ON THE PROFILE OF THE CHANGING SIGN MOUNTAIN PASS SOLUTIONS FOR AN ELLIPTIC PROBLEM

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Abstract. We consider nonlinear elliptic equations with small diffusion and Dirichlet boundary conditions. We construct changing sign solutions with peaks close to the boundary and consider the location of the peak.

1. Introduction

Consider

\[
\begin{aligned}
-\varepsilon^2 \Delta u &= f(u), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary, and \( \varepsilon > 0 \) is a small number.

In recent years, there have been a lot of results on the existence and the profile of solutions for (1.1). See for example [3, 5, 8, 9, 11, 12, 13, 19, 21, 25, 26, 28]. This problem arises from the biological sciences [18, 22]. It is observed that solutions of (1.1) may exhibit sharp peaks near a certain number of points. In biology, the locations of the peaks correspond to the higher concentration places of chemicals, certain populations, etc. Therefore, it is important to know the locations of the peaks of the solutions for (1.1).

In this paper, we consider a kind of nonlinearity \( f(u) \), such that the mountain pass type solution for (1.1) will exhibit a new concentration phenomenon. Assume that \( f(t) \) satisfies the following conditions:

\((f_1)\) there exists \( a < b \), \( a < 0 \), such that \( f(a) = f(b) = 0 \) and \( f(t) < 0 \) for \( t \in (a, b) \); 
\((f_2)\) \( \int_a^b f(s) \, ds < 0 \) if \( b < 0 \); 
\((f_3)\) \( f \in C^1([a, +\infty)) \cap C^2((a, +\infty)) \) and \( f''(t) > 0 \) for all \( t > a \); 
\((f_4)\) there is \( \alpha > 0 \), such that 
\[ (t - a)^{1-\alpha} f''(t) \to c_0 > 0, \quad \text{as } t \to a+, \]

for some \( c_0 > 0 \); 
\((f_5)\) there is a \( \mu > 1 \), such that 
\[ f'(t)t \geq \mu f(t), \]

for \( t > 0 \) large.

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(f_6) \[ |f(t)| \leq C(1+|t|^{p-1}) \text{ for some } p \in (2, 2^*) \text{, where } 2^* = 2N/(N-2) \text{ if } N \geq 3, \] 
\[ 2^* = +\infty \text{ if } N = 1, 2. \]

A typical example of a function satisfying (f_1)-(f_6) is
\[ f(t) = (t-a)^{p-1} - (t-a), \]
where \( 2 < p < 2^* \), \( a \in \left(-\left(\frac{p}{2}\right)^{1/(p-2)}, 0\right) \).

Note that (f_1) and \( f''(t) > 0 \) for all \( t > a \) imply \( f'(a) < 0 \). Moreover, from \( f''(t) > 0 \), we know that \( f(t) \) has exactly two zero points \( a \) and \( b \).

By [5], we know that (f_1), (f_2) and (f_3) guarantee the existence of a solution \( u_\varepsilon \), which is a local minimum of the corresponding functional of (1.1), for \( \varepsilon > 0 \) small, with \( a \leq u_\varepsilon \leq 0 \) and \( u_\varepsilon \to a \) as \( \varepsilon \to 0 \) on any compact subset \( K \) of \( \Omega \). Since \( u_\varepsilon \) is a local minimum, we can expect that (1.1) has a mountain pass solution \( u_\varepsilon \).

In this paper, we shall analyse the profile of the mountain pass type solution. Let \( v_\varepsilon = u_\varepsilon - u_\varepsilon \). Then \( v_\varepsilon \) satisfies
\[
\begin{align*}
\left\{ 
-\varepsilon^2 \Delta v - f'(u_\varepsilon) v &= g_\varepsilon(x, v), & \text{in } \Omega, \\
 v &= 0, & \text{on } \partial \Omega,
\end{align*}
\]
where
\[ g_\varepsilon(x, t) = f(u_\varepsilon + t) - f(u_\varepsilon) - f'(u_\varepsilon)t. \]

Let
\[
I_\varepsilon(v) = \frac{1}{2} \int_\Omega (\varepsilon^2 |Dv|^2 - f'(u_\varepsilon)v^2) - \int_\Omega G_\varepsilon(x, v), \quad v \in H^1_0(\Omega),
\]
\[ G_\varepsilon(x, t) = F(u_\varepsilon + t) - F(u_\varepsilon) - f(u_\varepsilon)t - \frac{1}{2}f'(u_\varepsilon)t^2, \quad F(t) = \int_0^t f(s) ds. \]

We define
\[
\gamma(t) = \inf_{\gamma_{\varepsilon \in C([-1, 1], H^1_0(\Omega))}, \gamma(0) = 0, \gamma(1) = \varepsilon} \max_{\gamma_{\varepsilon \in C([-1, 1], H^1_0(\Omega))}} I_\varepsilon(\gamma(t)),
\]
where \( \Gamma = \{ \gamma(t) \in C([0, 1], H^1_0(\Omega)), \gamma(0) = 0, \gamma(1) = \varepsilon \} \), \( \varepsilon \in H^1_0(\Omega) \) is a point with \( I_\varepsilon(\varepsilon) < 0 \).

It follows from (f_4) and (f_5) that there is \( \bar{\mu}_1 > 2 \), such that \( \bar{\mu}_1 G_\varepsilon(x, t) \leq t g_\varepsilon(x, t) \) for \( t \geq 0 \) (see the proof of (g.3) in section 3). By the mountain pass lemma of Ambrosetti and Rabinowitz [1], we know that (1.2) has a positive solution \( v_\varepsilon \) with \( I_\varepsilon(v_\varepsilon) = c_\varepsilon \). Thus (1.1) has a mountain pass type solution \( u_\varepsilon = u_\varepsilon + v_\varepsilon \). For the profile of \( u_\varepsilon \), we have

**Theorem 1.1.** Let \( u_\varepsilon = u_\varepsilon + v_\varepsilon \) be a mountain pass solution of (1.1). There is an \( \varepsilon_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0] \), we have

(i) There is a \( \bar{c} > 0 \), which is independent of \( \varepsilon \), such that \( \max_{x \in \Omega} u_\varepsilon(x) \geq \bar{c} \).

(ii) For any local maximum point \( x_\varepsilon \) of \( u_\varepsilon \) with \( u_\varepsilon(x_\varepsilon) \geq \tau > 0 \), where \( \tau > 0 \) is some constant independent of \( \varepsilon \), we have \( d(x_\varepsilon, \partial \Omega) \leq C\varepsilon \), and for any \( \theta > 0 \), \( |u_\varepsilon(x) - u_\varepsilon(x_\varepsilon)| \leq C\varepsilon^{-1/2} |x - x_\varepsilon| \text{ if } x \in \Omega \setminus B_\theta(x_\varepsilon) \). Here \( \nu > 0 \) is a constant.

(iii) For any sequence of \( \varepsilon \), there is a subsequence \( \varepsilon_j \to 0 \), such that \( x_j \to x_0 \in \partial \Omega \) with \( H(x_0) = \max_{x \in \partial \Omega} H(x) \), where \( x_j \) is any local maximum point of \( u_\varepsilon \) with \( u_\varepsilon(x_j) \geq \gamma_{\varepsilon_j}(1) > 0 \). \( H(x) \) is the mean curvature of \( \partial \Omega \) at \( x \).

(iv) \( \{ x : u_\varepsilon(x) > 0 \} \cap \partial \Omega \neq \emptyset \) and for any \( \theta > 0 \), \( \{ x : u_\varepsilon(x) > 0 \} \subseteq B_\theta(x_0), \) where \( x_0 \in \partial \Omega \) is the point in (iii) and \( \varepsilon_j \) is the subsequence defined in (iii).
By Theorem 1.1, we see that (1.1) has a changing sign mountain pass type solution which has a positive peak near a global maximum point $x_0$ of the mean curvature of the boundary and is negative away from a small neighbourhood of $x_0$.

If $f(t)$ has two zero points $b > 0$ and $0$, and $f(t) < 0$ for $t \in (0, b)$, Ni and Wei 26. Del Pino and Felmer 13 proved that a positive mountain pass solution of (1.1) has a unique local maximum point which locates near the center of the domain. The same result is still true if $f(t)$ has two zero points $b > a > 0$, and $f(t) < 0$ for $t \in (a, b)$. See 21. Our results here look very similar to those for the Neumann problem obtained in 23, 24. The main reason that the local maximum point of $u_\varepsilon$ is close to the boundary is that the corresponding problem in a half space possesses a mountain pass solution. See the result in Section 3. But unlike the Neumann problem, the local maximum point of $u_\varepsilon$ is inside the domain.

Our result does not claim that $u_\varepsilon$ has exactly one local maximum point, because, except for $N = 1$, it is not clear that the solution of the form $U(x) = u(x_N) + u(x)$ for problem (3.1) has exactly one local maximum point. But from the proof of the main result, it is easy to see that if $u_\varepsilon$ has two local maximum points $x_\varepsilon^{(1)}$ and $x_\varepsilon^{(2)}$, then $|x_\varepsilon^{(1)} - x_\varepsilon^{(2)}| \leq C\varepsilon$. Moreover, both $x_\varepsilon^{(1)}$ and $x_\varepsilon^{(2)}$ lie near (compare with $\varepsilon$) the normal direction of $\partial \Omega$ at $\bar{x}_\varepsilon$, where $\bar{x}_\varepsilon \in \partial \Omega$ with $|\bar{x}_\varepsilon - x_\varepsilon^{(1)}| = d(x_\varepsilon^{(1)}, \partial \Omega)$.

If $\Omega$ is the unit ball centered at the origin, then $u_\varepsilon$ is radially symmetric. Using the Mountain Pass Lemma in the radially symmetric class, we know that (1.2) has a radial solution $\bar{v}_\varepsilon(\vert x \vert)$. Moreover, it is easy to check that $\bar{v}_\varepsilon$ has a peak at the origin. In other words, $\bar{v}_\varepsilon$ has a peak at the global maximum of the distance function $d(x, \partial \Omega)$. So we may ask the question whether (1.2) always has a single peak solution with its peak near the global maximum of the distance function $d(x, \partial \Omega)$ in a general bounded domain. By Theorem 1.1 we see that this solution cannot be obtained by using the Mountain Pass Lemma in the whole space $H_0^1(\Omega)$.

We call a solution $u$ of (3.1) nondegenerate if the kernel $\text{Ker} L$ of the linear operator $L$ defined as

$$Lw =: -\Delta w - f'(u(x_N))w - \frac{\partial g_1(x_N, u)}{\partial u}w, \quad w \in H_0^1(R_N^+),$$

satisfies $\text{Ker} L = \text{span}\{ \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_{N-1}} \}$. It is an open problem whether the mountain pass solution of (3.1) is nondegenerate. If we can prove that (3.1) has a positive solution which is nondegenerate, then using the reduction method, we can easily construct various kinds of boundary peak solutions for (1.2) as in 14, 10, 17, 21. On the other hand, we know that for certain nonlinearities $f(t)$, such as $f(t) = (t - a)^{p-1} - (t - a)$, the positive solution of

$$-\Delta u = f(u + a), \quad u \in H_0^1(R_N)$$

is unique and nondegenerate. So we can prove the existence of positive interior peak solutions for (1.2) and find a lower estimate for the number of such solutions as in 7, 12, 16, 29. Especially, (1.2) has a single peak solution with its peak near the global maximum point of $d(x, \partial \Omega)$. We shall discuss this problem briefly in this paper. Here we stress that it is the nondegeneracy not the uniqueness which is important to us. Although the results on existence of positive interior peak solutions for (1.2) look similar to those for the Neumann problem 16, 29, the locations of the peaks of the positive interior peak solutions for these two problems are different.
It is worth pointing out that by the moving plane method of Gidas, Ni and Nirenberg \cite{14}, if $\Omega$ is convex, the distance of any local maximum point of a positive solution for \eqref{1.1} to the boundary of $\Omega$ has a positive lower bound which is independent of the nonlinearities and the solutions. So it is only possible for a changing sign solution to have a positive local maximum point close to the boundary of a convex domain. Therefore, the assumption $a < 0$ is essential in this paper.

