THE BERGMAN METRIC ON A STEIN MANIFOLD
WITH A BOUNDED PLURISUBHARMONIC FUNCTION

BO-YONG CHEN AND JIN-HAO ZHANG

Abstract. In this article, we use the pluricomplex Green function to give a sufficient condition for the existence and the completeness of the Bergman metric. As a consequence, we proved that a simply connected complete Kähler manifold possesses a complete Bergman metric provided that the Riemann sectional curvature $\leq -A/p^2$, which implies a conjecture of Greene and Wu. Moreover, we obtain a sharp estimate for the Bergman distance on such manifolds.

1. Introduction

Let $M$ be a complex $n$-dimensional manifold. Let $\mathcal{H}$ be the space of holomorphic $n$-forms on $M$ such that $|\int_M f \wedge \bar{f}| < \infty$. This space is a separable complex Hilbert space with an inner product $\langle f_1, f_2 \rangle = i^{n^2} \int_M f_1 \wedge \bar{f}_2$. Let $h_0, h_1, \ldots$ be a complete orthonormal basis for $\mathcal{H}$. Then the $2n$-form defined on $M \times M$ given by $K = \sum_{j=0}^{\infty} h_j \wedge h_j$ is called the Bergman kernel form of $M$. Let $z = (z_1, \ldots, z_n)$ be a local coordinate system in $M$ and let $K(z) = K^*(z)dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$, where $K^*$ is a locally defined function. Then $\beta := \partial \bar{\partial} \log K^*$ is a well-defined Hermitian form of bidegree $(1,1)$, whenever $K^*$ is nonzero. We say that $M$ possesses a Bergman metric iff $\beta$ is everywhere positive definite. In 1959, Kobayashi [11] began to investigate the completeness of the Bergman metric. After that, there are a lot of papers concerning the Bergman completeness for bounded pseudoconvex domains in $\mathbb{C}^n$ (see [3], [18] for a review). There are two general results: One says that any bounded hyperconvex domain is Bergman complete (cf. [1], [7]); the other states that any bounded pseudoconvex domain whose boundary can be locally described as the graph of a continuous function is also Bergman complete (cf. [4]). However, little is known for the Bergman metric of manifolds except the early work of Greene and Wu [6]. They proved that a simply connected complete Kähler manifold possesses a complete Bergman metric if the sectional curvature is pinched between two negative constants, or the curvature is nonpositive and the
following estimate holds outside a compact subset of $M$:
\[
\frac{B}{\rho^2} \leq \text{curvature} \leq \frac{A}{\rho^2}
\]
for some positive constants $A, B$ (in this paper, curvature will mean sectional curvature). Here $\rho$ denotes the distance function relative to some fixed point $o$ in $M$. They conjectured that the lower estimate $-\frac{B}{\rho^2}$ is unnecessary. We will solve this conjecture in the present paper.

To formulate our results precisely, we need some notions. Let $M$ be any complex manifold. We denote by $\text{PSH}(M)$ the set of all plurisubharmonic (psh) functions on $M$. According to Klimek [10], we define the pluricomplex Green function with a logarithmic pole on $M$ by
\[
g_M(x, y) = \sup \{u(x) \mid u \in \text{PSH}(M) \text{ satisfies the property that } u - \log |z| \text{ is bounded from above in a deleted neighborhood of } y \text{ for some holomorphic local coordinates } z \text{ centered at } y, z(y) = 0\}.
\]
It is known from [10] that for any $y \in M$ the function $g_M(\cdot, y)$ belongs to the above class, and it coincides with the classical (negative) Green function on hyperbolic Riemann surfaces (a Riemann surface is called hyperbolic if there is a negative nonconstant subharmonic function).

**Definition.** A complex manifold $M$ is said to satisfy the property $(B1)$ if for any $y \in M$ there is a positive number $a > 0$ such that the sublevel set $A(y, a) := \{x \in M : g_M(x, y) < -a\}$ is relatively compact in $M$.

It is easy to see that any bounded domain $D$ in $\mathbb{C}^n$ has the property $(B1)$ because of the trivial estimate $g_D(x, y) \geq \log \frac{R - |x|}{R}$, where $R_D$ is the diameter of $D$. We will also show in section 2 that any hyperbolic Riemann surface, any complex manifold carrying a bounded continuous strictly psh function, and any hyperconvex Stein manifold have the property $(B1)$. Following Stehle [17], we called a complex manifold $M$ hyperconvex if there exists a negative psh function $u$ such that the sublevel set $\{u < -c\}$ is relatively compact in $D$ for every $c > 0$.

**Definition.** A complex manifold $M$ is said to satisfy the property $(B2)$ if for any sequence of points $\{y_k\}, k = 1, 2, \cdots$, which has no adherent point in $M$ there exist a subsequence $\{y_{k_j}\}, j = 1, 2, \cdots$, and a number $a > 0$ such that for any compact subset $K$ one has $A(y_{k_j}, a) \subset M \setminus K$ for all sufficiently large $j$.

**Theorem 1.** If $M$ is a Stein manifold which satisfies the property $(B1)$, then it possesses a Bergman metric. If furthermore, $M$ satisfies the property $(B2)$, then the Bergman metric is complete.

