ON CERTAIN CO–H SPACES RELATED TO MOORE SPACES

MANFRED STELZER

Abstract. We show that certain co–H spaces, constructed by Anick and Gray, carry a homotopy co–associative and co–commutative co–H structure.

1. Introduction and statement of results

In a series of papers, Anick and Gray constructed and studied a family of co–H spaces $G^{2n}_{2^n}(p^r)$, $p > 5$, with some remarkable properties [1], [2], [7], [8]. The limit space, or rather its loop space $\Omega G^{2n}_{2^n}(p)$, shows up in a secondary version of the EHP sequence. Furthermore, Gray used the $G^{2n}_{2^n}(p^r)$ to construct $v_2$–periodic families in the homotopy groups of Moore spaces [8], and Anick used them to decompose loop spaces of finite complexes [1]. It is the aim of this paper to prove the following theorem, which was conjectured in part in [2]:

Theorem 1. The spaces $G^{2n}_{2^n}(p^r)$ carry a co–associative and co–commutative co–H structure.

In section 2 we define the notions of co–A and co–C deviation. Some properties are established, which are used in section 3 to prove Theorem 1.

After the results in this paper were obtained, I learned of Stephen Theriault’s thesis, in which he proved the homotopy co–associativity of $G_k$ (and much more) by using Ganea’s characterisation of co–associativity. But the argument given there has a gap. It is assumed during the proof that, given a homotopy commutative square of co–H spaces and co–H maps, the map induced on the cofibers is also a co–H map. But this is clearly false (see for example the discussion of Zabrodsky’s formula in [10], p. 228).

2. The co–A and co–C deviation

The co–A deviation was first defined by Harper [11]. It is an obstruction for the existence of a co–associative co–H structure on the mapping cone of a co–H map between co–associative co–H spaces. The dual A–deviation and C–deviation were studied by Zabrodsky in [13].

Let $f : X, \tau \rightarrow Y, \sigma$ be a co–H map between the co–H spaces $X, Y$ with structure maps $\tau, \sigma$. So there are based homotopies

$M_x : j \circ \tau \rightarrow \Delta_x$,

$M_y : j \circ \sigma \rightarrow \Delta_y$,

$F : (f \lor f) \circ \tau \rightarrow \sigma \circ f$.

Received by the editors December 1, 2001.

2000 Mathematics Subject Classification. Primary 55P45; Secondary 55S35.
Here $\triangle$ is the diagonal, and $j$ the inclusion of the wedge into the product.

Suppose $X, \tau$ and $Y, \sigma$ are co–associative resp. co–commutative up to homotopy. In this case we have based homotopies:

$$L_x : (1 \lor \tau) \circ \tau \longrightarrow (\tau \lor 1) \circ \tau, \quad L_y : (1 \lor \sigma) \circ \sigma \longrightarrow (\sigma \lor 1) \circ \sigma$$

and

$$G_x : \tau \longrightarrow T \circ \tau, \quad G_y : \sigma \longrightarrow T \circ \sigma,$$

where $T$ denotes the switch map.

We will use the following notation for stringing together a list of homotopies. Given $H_1, \ldots, H_n : Z \times I \to W$ such that $H_i(Z, 1) = H_{i+1}(Z, 0)$, define

$$\{H_1, \ldots, H_n\}(z, t) = H_i(z, nt - i + 1), \quad \text{if} \ (i - 1) \leq nt \leq i.$$

We also write $H^{-1}$ for the homotopy with $H^{-1}(z, t) = H(z, 1 - t)$.

Suppose $X, \tau$ and $Y, \sigma$ are homotopy co–associative, and fix $L_x, L_y$. Following Harper we define the co–$A$ deviation with respect to $F$ as follows:

$$A(f, F) = \{(f \lor F) \circ \tau, \ (1 \lor \sigma) \circ F, \ L_y \circ f, \ (\sigma \lor 1) \circ F^{-1}, \ (F^{-1} \lor f) \circ \tau, \ f(3) \circ L_x^{-1}\}.$$

This defines a map

$$A(f, F) : X, \ast \longrightarrow \Lambda Y(3), c,$$

where $\Lambda$ denotes the free loop space, $c$ is the map sending the whole circle to the base point, and $X(k)$ and $f(k)$ denote the $k$–fold wedge of the space and the map, respectively.