This paper is organized as follows. In section 2, we obtain an asymptotic expansion of the local minimum solution $u_\varepsilon$ near the boundary of $\Omega$. The estimates in section 2 are essential to the proof of the main results of this paper. In section 3, we study the existence of a mountain pass solution for an elliptic problem on a half space, which corresponds to the limit problem when we blow up \eqref{1.2} at a boundary point of $\Omega$. From this mountain pass solution, we can construct an approximate solution for \eqref{1.2} and thus obtain an upper bound for $c_\varepsilon$. Section 4 is devoted to the proof of Theorem 1.1. In section 5, we discuss briefly the existence of interior peak solutions for \eqref{1.2}. Appendix A contains a decay estimate of any positive solution of \eqref{3.1}.

2. The expansion of the local minimum near the boundary

Let $u_\varepsilon$ be the solution of \eqref{1.1} with $a \leq u_\varepsilon \leq 0$ and $u_\varepsilon \to a$ in any compact subset of $\Omega$ as $\varepsilon \to 0$. In this section, we shall obtain an asymptotic expansion for $u_\varepsilon$ near the boundary of $\Omega$.

Let $u(t)$ be the solution of

\begin{equation}
\begin{cases}
-u'' = f(u), & t \geq 0, \\
a \leq u(t) \leq 0, & t \geq 0, \\
u(0) = 0, & u(+\infty) = a.
\end{cases}
\end{equation}

Then $u(t)$ is decreasing and $|u(t) - a| \leq Ce^{-\sqrt{-f'(a)t}}$.

**Lemma 2.1.** We have that there is a $c_0 > 0$, such that

\[\int_0^{+\infty} (|v'|^2 - f'(u)v^2) \geq c_0 \|v\|^2_{H_0^1(R_+^1)},\]

for any $v \in H_0^1(R_+^1)$.

**Proof.** First, the proof of Proposition 2 in \cite{6} shows that $-y'' - f'(y)y$ with zero Dirichlet boundary condition has no nonpositive eigenvalue and this operator is seen to be Fredholm at points of the spectrum less than $-f'(a)$. Hence we see that

\[\inf \left\{ \int_0^{+\infty} (|v'|^2 - f'(u)v^2) : v \in H_0^1(R_+^1), \int_0^{+\infty} v^2 = 1 \right\} = \lambda > 0.\]

Thus,

\[\int_0^{+\infty} (|v'|^2 - f'(u)v^2) \geq \lambda \int_0^{+\infty} v^2.\]
If $c_0 > 0$ is small enough, we see
\[
\int_0^{+\infty} (|v'|^2 - f'(u)v^2) = c_0 \int_0^{+\infty} |v'|^2 + (1 - c_0) \int_0^{+\infty} |v'|^2 - \int_0^{+\infty} f'(u)v^2
\]
\[
\geq c_0 \int_0^{+\infty} |v'|^2 + (1 - c_0) \int_0^{+\infty} v^2 + \int_0^{+\infty} f'(u)v^2 - \int_0^{+\infty} f'(u)v^2
\]
\[
= c_0 \int_0^{+\infty} |v'|^2 + (1 - c_0) \int_0^{+\infty} v^2 - c_0 \int_0^{+\infty} f'(u)v^2
\]
\[
\geq c_0 \int_0^{+\infty} |v'|^2 + (1 - c_0) \int_0^{+\infty} v^2 - c_0 \int_0^{+\infty} v^2
\]
\[
\geq c'_0 \|v\|_{H^1(R_1)}^2.
\]

**Lemma 2.2.** Let $v_\varepsilon$ be a solution of
\[
(2.3) \quad \begin{cases} -\varepsilon^2 \Delta v - f'(u \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right)) v = h, & \text{in } \Omega, \\ v = 0, & \text{on } \partial \Omega. \end{cases}
\]

Then $|v_\varepsilon|_{\infty} \leq C|h|_{\infty}$ for some $C > 0$, independent of $\varepsilon$, $h$ and $v_\varepsilon$.

**Proof.** We argue by contradiction. Suppose that there are $\varepsilon_m \to 0$, $h_m$, such that the solution $v_m$ of (2.3) satisfies $|v_m|_{\infty} \geq m|h_m|_{\infty}$. Without loss of generality, we assume $|v_m|_{\infty} = 1$. Then, $|h_m|_{\infty} \to 0$ as $m \to +\infty$. Let $x_m \in \Omega$ be such that $|v_m(x_m)| = |v_m|_{\infty}$. Let $u_m(y) = v_m(\varepsilon y + x_m)$, $\varepsilon y + x_m \in \Omega$.

If $\frac{d(x_m, \partial \Omega)}{\varepsilon_m} \to +\infty$, then we see that
\[-\Delta u - f'(u)u = 0, \quad \text{in } R^N,
\]
has a nontrivial bounded solution. By averaging over the unit sphere, we see that
\[-u'' - \frac{\Delta u}{\varepsilon^2} u' - f'(u)u = 0 \quad \text{has a bounded nontrivial solution in } R^1, \]
which is impossible since $f'(u) < 0$.

If $\frac{d(x_m, \partial \Omega)}{\varepsilon_m} \leq C$ for some $C > 0$, then
\[-\Delta u - f'(u)u = 0, \quad \text{in } R^N,
\]
has a nontrivial bounded solution $u$ with $u|_{x_N = 0} = 0$. This is impossible by Proposition 2 of [6].

Let $\psi(t)$ be the solution of
\[
(2.4) \quad \begin{cases} -\psi'' - f'(u(t))\psi = -u', & t > 0, \\ \psi(0) = \psi(+\infty) = 0. \end{cases}
\]
The existence of such a solution is guaranteed by Lemma 2.1. Then $\psi(t) > 0$ if $t > 0$ by the positivity of the operator and $u' < 0$.

Now we are ready to obtain an asymptotic expansion for $u_\varepsilon$.

**Proposition 2.3.** Let $u_\varepsilon$ be the local minimum near $a$. Then
\[
u_\varepsilon(x) = u \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) + \varepsilon(N - 1) \psi \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) H(\bar{x}) + O(\varepsilon^{1+\sigma}),
\]
for some $\sigma > 0$, where $u$ and $\psi$ are defined by (2.1) and (2.4) respectively, and $x \in \partial \Omega$ is a point such that $|x - \bar{x}| = d(x, \partial \Omega)$.

Proof. Define

$$u^* = \begin{cases} u(t), & t \in [0, \delta/\varepsilon], \\ \text{smooth}, & t \in [\delta/\varepsilon, 2\delta/\varepsilon], \\ u(2\delta/\varepsilon), & t \in [2\delta/\varepsilon, +\infty). \end{cases}$$

In $[\delta/\varepsilon, 2\delta/\varepsilon]$, we can choose $u^*$ such that $a \leq u^*(t) \leq 0$, $|u^*(t) - a| \leq Ce^{-\sqrt{-f''(a)\delta/\varepsilon}}$, $|D^*u^*(t)| \leq C\varepsilon^{-2} e^{-\sqrt{-f''(a)\delta/\varepsilon}}$, $i = 1, 2$.

Let $u_{**}^*(x) = u^*(\frac{d(x, \partial \Omega)}{\varepsilon})$. Then $u_{**}^*(x)$ is constant for $d(x, \partial \Omega) \geq 2\delta$. By a simple calculation, using the equation satisfied by $u(t)$, we see that

$$-\varepsilon^2 \Delta u_{**}^* = -u_{**}''(\frac{d(x, \partial \Omega)}{\varepsilon}) - \varepsilon u_{**}'(\frac{d(x, \partial \Omega)}{\varepsilon}) \Delta d(x, \partial \Omega)$$

$$= \begin{cases} f(u^*(\frac{d(x, \partial \Omega)}{\varepsilon})) - \varepsilon u_{**}'(\frac{d(x, \partial \Omega)}{\varepsilon}) \Delta d(x, \partial \Omega), & \text{if } d(x, \partial \Omega) \leq \delta, \\ O(\varepsilon^{-2} e^{-\delta'/\varepsilon}) - \varepsilon u_{**}'(\frac{d(x, \partial \Omega)}{\varepsilon}) \Delta d(x, \partial \Omega), & \text{if } \delta \leq d(x, \partial \Omega) \leq 2\delta, \\ -\varepsilon u_{**}'(\frac{d(x, \partial \Omega)}{\varepsilon}) \Delta d(x, \partial \Omega), & \text{if } d(x, \partial \Omega) \geq 2\delta, \end{cases}$$

since for $d(x, \partial \Omega) \geq \delta$,

$$f(u_{**}^*) = f(a) + O(|u_{**}^* - a|) = O(\varepsilon^{-2} e^{-\delta'/\varepsilon}).$$

Let $\xi_\varepsilon = u_{**}^* - u^*$. Then

$$-\varepsilon^2 \Delta \xi_\varepsilon = f(u_{**}^*) - f(u^*) - \varepsilon u_{**}'(\frac{d(x, \partial \Omega)}{\varepsilon}) \Delta d(x, \partial \Omega) + O(\varepsilon^{-2} e^{-\delta'/\varepsilon})$$

$$= -c_\varepsilon(\xi_\varepsilon - \varepsilon u_{**}'(\frac{d(x, \partial \Omega)}{\varepsilon})) \Delta d(x, \partial \Omega) + O(\varepsilon^{-2} e^{-\delta'/\varepsilon}),$$

where $c_\varepsilon = -\int_0^1 f'(\tau u_{**}^*) + (1 - \tau)u_{**}' \, d\tau$.

Let $\xi_\varepsilon = \varepsilon^2 \psi_\varepsilon$. Then

$$\begin{cases} -\varepsilon^2 \Delta \psi_\varepsilon + c_\varepsilon \psi_\varepsilon = -u_{**}'(\frac{d(x, \partial \Omega)}{\varepsilon}) \Delta d(x, \partial \Omega) + O(\varepsilon^{-3} e^{-\delta'/\varepsilon}), & \text{in } \Omega, \\ \psi_\varepsilon = 0, & \text{on } \partial \Omega. \end{cases}$$

Similar to the proof of Lemma 2.2 it is not difficult to prove that $|\psi_\varepsilon|_\infty \leq C$.

(Here we need to use Proposition 2.4 in [5] and Theorem 2 in [27] to prove that $c_\varepsilon(\varepsilon y + x_\varepsilon) \to -f'(u(x_\varepsilon))$ as $\varepsilon \to 0$ if $d(x_\varepsilon, \partial \Omega) \leq C\varepsilon$.)