With an application of Theorem 1, we solve the conjecture of Greene and Wu in the sequel.

**Theorem 2.** Let $M$ be a simply-connected complete Kähler manifold of dimension $n$ with nonpositive sectional curvature such that the inequality
\[
\text{curvature} \leq -\frac{A}{\rho^2}
\]
holds outside a compact subset of $M$ for a suitable positive constant $A$. Then $M$ possesses a complete Bergman metric.
We also have the following consequences of Theorem 1:

**Corollary 3.** Any hyperconvex Riemann surface is Bergman complete.

**Corollary 4.** Let \( D \) be a domain in \( \mathbb{C}^n \), not necessarily bounded. Suppose that there exists a negative \( C^2 \) psh exhaustion function \( \psi \) on \( D \), such that
\[
\partial \bar{\partial} \psi \geq \partial \bar{\partial} |z|^2.
\]
Then \( D \) is Bergman complete.

This domain was introduced by Sibony in [13], where he obtained an estimate of the Kobayashi metric for this domain.

In fact, this paper is a continuation of the paper [2], where the first-named author proved the Bergman completeness under the assumption of curvature \( c \) for some positive constant. Greene and Wu used the geometry method of Siu and Yau [16] to get a comparison of the Bergman metric and the Kähler metric of the manifold which implies the completeness of the Bergman metric; hence the hypothesis that the curvature is bounded from below is essential. In this paper we just verify Kobayashi’s criterion for Theorem 1 with help of the \( L^2 \) estimates of Hörmander type. Under the curvature condition of Theorem 2, Greene and Wu [6] constructed some special bounded psh exhaustion functions on the manifold. These functions enable us to show that the manifold satisfies the properties \((B1)\) and \((B2)\).

Using a recent result of Jost and Zuo [9] together with Theorem 1, we obtain a vanishing theorem for the \( L^2 \)-cohomology groups with respect to the Bergman metric. Let \((M, ds^2)\) be a complete Kähler manifold of dimension \( n \) and let \( \mathcal{H}^{p,q}(M) \) denote the space of square-integrable harmonic \((p,q)\) forms on \( M \). The result of Jost and Zuo says that if the sectional curvature is nonpositive, then \( \mathcal{H}^{p,q}(M) = \{0\} \) for \( p + q \neq n \). This implies the following

**Corollary 5.** Let \( M \) be as in Theorem 1. Suppose that the sectional curvature of the Bergman metric is nonpositive. Then one has \( \mathcal{H}^{p,q}(M) = \{0\} \) for \( p + q \neq n \), where \( \mathcal{H}^{p,q}(M) \) denotes the space of harmonic \((p,q)\) forms on \( M \) which are square-integrable with respect to the Bergman metric \( \beta \).

In 1995, Diederich and Ohsawa [5] introduced a method of estimating the Bergman distance, which is based on Kobayashi’s alternative definition of the Bergman metric. Inspired by their work, we are able to improve Theorem 2 as follows:

**Theorem 6.** Let \( M \) be a simply-connected complete Kähler manifold of dimension \( n \) with nonpositive sectional curvature.

1) If the inequality (1) holds outside a compact subset of \( M \) for suitable positive constant \( A \), then there exists a positive constant \( C' \) such that
\[
\text{dist}_\beta(o,x) \geq C' \log \rho(x),
\]
where \( \text{dist}_\beta(o,x) \) denotes the distance between \( o \) and \( x \) with respect to the Bergman metric.

2) If the curvature is bounded from above by a negative constant \(-A\), then
\[
\text{dist}_\beta(o,x) \geq C'' \rho(x)
\]
for a suitable constant \( C'' > 0 \).
2. Proof of Theorem 1

We assume first that $M$ satisfies the property $(B1)$, that is, for any $y \in M$ there is a number $a > 0$ so that $A(y, a) \subset M$. To prove the existence of the Bergman metric, it suffices to show, according to [11], the following two statements:

(i) Given any point $y$ of $M$, there exists a form $f \in \mathcal{H}$ such that $f(y) \neq 0$.

(ii) For any holomorphic vector $X$ at $y$, there exists a form $f \in \mathcal{H}$ such that $f^*(0) = 0$ and $Xf^*(0) \neq 0$, where $f(z) = f^*(z)dz_1 \wedge \cdots \wedge dz_n$ in a local coordinate system centered at $y$.