Let $u : X, \ast \longrightarrow \Lambda Y(3), c$ be the map which sends $x$ to the constant map $f(3) \circ (1 \lor \tau) \circ \tau(x)$.

Then $A(f, F) - u$ factors uniquely over a map from $X, \ast \to \Omega Y(3), \ast$, and the homotopy class of the adjoint of this map

$$A_*(f, F) \in \left[\Sigma X, Y(3)\right]$$

is the co–$A$ deviation.

The homotopy $F$ is called primitive, if

$$j \circ F = \{(f \times f) \circ M_x, \ M_y^{-1} \circ f\}$$

as tracks, i.e., as homotopy classes relative to $X \times \partial I$. Berstein and Harper proved in [3] that one can choose $F$ to be primitive.

The map $f$ is called a co–$A$ map, if there is a primitive homotopy $F$ such that $A_*(f; F)$ is the class of the constant map.

The definition of the co–$C$ deviation is similar. Suppose $X, \tau$ and $Y, \sigma$ are homotopy co–commutative, and fix $G_x, G_y$. Define

$$C(f, F) = \{F, G_y \circ f, T \circ F^{-1}, (f \lor f) \circ G_x\}.$$

This gives us a map $C(f, F) : X, \ast \longrightarrow \Lambda Y(2), c$.

Again there is a map unique up to homotopy from $X$ to $\Omega Y(2)$ such that the composite with the inclusion $\Omega Y(2) \hookrightarrow \Lambda Y(2)$ is $C(f, F) - v$ with $v(x)$ constant of value $(f \lor f) \circ \tau(x)$. The homotopy class of the adjoint $C_*(f, F) \in \left[\Sigma X, Y(2)\right]$ is called the co–$C$ deviation of $f, F$.

The map $f$ is called a co–$C$ map, if there is a primitive homotopy $F$ such that $C_*(f, F)$ is the class of the constant map.
A result from [13], which we wish to dualize, is that the $A$–deviation, which is a map from $X \times X \times X$ to $\Omega Y$, can be chosen to be constant on the fat wedge in $X \times X \times X$. The proof used the existence of a strict unit for $H$–spaces. Since there is never a strict counit for a co–$H$ space unless the space is a point, we have not been able to prove the strictly dual theorem, i.e., that $A_*(f, F)$ lifts to the homotopy fiber of $Y_{(3)} \to R$, where $R$ is the homotopy limit of the diagram

\[
\begin{array}{ccc}
Y_1 \vee Y_2 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow \\
Y_2 & \leftarrow & Y_2 \vee Y_3 \\
\downarrow & & \downarrow \\
Y_3 & \leftarrow & \end{array}
\]

with the obvious maps. However, the following weak version will suffice for the applications which we have in mind.

**Theorem 2.1.** The homotopies $L_x, L_y$ and $G_x, G_y$ can be chosen such that

1. $j \circ A_*(f, F) \simeq \ast$, where $j : Y_{(3)} \to Y \times Y \times Y$, and
2. $j \circ C_*(f, F) \simeq \ast$, where $j : Y_{(2)} \to Y \times Y$.

For a map $g : X \to Y_{(i)}$ we denote the projection onto the $i$–th component by $g_i$.

The proof of Theorem 2.1 will need

**Lemma 2.2.** One can choose $L_x$ and $L_y$ such that

\[
\begin{align*}
L_{y,1} &= \sigma_1 \circ M^{-1}_{y,1}, & L_{x,1} &= M^{-1}_{x,1} \circ \tau_1, \\
L_{y,2} &= \{ M_{y,1} \circ \sigma_2, \sigma_2 \circ M^{-1}_{y,1} \}, & L_{x,2} &= \{ M_{x,1} \circ \tau_2, \tau_2 \circ M^{-1}_{x,1} \}, \\
L_{y,3} &= \sigma_2 \circ M_{y,2}, & L_{x,3} &= M_{x,2} \circ \tau_2
\end{align*}
\]

as tracks, i.e., as homotopy classes relative to $Y \times \partial I$ and $X \times \partial I$.

**Proof.** Since the loop of the homotopy fiber sequence $F \to Y_{(3)} \to Y \times Y \times Y$ splits, the lemma follows from [5, 1.8].

