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SASAKIAN-EINSTEIN STRUCTURES ON $9\#(S^2 \times S^3)$

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Abstract. We show that $9\#(S^2\times S^3)$ admits an 8-dimensional complex family of inequivalent non-regular Sasakian-Einstein structures. These are the first known Einstein metrics on this 5-manifold. In particular, the bound $b_2(M)\leq 8$ which holds for any regular Sasakian-Einstein M does not apply to the non-regular case. We also discuss the failure of the Hitchin-Thorpe inequality in the case of 4-orbifolds and describe the orbifold version.

0. Introduction

Recently Demailly and Kollár have developed some new techniques to study the existence of Kähler-Einstein metrics on compact Fano orbifolds [DK]. Johnson and Kollár applied these techniques to study Kähler-Einstein metrics on certain log del Pezzo surfaces in weighted projective 3-spaces [JK1] as well as anti-canonically embedded orbifold Fano 3-folds in weighted projective 4-spaces [JK2]. In [BG3], [BGN1] we have extended some of the results of [JK1] to the case of higher index and have studied their implications in the realm of Sasakian-Einstein metrics on simply connected smooth 5-manifolds. These arise as links of isolated hypersurface singularities given by quasi-homogeneous polynomials in $\mathbb{C}^4$ [BG3]. In [BGN1] we showed that there are many families of non-regular Sasakian-Einstein structures on k-fold connected sums of $S^2 \times S^3$ for $k = 1, 2, 3, 4, 5, 6, 7$. When $k = 8$ we have not found any non-regular Sasakian-Einstein structures. However, $8\#(S^2 \times S^3)$, viewed as a circle bundle over a blow-up of $\mathbb{CP}^2$ at eight points, comes with an eight complex parameter family of regular Sasakian-Einstein structures. It follows from the well-known classification of smooth del Pezzo surfaces that for regular Sasakian-Einstein 5-manifolds $\mathcal{S}$ we must have $b_2(\mathcal{S}) \leq 8$. Moreover, it follows from the Hitchin-Thorpe inequality that $\mathbb{CP}^2$ blown-up at 9 or more points in general position is excluded from admitting any Einstein metric whatsoever. Neither of these bounds holds in the orbifold case. For the Hitchin-Thorpe type result the topological Euler number and Hirzebruch signature must be replaced by their corresponding orbifold analogues. Hence, no such topological bound holds in the orbifold case. Nevertheless, all the non-regular examples found in [BGN1] still satisfy this bound. The main purpose of this note is to show that the bound does not hold for non-regular Sasakian-Einstein 5-manifolds. We achieve this by a careful analysis of the question of the existence of Kähler-Einstein metrics for the log del Pezzo surface $\mathcal{Z}_{16} \subset \mathbb{CP}^2(1, 3, 5, 8)$ of index 1 and degree 16 found by Johnson and

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Kollár [JK1]. Johnson and Kollár leave the question of existence of Kähler-Einstein metrics on $\mathcal{Z}_{16}$ open. Using well-known techniques of Milnor and Orlik [MO], one can easily compute the characteristic polynomial of the associated link $L_{16} \subset \mathbb{C}^4$, and from this one sees that the second Betti number $b_2(L_{16}) = 9$. As a consequence of this, a general argument concerning the absence of torsion in H_2 [BG3], and a classification theorem of Smale [Sm], the link L_{16}, which is naturally a V-bundle over $L_{16} \to \mathcal{Z}_{16}$, must be diffeomorphic to $9\#(S^2 \times S^3)$. In section 3, we show that $\mathcal{Z}$ does satisfy the sufficient conditions of [DK] and does admit a Kähler-Einstein metric. This is the key to establishing the following result.

Theorem A. *Let $\mathcal{S} = 9\#(S^2 \times S^3)$. Then $\mathcal{S}$ admits an 8-dimensional family of inequivalent non-regular Sasakian-Einstein structures. Furthermore, the Einstein metrics in the family are inequivalent as Riemannian metrics.*

It follows that the metric cone on $\mathcal{S} = 9\#(S^2 \times S^3)$ is a singular Calabi-Yau 3-fold [BG1]–[BG2]. Such cones arise naturally in the context of supersymmetric string theory (see, for example, [Y]). In view of Theorem A and Smale's classification result [Sm] one can ask the following important questions:

Problem B. *Suppose M is a compact simply-connected Sasakian-Einstein 5-manifold. Can $b_2(M)$ be arbitrarily large? And, if so, is there a Sasakian-Einstein structure on any connected sum of k copies of $S^2 \times S^3$?*

We are unable to answer this question here; however, in a forthcoming note [BGN2] the authors show that there is a positive Sasakian structure on $k\#(S^2 \times S^3)$ for $k \geq 1$.

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1. Sasakian Structures on Links of Isolated Hypersurface Singularities

Although the purpose of this note is to describe a solitary example of the Sasakian-Einstein geometry of $9\#(S^2 \times S^3)$ we shall begin with a very brief summary of the Sasakian and Sasakian-Einstein geometry of links of isolated hypersurface singularities defined by weighted homogeneous polynomials. For more details we refer the reader to [BG3], [BGN1]. Consider the affine space $\mathbb{C}^{n+1}$ together with a weighted $\mathbb{C}^*$-action $\mathbb{C}^*_{\mathbf{w}}$ given by $(z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n)$, where the *weights* w_j are positive integers. It is convenient to view the weights as the components of a vector $\mathbf{w} \in (\mathbb{Z}^+)^{n+1}$, and we shall assume that $\gcd(w_0, \ldots, w_n) = 1$. Let f be a quasi-homogeneous polynomial; that is, $f \in \mathbb{C}[z_0, \ldots, z_n]$ and satisfies

$$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n), \tag{1.1}$$

where $d \in \mathbb{Z}^+$ is the degree of f. We are interested in the *weighted affine cone* C_f defined by the equation $f(z_0, \ldots, z_n) = 0$. We shall assume that the origin in $\mathbb{C}^{n+1}$ is an isolated singularity, in fact the only singularity, of f. Then the link L_f defined by

$$L_f = C_f \cap S^{2n+1}, \tag{1.2}$$

where

$$S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} | \sum_{j=0}^{n} |z_j|^2 = 1\}$$

is the unit sphere in $\mathbb{C}^{n+1}$, is a smooth manifold of dimension $2n-1$. Furthermore, it is well-known [Mil] that the link L_f is $(n-2)$-connected.