Since $M$ is Stein, there exist $n$ holomorphic functions $\zeta_1, \cdots, \zeta_n$ on $M$ which form a local coordinate system centered at $y$. Without loss of generality, we assume $X = \partial/\partial \zeta_1$. We take a cut-off function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi \equiv 1$ on $(-\infty, 1]$ and $\chi \equiv 0$ on $[0, \infty)$. We set

$$
\eta = \left\{ \begin{array}{ll}
\chi (-\log (-g_M(\cdot, y) + a) + \log (2a)) & \text{for case (i)}, \\
\chi (-\log (-g_M(\cdot, y) + a) + \log (2a)) \zeta_1 & \text{for case (ii)},
\end{array} \right.
$$

and

$$
\varphi = 2(n + 1)g_M(\cdot, y) - \log (-g_M(\cdot, y) + a).
$$

Clearly $\varphi \in PSH(M)$. We will show that there exists a constant $C = C(y, a)$ so that the equation $\overline{\partial} u = \overline{\partial} \eta \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$ has a solution in the distribution sense such that the following inequality holds:

$$
\left| \int_M u \wedge \overline{\partial} e^{-\varphi} \right| \leq C.
$$

If we have proved the above fact under the assumption that the function $g_M(\cdot, y)$ is $C^\infty$, then for the general case, we can exhaust $M$ by an increasing sequence of relatively compact Stein domains $M_j, j = 1, 2, \cdots$, and for each $j$ the psh function $g_M(\cdot, y)$ can be approximated uniformly on $M_j$ by negative strictly psh functions $\psi_{j,k}, k = 1, 2, \cdots$. We replace $g_M(\cdot, y)$ by $\psi_{j,k}$. It follows that there is a solution to the equation $\overline{\partial} u_j = \overline{\partial} \eta \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$ on $M_j$ together with the estimate

$$
\left| \int_{M_j} u_{j,k} \wedge \overline{\partial} e^{-\varphi} \right| \leq C
$$

for a suitable constant $C > 0$ depending only on $y$ and $a$. To obtain the desired solution, we only need to take a weak limit as $k, j \to \infty$. This limiting procedure is standard (cf. [4]). Hence we may assume $g_M(\cdot, y)$ is $C^\infty$. Next, we need an $L^2$ estimate of Hörmander type for the $\partial$-equation on complete Kähler manifolds.

Proposition 7 (cf. [13]). Let $M$ be a complete Kähler manifold and let $\varphi$ be a $C^\infty$ strictly psh function on $M$. Then for any $\partial$-closed $(n, 1)$ form with $\int_M |\nabla_\varphi|^2 e^{-\varphi} dV_\varphi < \infty$, there is an $n$-form $u$ on $M$ such that $\partial u = v$ and

$$
\left| \int_M u \wedge \partial \overline{\partial} e^{-\varphi} \right| \leq \int_M |\nabla_\varphi|^2 e^{-\varphi} dV_\varphi,
$$

where $dV_\varphi$ denotes the volume with respect to $\partial \overline{\partial} \varphi$. 

This proposition gives us a solution to the equation \( \overline{\partial}u = \overline{\partial}\eta \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n \) with the following inequality:

\[
\left| \int_M u \wedge \bar{u} e^{-\varphi} \right| \leq \int_M |\overline{\partial}\eta|^2_{\overline{\partial}\varphi} e^{-\varphi} dV \varphi \leq C_1,
\]

noting that \( |\overline{\partial}\chi|_{\overline{\partial}\varphi} \leq \sup |\chi'| \) because
\[
\overline{\partial}\varphi \geq -\overline{\partial}\varphi \log(-g_M(\cdot, y) + a) \geq \overline{\partial}\log(-g_M(\cdot, y) + a) \overline{\partial}\log(-g_M(\cdot, y) + a).
\]

Here \( C_1 \) is a positive constant depending only on \( y, a \) and the choice of \( \chi; | \cdot |_{\overline{\partial}\varphi} \) denotes the point norm w.r.t. the metric \( \overline{\partial}\varphi \). This implies that the form \( f := \eta \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n - u \in H \), because \( \varphi \) is bounded from above. Moreover, the singularity of \( \varphi \) shows that \( f^*(0) = 1 \) for case (i), while \( f^*(0) = 0 \), \( \partial f^*/\partial\zeta_1(0) = 1 \) for case (ii). This completes the first part of the proof.

To prove the second part, we need the criterion of Kobayashi for Bergman completeness:

**Proposition 8 (cf. [12]).** Let \( M \) be a complex manifold which possesses a Bergman metric. Assume that there exists a dense subspace \( H' \) of \( H \) such that for any \( f \in H' \) and for any sequence of points \( \{y_k\}_{k=1}^{\infty} \) of \( M \) which has no adherent point in \( M \), there is a subsequence \( \{y_{k_j}\}_{j=1}^{\infty} \) such that

\[
\lim_{j \to -\infty} \frac{f(y_{k_j}) \wedge f(y_{k_j})}{K(y_{k_j})} = 0.
\]

Then \( M \) is Bergman complete.

Let \( f \in H \), and let \( \{y_k\}_{k=1}^{\infty} \) be a sequence of points which has no adherent point in \( M \). For any \( \epsilon > 0 \) one can find a compact subset \( M_\epsilon \) of \( M \) such that

\[
| \int_{M \setminus M_\epsilon} f \wedge \bar{f} | < \epsilon.
\]

By the hypothesis (B2), one can find a subsequence \( \{y_{k_j}\} \) with the following property: there exists a positive number \( a \) (independent of \( \epsilon \) and \( j \)) such that \( A(y_{k_j}, a) \subset M \setminus M_\epsilon \) for all sufficiently large \( j \). Let \( \chi \) be as before, and set
\[
\eta_j = \chi (-\log(-g_M(\cdot, y_{k_j}) + a) + \log(2a)) f,
\]
\[
\varphi_j = 2ng_M(\cdot, y_{k_j}) - \log(-g_M(\cdot, y_{k_j}) + a).
\]