Let

$$\psi_\varepsilon^*(t) = \begin{cases} \psi(t), & \text{if } t \in [0, \delta/\varepsilon], \\ \text{smooth}, & \text{if } t \in [\delta/\varepsilon, 2\delta/\varepsilon], \\ 0, & \text{if } t \in [2\delta/\varepsilon, +\infty). \end{cases}$$

For any $x$ with $d(x, \partial \Omega) \leq \delta$, let $\bar{x}$ be the unique point on $\partial \Omega$ such that $|x - \bar{x}| = d(x, \partial \Omega)$. Let

$$\psi_{**}^*(x) = -(N - 1)\psi_\varepsilon^*(\frac{d(x, \partial \Omega)}{\varepsilon}) H(\bar{x}),$$
where $H(x)$ is the mean curvature of $\partial \Omega$ at $\bar{x}$. Noting that
\[
\Delta \psi_\varepsilon'' = \varepsilon^{-2} \psi_\varepsilon'' + \varepsilon^{-1} \Delta d(x, \partial \Omega) \psi_\varepsilon'' = \varepsilon^{-2} \psi_\varepsilon'' + O(\varepsilon^{-1}),
\]
we see
\[
-\varepsilon^2 \Delta \psi_\varepsilon'''' = \varepsilon^2 (N-1) \left( H(\bar{x}) \Delta \psi_\varepsilon + 2\varepsilon^{-1} \Delta d(x, \partial \Omega) + \varepsilon \Delta H(\bar{x}) \right) + \psi_\varepsilon \Delta H(\bar{x})
\]
\[
= (N-1) H(\bar{x}) \Delta \psi_\varepsilon'' + O(\varepsilon)
\]
\[
= (N-1) \left( -f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) \psi_\varepsilon'' + f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) \right) H(\bar{x}) + O(\varepsilon)
\]
\[
= f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) (\psi_\varepsilon'' + (N-1) H(\varepsilon) + O(\varepsilon)).
\]

Thus,
\[
-\varepsilon^2 \Delta (\psi_\varepsilon'' - \psi_\varepsilon) = f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) (\psi_\varepsilon'' - \psi_\varepsilon)
\]
\[
= (f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right)) - c_\varepsilon \psi_\varepsilon + f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) \left( (N-1) H(\bar{x}) + \Delta d(x, \partial \Omega) \right) + O(\varepsilon)
\]
\[
= O(\varepsilon^\alpha),
\]
since
\[
f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) - c_\varepsilon
\]
\[
= \int_0^1 \left( f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) - f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) \left( u_{\varepsilon}'' - u_{\varepsilon} \right) \right) d\tau
\]
\[
= O \left( \left| f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) - f' \left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) \right| \right) = O(\varepsilon^\alpha)
\]
and
\[
\frac{d(x, \partial \Omega)}{\varepsilon} \left( \Delta d(x, \partial \Omega) + (N-1) H(\bar{x}) \right)
\]
\[
= O \left( \left| \frac{d(x, \partial \Omega)}{\varepsilon} \right| \right) = O(\varepsilon^{1-\tau}), \quad \text{for any } \tau > 0.
\]

Here, in the last equality, we have used Lemma 14.17 in [15]. By Lemma 2.2, we obtain
\[
|\psi_\varepsilon'' - \psi_\varepsilon| = O(\varepsilon^\sigma),
\]
for some $\sigma > 0$. \hfill $\square$

3. Existence of the mountain pass solution in half space

Let $u(t)$ be a solution of (2.1). Consider
\[
(3.1) \quad \begin{cases}
-\Delta u - f'(u(x_N)) u = g_1(x_N, u), & \text{in } R^N_+,
\end{cases}
\]
\[
u \in H^1_0(R^N_+),
\]
where $g_1(x_N, t) = f(u(x_N) + t) - f(u(x_N)) - f'(u(x_N)) t$.

Define
\[
J(u) = \frac{1}{2} \int_{R^N_+} \left| Du \right|^2 - f'(u(x_N)) u^2 - \int_{R^N_+} G_1(x_N, u),
\]
where
\[ G_1(x_N, t) = \int_0^t g_1(x_N, \tau) \, d\tau \]
\[ = F(w(x_N) + t) - F(w(x_N)) - f(w(x_N))t - \frac{1}{2}f''(w(x_N))t^2, \]
and \( F(t) = \int_0^t f(\tau) \, d\tau. \)

We summarize the properties of \( g_1(x_N, t) \) as follows:

1. \( g_1(x_N, t) \geq 0 \) for \( t \geq 0. \)
   
   In fact, by \((f_3)\), we have
   \[ g_1(x_N, 0) = 0, \quad \frac{\partial}{\partial t} g_1(x_N, 0) = 0, \]
   \[ \frac{\partial^2}{\partial t^2} g_1(x_N, t) = f''(w(x_N)) > 0. \]

2. \( g_1(x_N, t) \) is strictly increasing in \( t > 0. \)
   
   In fact, we have
   \[ \frac{\partial}{\partial t} \left( \frac{g_1(x_N, t)}{t} \right) = \left( \frac{\partial}{\partial t} g_1(x_N, t) - g_1(x_N, t) \right)/t^2. \]
   But
   \[ \left. \left( t \frac{\partial}{\partial t} g_1(x_N, t) - g_1(x_N, t) \right) \right|_{t=0} = 0, \]
   \[ \frac{\partial}{\partial t} \left( t \frac{\partial}{\partial t} g_1(x_N, t) - g_1(x_N, t) \right) = \frac{\partial^2}{\partial t^2} g_1(x_N, t) = tf''(w(x_N) + t) > 0, \]

   for all \( t > 0. \) Thus \( t \frac{\partial}{\partial t} g_1(x_N, t) - g_1(x_N, t) > 0 \) for all \( t > 0. \)

3. There is a constant \( \bar{\mu} > 1, \) such that

\[ \bar{\mu} g_1(x_N, t) \leq \frac{\partial g_1(x_N, t)}{\partial t}, \quad \forall \ t \geq 0. \]

First, by \((f_5)\), we see that there is a large \( T > 0 \) such that \((3.2)\) holds for \( t \geq T. \)

Next, we claim that there is a \( \theta > 0, \) such that

\[ tf''(\tau + t) \geq \theta (f'(\tau + t) - f'(\tau)), \quad \forall \ t \in (0, T], \ a \leq \tau \leq 0. \]

It follows from \( f''(t) > 0 \) that \((3.3)\) holds if \( t \geq t_0 > 0 \) or \( \tau - a > \tau_0 > 0, \) where \( t_0 \) and \( \tau_0 \) are fixed numbers. Thus it remains to prove that \((3.3)\) holds for \( t \in (0, t_0] \) and \( \tau \in (a, a + \tau_0]. \)

If \( t \leq \tau - a, \) then it follows from \((f_4)\) that

\[ tf''(\tau + t) - \theta (f'(\tau + t) - f'(\tau)) = t(f''(\tau + t) - \theta f''(\tau + \eta t)) \]
\[ \geq t \left( \frac{c_0}{(t + \tau - a)^{1-\alpha}} - \frac{\theta c_0''}{(\eta t + \tau - a)^{1-\alpha}} \right) \geq t \left( \frac{c_0'}{(t + \tau - a)^{1-\alpha}} - \frac{\theta c_0''}{(\tau - a)^{1-\alpha}} \right) \]
\[ \geq t \left( \frac{1}{(\tau - a)^{1-\alpha}} - \frac{\theta c_0''}{(\tau - a)^{1-\alpha}} \right) > 0 \]

if \( \theta > 0 \) is small enough.

Suppose that \( \tau - a \leq t. \) From \((f_4)\), we can deduce

\[ |f'(t_1) - f'(t_2)| \leq C|t_1 - t_2|^\alpha, \quad t_1, t_2 \in [a, a + \eta]. \]
Thus,
\[ tf''(\tau + t) - \theta(f'(\tau + t) - f'(\tau)) \geq tf''(\tau + t) - C\theta t^\alpha \]
\[ \geq t \left( \frac{c_0'}{(t + \tau - a)^{2-\alpha}} - \frac{\theta C}{t^{1-\alpha}} \right) \geq t \left( \frac{c_0'}{2^{1-\alpha}t^{1-\alpha}} - \frac{\theta C}{t^{1-\alpha}} \right) > 0. \]
Thus we have proved (3.3).

Now take \( \bar{\mu} = 1 + \theta > 1 \). Let
\[ \eta(x_N, t) = t \frac{\partial g_1(x_N, t)}{\partial t} - \bar{\mu} g_1(x_N, t). \]
Then \( \eta(x_N, 0) = 0 \). We have
\[ \frac{\partial}{\partial t} \eta(x_N, t) = t \frac{\partial^2}{\partial t^2} g_1(x_N, t) + (1 - \bar{\mu}) \frac{\partial}{\partial t} g_1(x_N, t) \]
\[ = tf''(w(x_N) + t) + (1 - \bar{\mu})(f'(w(x_N) + t) - f'(w(x_N))) \geq 0, \]
if \( t \in (0, T] \). As a result, \( \eta(x_N, t) > 0 \) for \( t \in (0, T] \).

(3.4) We have \( g_1(x_N, t) = O(t^2) \) as \( t \to 0 \), and \( |g_1(x_N, t)| \leq C|t|^{p-1} + 1 \).

Define
\[ c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(u), \]
where \( \Gamma = \{ \gamma(t) \in C([0,1], H_0^1(R_N^+)), \gamma(0) = 0, \gamma(1) = e \} \), \( e \in H_0^1(R_N^+) \) with \( J(e) < 0 \). By (3.2), it is easy to check that \( c \) is independent of \( e \).

It is easy to see that (3.1) is translation invariant in the \( x_i \) direction, \( i = 1, \cdots, N - 1 \). Since \( u(t) \to a \) as \( t \to +\infty \), we see that the corresponding limit problem in the \( x_N \) direction is
\[ \begin{cases} -\Delta w = f(w + a), & \text{in } R^N, \\ w \in H^1(R^N). \end{cases} \]
(3.5)

Let
\[ A = \frac{1}{2} \int_{R^N} |Dw|^2 - \int_{R^N} \left( \int_0^w f(\tau + a) d\tau \right), \]
where \( w \) is the least energy solution of (3.5).

Using the standard concentration compactness argument [20], we can prove that (3.1) has a solution with critical value \( c \) once we prove the following lemma:

**Lemma 3.1.** We have \( c < A \).

**Proof.** Let \( x_l = (0, \cdots, 0, l) \), \( w_l(x) = w(x - x_l) \). Let \( Pw_l \) be the solution of
\[ \begin{cases} -\Delta v - f'(a)v = f(w_l + a) - f'(a)w_l, & \text{in } R^N, \\ v \in H_0^1(R_N^+). \end{cases} \]
(3.6)
By the maximum principle, we have \( |Pw_l - w_l| \leq \max_{x_N=0} w_l \leq C e^{-\sqrt{-f'(a)t}} \). By the definition of \( c \) (see (3.4)), we have
\[ c \leq \max_{t \geq 0} J(tPw_l). \]
Now we estimate \( J(tPw_l) \).

Step 1. The estimate of \( J(Pw_l) \).

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Then we claim that there are $C > 0$, such that

\[ J(Pw_1) = \int_{R^N_+} |DPw_1|^2 - f'(u(x_N))|Pw_1|^2 - \int_{R^N_+} G_1(x_N, Pw_1) \]

\[ = \frac{1}{2} \int_{R^N_+} \left( f'(a) - f'(u(x_N)) \right)|Pw_1|^2 + \int_{R^N_+} \left( \frac{1}{2} \bar{g}(w_1) Pw_1 - G_1(x_N, Pw_1) \right) \]

\[ = \frac{1}{2} \int_{R^N_+} \left( f'(a) - f'(u(x_N)) \right)|Pw_1|^2 + \int_{R^N_+} \left( \frac{1}{2} \bar{g}(w_1) Pw_1 - \bar{G}(Pw_1) \right) \]

\[ + \int_{R^N_+} \left( \bar{G}(Pw_1) - G_1(x_N, Pw_1) \right) \]

\[ = \int_{R^N_+} \left( \frac{1}{2} \bar{g}(w_1) Pw_1 - \bar{G}(Pw_1) \right) \]

\[ + \int_{R^N_+} \left( F(a + Pw_1) - F(a) - (F(u(x_N) + Pw_1) - F(u(x_N)) - f(u(x_N)Pw_1) \right) \]

\[ = J_1 + J_2, \]

where $J_1$ and $J_2$ are defined by the last equality of the above relation.