Recall (cf. [BG1]) that a Riemannian manifold (M, g) of dimension m is *Sasakian* if the holonomy group of the metric cone $(C(M), \bar{g}) = (M \times \mathbb{R}^+, dr^2 + r^2 g)$ is contained in $U(\frac{m+1}{2})$, i.e. if $(C(M), \bar{g})$ is Kähler. The symplectic form ω on a Kähler cone is exact and takes the form $\omega = d(r\eta)$, where η is a contact 1-form on M. So if I is the almost complex structure on $C(M)$, the Kähler structure (ω, I) on $C(M)$ together with the "Liouville vector field" $\Psi = r\frac{\partial}{\partial r}$ induces a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on $M \simeq M \times \{1\}$, where the Reeb vector field is $\xi = I\Psi$, and the $(1,1)$ tensor field Φ is the restriction of I to TM.

On S^{2n+1} there is a well-known "weighted" Sasakian structure $\mathcal{S}_{\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$, where the vector field $\xi_{\mathbf{w}}$ is the infinitesimal generator of the circle subgroup $S^1_{\mathbf{w}} \subset \mathbb{C}^*_{\mathbf{w}}$. This Sasakian structure on S^{2n+1} induces a Sasakian structure, also denoted by $\mathcal{S}_{\mathbf{w}}$, on the link L_f. (See [YK], [BG1], [BGN1] for details.) The quotient space $\mathcal{Z}_f$ of S^{2n+1} by $S^1_{\mathbf{w}}$, or equivalently the space of leaves of the characteristic foliation $\mathcal{F}_\xi$ of $\mathcal{S}_{\mathbf{w}}$, is a compact Kähler orbifold which is a projective algebraic variety embedded in the weighted projective $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, w_1, \cdots, w_n)$, in such a way that there is a commutative diagram

$$\begin{tikzcd} L_f \arrow[r] \arrow[d, "\pi"] & S^{2n+1}_{\mathbf{w}} \arrow[d] \\ \mathcal{Z}_f \arrow[r] & \mathbb{P}(\mathbf{w}), \end{tikzcd}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, and the vertical arrows are principal S^1 V-bundles and orbifold Riemannian submersions.

As with Kähler structures, there are many Sasakian structures on a given Sasakian manifold. In fact, there are many Sasakian structures which have ξ as its characteristic vector field. Such deformations of a given Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ are obtained by adding to η a continuous one parameter family of basic 1-forms ζ_t. We require that the 1-form $\eta_t = \eta + \zeta_t$ satisfy the conditions

$$\eta_0 = \eta, \qquad \zeta_0 = 0, \qquad \eta_t \wedge (d\eta_t)^n \neq 0 \quad \forall\, t \in [0,1]. \tag{1.5}$$

Since ζ_t is basic, ξ is the Reeb (characteristic) vector field associated to η_t for all t. Now let us define

$$\begin{aligned} &\Phi_t = \Phi - \xi \otimes \zeta_t \circ \Phi, \\ &g_t = g + d\zeta_t \circ (\Phi \otimes \mathrm{id}) + \zeta_t \otimes \eta + \eta \otimes \zeta_t + \zeta_t \otimes \zeta_t. \end{aligned} \tag{1.6}$$

Then $\mathcal{S}_t = (\xi, \eta_t, \Phi_t, g_t)$ is a Sasakian structure for all $t \in [0,1]$ that has the same underlying contact structure and the same characteristic foliation. In general these structures are inequivalent, and the moduli space of Sasakian structures having the same characteristic vector field is infinite dimensional.

Suppose now we have a link L_f with a given Sasakian structure (ξ, η, Φ, g). When can we find a 1-form ζ such that the deformed structure $(\xi, \eta + \zeta, \Phi', g')$ is Sasakian-Einstein? This is a Sasakian version of the Calabi problem, and its solution is equivalent to solving the corresponding Calabi problem on the space of leaves $\mathcal{Z}_f$. Since a Sasakian-Einstein manifold necessarily has positive Ricci tensor, its Sasakian structure is necessarily positive. This also implies that the Kähler structure on $\mathcal{Z}_f$ is positive, i.e., $c_1(\mathcal{Z}_f)$ can be represented by a positive definite $(1,1)$ form. In this case there are well-known obstructions to solving the Calabi problem. These obstructions for finding a solution to the Monge-Ampère equations involve the non-triviality of certain *multiplier ideal sheaves* [Na], [DK] associated with effective canonical $\mathbb{Q}$-divisors on the space of leaves $\mathcal{Z}_f$. Consequently, if one can show that these multiplier ideal sheaves coincide with the full structure sheaf, one obtains the existence of a positive Kähler-Einstein metric on $\mathcal{Z}_f$, and hence a Sasakian-Einstein metric on L_f. Sufficient conditions that guarantee this come from Mori theory and can be phrased as follows [JK1]:

Sufficient Conditions for a K-E metric on $\mathcal{Z}_f$ and an S-E metric on L_f (1.7):

I. For some $\epsilon > 0$ and every effective $\mathbb{Q}$-divisor D on $\mathcal{Z}_f$ numerically equivalent to $-K_{\mathcal{Z}_f}$, the pair $(\mathcal{Z}, \frac{n+\epsilon}{n+1}D)$ is Kawamata log terminal (klt).