We can solve the equation \( \overline{\partial}u_j = \overline{\partial}\eta_j \) essentially as above together with the following estimate:

\[
\left| \int_M u_j \wedge \bar{u}_j e^{-\varphi_j} \right| \leq \int_M |\overline{\partial}\eta_j|^2_{\overline{\partial}\varphi_j} e^{-\varphi_j} dV \varphi_j \leq C_2 \int_{\supp \overline{\partial}\eta_j} f \wedge \bar{f} \leq C_2 \epsilon,
\]

because \( \supp \overline{\partial}\eta_j \subset A(y_{k_j}, a) \subset M \setminus M_\epsilon \) for \( j \) sufficiently large. Here \( C_2 \) is a constant depending only on \( a \) and the choice of \( \chi \). We set \( f_j = \eta_j - u_j \). Then \( f_j(y_{k_j}) = f(y_{k_j}) \)
and \( |\int_M f_j \wedge \overline{f_j}| \leq C_M \epsilon \). It follows that
\[
\frac{f(y_\kappa) \wedge \overline{f(y_\kappa)}}{K(y_\kappa)} \leq \left| \int_M f_j \wedge \overline{f_j} \right| \leq C_M \epsilon.
\]
Hence the assertion follows immediately from Proposition 8.

Let us see that there are various complex manifolds satisfying (B1):

1. \( D \) is a hyperbolic Riemann surface, that is, \( M \) carries a bounded non-constant subharmonic function. It is well known that this condition is equivalent to the fact that \( M \) carries a (negative) Green function. Since \( g_M(x, y) \) and \( g_M(x, y) - \log |z| \) is harmonic in a local coordinate chart at \( y \), we see that \( M \) satisfies the property (B1).

2. Let \( M \) be a complex manifold carrying a bounded continuous strictly psh function \( \psi \). By the well-known theorem of Richberg \[14\], we may assume that \( \psi \) is \( C^\infty \). For any \( y \in M \), we take a function \( \kappa \) which is compactly supported in a coordinate chart at \( y \) and identically equal to 1 in a neighborhood of \( y \). One can find a constant \( a_y \) such that \( \kappa \log |z| + a_y \) is a psh function on \( M \) with a logarithmic pole at \( z(y) = 0 \) which is bounded above by a constant depending only on \( y \). It follows from the definition of the pluricomplex Green function that \( M \) satisfies (B1).

3. Let \( M \) be a hyperconvex Stein manifold. We will show that \( M \) satisfies (B1).

We first prove the following fact.

**Claim.** Let \( M \) be a Stein manifold and let \( y \neq y' \) be two points of \( M \). Then there is a holomorphic function \( f \) on \( M \) such that \( f(y) = 0 \), \( df(y) = 0 \) and \( f(y') = 1 \).

**Proof.** The proof is standard. Let \( \psi \) be a \( C^\infty \) strictly psh exhaustion function on \( M \). Similarly as before, one can find \( \psi_y \), \( \psi_{y'} \) on \( M \) with a logarithmic pole at \( y \), \( y' \) respectively. We choose a cutoff function \( \tau \) which is compactly supported in \( M \) and is such that \( \tau \equiv 0 \) in a neighborhood of \( y \) and \( \tau \equiv 1 \) in a neighborhood of \( y' \). Now we take a convex, rapidly increasing function \( \gamma \) such that there is, according to Theorem 5.2.4 in [8], a solution to the equation \( \overline{\partial}u = \overline{\partial}\tau \) satisfying the following estimate:
\[
\int_M |u|^2 e^{-\varphi} dV \leq \int_M |\overline{\partial}\tau|^2 e^{-\varphi} dV \leq C,
\]
where \( \varphi = \gamma \circ \psi + 2(n+1)\psi_y + 2n\psi_{y'} \); the point norm and the volume are associated to some fixed Kähler metric on \( M \). Then the function defined by \( f = \tau - \mu \) is holomorphic on \( M \) and satisfies \( f(y) = 0 \), \( df(y) = 0 \) and \( f(y') = 1 \).

Now let \( \mu \) be a negative psh exhaustion function on \( M \). Again we take \( n \) globally defined holomorphic functions \( \zeta_1, \ldots, \zeta_n \) which form a local coordinate system centered at \( y \), and denote by \( U \) the coordinate neighborhood of \( y \). We set
\[
K = \{ x \in M : \mu(x) \leq \mu(y)/8 \} \setminus U.
\]
Since it is compact in \( M \), we obtain from the above claim finite holomorphic functions \( \zeta_{n+1}, \ldots, \zeta_{n+m} \) on \( M \) such that \( \zeta_{n+j}(y) = 0 \), \( d\zeta_{n+j}(y) = 0 \) for all \( 1 \leq j \leq m \), and the function \( \sum_{j=1}^m |\zeta_{n+j}|^2 \) is nowhere vanishing on \( K \). We denote \( \zeta = (\zeta_1, \ldots, \zeta_{n+m}) \) and set
\[
\lambda = \inf_{\{\mu(x) = \mu(y)/2\}} \log |\zeta(x)|/R_y,
\]
\[
\tilde{\mu}(x) = \lambda \log(-\mu(x) - \mu(y)/4) - \log(-\mu(y)/2) / \log 3/2,
\]
where
\[ R_y = \sup_{\{\mu(x) = \mu(y)/2\}} |\zeta(x)| + 1. \]

