) + (15,0) = (30 - 2s, s)$, so $s = 0$). Hence, by Cor. 3.3,

$$(d_2 + d_3) \geq \frac{1}{6}(3^6 - 9) = 120. \tag{4.2}$$

If $3\mathcal{L}_1 \neq 3\mathcal{L}_2 \subset \Lambda$ are $(9,9)$-divisors, where $\mathcal{L}_1, \mathcal{L}_2$ are the divisors described in Lemma 2.6, and $\mathrm{supp}(\mathcal{L}_1) \cap \mathrm{supp}(\mathcal{L}_2)$ contains a line, then one of the divisors $3\mathcal{L}_1 \pm 3\mathcal{L}_2$ is a $(9 - r, 9 - r)$-divisor. Thus $r = 0$, which contradicts Lemma 4.3.

Let $3\mathcal{L}_1 \neq 3\mathcal{L}_2 \subset \Lambda$ be $(12,3)$-divisors, $C_i \in |\mathcal{K}_Y - \mathcal{L}_i|$, and let $C_1 \cap C_2$ contain a line. Then, by Lemma 2.4, $\mathrm{supp}(3\mathcal{L}_1) \cap \mathrm{supp}(3\mathcal{L}_2)$ consists of either 3 or 6 lines. In both cases, Λ contains a $(12,12)$-divisor Λ_3 (given by $3\mathcal{L}_1 - 3\mathcal{L}_2$, resp. $3\mathcal{L}_1 + 3\mathcal{L}_2$), so $\Lambda - \Lambda_3 = (6,12)$. Contradiction.

Let $3\mathcal{L}_1$ (resp. $3\mathcal{L}_2$) $\subset \Lambda$ be a $(9,9)$-divisor (resp. $(12,3)$), where $\mathcal{L}_1 := N_2 - N_1$ and N_1, N_2 are lines. Let $C \in |\mathcal{K}_Y - \mathcal{L}_2|$. If $\mathrm{supp}(\mathcal{L}_1) \cap C$ contains a line, then either

N_1 or N_2 meets at least twelve lines L_i. By Lemma 2.6 there exists a $(9,9)$-divisor Λ_4 such that $3\mathcal{L}_1 - \Lambda_4$ is a $(3,3)$-divisor.

Hence (4.2) implies that Y contains at least $2\cdot 120+30 = 270$ lines. Contradiction (see [13, §.4], [5, p.28]). □

Let W be the surface obtained by blowing down the (-1)-curves $\tilde{L}_i$. Then

$$\mathcal{K}_W^2 = 15 + (p+q), \text{ and } \mathrm{c}_2(W) = 165 - 5(p+q).$$

One can check that a $(21-s,s)$-divisor Λ contains no $(15-s,s)$-divisors, so $\dim_{\mathbb{F}_3}(\ker\varphi) = 1$. We get the following table of invariants.

$p+q$	(p,q)	$\mathrm{h}^1(\mathcal{O}_W)$	$\mathrm{p_g}(W)$	$\mathrm{c}_1^2(W)$	$\mathrm{c}_2(W)$
15	$(12,3)$, $(15,0)$	0	9	30	90
18	$(9,9)$, $(12,6)$, $(15,3)$	0	8	33	75
21	$(12,9)$, $(15,6)$	0	7	36	60

For $p+q \geq 24$, Lemma 3.2 no longer implies $\mathrm{h}^1(\mathcal{O}_W) = 0$. Observe that for $p+q = 30$, the equality $\mathrm{c}_1^2(W) = 3\mathrm{c}_2(W)$ holds.

$p+q$	(p,q)-divisors	$\mathrm{c}_1^2(W)$	$\mathrm{c}_2(W)$
24	$(12,12)$, $(15,9)$, $(18,6)$	39	45
27	$(15,12)$, $(18,9)$, $(21,6)$, $(24,3)$	42	30
30	$(15,15)$, $(18,12)$, $(21,9)$, $(24,6)$, $(27,3)$	45	15

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