For $J_1$, we have

\[ J_1 = \int_{R^N_+} \left( \frac{1}{2} \bar{g}(w_1) w_1 - \bar{G}(w_1) \right) + \int_{R^N_+} \bar{g}(w_1)(w_1 - Pw_1) \]

\[ + O(e^{-2\sqrt{-f'(a)l}}) \]

\[ = \int_{R^N_+} \left( \frac{1}{2} \bar{g}(w_1) w_1 - \bar{G}(w_1) \right) + O(e^{-(1+\sigma)\sqrt{-f'(a)l}}) \]

\[ + \frac{1}{2} \int_{R^N_+} \left( -\Delta w_1 - f'(a)w_1 \right)(w_1 - Pw_1) + O(e^{-2\sqrt{-f'(a)l}}) \]

\[ = A + \frac{1}{2} \int_{x_N=0} \left( \frac{\partial w_1}{\partial x_N}(w_1 - Pw_1) \right) \]

\[ + O(e^{-(1+\sigma)\sqrt{-f'(a)l}}) \]

\[ = A + O(e^{-(1+\sigma)\sqrt{-f'(a)l}}). \]

Now we estimate $J_2$. Let

\[ K(t_1, t_2) = F(a + t_1 + t_2) - F(a + t_1) - f(a + t_1)t_2 \]

\[ - (F(a + t_2) - F(a) - f(a)t_2) \]

\[ - (f(a + t_2)t_1 - f(a)t_1 - f'(a)t_1t_2). \]

We claim that there are $C > 0$ and $\sigma > 0$, such that

\[ |K(t_1, t_2)| \leq C|t_1|^{1+\sigma}|t_2|^{1+\sigma}. \]
In fact, if $|t_2| \leq |t_1|$, then
\[
K(t_1, t_2) = \left[ (f(a + t_2 + \xi t_1) - f(a + \xi t_1) - f'(a + \xi t_1)t_2) \right. \\
\left. - (f(a + t_2) - f(a) - f'(a)t_2) \right] t_1 \\
= \left[ (f'(a + \xi t_1 t_2 + \xi t_1) - f'(a + \xi t_1)) - (f'(a + t_1 t_2) - f'(a)) \right] t_1 t_2.
\]

By (f4), we see $|f'(a) - f'(t_2)| \leq C|t_1 - t_2|^\alpha$. As a result,
\[
|K(t_1, t_2)| \leq C|t_2|^\alpha|t_1 t_2| \leq C|t_1|^{1+\alpha/2}|t_2|^{1+\alpha/2}.
\]

If $|t_1| \leq |t_2|$, then
\[
K(t_1, t_2) = \frac{1}{2}(f'(a + t_2 + \xi t_1) - f'(a + \xi t_1)t_2^2) \\
- (f'(a + \xi t_1) - f'(a))t_1 t_2.
\]

Thus,
\[
K(t_1, t_2) = C|t_2|^\alpha t_1^2 + C|t_1|^\alpha |t_1 t_2| \\
\leq C|t_1|^{1+\sigma} |t_2|^{1+\sigma}.
\]

We have
\[
J_2 = - \int_{R_+^N} \left( f(a + Pw) - f(a) - f'(a)Pw \right)(\tilde{u}(x_N) - a) \\
- \int_{R_+^N} K(u(x_N) - a, Pw) \\
= - \int_{R_+^N} \left( f(a + Pw) - f(a) - f'(a)Pw \right)(\tilde{u}(x_N) - a) \\
+ O \left( \int_{R_+^N} |u(x_N) - a|^{1+\sigma} |Pw|^{1+\sigma} \right) \\
= - \int_{R_+^N} \left( f(a + Pw) - f(a) - f'(a)Pw \right)(\tilde{u}(x_N) - a) \\
+ O \left( \int_{R_+^N} e^{-(1+\sigma)\sqrt{-f'(a)x_N}} e^{-(1+\sigma)\sqrt{-f'(a)|x-x_1|}} \right) \\
= - \int_{R_+^N} \left( f(a + Pw) - f(a) - f'(a)Pw \right)(\tilde{u}(x_N) - a) + O(e^{-(1+\sigma)\sqrt{-f'(a)t}}).
\]

So we have
\[
J(Pw) = A - \int_{R_+^N} \left( f(a + Pw) - f(a) - f'(a)Pw \right)(\tilde{u}(x_N) - a) \\
+ O(e^{-(1+\sigma)\sqrt{-f'(a)t}}).
\]

Using the convexity of $f$, we see
\[
f(a + t) - f'(a)t > 0, \quad \forall \ t > 0.
\]

Noting that $\tilde{u}(x_N) - a > 0$, we obtain
\[
\int_{R_+^N} \left( f(a + Pw) - f(a) - f'(a)Pw \right)(\tilde{u}(x_N) - a) \\
\geq \int_{B_1(x_1)} \left( f(a + Pw) - f(a) - f'(a)Pw \right)(\tilde{u}(x_N) - a) \\
\geq c_0 e^{-\sqrt{-f'(a)t}}
\]
for some \( c_0 > 0 \). So we have proved that \( J(Pw_t) < A \).

Step 2. The estimate of \( \max_{t \geq 0} J(tPw_t) - J(Pw_t) \).

Let \( t_l \) be the maximum point of \( \max_{t \geq 0} J(tPw_t) \). Then

\[
J'(t_l Pw_t, Pw_t) = 0.
\]

We claim that \( t_l \to 1 \) as \( l \to +\infty \).

In fact, from \( g_1(x_N, t) = O(t^2) \) as \( t \to 0 \), and \( \frac{g_1(x_N, t)}{t^{N-1}} \to +\infty \) as \( t \to +\infty \), we see \( 0 < t_0 \leq t_l \leq T, \forall l \). Suppose that \( t_l \to t_\infty \). Letting \( l \to +\infty \) in \( 3.9 \), we see

\[
t_\infty \int_{\mathbb{R}^N} |Dw|^2 = \int_{\mathbb{R}^N} f(a + t_\infty w)w.
\]

It is easy to see that from \((g2)\) there is exactly one \( t > 0 \) satisfying \( 3.10 \). On the other hand, \( t = 1 \) satisfies \( 3.10 \). Thus, \( t_\infty = 1 \).

But

\[
\begin{align*}
J'(Pw_t, Pw_t) &= \int_{\mathbb{R}^N} f'(a)Pw_t(Pw_t - w_t) + \int_{\mathbb{R}^N} (f(u(x_N)) - f(a))Pw_t \\
&\quad + \int_{\mathbb{R}^N} (f(w + a) - f(Pw_t + u(x_N)))Pw_t = O(e^{-\sqrt{-J'(a)}}),
\end{align*}
\]

\[
J''(Pw_t)(Pw_t, Pw_t) = \int_{\mathbb{R}^N} (|DPw_t|^2 - f'(a)|Pw_t|^2) - \int_{\mathbb{R}^N} \frac{\partial g_1(x_N, Pw_t)}{\partial t} |Pw_t|^2 \leq -c_0 < 0
\]

since \( t^2 \frac{\partial g_1(x_N, \cdot)}{\partial t} - t g_1(x_N, t) \geq (\mu - 1) t g_1(x_N, t) \). Thus we see \( t_l - 1 = O(e^{-\sqrt{-J'(a)}}) \).

As a result,

\[
J(t_l Pw_t) - J(Pw_t) = (t_l - 1)\langle J'(Pw_t), Pw_t \rangle + O(|t_l - 1|^2) = O(e^{-2\sqrt{-J'(a)}}).
\]

Combining Step 1 and Step 2, we obtain

\[
\max_{t \geq 0} J(tPw_t) = J(t_l Pw_t) = J(Pw_t) + O(e^{-2\sqrt{-J'(a)}}) \leq A - c_0 e^{-\sqrt{-J'(a)}} + O(e^{-(1+\sigma)\sqrt{-J'(a)}}) < A.
\]

Using Lemma 3.1, we have

**Theorem 3.2.** Problem \( 3.1 \) has a least energy solution \( u \) with \( J(u) = c \).

**Proof.** Let \( u_m \in H_0^1(R_+^N) \) be a sequence with \( J(u_m) \to c \) and \( J'(u_m) \to 0 \) as \( m \to +\infty \). By Lemma 2.1, we see that

\[
\int_{R_+^N} (|Du_m|^2 - f'(a(x_N))u_m^2) \geq c_0 \|u_m\|^2
\]

for some \( c_0 > 0 \). Then by \((g3)\), we see that \( u_m \) is bounded in \( H_0^1(R_+^N) \). Using the concentration compactness argument [20] and Lemma 3.1, we can deduce that there are \( x_m \in \{ x_N = 0 \}, m = 1, \ldots \), such that \( u_m(x + x_m) \) is compact in \( H_0^1(R_+^N) \). As a result, \( 3.1 \) has a solution with \( J(u) = c \). On the other hand, by using \((g2)\), it is easy to check that

\[
c = \inf \{ J(u) : u \in H_0^1(R_+^N), u \neq 0, \langle J'(u), u \rangle = 0 \}.
\]
See, for example, [24]. Thus for any nontrivial solution \( v \) of (3.1), \( J(v) \geq c \). So \( u \) is the least energy solution.

**Remark 3.3.** By Proposition \[ A.1 \], \( u(x) \leq C e^{-\sqrt{\lambda_0}|x|} \), for some \( \lambda_0 > 0 \). Since (3.1) does not depend on \( x' \), we can use the moving plane method of [14] in the directions \( x_i, i = 1, \cdots, N - 1 \), to prove that (3.1) has a least energy solution \( u(x) = u(|x'|, x_N), x' = (x_1, \cdots, x_{N-1}) \).

**Remark 3.4.** Let \( U = u(x_N) + u \). Then \( U \) is a solution of

\[
(3.11) \begin{cases}
-\Delta U = f(U), & \text{in } \mathbb{R}^N_+,
U(x', 0) = 0, & U(x', x_N) \rightarrow a, \text{uniformly in } x' \text{ as } x_N \rightarrow +\infty.
\end{cases}
\]

Clearly, \( U \geq u \) and \( U < 0 \) if \( |x'| \) is sufficiently large and \( x_N > 0 \) is small enough. Suppose that \( U \) has fixed sign. Then \( u \leq U \leq 0 \). By Proposition 2.5 in [5], we have \( U = u \). This is a contradiction. So \( U \) is a changing sign solution. We can also derive this result from [2] if \( N = 2 \), or \( N = 3, f(0) < 0 \). In the next section, we shall give a direct and simple proof of this fact. See Remark 4.2.

**Remark 3.5.** To prove that the mountain pass type solution \( u_\epsilon = u_\epsilon + v_\epsilon \) for (1.1) has exactly one positive local maximum point, it is important to show that \( U \) has exactly one positive local maximum point. It is easy to prove that this is true if \( N = 1 \) by using the relation \((U'(t))^2 - (U'(0))^2 = -2 \int_0^{U'(t)} f(\tau) d\tau \). It is an open problem whether \( U \) has exactly one positive local maximum point if \( N \geq 2 \).

**Remark 3.6.** It is possible to replace the convexity assumption on \( f \) by other conditions to obtain the existence result for (3.1). Under the condition that \( \frac{f(t)}{t^{N+\frac{q}{p}} t} \rightarrow a_\infty \in (0, +\infty) \), as \( t \rightarrow +\infty \), where \( 1 < p < \frac{N+2}{N+\frac{q}{p}} \) and \( q \) is finite, or \( 0 < A < t^{-p} f(t) < B < +\infty \) for large \( t \), where \( 1 < p < \frac{N+2}{N+m} \), we can prove that the Dirichlet problem on \( \{|x'| < M, x_N \in (0, m)\} \) has a positive mountain pass solution \( u_{M, m}(|x'|, x_N) \), which are uniformly bounded (by using a blow-up argument). The idea is to let \( m \rightarrow +\infty \) first, and then let \( M \rightarrow +\infty \) to obtain a decaying solution for (3.1). To make this work, we need to assume \( f(a + t) > f'(a)t \) for \( t > 0 \) and \( f(t)(t - \tilde{t}) > 2F(t) \) for \( t > \tilde{t} \), where \( F(t) \leq 0 \) for \( t \leq \tilde{t} \). Using the second condition, we can rewrite the energy of the solution to an integral, where the integrand is positive. Thus we can use the first condition to stop part of the solution moving to infinity in the \( x_N \) direction. For this remark, we choose \( F \) with \( F(a) = 0 \).