Now conditions on the weights that guarantee that the hypersurface $C_f \subset \mathbb{C}^{n+1}$ have only an isolated singularity at the origin are well-known [Fle], [JK1]. These conditions, known as *quasi-smoothness conditions*, guarantee that $\mathcal{Z}_f$ is smooth in the orbifold sense, that is, at a vertex $P_i \in \mathbb{P}(\mathbf{w})$ the preimage of $\mathcal{Z}_f$ in the orbifold chart of $\mathbb{P}(\mathbf{w})$ is smooth. It is easy to see that one can formulate all these conditions as follows [Fle], [JK1]:

Quasi-Smoothness Conditions (1.8):

II. For each $i = 0, \ldots, 3$ there is a j and a monomial $z_i^{m_i} z_j \in \mathcal{O}(d)$. Here $j = i$ is possible.

III. If $\gcd(w_i, w_j) > 1$ then there is a monomial $z_i^{b_i} z_j^{b_j} \in \mathcal{O}(d)$.

IV. For every i, j either there is a monomial $z_i^{b_i} z_j^{b_j} \in \mathcal{O}(d)$, or there are monomials $z_i^{c_i} z_j^{c_j} z_k$ and $z_i^{d_i} z_j^{d_j} z_l \in \mathcal{O}(d)$ with $\{k, l\} \neq \{i, j\}$.

In condition I the $i = j$ case corresponds to the case when $\mathcal{Z}_f$ does not pass through the point P_i. The second condition is equivalent to $\mathcal{Z}_f$ not containing any of the singular lines in $\mathbb{P}(\mathbf{w})$. If $\mathcal{Z}_f$ contains a coordinate axis (say $z_j = z_j = 0$) then the condition III forces $\mathcal{Z}_f$ to be smooth along it, except possibly at the vertices.

There is another condition apart from quasi-smoothness that assures us that the adjunction theory behaves correctly, and that $\mathbb{P}(\mathbf{w})$ does not have any orbifold singularities of codimension 1. It is [Do1], [Fle]

Well-formedness Condition (1.9)

V. For any triple of distinct integers i, j, k we have $\gcd(w_i, w_j, w_k) = 1$.

Condition IV guarantees that the canonical V-bundle $K_{\mathcal{Z}}$ is determined in terms of the degree and index by

$$K_{\mathcal{Z}} \simeq \mathcal{O}(-I) = \mathcal{O}(d - |\mathbf{w}|), \tag{1.10}$$

where $|\mathbf{w}| = \sum_i w_i$.

2. Sufficient Conditions for Kähler-Einstein Metrics

In this section we consider hypersurfaces $\mathcal{Z}_{16}$ of degree 16 and weights $(1, 3, 5, 8)$. As noted in [BGN1], there are 20 monomials in $H^0(\mathbb{P}(1, 3, 5, 8), \mathcal{O}(16))$. To begin we consider the hypersurface $\mathcal{Z}_{16}$ given by the zero set of

$$f(z_0, z_1, z_2, z_3) = z_0^{16} + z_1^5 z_0 + z_2^3 z_0 + z_1 z_2 z_3 + z_3^2$$

in the weighted projective space $\mathbb{P}(1, 3, 5, 8)$. Thus $\mathcal{Z}_{16}$ is a hypersurface of degree 16 and index 1. This hypersurface passes through only two singular points of $\mathbb{P}(1, 3, 5, 8)$, namely $P_1 = (0, 0, 1, 0)$ and $P_2 = (0, 1, 0, 0)$. We will prove

Theorem 2.1. *For any effective divisor*

$$D \equiv \mathcal{O}_{\mathcal{Z}_{16}}\left(-\frac{11}{16} K_{\mathcal{Z}_{16}}\right)$$

the pair $(\mathcal{Z}_{16}, D)$ *is klt.*

Proof. Suppose $D \equiv -K_{\mathcal{Z}_{16}}$. If $Q \neq P_1, P_2$, then we see, since the linear series $\mathcal{O}_{\mathcal{Z}_{16}}(5)$ has only isolated base points, that

$$\operatorname{mult}_Q(D) \leq D \cdot \mathcal{O}_{\mathcal{Z}_{16}}(5) = \frac{5 \cdot 16}{3 \cdot 5 \cdot 8} = \frac{2}{3} < 1$$

and hence D is klt at Q (see [BGN1], Lemma 2.11). □

We next consider the point P_1, which is the most difficult. Let $\pi : (\mathbb{C}^2, 0) \to (\mathcal{Z}_{16}, P_1)$ be a local cover of the index 5 quotient singularity at P_1. Intersecting with a general member of $\pi^* \mathcal{O}_{\mathcal{Z}_{16}}(3)$, all of which pass through the point P_1, we see that

$$\operatorname{mult}_0(\pi^* D) \leq \frac{5 \cdot 3 \cdot 16}{3 \cdot 5 \cdot 8} = 2.$$

Thus we may apply Shokurov's inversion of adjunction ([KM], Theorem 5.50) which says that the pair $(\mathcal{Z}_{16}, D)$ is log–canonical at P_1 provided the pair $p_*^{-1} \pi^* D$ is a sum of points with multiplicity at most one (here $p : S \to \mathbb{C}^2$ is the blow-up at the origin).

In order to analyze $p_*^{-1}(\pi^* D)$ we first look at the natural candidate for "worst case scenario," namely $D = Z(z_0)$. This is a curve $C \subset \mathbb{P}(3, 5, 8)$ with two components, namely $C_1 = Z(z_3)$ and $C_2 = Z(z_3 + z_1 z_2)$. Let $D_1 = \pi^* C_1$ and $D_2 = \pi^* C_2$. We claim that D_1 and D_2 are both smooth at 0. To see this, consider $F = Z(z_1)$. Then $F \cap C$ is proper, and hence

$$\pi^* F \cdot \pi^* C \geq 2,$$

with equality holding if and only if $\pi^* F$ meets D_1 and D_2 transversally at 0 (and at no other point). It is easy to check that F and C meet only at P_1, and hence $\pi^* F$ meets D_1 and D_2 only at 0. We compute the intersection multiplicity

$$i_0(\pi^* F \cdot \pi^* C) = \frac{5 \cdot 3 \cdot 16}{3 \cdot 5 \cdot 8} = 2.$$

Thus $\pi^* F$ meets D_1 and D_2 transversally at 0, and so D_1 and D_2 are both smooth at 0.