It follows that $\mu$ is a psh function on $M$ satisfying
\[ \mu(x) = \lambda \leq \log |\zeta(x)|/R_y \quad \text{if } \mu(x) = \mu(y)/2, \]
\[ \mu(x) = 0 \geq \log |\zeta(x)|/R_y \quad \text{if } \mu(x) = \mu(y)/4. \]

Hence the function defined by
\[ v(x) = \begin{cases} 
\log |\zeta(x)|/R_y & \text{if } \mu(x) < \mu(y)/2, \\
\max\{\log |\zeta(x)|/R_x, \mu(x)\} & \text{if } \mu(y)/2 \leq \mu(x) \leq \mu(y)/4, \\
\mu(x) & \text{if } \mu(x) > \mu(y)/4 
\end{cases} \]
is a psh function on $M$ with a logarithmic pole at $y$ which is bounded from above by a constant depending only on $y$. Then, similarly as above, $M$ satisfies (B1).

**Proof of Corollary 3.** It suffices to show that $M$ satisfies (B2). Let $\mu$ be a negative psh exhaustion function on $M$. We denote
\[ M_c = \{ x \in M : \mu(x) < -c \} \]
for any $c > 0$. Now let $c$ be fixed and let $y \in M_{2c}$ be arbitrary. We set
\[ \psi_y(x) = \begin{cases} 
\max\{C\mu(x), g_M(x, y) - 1\} & \text{if } x \in M\setminus M_c, \\
g_M(x, y) - 1 & \text{if } x \in M_c, 
\end{cases} \]
where
\[ C = -e^{-1} \min_{\{\mu(x) = -c\}} (g_M(x, y) - 1) \]
is a constant depending only on $c$ because $g_M$ is a continuous function off the diagonal on $M \times M$. Clearly, $\psi_y$ is a negative psh function with a logarithmic pole at $y$, and furthermore, there is a positive constant $c' < c$ such that $\psi_y(x) \geq -1$ on $M\setminus M_{c'}$. It follows from the extremal property of the Green function that the inequality $g_M(x, y) \geq -1$ holds there. Since $g_M$ is symmetric, we have
\[ g_M(x, y) \geq -1, \forall x \in M_{2c}, y \in M\setminus M_{c'}. \]
It follows that $A(y, -1) \subset M\setminus M_{2c}$ for any $y \in M\setminus M_{c'}$, which implies the property (B2). The proof is complete.

**Proof of Corollary 4.** Since $\psi$ is a negative $C^2$ strictly psh function on $D$, according to the above facts, $D$ carries a Bergman metric. Moreover, it is the standard Bergman metric since $D$ is a domain in $C^n$. Let $\{y_k\}$ be an arbitrary sequence of points which has no adherent point in $D$. We distinguish two cases:

(a) There is a subsequence $\{y_{kj}\}$ such that $|y_{kj}| \to \infty$ as $j \to +\infty$. We take a cutoff function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi|_{(-\infty, 1/2]} = 1$ and $\chi|_{[1, +\infty)} = 0$. Since $\partial\bar{\partial}\psi \geq \partial\bar{\partial}|z|^2$, there is a constant $C' > 0$, depending only on the choice of $\chi$, such that the function $\varphi := C'\psi + \chi(|z - y_{kj}|)\log |z - y_{kj}|$ is a negative psh function on $D$ with a logarithmic pole at $y_{kj}$. Hence for any compact subset $K$ of $D$, one has $A(y_{kj}, -1) \subset D\setminus K$ for all sufficiently large $j$. Similarly as in the proof of Theorem 1, the criterion of Kobayashi holds for $\{y_k\}$.

(b) Otherwise, there is a subsequence $\{y_{kj}\}$ such that $y_{kj}$ converges to a boundary point $y_0$. Take a ball $B(y_0, 1)$ and set $D' = D \cap B(y_0, 1)$. Clearly $D'$ is a bounded
hyperconvex domain. Without loss of generality, we may assume \( y_k \in B(y_0, 1/4) \). If we have proved that
\[ K_D(y_k) \geq C'' K_D(y_k) \]
for some constant \( C'' \) independent of \( j \), then for any \( f \in \mathcal{H}(D) \)
\[
\lim_{k \to \infty} \frac{|f(y_k)|^2}{K_D(y_k)} \leq C'' \lim_{k \to \infty} \frac{|f(y_k)|^2}{K_D(y_k)} = 0,
\]
where the last equality was shown in \([1, 7]\). Hence Kobayashi’s criterion holds for \( \{y_k\} \). The proof is reduced to showing the localization property of the Bergman kernel. Let \( \varphi \) be as above. We solve the equation
\[
\dot{\varphi} u_j = \partial \chi(|z - y_0|) K_D'(z, y_k)/K_D^{1/2}(y_k)
\]
together with the following estimate:
\[
\int_D |u_j|^2 e^{-2n\varphi_j - \psi} dV \leq \int_D \partial \chi(|z - y_0|)^2 \partial \varphi \circ e^{-2n\varphi_j - \psi} dV \leq C''',
\]
where \( C''' \) is a constant independent of \( j \). Set again
\[
f_j = \chi(|z - y_0|) K_D'(z, y_k)/K_D^{1/2}(y_k) - u_j.
\]
We obtain
\[
K_D(y_k) \geq \frac{|f(y_k)|^2}{\int_D |f|^2 dV} \geq C'' K_D(y_k)
\]
for a suitable constant \( C'' \) independent of \( j \).