4. The Location of the Peak of the Mountain Pass Solution

Let \( u_\epsilon(x) \) be a solution of (1.1) such that \( |u_\epsilon - a| \) is small on any \( K \subset \subset \Omega \). Now we consider (1.2). Similar to the proof of Proposition 2 of [6], we can check that the first eigenvalue \( \lambda_{e, 1} \) of \(-\epsilon^2 \Delta - f'(u_\epsilon)I \text{ in } H_0^1(\Omega) \) satisfies \( \lambda_{e, 1} \geq \lambda_0 > 0 \). Thus by a similar argument to that in the proof of Lemma 2.1, we see

\[
\int_\Omega (\epsilon^2|Du|^2 - f'(u_\epsilon)u^2) \geq c' \int_\Omega (\epsilon^2|Du|^2 + u^2), \quad \forall u \in H_0^1(\Omega),
\]

for some \( c' > 0 \). So it is easy to check that (1.2) has a positive solution \( v_\epsilon \) with \( \int u(x) = c_\epsilon \). In this section, we shall prove that all the local maximum points of \( v_\epsilon \) tend to the same point \( x_0 \) on the boundary, at which the mean curvature \( H(x) \) attains its global maximum.

First, we have an upper bound for \( c_\epsilon \).
Proposition 4.1. Let \( u \in H_0^1(R^N_+) \) be any solution of \((3.1)\) with \( J(u) = c \). Then we have
\[
c_\varepsilon \leq \varepsilon^N (c - \varepsilon B(u) H_M + O(\varepsilon^{1+\sigma})),
\]
where \( H_M = \max_{x \in \partial \Omega} H(x) \), and
\[
B(u) = -\frac{1}{2} \int_{x_N=0} D_{x_N} u u'(x_N) |x'|^2 - \frac{1}{4} \int_{x_N=0} |D_{x_N} u|^2 |x'|^2.
\]
Moreover, we have \( B(u) > 0 \).

Proof. Take any solution \( u(r, x_N) \) of \((2.1)\) with \( J(u) = c \), where \( r = |x'| \). Let \( x_0 \in \partial \Omega \) be a point such that \( H(x_0) = \max_{x \in \partial \Omega} H(x) \). After translation and rotation, we may assume that \( x_0 = 0 \) and
\[
\begin{align*}
\Omega \cap B_\delta(0) &= \{ x : x_N > \varphi(x') \} \cap B_\delta(0), \\
\partial \Omega \cap B_\delta(0) &= \{ x : x_N = \varphi(x') \} \cap B_\delta(0),
\end{align*}
\]
where \( \varphi(x') \in C^2(R^{N-1}) \), \( \varphi(0) = 0 \), \( D \varphi(0) = 0 \) and
\[
\varphi(x') = \frac{1}{2} \sum_{i=1}^{N-1} a_i x_i^2 + O(|x'|^2).
\]
Let \( \eta \in C_0^\infty(B_\delta(0)) \), \( \eta = 1 \) for \( x \in B_{\delta/2}(0) \), \( 0 \leq \eta \leq 1 \). Define
\[
w_\varepsilon = \eta(x) u(\varepsilon^{-1} r, \varepsilon^{-1}(x_N - \varphi(x'))),
\]
where \( r = |x'| \). Then \( w_\varepsilon \in H_0^1(\Omega) \). So we have
\[
c_\varepsilon \leq \max_{t \geq 0} I_\varepsilon(t w_\varepsilon).
\]
Now we estimate \( I_\varepsilon(w_\varepsilon) \). We have
\[
I_\varepsilon(w_\varepsilon) = \frac{1}{2} \int_\Omega \varepsilon^2 |Dw_\varepsilon|^2 - \int_\Omega \left( F(u_\varepsilon + w_\varepsilon) - F(u_\varepsilon) - f(u_\varepsilon) w_\varepsilon \right)
= I_1 - I_2.
\]
Since \( u = \varepsilon^{-1}d(x, \partial \Omega) + \varepsilon(N - 1)H(x)\psi(\varepsilon^{-1}d(x, \partial \Omega)) + O(\varepsilon^{1+\sigma}) \), where \( \bar{x} \in \partial \Omega \) with \( |x - \bar{x}| = d(x, \partial \Omega) \), we have

\[
I_2 = \int_{\Omega} \left( F\left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) + w_x \right) - F\left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) \right) - f\left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) w_x \right) \\
+ \varepsilon(N - 1) \int_{\Omega} \left( f\left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) + w_x \right) - f\left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) \right) - f\left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) w_x \right) \psi\left( \frac{d(x, \partial \Omega)}{\varepsilon} \right) H(\bar{x}) \\
+ O(\varepsilon^{N+1+\sigma})
\]

where \( \Omega_\varepsilon = \{ x : \varepsilon x \in \Omega \} \), \( \bar{w}_x = w_x(\varepsilon x), x \in \Omega_\varepsilon \).

Let \( \varphi_\varepsilon(x') = \varepsilon^{-1}\varphi(\varepsilon x') \). Let \( \bar{x}_\varepsilon \in \partial \Omega_\varepsilon \) be such that \( |x - \bar{x}_\varepsilon| = d(x, \partial \Omega_\varepsilon) \). Then we have \( x_i - \bar{x}_{\varepsilon i} + (x_N - \varphi_\varepsilon(\bar{x}'))D_i\varphi_\varepsilon(\bar{x}') = 0, i = 1, \cdots, N - 1 \). Thus \( |x' - \bar{x}'| = O(\varepsilon) \).

As a result,

\[
d(x, \partial \Omega_\varepsilon) = (1 + |D\varphi_\varepsilon(\bar{x}')|^2)^{1/2}(x_N - \varphi_\varepsilon(\bar{x}')) = x_N - \varphi_\varepsilon(\bar{x}') + O(|D\varphi_\varepsilon(x')|^2) = x_N - \varphi_\varepsilon(x') + O(\varepsilon^2).
\]

Thus, making the change of variable \( y' = x', y_N = x_N - \varphi_\varepsilon(x') \), we obtain

\[
I_2 = \varepsilon^N \left[ \int_{R^N_\varepsilon} \left( F\left( u(x_N) + u \right) - F(u(x_N)) - f(u(x_N))u \right) \right) \\
+ \varepsilon(N - 1) \int_{R^N_\varepsilon} \left( f\left( u(x_N) + u \right) - f(u(x_N)) - f'(u(x_N))u \right) \psi(x_N)H(0) \\
+ O(\varepsilon^{N+1+\sigma}).
\]

Here, we have used the fact that \( u \) decays exponentially.
For the estimate of $I_1$, we have

$$
(4.6) \int_{\Omega} |Dw| \leq \epsilon^2 \int_{\Omega \cap B_{\delta/2}(0)} |Dw| + O(\epsilon^{-\delta/\epsilon})
$$

$$
= \epsilon^2 \int_{\Omega \cap B_{\delta/2}(0)} \left( |Du(x, x_N - \varphi(x'))| \right)^2
$$

$$
- 2Dx_N u(x, x_N - \varphi(x')) \sum_{i=1}^{N-1} Dx_i u(x, x_N - \varphi(x')) Dx_i \varphi(x')
$$

$$
+ |Dx_N u(x, x_N - \varphi(x'))|^2 \sum_{i=1}^{N-1} |Dx_i \varphi(x')|^2 + O(\epsilon^{-\delta/\epsilon})
$$

$$
= \epsilon^2 \left( \int_{\Omega} |Du|^2 - 2 \int_{\Omega} Dx_N u \sum_{i=1}^{N-1} a_i x_i Dx_i u + O(\epsilon^2) \right).
$$

In the last equality, we have used again the fact that $u$ decays exponentially. Since $u$ is a solution of $-\Delta u = f(u(x_N) + u) - f(u(x_N))$, we have

$$
- \int_{\Omega} \Delta u x_i^2 Dx_N u = \int_{\Omega} \left( f(u(x_N) + u) - f(u(x_N)) \right) x_i^2 Dx_N u
$$

for $i = 1, \cdots, N - 1$.

But for each fixed $i$, we have

$$
- \int_{\Omega} \Delta u x_i^2 Dx_N u
$$

$$
= - \int_{x_N=0} \frac{\partial u}{\partial n} x_i^2 Dx_N u + 2 \int_{\Omega} Dx_N u x_i Dx_i u + \sum_{j=1}^{N} \int_{\Omega} x_i^2 Dx_j u Dx_j x_N u
$$

$$
= \int_{x_N=0} Dx_N u x_i^2 Dx_N u + 2 \int_{\Omega} Dx_N u x_i Dx_i u + \frac{1}{2} \sum_{j=1}^{N} \int_{\Omega} x_i^2 Dx_j |Dx_j u|^2
$$

$$
= \frac{1}{2} \int_{x_N=0} x_i^2 |Dx_N u|^2 + 2 \int_{\Omega} Dx_N u x_i Dx_i u,
$$

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since $D_{x_j}u = 0$ if $x_N = 0$, for $j = 1, \cdots, N - 1$. Here we do not use the summation convention. So

$$2 \int_{R_N^+} D_{x_N} u x_i D_{x_i} u$$

$$= \int_{R_N^+} \left( f(u(x_N) + u) - f(u(x_N)) \right) x_i^2 D_{x_N} u - \frac{1}{2} \int_{x_N = 0} x_i^2 |D_{x_N} u|^2$$

$$= \int_{R_N^+} \frac{d}{dx_N} \left( F(u(x_N) + u) - F(u(x_N)) \right) x_i^2$$

$$- \int_{R_N^+} \left( f(u(x_N) + u) - f(u(x_N)) \right) \frac{d}{dx_N} \left( f(u(x_N)) u \right) x_N x_i^2$$

$$- \frac{1}{2} \int_{x_N = 0} x_i^2 |D_{x_N} u|^2$$

$(4.7)$

$$= - \int_{R_N^+} \left( f(u(x_N) + u) - f(u(x_N)) \right) \frac{d}{dx_N} \left( f(u(x_N)) u \right) x_N x_i^2$$

$$- \frac{1}{2} \int_{x_N = 0} x_i^2 |D_{x_N} u|^2$$

$$= - \frac{1}{N - 1} \int_{R_N^+} \left( f(u(x_N) + u) - f(u(x_N)) \right) \frac{d}{dx_N} \left( f(u(x_N)) u \right) x_N x_i^2$$

$$- \frac{1}{2(N - 1)} \int_{x_N = 0} |x_i|^2 |D_{x_N} u|^2.$$

Combining $(4.6)$ and $(4.7)$, we obtain

$(4.8)$

$$I_1 = \varepsilon^N \left( \frac{1}{2} \int_{R_N^+} |D u|^2 + \frac{\varepsilon H(0)}{4} \int_{x_N = 0} |x_i|^2 |D_{x_N} u|^2 + O(\varepsilon^2) \right)$$

$$+ \frac{\varepsilon H(0)}{2} \int_{R_N^+} \left( f(u(x_N) + u) - f(u(x_N)) \right) \frac{d}{dx_N} \left( f(u(x_N)) u \right) x_N x_i^2.$$