We now show that D_1 and D_2 meet transversally at 0. We already know that D_1 meets $\pi^* F$ transversally. Let f_1 be a local defining equation of D_1 near 0 and f_2 a local defining equation of D_2. Also, let m_0 denote the maximal ideal of the local ring of $\mathbb{C}^2$ at 0. Then f_1 and f_2 generate m_0 if and only if f_1 and $f_1 - f_2$ generate m_0. But $Z(f_1) = \pi^* Z(z_3)$ and $Z(f_2) = \pi^* Z(z_1 z_2 + z_3)$. Thus $Z(f_1 - f_2) = \pi^* Z(z_1 z_2)$. Since $0 \notin \pi^* Z(z_2)$, we see that D_1 and D_2 meet transversally at 0 if and only if D_1 and $\pi^*(Z(z_1)) = \pi^* F$ meet transversally at 0. Thus, using the previous paragraph, we have established that D_1 and D_2 are both smooth at 0 and meet transversally, i.e. the divisor $D = Z(z_0)$ is klt at P_1. Note that the same argument also holds with the point P_1 replaced by the other singular point P_2.

Now, following [JK1] and [BGN1], we deal with an arbitrary (effective) choice of $D \equiv -K_{\mathcal{Z}_{16}}$. Write

$$D = aC_1 + bC_2 + D', \tag{2.2}$$

where D' meets C_1 and C_2 properly. We compute intersection numbers:

$$\mathcal{O}_{\mathcal{Z}_{16}}(1) \cdot (C_1 + C_2) = \frac{16}{3 \cdot 5 \cdot 8} = \frac{2}{15}, \tag{2.3}$$

$$\mathcal{O}_{\mathcal{Z}_{16}}(1) \cdot C_1 = \frac{8}{3 \cdot 5 \cdot 8} = \frac{1}{15}, \tag{2.4}$$

$$\mathcal{O}_{\mathcal{Z}_{16}}(1) \cdot C_2 = \frac{1}{15}, \tag{2.5}$$

$$C_1 \cdot C_2 = \frac{8}{15}, \tag{2.6}$$

$$C_1^2 = C_2^2 = \frac{-7}{15}, \tag{2.7}$$

$$C_1 \cdot D' \leq \frac{2-a}{15}. \tag{2.8}$$

The first equality (2.3) follows since $z_0 = 0$ has $C_1 + C_2$ as divisor of zeroes; 2.4 follows since C_1 is the intersection of the two hypersurfaces $z_0 = 0$ and $z_3 = 0$. The third formula (2.5) follows from (2.3) and (2.4). Since C_1 and C_2 meet only at the two singular points P_1 and P_2 and since the corresponding pull–backs meet transversally at the origin, this implies (2.6). The next equation, (2.7), follows from (2.4), (2.5), and (2.6), using the fact that $\mathcal{O}_{\mathcal{Z}_{16}}(1) \cdot \mathcal{Z}_{16}$ is represented by $C_1 + C_2$. Finally, (2.8) follows since

$$C_1 \cdot D' \leq D \cdot D' \leq D \cdot D - aD \cdot C_1.$$

Intersecting (2.2) with C_1 and using (2.4), (2.6), and (2.7) gives

$$\begin{aligned} \frac{1}{15} &= C_1 \cdot \mathcal{O}_{\mathcal{Z}_{16}}(1) \\ &= C_1(aC_1 + bC_2 + D') \\ &= \frac{-7a}{15} + \frac{8b}{15} + C_1 \cdot D'. \end{aligned} \tag{2.9}$$

Applying (2.8) and using the fact that $a + b \leq 2$ (since $\mathrm{mult}_0(\pi^* D) \leq 2$), (2.9) implies

$$\frac{7a}{15} \leq \frac{8(2-a)}{15} + \frac{2-a}{15}. \tag{2.10}$$

Rearranging terms in (2.10) gives

$$a \leq \frac{9}{8}. \tag{2.11}$$

An identical argument shows that $b \leq \frac{9}{8}$.

It remains to deal with the final term D'. Since $D \cdot D' \leq \frac{2-a}{15}$, it follows that

$$\mathrm{mult}_0(\pi^* D') \leq \frac{2-a}{3}. \tag{2.12}$$

Thus, in the tangent direction of D_1 at 0 one obtains a divisor with coefficient at most

$$a + \frac{2-a}{3} = \frac{2a+2}{3} < \frac{16}{11}.$$

An identical bound holds with b in place of a, and so we conclude that $(\mathcal{Z}_{16}, \frac{11}{16}D)$ is klt at P_1.

We now turn to the point P_2, which will be simple as we have already performed the relevant computations. Let $\pi : (\mathbb{C}^2, 0) \to \mathcal{Z}_{16}$ be a local cover of the quotient singularity at P_2. Shokurov's inversion of adjunction still applies, as one sees by intersecting with a general member of $|\mathcal{O}_{\mathcal{Z}_{16}}(5)|$. We again let $Z(z_0) = C_1 \cup C_2$. Taking $F = Z(z_2)$ and arguing exactly as above establishes that $\pi^*(C_1)$ and $\pi^*(C_2)$ meet transversally at 0.