The proof follows immediately from Proposition 8.

3. Proof of Theorem 2

We will follow the argument of Greene and Wu \([6]\) throughout this section. Let \( M \) be as in Theorem 1. Suppose that (1) holds in \( M \setminus B(o, C) \) for some positive constant \( C \), where \( B(x, \delta) \) denotes the geodesic ball with radius \( \delta \) around \( x \). Let \( x_0 \) be any point in \( M \setminus B(o, 2C) \). Let \( \rho_0 \) denote the distance function relative to \( x_0 \). Let \( G \) be the complete Kähler metric of \( M \) and let \( K_G(x) \) denotes the maximum of the sectional curvatures at \( x \). It is not difficult to see that the inequality
\[
K_G(x) \leq -\frac{A}{4\rho_0(x)^2}
\]
holds for all \( x \in M \setminus B(x_0, 2\rho(x_0)) \). Consider the new Kähler metric \( H = \frac{G}{\rho(x_0)^2} \). Let \( \gamma_0 \) denote the distance function of \( H \) relative to \( x_0 \). Then \( \gamma_0 = \frac{\rho_0}{\rho(x_0)} \) and \( K_H = \rho(x_0)^2 K_G \). Hence inequality (3) is equivalent to
\[
K_H(x) \leq -\frac{A}{4\gamma_0(x)^2}
\]
for all \( x \in M \) with \( \gamma_0(x) \geq 2 \). Notice also that \( K_H \leq 0 \) everywhere. By Lemma 5.15 in \([7]\), there is a complete Hermitian metric \( h \) on the unit disc \( D \) which is rotationally symmetric, and its Gaussian curvature \( K_h \) satisfies (a) \( K_h \leq 0 \) and (b) if \( \bar{\rho} \) denotes the distance function of \( h \) relative to the origin, then
\[
K_h = \begin{cases} 
0 & \text{on } \{\bar{\rho} \leq 2\}, \\
-A/(4\bar{\rho}^2) & \text{on } \{\bar{\rho} \geq 3\},
\end{cases}
\]
and in the annulus \(2 < \rho < 3\), \(K_h\) is rotationally symmetric. Write \(h = d\bar{\rho}^2 + f(\bar{\rho})^2d\theta^2\) in terms of geodesic polar coordinates. Since \(f'' = -K_h f\), it follows that \(f'' \equiv 0\) on \([0, 2]\); hence \(f(\bar{\rho}) = \bar{\rho}\) there. Next we write \(h\) as follows:

\[
h = \eta(r) dz d\bar{\xi} = \eta(r)(dr^2 + r^2 d\theta^2),
\]
where \(r : D \to [0, 1)\) is the ordinary radial function on \(D\). Clearly, one has

\[
\eta(r) = [\bar{\rho}'(r)]^2, \tag{3}
\]
\[
r^2 \eta(r) = f(\bar{\rho}(r))^2. \tag{4}
\]

We will regard \(r\) as a function \(\rho\) so that \(r : [0, 1) \to [0, 1)\). By (3), (4), one has

\[
\frac{1}{r} = \frac{\bar{\rho}'(r)}{f(\bar{\rho}(r))}.
\]

Integrating both sides relative to \(dr\) from \(r\) to 1, we obtain

\[
r(\bar{\rho}) = \exp \left\{- \int_r^1 \frac{\bar{\rho}'(r)}{f(\bar{\rho}(r))} dr \right\} = \exp \left\{- \int_{\bar{\rho}}^1 \frac{1}{f} \right\}.
\]

Set \(\phi_{x_0} = r(\gamma_0)^2\). Using a Hessian comparison theorem, Greene and Wu proved that \(\phi_{x_0}\) is a bounded exhaustion function on \(M\) which is \(C^\infty\) strictly psh, and satisfies \(0 < \phi_{x_0} < 1, \phi_{x_0}^{-1}(0) = x_0, \phi_{x_0} = O(\gamma_0^2)\) near \(x_0\), and \(\log \phi_{x_0}\) is also psh. Observe that

\[
\log \phi_{x_0}(x) = 2 \log r(\gamma_0(x)) = -2 \int_{\gamma_0(x)}^1 \frac{1}{f}
\]
\[
= -2 \left( \int_{\gamma_0(x)}^1 \frac{1}{f} + \int_1^\infty \frac{1}{f} \right) = 2 \log \frac{\gamma_0(x)}{b}
\]
for any \(x \in M\) with \(\gamma_0(x) \leq 1\), since \(f(t) = t\) for \(t \leq 2\). Here \(b = \exp \left( \int_1^\infty \frac{1}{f} \right) > 1\), which is a constant depending only on \(A\). On the other hand, one has

\[
\log \phi_{x_0}(x) \geq -2 \int_1^\infty \frac{1}{f} = -2 \log b
\]
whenever \(\gamma_0(x) > 1\). If we set

\[
\tilde{A}(x_0, c) := \{ x \in M : \log \phi_{x_0}(x) < -c \}
\]
for any \(c > 0\), then we immediately obtain the following fact.