**Lemma 2.3.** One can choose $G_x$ and $G_y$ such that

\[
\begin{align*}
G_{x,1} &= \{ M_{x,1}, M^{-1}_{x,2} \}, & G_{y,1} &= \{ M_{y,1}, M^{-1}_{y,2} \}, \\
G_{x,2} &= \{ M_{x,2}, M^{-1}_{x,1} \}, & G_{y,2} &= \{ M_{y,2}, M^{-1}_{y,1} \}.
\end{align*}
\]

**Proof.** Same argument as above.

**Proof of Theorem 2.1.** a) We choose $F$ primitive and $L_x, L_y$ as in Lemma 2.2. We show that in this case $A(f, F)_i = u_i, i \in \{1, 2, 3\}$. (By a slight abuse of notation we also denote by $g_i$ the composition of a map to $A(Y_{(i)})$ with the $i$–th projection to $A(Y)$.)

We first have a hard look at $A(f, F)_1$:

\[
A(f, F)_1 = \{ f \circ M_{x,1}, M_{y,1}^{-1} \circ f, \sigma_1 \circ M_{y,1}^{-1} \circ f, \sigma_1 \circ M_{y,1} \circ f, \sigma_1 \circ f \circ M^{-1}_{x,1}, \\
M_{y,1} \circ f \circ \tau_1, f \circ M_{x,1}^{-1} \circ \tau_1, f \circ M_{x,1} \circ \tau_1 \}
\]

\[
= \{ f \circ M_{x,1}, M_{y,1}^{-1} \circ f, \sigma_1 \circ f \circ M^{-1}_{x,1}, M_{y,1} \circ f \circ \tau_1 \}.
\]
Claim.

\[
\{ f \circ M_{x,1}, M_{y,1}^{-1} \circ f \} = M_{y,1}^{-1} \circ f \circ M_{x,1},
\]
\[
\{ \sigma_1 \circ f \circ M_{x,1}, M_{y,1} \circ f \circ \tau_1 \} = M_{y,1} \circ f \circ M_{x,1}^{-1}.
\]

Here \( M_{y,1}^{-1} \circ f \circ M_{x,1} \) is shorthand for \( M_{y,1}^{-1}(f \circ M_{x,1}, \text{id}_1) \), and similarly for \( M_{y,1} \circ f \circ M_{x,1}^{-1} \).

Moreover, for compositions, the first argument may be replaced by – if the expression is complicated and forced by contest.

To see this, consider the homotopies of tracks

\[
K_1(x, t, s) := \begin{cases} 
  f \circ M_{x,1}(x, 2t) & \text{if } 0 \leq t \leq \frac{s}{2}, \\
  M_{y,1}^{-1}(f \circ M_{x,1}(x, t + \frac{s}{2}), t - \frac{s}{2}) & \text{if } \frac{s}{2} \leq t \leq 1 - \frac{s}{2}, \\
  M_{y,1}^{-1}(f(x), 2t - 1) & \text{if } 1 - \frac{s}{2} \leq t \leq 1.
\end{cases}
\]

\[
K_2(x, t, s) := \begin{cases} 
  \sigma_1 \circ f \circ M_{x,1}^{-1}(x, 2t) & \text{if } 0 \leq t \leq \frac{s}{2}, \\
  M_{y,1}(f \circ M_{x,1}^{-1}(x, t + \frac{s}{2}), t - \frac{s}{2}) & \text{if } \frac{s}{2} \leq t \leq 1 - \frac{s}{2}, \\
  M_{y,1}(f(x), 2t - 1) & \text{if } 1 - \frac{s}{2} \leq t \leq 1.
\end{cases}
\]

The picture is:

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) node[midway, above right]{$M_{y,1}^{-1} \circ f$};
\draw (0,0) -- (0,1) node[midway, left]{$f \circ M_{x,1}$};
\draw (0,0) -- (1,0) node[midway, below right]{$M_{y,1}^{-1} \circ f \circ M_{x,1}$};
\end{tikzpicture}
\end{array}
\]

where the path in the middle is \( K_1(\ldots, s) \). In conclusion, we have shown that \( A(f, F)_1 \simeq A(f, F)_1(0) = u_1 \). \( A(f, F)_2 \) is a little more complex:

\[
A(f, F)_2 = \{ F_1 \circ \tau_2, \sigma_1 \circ F_2, L_{y,2} \circ f, \sigma_2 \circ F_1^{-1}, F_2^{-1} \circ \tau_1, f \circ L_{x,2} \}$
\[
= \{ f \circ M_{x,1} \circ \tau_2, M_{y,1}^{-1} \circ f \circ \tau_2, \sigma_1 \circ f \circ M_{x,2}, \sigma_1 \circ M_{y,2}^{-1} \circ f, \\
\quad M_{y,1} \circ \sigma_2 \circ f, \sigma_2 \circ M_{y,1}^{-1} \circ f, \sigma_2 \circ M_{y,1} \circ f, \sigma_2 \circ f \circ M_{x,1}^{-1}, \\
\quad M_{y,2} \circ f \circ \tau_1, f \circ M_{x,2} \circ \tau_1, f \circ \tau_2 \circ M_{x,1}, f \circ M_{x,1}^{-1} \circ \tau_2 \}$
\[
= \{ \{ M_{y,1}^{-1} \circ f \circ \tau_2, \sigma_1 \circ f \circ M_{x,2}, \sigma_1 \circ M_{y,2}^{-1} \circ f, M_{y,1} \circ \sigma_2 \circ f \}, \\
\quad \{ \sigma_2 \circ f \circ M_{x,1}^{-1}, M_{y,2} \circ f \circ \tau_1, f \circ M_{x,2}^{-1} \circ \tau_1, f \circ \tau_2 \circ M_{x,1} \} \}
\]

As in the case above, one sees that

\[
\{ \sigma_1 \circ f \circ M_{x,2}, \sigma_1 \circ M_{y,2}^{-1} \circ f \} = \sigma_1 \circ M_{y,2}^{-1} \circ f \circ M_{x,2},
\]
\[
\{ M_{y,2} \circ f \circ \tau_1, f \circ M_{x,2}^{-1} \circ \tau_1 \} = M_{y,2} \circ f \circ M_{x,1}^{-1} \circ \tau_1.
\]

So it remains to show that

\[
\{ M_{y,1}^{-1} \circ f \circ \tau_2, \sigma_1 \circ M_{y,2}^{-1} \circ f \circ M_{x,2}, M_{y,1} \circ \sigma_2 \circ f \}
\]
\[
= \{ \sigma_2 \circ f \circ M_{x,1}^{-1}, M_{y,2} \circ f \circ M_{x,2}^{-1} \circ \tau_1, f \circ \tau_2 \circ M_{x,1} \}^{-1}.
\]
To see this, consider the homotopy between tracks:

\[ H_1(x, t, s) := \begin{cases} 
M_{y,1}^{-1}(f \circ \tau_2(x), 3t) & \text{if } 0 \leq t \leq \frac{4}{3}, \\
M_{y,1}^{-1}(-, s) \circ M_{y,2}^{-1}(-, \frac{t - \frac{4}{9}}{1 - \frac{2}{3}}) \circ f \circ M_{x,2}^{-1}(-, \frac{t - \frac{4}{9}}{1 - \frac{2}{3}}) & \text{if } \frac{4}{3} \leq t \leq 1 - \frac{4}{3}, \\
\sigma_2 \circ M_{y,1}(f(x), 3t - 2) & \text{if } 1 - \frac{4}{3} \leq t \leq 1,
\end{cases} \]

\[ H_2(x, t, s) := \begin{cases} 
\sigma_2 \circ f \circ M_{x,1}^{-1}(x, 3t) & \text{if } 0 \leq t \leq \frac{4}{3}, \\
M_{y,2}^{-1}(\frac{t - \frac{4}{9}}{1 - \frac{2}{3}}) \circ f \circ M_{x,2}^{-1}(\frac{t - \frac{4}{9}}{1 - \frac{2}{3}}) \circ M_{x,1}^{-1}(x, s) & \text{if } \frac{4}{3} \leq t \leq 1 - \frac{4}{3}, \\
f \circ \tau_2 \circ M_{x,1}(x, 3t - 2) & \text{if } 1 - \frac{4}{3} \leq t \leq 1.
\end{cases} \]

Since \( H_1(x, t, 0) = H_2(x, 1 - t, 0) = M_{y,2}^{-1}(-, t) \circ f \circ M_{x,2}(x, t) \), the identity above is proved.

The case \( A(f, F)_3 \) is like the first one.