If $D \equiv -K_{\mathcal{Z}_{16}}$ is an effective divisor, then as above we write

$$D = aC_1 + bC_2 + D',$$

where $a, b \le \frac{9}{8}$. The bound (2.12) still holds for $\mathrm{mult}_0(\pi^*(D'))$ (in fact, one can do better, since the ramification index of π is only 3), and so, exactly as before, we find that $\left(\mathcal{Z}_{16}, \frac{11}{16}D\right)$ is klt at P_2. $\square$

Note that in the proof of Theorem 2.1 the same argument works for a much more general hypersurface Y. In particular, any of the numerous monomials with a z_0 term can be added without changing the proof. The only monomial which presents a problem is $z_1^2 z_2^2$. Consider the hypersurface Y defined by

$$f(z_0, z_1, z_2, z_3) = z_0^{16} + z_1^5 z_0 + z_2^3 z_0 + z_3^2 + z_1^2 z_2^2.$$

The divisor $z_0 = 0$ again splits into two irreducible components, $C_1 = Z(z_1 z_2 + i z_3)$ and $C_2 = Z(z_1 z_2 - i z_3)$. Arguing precisely as above shows that the divisor $z_1 = 0$ meets C_1 and C_2 transversally on the local cover $\pi : (\mathbb{C}^2, 0) \to (Y, P_1)$. Hence $\pi^*(C_1')$ meets $\pi^*(C_2)$ transversally, where $C_1' = Z(z_1 z_2)$. Adding the equations for C_1 and C_2 then establishes that $\pi^*(C_1)$ and $\pi^*(C_2)$ meet transversally. An identical argument applies for the point P_2 as above. Thus the inversion of adjunction formula applies, and the proof of Theorem 2.1 can be copied to establish that $\left(Y, \frac{11}{16}D\right)$ is klt for any choice of effective $D \equiv -K_Y$. This argument always applies provided the quadratic form $as^2 + bst + ct^2$ has distinct roots, where a is the coefficient of z_3^2, b is the coefficient of $z_1 z_2 z_3$ and c is the coefficient of $(z_1 z_2)^2$. This then guarantees the existence of Kähler-Einstein metrics on the hypersurfaces $\mathcal{Z}_{16}$ which depend on 20 complex parameters. We thus obtain induced Sasakian-Einstein metrics on the link L_{16} depending on 20 complex parameters. However, these parameters are not all effective, and in section 5 we discuss the moduli problem for these hypersurfaces and links.

3. The Topology of the Link L_{16}

The main purpose of this section is to prove

Theorem 3.1. *The Sasakian-Einstein manifold L_{16} is diffeomorphic to $9\#(S^2 \times S^3)$.*

In the process of proving this theorem we study the geometry of the link, and in particular compute the characteristic polynomial, which is an important link invariant that generalizes the Alexander polynomial of a knot. Let us recall the well-known construction of Milnor [Mil] concerning isolated hypersurface singularities.

There is a fibration of $(S^{2n+1} - L_f) \to S^1$ whose fiber F is an open manifold that is homotopy equivalent to a bouquet of n-spheres $S^n \vee S^n \vee \cdots \vee S^n$. The *Milnor number* μ of L_f is the number of S^n's in the bouquet. It is an invariant of the link which can be calculated explicitly in terms of the degree d and weights $(w_0, \ldots, w_n)$ by the formula [MO]

$$\mu = \mu(L_f) = \prod_{i=0}^{n} \Big(\frac{d}{w_i} - 1\Big). \tag{3.1}$$

For our link L_{16} with weights $\mathbf{w} = (1, 3, 5, 8)$, one immediately obtains

Proposition 3.2. *The Milnor number of the simply connected Sasakian-Einstein 5-manifold L_{16} is $\mu(L_{16}) = 143$.*

The closure $\bar{F}$ of F has the same homotopy type as F and is a compact manifold whose boundary is precisely the link L_f. So the reduced homology of F and $\bar{F}$ is only non-zero in dimension n and $H_n(F, \mathbb{Z}) \approx \mathbb{Z}^{143}$. Using the Wang sequence of the Milnor fibration together with Alexander-Poincaré duality gives the exact sequence [Mil]

$$0 \to H_n(L_f, \mathbb{Z}) \to H_n(F, \mathbb{Z}) \stackrel{\mathbb{I}-h_*}{\to} H_n(F, \mathbb{Z}) \to H_{n-1}(L_f, \mathbb{Z}) \to 0, \tag{3.3}$$

where h_* is the *monodromy* map (or characteristic map) induced by the $S^1_{\mathbf{w}}$ action. From this we see that $H_n(L_f, \mathbb{Z}) = \ker(\mathbb{I} - h_*)$ is a free Abelian group, and $H_{n-1}(L_f, \mathbb{Z}) = \mathrm{coker}(\mathbb{I} - h_*)$ which in general has torsion, but whose free part equals $\ker(\mathbb{I} - h_*)$. There is a well-known algorithm due to Milnor and Orlik [MO] for computing the free part of $H_{n-1}(L_f, \mathbb{Z})$ in terms of the characteristic polynomial $\Delta(t) = \det(t\mathbb{I} - h_*)$, but finding the torsion is, in general, much more difficult. However, in the case at hand, links of dimension five, under a mild assumption a method of Randell [Ran] shows that the torsion vanishes [BG3]. The Betti number $b_n(L_f) = b_{n-1}(L_f)$ equals the number of factors of $(t-1)$ in $\Delta(t)$.

We now compute the characteristic polynomials $\Delta(t)$ for our example.

Proposition 3.4. *The characteristic polynomial $\Delta(t)$ of the link L_{16} is given by*

$$\Delta(t) = (t-1)^9 (t+1)^8 (t^2+1)^9 (t^4+1)^9 (t^8+1)^9.$$

Hence, the second Betti number is $b_2(L_{16}) = 9$.