**Lemma 9.** Under the condition of Theorem 2, one has

\[
\tilde{A}(x_0, 2 \log(2b)) \subset \left\{ x \in M : \gamma_0(x) < \frac{1}{2} \right\} = \left\{ x \in M : \rho_0(x) < \frac{1}{2} \rho(x_0) \right\}
\]
for any \(x_0 \in M \setminus B(o, 2c)\).
Proof of Theorem 2. For any \( x_0 \in M \setminus B(o, 2c) \), \( \phi_{x_0} - 1 \) is a negative \( C^\infty \) strictly psh exhaustion function of \( M \). It follows from the previous section that \( M \) satisfies the property \( (B1) \). By Lemma 9 we claim that, for any sequence of points \( y_k, k = 1, 2, \cdots \), which has no adherent point in \( M \),
\[
A(y_k, 2 \log(2b)) \subset A^*(y_k, 2 \log(2b))
\]
and
\[
\left\{ x \in M : \rho_k(x) < \frac{1}{2} \rho(y_k) \right\} \subset \left\{ x \in M : \rho(x) > \frac{1}{2} \rho(y_k) \right\}
\]
provided \( k \) is sufficiently large. Here \( \rho_k \) denotes the distance associated to \( y_k \). This implies that the property \( (B2) \) is also satisfied. Thus the assertion follows immediately from Theorem 1. \( \Box \)

4. Proof of Theorem 6

We first prove 1). Let \( x_1, x_2 \) be two arbitrary points which satisfy \( \rho(x_2) \geq 2c \) and \( \rho(x_1) = 4\rho(x_2) \). Take a complete orthonormal basis \( \{h_j\}_{j=0}^\infty \) for \( H \) such that \( h_j(x_2) = 0 \) for all \( j \geq 1 \). We claim that the following holds:

Lemma 10. There is a constant \( C_4 > 0 \) such that
\[
C_4 h_0(x_1) \wedge \tilde{h}_0(x_1)
\]
\[
\leq \sup \left\{ f(x_1) \wedge \tilde{f}(x_1) : f \in H, f(x_2) = 0, \left| \int_M f \wedge \tilde{f} \right| \leq 1 \right\},
\]
where for any two forms \( f(z) = f^*(z)dz_1 \wedge \cdots \wedge dz_n, g(z) = g^*(z)dz_1 \wedge \cdots \wedge dz_n, f(z) \wedge \tilde{f}(z) \leq g(z) \wedge \tilde{g}(z) \) if \( f^*(z) \leq |g^*(z)| \).

Proof. We will use Lemma 9 with \( x_0 = x_1, x_2 \) respectively. Set
\[
\varphi = n (\log \phi_{x_1} + \log \phi_{x_2}) + \phi_{x_1} - \log \left( -\log \frac{\phi_{x_1}}{2} \right).
\]
Clearly, it is a \( C^\infty \) strictly psh function on \( M \setminus \{x_1, x_2\} \) which satisfies the following estimate:
\[
\partial \bar{\partial} \varphi \geq \partial \bar{\partial} \left( -\log \left( -\log \frac{\phi_{x_1}}{2} \right) \right) \geq \partial \log \phi_{x_1} \partial \log \phi_{x_2}.
\]
Choose a \( C^\infty \) cutoff function \( \chi : \mathbb{R} \to [0, 1] \) such that \( \chi|_{(-\infty, -2)} = 1, \chi|_{(-1, \infty)} = 0 \).

Set
\[
\eta = \chi \left( \frac{\log \phi_{x_1}}{2 \log(2b)} \right) h_0.
\]
Clearly, \( \eta(x_1) = h_0(x_1) \), and it follows from Lemma 9 that
\[
\text{supp} \eta \subset \left\{ x \in M : \log \phi_{x_1}(x) < -2 \log(2b) \right\}
\]
\[
\subset \left\{ x \in M : \rho_1(x) \leq \frac{1}{2} \rho(x_1) \right\},
\]
where \( \rho_1(x) \) denotes the distance function relative to \( x_1 \). It follows that \( \eta(x_2) = 0 \) and
\[
\log \phi_{x_2}(x) \geq -2 \log(2b), \quad \forall x \in \text{supp} \eta,
\]
because \( \rho(x_1) = 4\rho(x_2) \). By (7), one has
\[
\left| \partial \chi \left( \frac{\log \phi_{x_1}}{2 \log(2b)} \right) \right| \leq C_5,
\]
where $|·|_{\partial\bar{\partial}c}$ denotes the point norm with respect to the metric $\partial\bar{\partial}\varphi$ and $C_6 > 0$ is a constant that only depends on $b$ and the choice of $\chi$.