Next we prove b), which is easier. Using the primitivity of \( F \) and Lemma 2.3, we find that

\[ \{F, G_y \circ f, T \circ F^{-1}, f \vee f \circ G_y^{-1}\}_3 \]

\[ = \{ f \circ M_{x,1}, M_{y,1} \circ f, M_{y,1} \circ f, M_{y,2} \circ f, M_{y,2} \circ f, M_{x,2} \circ f \circ M_{x,1} \}, \]

which is the constant track.

A similar argument works for the second coordinate. \( \square \)

The following two lemmas were proved in [5].

**Lemma 2.4.** Let \((f, F) : \Sigma X, \tau \rightarrow Y, \sigma \) be a \( \text{co-H} \) map with primitive homotopy \( F \). Suppose \( Y, \sigma \) is homotopy co-associative and \( \Sigma X, \tau \) is the suspension co-\( \text{H} \) structure. Then for each suspension \( \Sigma g : \Sigma Z \rightarrow \Sigma X \) the co-\( A \) deviation satisfies

\[ A_*(f \circ \Sigma g, F \circ \Sigma g) = A_*(f, F) \circ \Sigma^2 g. \]

\( \square \)

Recall from [5] that for a co-\( \text{H} \) map \( f : X, \tau \rightarrow Y, \sigma \) with primitive homotopy \( F \), the mapping cone \( C_f \) carries a co-\( \text{H} \) structure \( \overline{\sigma} \) defined by

\[ \overline{\sigma}(x, t) = \begin{cases} 
F^{-1}(x, 2t) & \text{for } 0 \leq 2t \leq 1, \\
\tau(x), 2t - 1 & \text{for } 0 \leq 2t \leq 1,
\end{cases} \]

\[ \overline{\sigma}(y) = \sigma(y). \]

**Lemma 2.5.** Let \((f, F) \) be a co-\( A \) map. Then \( \overline{\sigma} \) is homotopy co-associative. \( \square \)

There are similar results for the co-\( C \) deviations:

**Lemma 2.6.** Let \((f, F_1) : X, \tau \rightarrow Y, \sigma \) and \((g, F_2) : Y, \sigma \rightarrow Z, \eta \) be co-\( \text{H} \) maps and primitive homotopies, and suppose all spaces are homotopy co-commutative. Then

\[ C_*(f \circ g, \{g \vee g \circ F_1, F_2 \circ f\}) = (g \vee g) \circ C_*(f, F_1) + C_*(g, F_2) \circ \Sigma f. \]
Proof. The proof is given by looking at the diagram

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
g \circ f \circ g \circ f \circ G_x \\
T \circ g \circ g \circ F_1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
g \circ g \circ G_x \circ f \\
G_x \circ g \circ f
\end{array}
\end{array}
\end{array}
\end{array}
\]

\[T \circ F_2^{-1} \circ f \]

\[T \circ g \circ F_1^{-1} \]

\[T \circ g \circ F_2^{-1} \circ f \]

Lemma 2.7. Let \((f, F)\) be a co–C map and a primitive homotopy. Then the co–H structure on \(C_f\) defined by \(F\) is homotopy co–commutative.

Proof. Consider the diagram

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
G_y \\
G_y \circ f \\
G_y \circ f
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
T \circ F^{-1}
\end{array}
\end{array}
\end{array}
\end{array}
\]

The square in the middle can be filled since \(f\) is a co–C map, and the triangle is extendable by the cone on \(G_x\). \(\square\)

3. Review on \(G_k^{2n}(p^r)\) and proof of theorem 1

Recall from [2] that the space \(G_k^{2n}(p^r)\) has a CW structure as follows:

\[G_k^{2n}(p^r) = S^{2n} \cup_{p^r} e^{2n+1} \cup_{g_1} \ldots \cup_{g_k} S^{2np^k} \cup_{p^{r+k}} e^{2np^k+1};\]

the attaching maps \(g_i\) are divisible by \(p^{r+i-1}\) in homotopy.

One also has \(G_{k-1}^{2n}(p^r) \cup_{g_k} S^{2np^k} \cup_{p^{2np^k+1}} = G_k^{2n}(p^r)\), and there is a compatible co–H structure on \(G_k^{2n}(p^r)\). Denote by \(W_k^{r+k}\) the class of spaces that are of finite type and of the homotopy type of bouquets of Moore spaces of type \(p^{r+s}\) with \(s \in \{0, \ldots, k\}\).