Proof. The Milnor and Orlik [MO] algorithm for computing the characteristic polynomial of the monodromy operator for weighted homogeneous polynomials is as follows: first associate to any monic polynomial F with roots $\alpha_1, \ldots, \alpha_k \in \mathbb{C}^*$ its divisor

$$\div F = \langle \alpha_1 \rangle + \cdots + \langle \alpha_k \rangle$$

as an element of the integral ring $\mathbb{Z}[\mathbb{C}^*]$, and let $\Lambda_n = \div(t^n - 1)$. The rational weights w_i' used in [MO] are related to our integer weights w_i by $w_i' = \frac{d}{w_i}$, and we write the $w_i' = \frac{u_i}{v_i}$ in irreducible form. So we have $w_0' = 16, w_1' = \frac{16}{3}, w_2' = \frac{16}{5}, w_3' = 2$. Then the divisor of the characteristic polynomial is

$$\div\Delta(t) = \Big(\frac{\Lambda_{16}}{5} - 1\Big)\Big(\frac{\Lambda_{16}}{3} - 1\Big)(\Lambda_{16} - 1)(\Lambda_2 - 1) \tag{3.5}$$

which upon using the relations $\Lambda_a \Lambda_b = \gcd(a, b) \Lambda_{lcm(a,b)}$ reduces to

$$\div\Delta(t) = 9\Lambda_{16} - \Lambda_2 + 1.$$

The characteristic polynomial $\Delta(t)$ is then determined from its divisor by

$$\Delta(t) = (t-1)\prod(t^j-1)^{a_j}, \tag{3.6}$$

where $\div\Delta(t) = 1 + \sum_j a_j\Lambda_j$. This gives the result. □

Now the proof of Theorem 3.1 follows by the Smale Classification Theorem [Sm], since L_{16} has no torsion by Lemma 5.8 of [BG3], and any Sasakian-Einstein manifold is spin.

4. Topological versus Orbifold Invariants

Dimensions three and four are the only dimensions where there are known topological obstructions to the existence of Einstein metrics on smooth manifolds. The first example of this in dimension four is due to Marcel Berger [Be], who noticed that a compact four dimensional Einstein manifold must have non-negative Euler characteristic. Later Thorpe [Be] and then Hitchin [Hit] noticed a much sharper inequality obstructing Einstein metrics.

In this section we discuss both topological and orbifold invariants of compact complex four dimensional orbifolds with isolated orbifold singularities, and the failure of the Hitchin-Thorpe (cf. [Be], [Hit], [LeB]) inequality in the singular case. Of course, we certainly don't expect this to hold as is in the orbifold case. There are orbifold correction terms as indicated by Satake [Sat] for the Euler characteristic. We note that for any log del Pezzo surface $\mathcal{Z}$, both the topological Euler characteristic and the Hirzebruch signature are determined in terms of the second Betti number $b_2(\mathcal{Z})$ by

$$\chi_{top}(\mathcal{Z}) = 2 + b_2(\mathcal{Z}), \qquad \tau_{top}(\mathcal{Z}) = 2 - b_2(\mathcal{Z}). \tag{4.1}$$

This gives

$$2\chi_{top}(\mathcal{Z}) + 3\tau_{top}(\mathcal{Z}) = 10 - b_2(\mathcal{Z}). \tag{4.2}$$

In the smooth case this is positive for del Pezzo surfaces, and any smooth rational surface with $b_2 \geq 10$ cannot admit any Einstein metric. However, for our singular $\mathcal{Z}_{16}$ we have

$$b_2(\mathcal{Z}_{16}) = 10, \qquad \chi_{top}(\mathcal{Z}_{16}) = 12, \qquad \tau_{top}(\mathcal{Z}_{16}) = -8. \tag{4.3}$$

Thus, we have

$$2\chi_{top}(\mathcal{Z}_{16}) + 3\tau_{top}(\mathcal{Z}_{16}) = 0 \tag{4.4}$$

even though $\mathcal{Z}_{16}$ has a positive Kähler-Einstein metric. Thus, the Hitchin-Thorpe inequality clearly fails in the singular category.

The point is, of course, that for orbifolds both the Gauss-Bonnet [Sat] and signature theorems [Kaw] relate integrals of curvature invariants to rational numbers that are orbifold invariants. In particular, the orbifold Euler characteristic $\chi_{orb}(\mathcal{Z})$ for a compact 4-orbifold $\mathcal{Z}$ with at most isolated orbifold singularities is given by [Sat]

$$\chi_{orb}(\mathcal{Z}) = \frac{1}{8\pi^2}\int_{\mathcal{Z}}\Big(|W_+|^2 + |W_-|^2 + \frac{s^2}{24} - \frac{|\mathrm{Ric}^0|^2}{2}\Big), \tag{4.5}$$

where $W_\pm, s, \mathrm{Ric}^0$ are the self-dual (anti-self-dual) pieces of the Weyl tensor, the scalar curvature, and traceless Ricci curvature, respectively. In the case that $\mathcal{Z}$ is a compact complex surface, $\chi_{orb}(\mathcal{Z})$ equals [Bla] the top Chern number $c_2(\mathcal{Z})$.

Moreover, χ_{orb} is related to the topological Euler characteristic $\chi_{top}(\mathcal{Z})$ by [Bla], [Sat]

$$\chi_{orb}(\mathcal{Z}) = \chi_{top}(\mathcal{Z}) - \sum_x \Big(1 - \frac{1}{|\Gamma_x|}\Big), \tag{4.6}$$

where $|\Gamma_x|$ denotes the order of the local uniformizing group Γ_x at the singular point x and the sum is taken over all singular points. Notice that for orbifolds the analogue of Berger's result is a bit stronger:

Proposition 4.7. *Let $\mathcal{Z}$ be a compact 4-dimensional orbifold with precisely N isolated orbifold singularities. Suppose further that $\mathcal{Z}$ admits an orbifold Einstein metric. Then*

$$\chi_{top}(\mathcal{Z}) \geq \sum_x (1 - \frac{1}{|\Gamma_x|}) \geq \frac{N}{2}.$$