Observe that $M \setminus \{x_1, x_2\}$ still carries a complete Kähler metric, defined as follows:

$$\partial\bar{\partial}(-\log (-\log \phi_{x_1}) - \log (-\log \phi_{x_2}) + \phi_{x_1}).$$

By Proposition 7, there is a solution to the equation $\partial u = \bar{\partial}u$ on $M \setminus \{x_1, x_2\}$ which satisfies

$$\left| \int_{M \setminus \{x_1, x_2\}} u \wedge \bar{u} e^{-\varphi} \right| \leq \int_{M \setminus \{x_1, x_2\}} |\partial\eta|_{\partial\bar{\partial}c}^2 e^{-\varphi} d\varphi \quad \leq \int_{M \setminus \{x_1, x_2\}} |\partial\chi (\log \phi_{x_1})^2 h_0 \wedge \bar{h}_0 e^{-\varphi}| \quad \leq C_6$$

because of (8)–(10). Set $f = \eta - u$. It is a holomorphic $n$-form on $M \setminus \{x_1, x_2\}$, and by (11), $f$ can be extended holomorphically across $x_1, x_2$; moreover, $f(x_1) = h_0(x_1), f(x_2) = 0$ because of the singularity of $\varphi$ at $x_1, x_2$. Since $\varphi$ is bounded from above on $M$, one has

$$\left| \int_M f \wedge \bar{f} \right| \leq 2 \left| \int_M \eta \wedge \bar{\eta} \right| + 2 \left| \int_M u \wedge \bar{u} \right| \leq 2 \left| \int_M h_0 \wedge \bar{h}_0 \right| + C_7 \left| \int_M u \wedge \bar{u} e^{-\varphi} \right| \leq C_8,$$

from which the assertion immediately follows with the constant $C_4 = C_8^{-1}$. 

We proceed to prove the theorem. According to Kobayashi’s alternative definition of the Bergman metric, $\beta$ is nothing but the pullback of the Fubini–Study metric of the infinite-dimensional complex projective space $\mathbb{CP}(\mathcal{H})$ (cf. [11]). It follows that the Bergman distance $\text{dist}_\beta(x_1, x_2)$ is no less than the Fubini–Study distance between the points $p_1 = (a_0 : a_1 : \cdots)$ and $p_2 = (1 : 0 : \cdots)$, where the $a_j$ are given by

$$|a_j|^2 = \frac{h_j(x_1) \wedge \bar{h}_j(x_1)}{\sum_{j=0}^\infty h_j(x_1) \wedge \bar{h}_j(x_1)} = \frac{h_j(x_1) \wedge \bar{h}_j(x_1)}{K(x_1)}.$$

This implies that

$$\text{dist}_\beta(x_1, x_2) \geq \sqrt{|1 - a_0|^2 + \sum_{j=1}^\infty |a_j|^2}.$$

Assume that the supremum on the right side of (5) is realized by a certain $n$-form $f$. Then, without loss of generality, we can take $h_1 = f$. If $|a_0| \geq 1/2$, we have

$$\text{dist}_\beta(x_1, x_2) \geq |a_1| = \sqrt{\frac{f(x_1) \wedge \bar{f}(x_1)}{K(x_1)}} \geq \sqrt{C_4} \sqrt{\frac{h_0(x_1) \wedge \bar{h}_0(x_1)}{K(x_1)}} = \sqrt{C_4} |a_0| \geq \frac{1}{2} \sqrt{C_4}. $$
Otherwise, it is clear that \( \text{dist} \beta(x_1, x_2) \geq 1 - |a_0| \geq \frac{1}{2} \). Therefore, there is a positive constant \( C_9 > 0 \) such that
\[
\text{dist} \beta(x_1, x_2) \geq C_9
\]
holds for any \( x_1, x_2 \in M \) satisfying \( \rho(x_1) = 4\rho(x_2) \). From this the inequality (2) immediately follows.

Next we prove 2). The idea is similar, but simpler. It is known from page 109 of [6] that the bounded psh exhaustion function has an explicit form:
\[
\phi_{x_0} = \left( \tanh \frac{\sqrt{\rho_0}}{2} \right)^2
\]
for any \( x_0 \in M \). Hence there exists a constant \( b_1 > 0 \) such that
\[
\tilde{A}(x_0, b_1) \subset \{ x \in M : \rho_0(x) < 1 \}.
\]
Repeating the argument as above, one can find a positive constant \( C_{10} \) such that for any two points \( x_1, x_2 \in M \) with \( \rho(x_1) = \rho(x_2) + 3 \), we have
\[
\text{dist} \beta(x_1, x_2) \geq C_{10},
\]
from which the assertion immediately follows.

References


Department of Applied Mathematics, Tongji University, Shanghai 200092, China

E-mail address: chenboy@online.sh.cn

Department of Mathematics, Fudan University, Shanghai 200433, China

E-mail address: zhangjhhk@online.sh.cn