Lemma 3.1. The following statements hold:

a) \(\Sigma^2 \Omega G_k^{2n}(p^r) \in W_k^{r+k}\).

b) \(\Sigma \Omega G_k^{2n}(p^r) \wedge \Omega G_k^{2n}(p^r) \in W_k^{r+k}\).
c) Let $F$ be the homotopy fiber of
$$j : \bigvee_{i=1}^{3} G_{k}^{2n}(p^r) \longrightarrow \prod_{i=1}^{3} G_{k}^{2n}(p^r).$$

Then $\Omega F$ is homotopy equivalent to a weak product of spaces of the form $\Omega W_{j}$, $W_{j} \in W_{r+k}^{r}$. 

Proof. Parts a) and b) are in [2]. For c), recall the two homotopy equivalences

i) $\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee A \wedge B$,

ii) $\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \wedge \Omega(\Sigma X \wedge \Omega Y)$. 

Apply i) to $\Omega(Y \vee Y)$ to find that
$$\Omega F \simeq \Omega \Sigma(\Omega Y)^{\wedge 2} \times \Omega(\Sigma \Omega Y \wedge (\Omega Y \times \Omega Y \wedge \Omega(\Omega Y)^{\wedge 2})), $$

and ii) to see that
$$\Sigma(\Omega Y \times \Omega Y \times \Omega(\Omega Y)^{\wedge 2})$$

$$\simeq \Sigma \Omega Y \vee \Sigma \Omega Y \vee \Sigma(\Omega Y)^{\wedge 2} \vee \Sigma \Omega \Sigma(\Omega Y)^{\wedge 2} \vee \Sigma \Omega Y \wedge \Omega(\Sigma(\Omega Y)^{\wedge 2}) \wedge \Omega(\Omega(\Omega Y)^{\wedge 2}) \wedge \Omega(\Omega Y)^{\wedge 2}. $$

By Hilton–Milnor, and the fact that $W_{r+k}^{r}$ is closed under smash products [12], it is enough to show that each wedge factor is in $W_{r+k}^{r+k}$ for $Y = G_{k}^{2n}(p^r)$. By b) $\Sigma(\Omega G_{k}^{2n}(p^r))^{\wedge 2} \in W_{r+k}^{r}$, and since this space is 2-connected it is a double suspension. By a) the smash with $\Omega G_{k}^{2n}(p^r)$ is in $W_{r+k}^{r+k}$. The splitting of $\Sigma \Omega \Sigma X$ shows that also $\Sigma \Omega \Sigma \Omega G_{k}^{2n}(p^{r+k}) \in W_{r+k}^{r+k}$.

Proof of Theorem 1. The proof is by induction on $k$. For $k = 0$, $G_{k}^{2n}(p^r)$ is a simply connected Moore space of type $p^r$, $p$ odd. Hence it is a suspension, and so homotopy co-associative. If $n \geq 2$ it is a double suspension, and the homotopy co-commutativity for $n = 1$ was proved in [4]. So suppose the assertion holds for $k - 1, k \geq 1$. We show that the co–$A$ and the co–$C$ deviations of $g_k$ vanish. This suffices by Lemmas 2.5 and 2.7. Since $g_k$ is divisible by $p^{r+k-1}$, it follows from Lemmas 2.4 and 2.6 that $A_*(g_k, F)$ and $C_*(g_k, F)$ are also divisible by $p^{r+k-1}$.

By Theorem 2.1 the maps $A_*(g_k, F)$ and $C_*(g_k, F)$ lift, uniquely up to homotopy, to the homotopy fibers $F_1$ and $F_2$, respectively, of
$$j : \bigvee_{i=1}^{s} G_{k}^{2n}(p^r) \longrightarrow \prod_{i=1}^{s} G_{k}^{2n}(p^r), \quad s \in \{2, 3\}. $$

Moreover, the lift is also divisible by $p^{r+k-1}$, since the loop of this fibration splits. We claim that in fact
$$p^{r+k-1}[B^{2np^k}(p^r), \Omega F_i] = 0. $$

The homotopy type of $F_2$ is well known to be $\Sigma \Omega G_{k-1}^{2n}(p^r) \wedge \Omega G_{k-1}^{2n}(p^r)$, and the type of $\Omega F_1$ was determined during the proof of Lemma 3.1.