A similar analysis can be done for the signature. The Hirzebruch signature theorem says that for smooth manifolds whose dimension is a multiple of four the signature τ can be obtained by evaluating the L-function of the tangent bundle on the fundamental class of the manifold. In the orbifold case this is replaced by Kawasaki's signature theorem [Kaw], which gives

$$\tau_{top}(\mathcal{Z}) = \langle L(T\mathcal{Z}), [\mathcal{Z}] > \rangle + \sum_x \frac{\mathcal{L}_x}{|\Gamma_x|}, \tag{4.8}$$

where again the sum is over all singular points, and $\mathcal{L}_x$ denotes the evaluation of the residual L-class at x. Here we have

$$\langle L(T\mathcal{Z}), [\mathcal{Z}]\rangle = \frac{1}{12\pi^2}\int_{\mathcal{Z}} \Big(|W_+|^2 - |W_-|^2\Big). \tag{4.9}$$

We can view this expression as an "orbifold signature" which is a rational number, and write $\tau_{orb}(\mathcal{Z}) = \langle L(T\mathcal{Z}), [\mathcal{Z}]\rangle$. Then we rewrite equation (4.8) as

$$\tau_{top}(\mathcal{Z}) = \tau_{orb}(\mathcal{Z}) + \tau_{res}(\mathcal{Z})$$

and the analogue for orbifolds of the Hitchin-Thorpe curvature integral becomes

$$2\chi_{orb}(\mathcal{Z}) \pm 3\tau_{orb}(\mathcal{Z}) = \frac{1}{4\pi^2}\int_{\mathcal{Z}} \Big(2|W_\pm|^2 + \frac{s^2}{24} - \frac{|\mathrm{Ric}^0|^2}{2}\Big), \tag{4.10}$$

so we obtain

Proposition 4.11. *Let $\mathcal{Z}$ be a compact 4-dimensional orbifold with only isolated orbifold singularities. Suppose further that $\mathcal{Z}$ admits an orbifold Einstein metric. Then*

$$\chi_{orb}(\mathcal{Z}) \geq \frac{3}{2}|\tau_{orb}(\mathcal{Z})|.$$

Thus, we have

$$\chi_{top}(\mathcal{Z}) \geq \frac{3}{2}|\tau_{orb}(\mathcal{Z})| + \sum_x \left(1 - \frac{1}{|\Gamma_x|}\right) \geq \frac{3}{2}|\tau_{orb}(\mathcal{Z})| + \frac{N}{2}.$$

In general Proposition 4.11 contains much less information than the Hitchin-Thorpe result, since it involves orbifold invariants and not topological invariants. Nevertheless, it does give an improvement of Proposition 4.7. We also have

Proposition 4.12. *Let $\mathcal{Z}$ be a compact complex 4-orbifold. Then*

$$c_1^2(\mathcal{Z}) = 2\chi_{orb}(\mathcal{Z}) + 3\tau_{orb}(\mathcal{Z}).$$

Thus, if $\mathcal{Z}$ admits an Einstein metric compatible with the orbifold structure, then $c_1^2 \geq 0$.

Proof. The Chern number c_1^2 is obtained by evaluating the square of the Ricci form ρ on the fundamental class of the orbifold, and the usual proof holds once we replace the topological invariants χ_{top} and τ_{top} by their orbifold counterparts. The last statement then follows immediately from Proposition 4.11. □

Remark 13. If $\mathcal{Z}$ is a compact complex 4-orbifold embedded as a hypersurface of degree d in the weighted projective space $\mathbb{P}(\mathbf{w})$, then it is easy to see that

$$c_1^2 = \frac{d(|\mathbf{w}| - d)^2}{w_0 w_1 w_2 w_3}. \tag{4.14}$$

Thus, we see that if a compact complex 4-orbifold $\mathcal{Z}$ has $2\chi_{orb} + 3\tau_{orb} = c_1^2 < 0$ it cannot admit a compatible Einstein metric nor be embedded as a hypersurface in $\mathbb{P}(\mathbf{w})$. We also mention that the case that the canonical V-bundle is trivial, so $c_1^2 = 0$ and $\mathcal{Z}$ is simply connected, gives Reid's list [Fle] of 95 mostly singular K3 surfaces.

Let us now evaluate the orbifold invariants for the case currently under study, namely the singular complex surface $\mathcal{Z}_{16}$. In this case it is easy to evaluate both c_1^2 and χ_{orb} directly giving

$$c_1^2(\mathcal{Z}_{16}) = \frac{16}{3\cdot 5\cdot 8} = \frac{2}{15}, \qquad \chi_{orb}(\mathcal{Z}_{16}) = 12 - \frac{2}{3} - \frac{4}{5} = 10 + \frac{8}{15}.$$

We have not evaluated $\tau_{orb}(\mathcal{Z}_{16})$ directly, but 4.12 gives $\tau_{orb}(\mathcal{Z}_{16}) = -7 + \frac{1}{45}$.

5. The Moduli of Sasakian-Einstein Structures on L_{16}

To analyze the moduli problem in the non-regular case, we begin by describing the $G_{\mathbf{w}}$ group of complex automorphisms of the weighted projective 3-space $\mathbb{P}(\mathbf{w})$. We shall assume that $\mathbb{P}(\mathbf{w})$ is well-formed. Recall that $\mathbb{P}(\mathbf{w})$ can be defined as a scheme $\mathrm{Proj}(S(\mathbf{w}))$, where

$$S(\mathbf{w}) = \bigoplus_d S^d(\mathbf{w}) = \mathbb{C}[z_0, z_1, z_2, z_3].$$