Putting $(4.5)$ and $(4.8)$ together, we see

$(4.9)$

$$I_2(w_\varepsilon) = \varepsilon^N \left( c + \frac{\varepsilon H(0)}{4} \int_{x_N = 0} |x_i|^2 |D_{x_N} u|^2 + O(\varepsilon^{1+\sigma}) \right)$$

$$+ \frac{\varepsilon H(0)}{2} \int_{R_N^+} \left( f(u(x_N) + u) - f(u(x_N)) \right) \frac{d}{dx_N} \left( f(u(x_N)) u \right) x_N x_i^2$$

$$- (N - 1)\varepsilon H(0) \int_{R_N^+} \left( f(u(x_N) + u) - f(u(x_N)) \right) \frac{d}{dx_N} \left( f(u(x_N)) u \right) \psi(x_N).$$
Using the exponential decay of the solution \( u, u = \psi = 0 \) if \( x_N = 0 \), and self-adjointness of the Laplacian, we see
\[
\int_{R_+^N} \left( f(u(x_N) + u) - f(u(x_N)) - f'(u(x_N))u \right) \psi(x_N)
= \int_{R_+^N} \left( -\Delta \psi - f'(u(x_N))\psi \right) u
- \int_{R_+^N} u'(x_N)u.
\]
(4.10)

Similarly, we have
\[
\int_{R_+^N} \left( f(u(x_N) + u) - f(u(x_N)) - f'(u(x_N))u \right) u'(x_N)|x'|^2
= -\int_{R_+^N} \Delta uu'(x_N)|x'|^2
= -\int_{x_N=0} \frac{\partial u}{\partial x} u'(x_N)|x'|^2 - \int_{R_+^N} u\Delta(u'(x_N)|x'|^2)
= \int_{x_N=0} D_{x_N} u u'(x_N)|x'|^2 - \int_{R_+^N} u(u''|x'|^2 + 2(N-1)u'(x_N))
= \int_{x_N=0} D_{x_N} u u'(x_N)|x'|^2 + \int_{R_+^N} u(f'(u(x_N))u'(x_N)|x'|^2 - 2(N-1)u'(x_N)).
\]

As a result,
\[
\int_{R_+^N} \left( f(u(x_N) + u) - f(u(x_N)) - f'(u(x_N))u \right) u'(x_N)|x'|^2
= \int_{x_N=0} D_{x_N} u u'(x_N)|x'|^2 - 2(N-1) \int_{R_+^N} u u'(x_N).
\]
(4.11)

Combining (4.9), (4.10) and (4.11), we obtain
\[
I_\varepsilon(w_\varepsilon) = \varepsilon^N \left(c - \varepsilon B(u)H(0) + O(\varepsilon^{1+\sigma}) \right),
\]
(4.12)

where
\[
B(u) = -\frac{1}{2} \int_{x_N=0} D_{x_N} u u'(0)|x'|^2 - \frac{1}{4} \int_{x_N=0} |D_{x_N} u|^2|x'|^2.
\]

It is easy to check that \( \langle I'_\varepsilon(w_\varepsilon), w_\varepsilon \rangle = O(\varepsilon^{N+1}), I''_\varepsilon(w_\varepsilon)(w_\varepsilon, w_\varepsilon) \leq -c_0 \varepsilon^N < 0 \). Thus, similar to the proof of step 2 of Lemma 3.1, we see that if \( t_\varepsilon \) achieves \( \max_{t \geq 0} I_\varepsilon(tw_\varepsilon) \), then \( t_\varepsilon = 1 + O(\varepsilon) \). So
\[
I_\varepsilon(t_\varepsilon w_\varepsilon) = I_\varepsilon(w_\varepsilon) + (t_\varepsilon - 1)\langle I'_\varepsilon(w_\varepsilon), w_\varepsilon \rangle + O(\varepsilon^N |t_\varepsilon - 1|^2)
= I_\varepsilon(w_\varepsilon) + O(\varepsilon^{N+2}).
\]
(4.13)

Thus, the upper bound follows from (4.12) and (4.13).

It remains to prove that \( B(u) > 0 \). Let \( U = u + u \). Then \( U \) satisfies \(-\Delta U = f(U)\) on \( R_+^N \), \( U = 0 \) on \( x_N = 0 \) and \( U \to a < 0 \) as \( x_N \to +\infty \). It is easy to check that
\[
B(u) = \frac{1}{4} \int_{x_N=0} \left( u'(0)^2 - |D_{x_N} U(x', 0)|^2 \right)|x'|^2.
\]
First we claim that
\[(4.14) \quad \int_{x_N=0} (u'(0)^2 - |D_{xN}U(x', 0)|^2) = 0.\]

Multiplying $-\Delta U = f(U)$ by $D_{xN}U$ and integrating over $B_R^{N-1}(0) \times [0, +\infty)$, where $B_R^{N-1}(0)$ is the ball in $R^{N-1}$, centered at the origin with radius $R$, we obtain
\[- \sum_{i=1}^{N-1} \int_{B_R^{N-1}(0) \times [0, +\infty)} U_{x_i} U_{x_N} + \frac{1}{2} \int_{B_R^{N-1}(0)} U_{x_N}^2 (x', 0) \int_{B_R^{N-1}(0)} \int_0^\infty \frac{d}{dx_N} F(U) dx_N dx' = \int_{B_R^{N-1}(0)} F(a) = \int_{B_R^{N-1}(0)} \int_0^a f(\tau) d\tau.\]

But $\frac{1}{2} |u'(0)|^2 = \int_0^a f(\tau) d\tau$. So we have
\[\int_{B_R^{N-1}(0)} (u'(0)^2 - |D_{xN}U(x', 0)|^2) = -2 \sum_{i=1}^{N-1} \int_{B_R^{N-1}(0) \times [0, +\infty)} U_{x_i} U_{x_N}.\]

On the other hand, we have
\[\int_{\partial B_R^{N-1}(0)} U_{x_i} U_{x_N} = \int_{\partial B_R^{N-1}(0)} n_i U_{x_i} U_{x_N} - \int_{\partial B_R^{N-1}(0)} U_{x_i} U_{x_N} dx' = \int_{\partial B_R^{N-1}(0)} n_i U_{x_i} U_{x_N} - \frac{1}{2} \int_{\partial B_R^{N-1}(0)} U_x^2 dx'.\]

So
\[\int_{B_R^{N-1}(0) \times [0, +\infty)} U_{x_i} U_{x_N} = \int_{\partial B_R^{N-1}(0) \times [0, +\infty)} n_i U_{x_i} U_{x_N} \rightarrow 0,\]
as $R \rightarrow +\infty$, since $U_{x_i} = u_{x_i}$ decays exponentially as $|x| \rightarrow +\infty$. Thus (4.14) follows.

Now we prove that $B(u) > 0$. Since $U(|x'|, x_N)$ is a decreasing function of $|x'|$, we see that $U_{x_N}(|x'|, 0)$ is nonincreasing in $|x'|$. Let $r_0$ be such that $u'(0)^2 - |D_{xN}U(x', 0)|^2 \geq 0$ if $|x'| \geq r_0$, $u'(0)^2 - |D_{xN}U(x', 0)|^2 \leq 0$ if $|x'| \leq r_0$. Since $u_{x_N}(x', 0)$ tends to zero as $|x'| \rightarrow +\infty$ and $u_{x_N}(x', 0) \geq 0$, we see that $u'(0)^2 - |D_{xN}U(x', 0)|^2 > 0$ if $|x'|$ is large. So $r_0 \rightarrow +\infty$. Hence, by (4.14), we obtain
\[B(u) = \int_{|x'| \leq r_0} (u'(0)^2 - |D_{xN}U(x', 0)|^2)|x'|^2 + \int_{|x'| \geq r_0} (u'(0)^2 - |D_{xN}U(x', 0)|^2)|x'|^2 \geq \int_{|x'| \leq r_0} (u'(0)^2 - |D_{xN}U(x', 0)|^2)|x'|^2 + \int_{|x'| \geq r_0} (u'(0)^2 - |D_{xN}U(x', 0)|^2)|x'|^2 = \int_{|x'| \geq r_0} (u'(0)^2 - |D_{xN}U(x', 0)|^2)(|x'|^2 - r_0^2) > 0.\]

Remark 4.2. Since $U_{x_N}(x', 0) \geq u'(0)$, it is easy to see from (4.14) that
\[U_{x_N}(0) = \max_{x_N=0} U_{x_N}(x', 0) \geq -u'(0) > 0.\]
Moreover, it is easy to see that $U_{xy}(x', 0)$ is negative if $|x'|$ is large. Thus $U$ is positive near the origin and is negative if $|x'|$ is large and $x_N > 0$ is small. So we have proved that $U$ is a changing sign solution.

Next, we shall obtain a lower bound for $c_\varepsilon$.

**Lemma 4.3.** Suppose that $v_\varepsilon$ is a solution of (1.2) with $I_\varepsilon(v_\varepsilon) = c_\varepsilon$. Let $x_\varepsilon$ be the location of a maximum point of $u_\varepsilon$.

(i) we have $d(x_\varepsilon, \partial \Omega) \leq C \varepsilon$.

(ii) Suppose that $u_\varepsilon$ has another local maximum point $x^{(1)}_\varepsilon$. Then $|x^{(1)}_\varepsilon - x_\varepsilon| \leq C \varepsilon$.

(iii) For any $\theta > 0$, there is a $\nu > 0$, such that $v_\varepsilon(x) \leq C e^{-\nu |x - x_\varepsilon|^2}/\varepsilon$, $\forall x \in \Omega \setminus B_\theta(x_\varepsilon)$.

**Proof.** Suppose that $\varepsilon^{-1}d(x_\varepsilon, \partial \Omega) \to +\infty$. Let $\bar{v}_\varepsilon(y) = v_\varepsilon(\varepsilon y + x_\varepsilon)$, $y \in \Omega, x = \{\varepsilon \varepsilon y + x_\varepsilon \in \Omega\}$. Then, from the upper bound for $c_\varepsilon$, we see that $\bar{v}_\varepsilon$ is bounded in $H^1(R^N)$. So we may assume (up to a subsequence) that

$$\bar{v}_\varepsilon \to v_0, \text{in } C^1_{\text{loc}}(R^N)$$

and

$$-\Delta v_0 = f(v_0 + a), \text{in } R^N.$$

Thus, noting that $\frac{1}{2} g_\varepsilon(x, t) t - G_\varepsilon(x, t) \geq 0$, we see

$$I_\varepsilon(v_\varepsilon) = \int_{\Omega} \left( \frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon) \right)$$

$$\geq \int_{B_{\varepsilon R}(x_\varepsilon)} \left( \frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon) \right)$$

$$= \varepsilon^N \int_{B_{\varepsilon R}(0)} \left( \frac{1}{2} g_\varepsilon(\varepsilon y + x_\varepsilon, \bar{v}_\varepsilon) \bar{v}_\varepsilon - G_\varepsilon(\varepsilon y + x_\varepsilon, \bar{v}_\varepsilon) \right)$$

$$= \varepsilon^N \left( \int_{B_{\varepsilon R}(0)} \left( \frac{1}{2} f(a + v_0) v_0 - (F(a + v_0) - F(a)) \right) + o_\varepsilon(1) \right)$$

$$\geq \varepsilon^N (A + o_{\varepsilon, R}(1)),$$

where $o_{\varepsilon, R} \to 0$ as $\varepsilon \to 0$ and $R \to +\infty$. This is a contradiction to Proposition 4.1.

Next, we prove (ii). Suppose that $\frac{|x^{(1)}_\varepsilon - x_\varepsilon|}{\varepsilon} \to +\infty$. Then it is easy to check that

$$I_\varepsilon(v_\varepsilon) \geq \varepsilon^N (2c + o(1)),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. This is a contradiction.

It remains to prove (iii).