By [14] the homotopy exponent of $p^l(p^s)$ is $p^{s+1}$ for $p$ odd. It follows that the only factors in the product decomposition of $\Omega F_i$ which could contribute a class of order $p^{r+k}$ are of the form $\Omega p^{l}(p^{r+k-1})$. So we have to determine the least $\ell$ for which such a factor occurs. The first class in $H_*(\Omega G_{k}^{2n}(p^r); \mathbb{Z})$ of order $p^{r+k-1}$ shows up in dimension $2np^{k-1} - 1$ by [2, p. 864].

Consequently the first class of order $p^{r+k-1}$ in $H_*(\Omega \Sigma \Omega G_{k}^{2n}(p^r) \wedge \Omega G_{k}^{2n}(p^r); \mathbb{Z})$ shows up in dimension $4np^{k-1} - 2$. Inspection of the proof of Lemma 3.1 shows that
in the splitting of $\Omega F_1$ each factor is of the form loop of $\Sigma(\Omega G_{k}^{2n}(p^r)) \wedge^2 \wedge Z$. So also in $H_*(\Omega F_1, Z)$ and $H_*(\Omega F_2, Z)$ the first class of order $p^{r+k-1}$ is in dimension $\geq 4np^{k-1} - 2$. This class comes from a factor $\Omega P^{4np^{k-1}} \cong (p^{r+k-1})$.

By [3], the first element of order $p^{r+k}$ in $\pi_*(P^{2m+1}(p^{r+k-1}))$, respectively $\pi_*(P^{2m}(p^{r+k-1}))$ is in dimension $2mp - 1$, resp. $4mp - 2p - 1$. The universal coefficient sequence for homotopy groups splits at an odd prime. So it is enough to show that $\pi_*(P^{k}(p^{r+k-1}))$ does not contain a class of order $p^{r+k}$ for $s \leq 2np^{k-1} + 1$ and $\ell \geq 4np^{k-1} = 2m$. But this follows from what was said above and the trivial estimates

\[ 2np^{k} + 1 < 4mp - 2p - 1 = 8np^{k} - 2p - 1, \]
\[ 2np^{k} + 1 < 2mp - 1 = 4np^{k} - 1. \]

The next two corollaries are just mild strengthenings of two results from [2].

**Corollary 3.2.** Suppose $X$ is an $H$ space, and $\varphi_{k-1} : G_{k-1}^{2n}(p^r) \to \Sigma X$ is a map such that $\varphi_{k-1} |_{K'}$, where $K$ is a skeleton, is a co-$H$ map. Suppose also that $p^{r+k-1} \pi_{2np^{k-1}}(X; \mathbb{Z}/p^{r+k}) = 0$. Then $\varphi_{k-1}$ has an extension $\varphi_k : G_k^{2n}(p^r) \to \Sigma X$.

**Proof.** It was shown in [2] 4.1 that the corollary would follow if one could choose the coretration $\sigma : G_k^{2n}(p^r) \to \Sigma \Omega G_k^{2n}(p^r)$ corresponding to the co-$H$ structure to be a co-$H$ map. Since every 1-connected co-associative co-$H$ space is a cogroup [3], this follows from [3] 4.2 and Theorem 1.

**Corollary 3.3.** Let $0 \leq s \leq m \leq \infty$, and suppose that $X$ is an $H$-space such that $p^{r+k-1} \pi_{2np^{k-1}}(X; \mathbb{Z}/p^{r+k}) = 0$ for $s < k \leq m$. Let $\varphi : P^{2mnp^r}(p^{r+s}) \to X$ be a map. Then there is a map $\varphi_m : G_m^{2n}(p^r) \to \Sigma X$ which extends

\[ G_s^{2n}(p^r) \xrightarrow{\text{pinch}} G_s^{2n}(p^r)/G_{s-1}^{2n}(p^r) = P^{2mnp^r+1}(p^{r+s}) \xrightarrow{\Sigma \varphi} \Sigma X. \]

**Proof.** Since $G_s^{2n}(p^r)$ is a sub-co-$H$ space of $G_s^{2n}(p^r)$ the pinch map is a co-$H$ map for the induced co-$H$ structure on the quotient space. This co-$H$ structure is unique [4]. Thus the composite with $\Sigma \varphi$ is a co-$H$ map, and the assertion follows from Corollary 3.2.

**References**


Sesenheimerstrasse 20, 10627 Berlin, Germany