The ring of polynomials $\mathbb{C}[z_0, z_1, z_2, z_3]$ is graded with grading defined by the weights $\mathbf{w} = (w_1, w_1, w_2, w_3)$. As a projective variety we can embed $\mathbb{P}(\mathbf{w}) \subset \mathbb{CP}^N$, and then the group $G_{\mathbf{w}}$ is a subgroup of $PGL(N, \mathbb{C})$. $\mathbb{P}(\mathbf{w})$ is a toric variety, and we can describe $G_{\mathbf{w}}$ explicitly as follows: Let $\mathbf{w} = (w_0, w_1, w_2, w_3)$ be ordered as before. We consider the group $G(\mathbf{w})$ of automorphisms of the graded ring $S(\mathbf{w})$ defined on generators by

$$\varphi_{\mathbf{w}} = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} f_0^{(w_0)}(z_0, z_1, z_2, z_3) \\ f_1^{(w_1)}(z_0, z_1, z_2, z_3) \\ f_2^{(w_2)}(z_0, z_1, z_2, z_3) \\ f_3^{(w_3)}(z_0, z_1, z_2, z_3) \end{pmatrix}, \tag{5.1}$$

where $f_i^{(w_i)}(z_0,z_1,z_2,z_3)$ is an arbitrary weighted homogeneous polynomial of degree w_i in (z_0,z_1,z_2,z_3). This is a finite dimensional Lie group, and it is a subgroup of $GL(N,\mathbb{C})$. Projectivising, we get $G_{\mathbf{w}} = \mathbb{P}_{\mathbb{C}}(G(\mathbf{w}))$.

Note that when $\mathbf{w} = (1,1,1,1)$ then $G(\mathbf{w}) = GL(4,\mathbb{C})$. Except in this case, three weights are never the same if $\mathbb{P}(\mathbf{w})$ is well-formed. If two weights coincide, then $G(\mathbf{w})$ contains $GL(2,\mathbb{C})$ as a subgroup. Finally, when all weights are distinct we can write

$$\varphi_{\mathbf{w}}\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} a_0z_0 \\ a_1z_1 + f_1^{(w_0)}(z_1) \\ a_2z_2 + f_2^{(w_2)}(z_0,z_1) \\ a_3z_3 + f_3^{(w_3)}(z_0,z_1,z_2) \end{pmatrix}, \tag{5.2}$$

where $(a_0,a_1,a_2,a_3) \in (\mathbb{C}^*)^4$ and $f_i^{(w_i)}, \quad i=1,2,3$, are weighted homogeneous polynomials of degree w_i. The simplest situation occurs when $f_1 = f_2 = f_3$ are forced to vanish. Then $G_{\mathbf{w}} = (\mathbb{C}^*)^3$ is the smallest it can possibly be as $\mathbb{P}(\mathbf{w})$ is toric.

Let $S^d_{\mathbf{w}} \subset S(\mathbf{w})$ be the vector subspace spanned by all monomials in (z_0,z_1,z_2,z_3) of degree $d = |w| - I$, and let $\hat{S}^d(\mathbf{w}) \subset S^d(\mathbf{w})$ denote the subset of all quasi-smooth elements. Then we define $m^d_{\mathbf{w}}$ to be the dimension of the subspace generated by $\hat{S}^d_{\mathbf{w}}$. Now the automorphism group $G(\mathbf{w})$ acts on $S^d_{\mathbf{w}}$ leaving the subset $\hat{S}^d(\mathbf{w})$ of quasi-smooth polynomials invariant. Thus, for each log del Pezzo surface we define the moduli space

$$\mathcal{M}^d_{\mathbf{w}} = \hat{S}^d_{\mathbf{w}}/G(\mathbf{w}) = \mathbb{P}(\hat{S}^d_{\mathbf{w}})/G_{\mathbf{w}}, \tag{5.3}$$

with $n^d_{\mathbf{w}} = \dim(\mathcal{M}^d_{\mathbf{w}})$. Now there is an injective map

$$\mathcal{M}^d_{\mathbf{w}} \to \mathcal{M}^{\mathbb{C}}(\mathcal{Z}_{\mathbf{w}}), \tag{5.4}$$

and each element in $\mathcal{M}^d_{\mathbf{w}}$ corresponds to a unique homothety class of Kähler-Einstein metrics modulo $G_{\mathbf{w}}$, and hence to a unique Sasakian-Einstein structure on the corresponding 5-manifold $\mathcal{S}_l$ modulo the group $G_{\mathbf{w}}$ acting as CR automorphisms.

Let us now consider our $\mathcal{Z}_{16}$. It is easy to see that $S^{16}(1,3,5,8)$ is isomorphic to $\mathbb{C}^{20}$ and it is spanned by the monomials $z_3^2, z_1z_2z_3, z_1^2z_2^2, z_0z_2^3, z_0z_1^5, z_0^2z_1^2z_3, z_0^2z_1^3z_2, z_0^3z_2z_3, z_0^3z_1z_2^2, z_0^4z_1^4, z_0^5z_1z_3, z_0^5z_1^2z_2, z_0^6z_2^2, z_0^7z_1^3, z_0^8z_3, z_0^8z_1z_2, z_0^{10}z_1^2, z_0^{11}z_2, z_0^{13}z_1, z_0^{16}$. We take the open submanifold $\hat{S}^{16}(1,3,5,8) \subset S^{16}(1,3,5,8)$. This is acted on by the complex automorphism group, namely the group $G(1,3,5,8)$ generated by

$$\varphi_{\mathbf{w}}\begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} a_0^1z_0 \\ a_1^1z_1 + a_1^2z_0^3 \\ a_2^1z_2 + a_2^2z_0^5 + a_2^3z_0^2z_1 \\ a_3^1z_3 + a_3^2z_0^8 + a_3^3z_1z_0^5 + a_3^4z_1^2z_0^2 + a_3^5z_2z_0^3 + a_3^6z_2z_1 \end{pmatrix}, \tag{5.5}$$

where $a_i^1 \in \mathbb{C}^*$ and all other coefficients are in $\mathbb{C}$. $G(1,3,5,8)$ is a 12-dimensional complex Lie group acting on the open submanifold $\hat{S}^{16}(1,3,5,8) \subset S^{16}(1,3,5,8) \approx \mathbb{C}^{20}$. It follows that the quotient is an 8-dimensional complex manifold, which by the Bando-Mabuchi Theorem [BM] is the moduli space of positive Kähler-Einstein metrics on the underlying compact orbifold $\mathcal{Z}_{16}$.

Theorem A now follows from these results and Proposition 7.14 of [BGN1].

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