First, we prove that as $\varepsilon \to 0$, $v_\varepsilon \to 0$ uniformly in $\Omega \setminus B_\theta(x_\varepsilon)$, for any $\theta > 0$. In fact, suppose that there is $\tilde{x}_\varepsilon \in \Omega \setminus B_\theta(x_\varepsilon)$, such that $v_\varepsilon(\tilde{x}_\varepsilon) \geq c_0 > 0$. Then

$$v_\varepsilon(\varepsilon y + \tilde{x}_\varepsilon) \to \tilde{v}_0 \neq 0.$$

As a result,

$$I_\varepsilon(v_\varepsilon) = \int_{\Omega} \left( \frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon) \right)$$

$$\geq \int_{B_{\varepsilon R}(x_\varepsilon)} \left( \frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon) \right) + \int_{B_{\varepsilon R}(x_\varepsilon)} \left( \frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon) \right)$$

$$\geq \varepsilon^N (2c + o_{\varepsilon, R}(1)).$$

This is a contradiction.
Next, from \(-f'(u)\) \geq 2n > 0, for \(x \in \Omega_{\varepsilon R} =: \{x \in \Omega, d(x, \partial \Omega) \geq \varepsilon R\}\) if \(R > 0\) is large enough, we can check easily by using the standard comparison theorem that
\[
v_{\varepsilon}(x) \leq C e^{-\sqrt{m}|x-x_{\varepsilon}|/\varepsilon}, \quad x \in \Omega_{\varepsilon R}.
\]

Suppose now that \(x \in (\Omega \setminus \Omega_{\varepsilon R}) \setminus B_{\theta}(x_{\varepsilon})\). Without loss of generality, we may assume \(x_{\varepsilon} = 0\). Denote \(\bar{x} \in \partial \Omega\) such that \(d(x, \partial \Omega) = |x - \bar{x}|\). Let \(\xi(t)\) be the positive function defined in Appendix A. Let
\[
\bar{\nu} \geq \varepsilon > 0\) is a small number. Noting that \((De^{-|\bar{\nu}|/\varepsilon}, Dd(x, \partial \Omega) = 0\), we have
\[
\begin{align*}
-\varepsilon^2 \Delta w_{\varepsilon} + f'(u) - \tau w_{\varepsilon} &= -\varepsilon^2 \xi \left(\frac{d(x, \partial \Omega)}{\varepsilon}\right) - 2\varepsilon^2 (De^{-|\bar{\nu}|/\varepsilon}, D\xi \left(\frac{d(x, \partial \Omega)}{\varepsilon}\right)) \\
&\quad - \varepsilon^2 \xi \left(\frac{d(x, \partial \Omega)}{\varepsilon}\right) \Delta e^{-|\bar{\nu}|/\varepsilon} \\
&\quad + \left(-\varepsilon^2 \xi \left(\frac{d(x, \partial \Omega)}{\varepsilon}\right) - \varepsilon^2 \xi \left(\frac{d(x, \partial \Omega)}{\varepsilon}\right)\right) + O(\varepsilon) - \tau \xi \left(\frac{d(x, \partial \Omega)}{\varepsilon}\right) \Delta e^{-|\bar{\nu}|/\varepsilon} \\
&\quad - \xi \left(\frac{d(x, \partial \Omega)}{\varepsilon}\right) \left(-\varepsilon^2 \Delta e^{-|\bar{\nu}|/\varepsilon} + (3\varepsilon_0 + O(\varepsilon)) e^{-|\bar{\nu}|/\varepsilon}\right) > 0, \quad \forall x \in \Omega \setminus \Omega_{\varepsilon R}.
\end{align*}
\]

From \(-\Delta v_{\varepsilon} = f'(u) v_{\varepsilon} = O(|v_{\varepsilon}|^2)\) and \(v_{\varepsilon} \to 0\) as \(\varepsilon \to 0\) uniformly for \(x \in \Omega \setminus B_{\theta/2}(0)\), we see that \(-\Delta v_{\varepsilon} = f'(u) - \tau v_{\varepsilon} \leq 0\) if \(x \in \Omega \setminus B_{\theta/2}(0)\). So by comparison in the domain \((\Omega \setminus \Omega_{\varepsilon R}) \setminus B_{\theta/2}(0)\), we have
\[
v_{\varepsilon}(x) \leq C \max_{z \in \Omega \cap \partial B_{\theta/2}(0)} \left(\frac{w_{\varepsilon}(z)}{w_{\varepsilon}(x)}\right)^{-1} w_{\varepsilon}(x), \quad \forall x \in (\Omega \setminus \Omega_{\varepsilon R}) \setminus B_{\theta/2}(0),
\]

since
\[
v_{\varepsilon}(x) \leq C \max_{z \in \Omega \cap \partial B_{\theta/2}(0)} \left(\frac{w_{\varepsilon}(z)}{w_{\varepsilon}(x)}\right)^{-1} w_{\varepsilon}(x) \quad \text{if} \quad x \in \partial ((\Omega \setminus \Omega_{\varepsilon R}) \setminus B_{\theta/2}(0)).
\]

Thus, if \(x \in (\Omega \setminus \Omega_{\varepsilon R}) \setminus B_{2\theta}(0)\), then
\[
\begin{align*}
v_{\varepsilon}(x) &\leq C \max_{z \in \Omega \cap \partial B_{\theta/2}(0)} \left(\frac{w_{\varepsilon}(z)}{w_{\varepsilon}(x)}\right)^{-1} e^{-\varepsilon^2 \Delta \varepsilon^2 - 4\varepsilon e^{-|\bar{\nu}|/4\varepsilon}} \\
&\leq C' \max_{z \in \Omega \cap \partial B_{\theta/2}(0)} e^{\varepsilon^2 \varepsilon^2 - \varepsilon^2 e^{-|\bar{\nu}|/4\varepsilon}} \leq C' e^{-|\bar{\nu}|/4\varepsilon}.
\end{align*}
\]

\(\square\)

**Proposition 4.4.** Let \(x_{\varepsilon}\) be a global maximum point of \(v_{\varepsilon}\) and \(\bar{x}_{\varepsilon} \in \partial \Omega\) be such that \(|x_{\varepsilon} - \bar{x}_{\varepsilon}| = d(x_{\varepsilon}, \partial \Omega)\). Then,
\[
c_{\varepsilon} \geq \varepsilon N(\varepsilon) \frac{H(\bar{x}_{\varepsilon}) B(u) + o(\varepsilon)}{H(x_{\varepsilon}) B(u) + o(\varepsilon)}
\]
where \( u \in H^1_0(R^N_+) \) is a solution of \(-\Delta u = f(u + u) - f(u), B(u) \) is defined as in Proposition 4.1.

Proof. Let \( v_\varepsilon \) be a solution of (1.2) with \( I_\varepsilon(x) = c_\varepsilon \). Let \( \bar{x}_\varepsilon \in \partial \Omega \) be such that \(|\bar{x}_\varepsilon - x_\varepsilon| = d(x_\varepsilon, \partial \Omega)\). We assume that \( \bar{x}_\varepsilon = 0 \).

Similar to the estimate of the upper bound for \( c_\varepsilon \), using the fact that \( v_\varepsilon \) is exponentially small outside \( B_{\delta/2}(0) \), we can prove that

\[
\begin{align*}
\frac{1}{2} \varepsilon^2 \int_{\Omega} |Dv_\varepsilon|^2 &= \frac{1}{2} \varepsilon^2 \int_{\Omega \cap B_{\delta/2}(x_\varepsilon)} |Dv_\varepsilon|^2 + O(e^{-\delta'/\varepsilon}) \\
&= \varepsilon^N \left[ \frac{1}{2} \int_{R^N_+} |D\bar{v}_\varepsilon|^2 + \frac{\varepsilon H(\bar{x}_\varepsilon)}{4} \right] \\
&\quad + \frac{\varepsilon H(\bar{x}_\varepsilon)}{2} \int_{R^N_+} x' \left( f(u(x_N) + u) - f(u(x_N)) - f'(u(x_N))u \right) \psi(x_N) H(\bar{x}_\varepsilon) + o(\varepsilon).
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
\int_{\Omega} \left( F(\bar{u}_\varepsilon + t\bar{v}_\varepsilon) - F(\bar{u}_\varepsilon) - f(\bar{u}_\varepsilon)t\bar{v}_\varepsilon \right) \\
= \varepsilon^N \left[ \int_{R^N_+} \left( F(u(x_N) + t\bar{v}_\varepsilon) - F(u(x_N)) - f(u(x_N))t\bar{v}_\varepsilon \right) \\
+ (N - 1) \varepsilon \int_{R^N_+} \left( f(u(x_N) + tu) - f(u(x_N)) - f'(u(x_N))tu \right) \psi(x_N) H(\bar{x}_\varepsilon) + o(\varepsilon).\right]
\end{align*}
\]
It is easy to see (i) follows from Lemma 4.3. To prove (ii), we can combine
proofs.

\[ c \] converges in \( C \), suppose that there is a sequence of \( \bar{v}_\varepsilon \) and for another local maximum point \( \bar{x}_\varepsilon \) of \( \bar{v}_\varepsilon \), we can deduce easily that \( \bar{t}_\varepsilon \rightarrow 1 \) as \( \varepsilon \rightarrow 0 \). Thus, \( I_{c4}(\bar{t}_\varepsilon) = I_{c5}(\bar{t}_\varepsilon) = I_{c5}(1) + o(\varepsilon^{N+1}) \). As a result,

\[ c = \max_{t \geq 0} I_{c}(tv_{\varepsilon}) \geq I_{c}(\bar{t}_\varepsilon v_{\varepsilon}) \geq \varepsilon^{N} \left( c - \varepsilon H(\bar{t}_\varepsilon) B(u) + o(\varepsilon) \right) \]

where \( u \in H^{1}_{0}(R_{i}^{N}) \) is a solution of \( -\Delta u = f(u + u) - f(u) \). So we have completed the proof of this proposition.

Now we are ready to prove the following results.

**Theorem 4.5.** Let \( v_{\varepsilon} \) be a positive mountain pass solution of \( (1.2) \). We have

(i) For any local maximum point \( x_{\varepsilon} \) of \( v_{\varepsilon} \), we have \( d(x, \partial \Omega) \leq C \). If \( v_{\varepsilon} \) has another local maximum point \( x_{\varepsilon}^{(1)} \), then \( |x_{\varepsilon} - x_{\varepsilon}^{(1)}| \leq C \varepsilon \). Moreover, for any \( \theta > 0 \), there is a \( v > 0 \), such that \( |v_{\varepsilon}(x)| \leq C e^{-\frac{\varepsilon|x-x_{\varepsilon}|^2}{\varepsilon}} \) for \( x \in \Omega \setminus B_{\theta}(x_{\varepsilon}) \).

(ii) For any sequence of \( \varepsilon \), there is a subsequence \( \varepsilon_{j} \rightarrow 0 \), such that \( x_{j} \rightarrow x_{0} \in \partial \Omega \) with \( H(x_{0}) = \max_{x \in \partial \Omega} H(x) \), where \( x_{j} \) is any local maximum point of \( v_{\varepsilon} \), and \( H(x) \) is the mean curvature of \( \partial \Omega \) at \( x \).

Proof. It is easy to see (i) follows from Lemma 4.3. To prove (ii), we can combine Propositions 4.1 and 4.3 to obtain

\[ -H(x_{0}) \leq -\max_{x \in \partial \Omega} H(x) + o(1) \]

As a result, \( H(x_{0}) = \max_{x \in \partial \Omega} H(x) \).

**Proof of Theorem 4.5.** Let \( z_{\varepsilon} \) be a global maximum point of \( v_{\varepsilon} \). Then \( u_{\varepsilon}(\varepsilon y + z_{\varepsilon}) \) converges in \( C^{1}_{\text{loc}}(R^{N}_{i}) \) to \( U(y) = \overline{u}(yN) + u \), where \( u \) is the mountain pass solution of \( (3.1) \). By Remark 4.2, we know that \( \max_{y \in R^{N}_{i}} U(y) > 0 \). Thus (i) follows. It also follows from Remark 4.2 that \( U_{x_{\varepsilon}}(0) > 0 \). So we see that \( \frac{\partial}{\partial y} u_{\varepsilon}(\varepsilon y + z_{\varepsilon}) > 0 \) at \( \xi_{\varepsilon} \), where \( v \) is the inward unit normal of \( \partial \Omega \) at \( \xi_{\varepsilon} \); \( \varepsilon \in \partial \Omega \) satisfies \( |\xi_{\varepsilon} - \xi_{\varepsilon}| = d(\xi_{\varepsilon}, \partial \Omega) \); \( \Omega = \{ \xi : \varepsilon y + z_{\varepsilon} \in \Omega \} \). As a result, \( u_{\varepsilon}(\varepsilon y + z_{\varepsilon}) > 0 \) in a neighbourhood of \( \xi_{\varepsilon} \) and (vi) follows. On the other hand, (ii) follows from (i) of Theorem 4.5.

To prove (iii), we claim that for any local maximum point \( x_{\varepsilon} \) of \( u_{\varepsilon} \) with \( u_{\varepsilon}(x_{\varepsilon}) \geq \varepsilon_{0} > 0 \), we have \( |x_{\varepsilon} - z_{\varepsilon}| \leq C \varepsilon \), where \( z_{\varepsilon} \) is a local maximum point of \( v_{\varepsilon} \). In fact, suppose that there is a sequence of \( x_{\varepsilon} \), such that \( \varepsilon^{-1}|x_{\varepsilon} - z_{\varepsilon}| \rightarrow +\infty \) as \( \varepsilon \rightarrow 0 \) for
any local maximum point \( z_\varepsilon \) of \( v_\varepsilon \). Then \( v_\varepsilon \geq c_0 \) in a small neighbourhood of \( x_\varepsilon \). So we can blow up \( v_\varepsilon \) at \( x_\varepsilon \) and see that the energy of \( v_\varepsilon \) is strictly larger than \( c_\varepsilon \). So we get a contradiction. See the proof of Lemma 4.3. Thus (iii) follows from (ii) of Theorem 4.5.

5. Existence of interior peak solutions

In this section, we shall briefly prove the existence of interior peak solutions for \( (1.2) \) and estimate the number of such solutions. For simplicity, we only discuss the case \( f(t) = (t - a)^{p-1} - (t - a) \). So the mountain pass solution of \( (3.3) \) is nondegenerate. See [24].

As we see in Section 4, the main contribution to the energy of \( I_\varepsilon(v_\varepsilon) \) comes from the error term in the expansion of \( u_\varepsilon \) near the boundary of \( \Omega \). To construct the interior peak solution for \( (1.2) \), we need the following proposition.

**Proposition 5.1.** We have

\[
u_\varepsilon(x) = a + e^{-\sqrt{-f'(a)(d(x,\partial\Omega)+o(1))/\varepsilon}} \]

for any \( x \in \Omega \) with \( \varepsilon^{-1}d(x,\partial\Omega) \) large. Here, \( o(1) \to 0 \) as \( \varepsilon \to 0 \).

**Proof.** See the proof of Theorem 2.1 in [8].

Let \( w \) be the unique positive solution of \( (5.5) \), which is nondegenerate. Define

\[
w_{\varepsilon,x}(y) = w\left(\frac{y - x}{\varepsilon}\right).
\]

Let \( P_{\varepsilon,\Omega}w_{\varepsilon,x} \in H^1_0(\Omega) \) be the solution of

\[
-\Delta v - f'(a)v = f(a + w_{\varepsilon,x}) - f'(a)w_{\varepsilon,x}.
\]

Using Proposition 5.1, we can prove

**Proposition 5.2.**

\[
I(P_{\varepsilon,\Omega}w_{\varepsilon,x}) = \varepsilon^N A - \tau_{\varepsilon,x} + O(e^{-(1+\sigma)\sqrt{-f'(a)d(x,\partial\Omega)/\varepsilon}}),
\]

where \( \tau_{\varepsilon,x} = \int_{\Omega} (f(a + w_{\varepsilon,x}) - f'(a)w_{\varepsilon,x})(w_{\varepsilon,x} - a) \). Moreover, \( \tau_{\varepsilon,x} \) satisfies

\[
c_0 \varepsilon^{-(1+\theta)\sqrt{-f'(a)d(x,\partial\Omega)/\varepsilon}} \leq \tau_{\varepsilon,x} \leq c_1 \varepsilon^{-(1+\theta)\sqrt{-f'(a)d(x,\partial\Omega)/\varepsilon}},
\]

for any \( \theta > 0 \), and \( c_0 > 0 \) and \( c_1 > 0 \) are some constants.

**Proof.** The proof of this proposition is similar to that of Step 1 of Lemma 3.1 and thus we omit it.

By direct calculation, it is not difficult to prove the following proposition (see for example [10]):

**Proposition 5.3.**

\[
I\left(\sum_{j=1}^{k} P_{\varepsilon,\Omega}w_{\varepsilon,x_j}\right)
= \sum_{j=1}^{k} I(P_{\varepsilon,\Omega}w_{\varepsilon,x_j}) - \int_{\Omega} \left(F_a\left(\sum_{j=1}^{k} w_{\varepsilon,x_j}\right) - \sum_{j=1}^{k} F_a(w_{\varepsilon,x_j}) - \sum_{i<j} f_a(w_{\varepsilon,x_i})(w_{\varepsilon,x_j})\right)
+ O\left(\sum_{j=1}^{k} e^{-(1+\sigma)\sqrt{-f'(a)d(x_j,\partial\Omega)/\varepsilon}} + \sum_{i\neq j} e^{-(1+\sigma)\sqrt{-f'(a)|x_j - x_i|/\varepsilon}}\right),
\]

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where \( f_a(t) = f(a + t) \), \( F_a(t) = \int_0^t f_a(\tau) \, d\tau \).

Propositions 5.2 and 5.3 show that the energy of the approximate multipeak solution \( \sum_{j=1}^k P_{\varepsilon, \Omega \omega} w_{\varepsilon, x_j} \) will become larger if the peak \( x_j \) moves away from the boundary of \( \Omega \), or if a pair of peak \((x_i, x_j)\) moves away from each other. Thus this estimate is similar to that for the interior peak solution of the Neumann problem \( \textbf{20} \). But one should note that in Proposition 5.2, the first small term is \( e^{-\sqrt{-f'(\varepsilon)d(x, \partial \Omega)/\varepsilon}} \) instead of \( e^{-2\sqrt{-f'(\varepsilon)d(x, \partial \Omega)/\varepsilon}} \) as in the Neumann problem.

Arguing in exactly the same way as in \[\text{[7, 29]}\], we have

**Theorem 5.4.** For any positive integer \( k \), there is an \( \varepsilon_0 > 0 \), such that for any \( \varepsilon \in (0, \varepsilon_0] \), (1.2) has a solution of the form

\[
(5.1) \quad v_\varepsilon = \sum_{j=1}^k P_{\varepsilon, \Omega \omega} w_{\varepsilon, x_j} + \varphi_\varepsilon,
\]

where \( \varphi_\varepsilon \) satisfies

\[
\int_{\Omega} \varepsilon^2 |D\varphi_\varepsilon|^2 + \varphi_\varepsilon^2 \equiv \varepsilon^N O(\sum_{j=1}^k e^{-(1+\sigma)\sqrt{-f'(\varepsilon)d(x_{\varepsilon, j}, \partial \Omega)/\varepsilon}} + \sum_{i \neq j} e^{-(1+\sigma)\sqrt{-f'(\varepsilon)|x_{\varepsilon, j} - x_{\varepsilon, i}|/\varepsilon}}).
\]

Furthermore, we have

(i) if \( k = 1 \), the number of the interior single peak solutions is at least \( \text{Cat}(\Omega) \), and there is at least one single peak solution with its peak near the global maximum point of the distance function \( d(x, \partial \Omega) \);

(ii) if \( k \geq 2 \), the number of the interior \( k \) peak solutions is at least \( N \), and there is at least one \( k \) peak solution of the form \( (x_{\varepsilon, 1}, \ldots, x_{\varepsilon, k}) \rightarrow (x_{0, 1}, \ldots, x_{0, k}) \) and \( (x_{0, 1}, \ldots, x_{0, k}) \) is a maximum point of the function

\[
\min(d(x_j, \partial \Omega), |x_i - x_j|, i = 1, \ldots, k, i \neq j).
\]

**Remark 5.5.** The results here look similar to those in the Neumann problem. But for the interior multipeak solution of (1.2), we see that the locations of the peaks are different from those for the Neumann problem, where there is a positive interior peak solution, whose peaks are near a maximum point of the function

\[
\min(2d(x_j, \partial \Omega), |x_i - x_j|, i = 1, \ldots, k, i \neq j).
\]

**APPENDIX A.**

Let \( u \) be a positive solution of (3.1). In this section, we shall prove that \( u \) decays exponentially as \( |x| \to +\infty \). Since we do not assume that \( f'(t) \) is negative in \([a, 0]\), we cannot use the comparison theorem as usual to obtain the decay estimate. More work is needed.

**Proposition A.1.** Suppose that \( u \) is a positive solution of (3.1). Then there are \( C > 0 \) and \( \lambda_0 > 0 \), such that \( u(x) \leq Ce^{-\sqrt{\lambda_0}|x|} \).

**Proof.** Choose \( R > 0 \) large enough, such that \( f'(u(x_N)) < -2\mu_0 < 0 \) for \( x_N \geq R \). Here \( \mu_0 \) is a small constant. By a standard comparison argument, we can get easily that

\[
(A.1) \quad u(x) \leq Ce^{-\sqrt{\mu_0}|x|}, \quad \forall x_N \geq R.
\]
On the other hand, by Lemma 2.1, we have
\[
\int_0^{+\infty} (|\xi'|^2 - f'(\xi(t))\xi^2) \, dt \geq 4\lambda_0 \int_0^{+\infty} (|\xi'|^2 + \xi^2) \, dt, \quad \forall \xi \in H^1_0((0, +\infty)),
\]
for some \( \lambda_0 > 0 \). So we see that the following problem has a positive solution \( \xi(t) \):
\[
(A.2)
\begin{align*}
-\xi'' - f'(\xi(t))\xi &= 3\lambda_0 \xi, \quad t \in (0, R), \\
\xi(0) &= \xi(R) = 1.
\end{align*}
\]
In fact, let \( \bar{\xi}(t) = \xi(t) - 1 \). By the Lax-Milgram theorem, we see that the following problem has a unique solution:
\[
\begin{align*}
-\bar{\xi}'' - f'(\bar{\xi}(t))\bar{\xi} &= -f'(\xi(t)) - 3\lambda_0, \quad t \in (0, R), \\
\bar{\xi}(0) &= \bar{\xi}(R) = 0.
\end{align*}
\]
Thus \( A.2 \) has a unique solution. Besides, using \( \varphi = \xi^- = \min(0, \xi) \in H^1_0((0, 1)) \) as a test function, we see \( \xi^- = 0 \). Hence, \( \xi \) is nonnegative. Suppose that the minimum of \( \xi \) equals 0. Then \( \xi' = 0 \) at the minimum point. By the uniqueness of the ordinary differential equation, \( \xi = 0 \). This is a contradiction. So \( \xi \) is positive.

Let \( v = Ce^{-\sqrt{\lambda_0}|x|} \xi(x_N), \, x_N \in [0, R] \). Then
\[
-\Delta v - (f'(v(t)) + \lambda_0)v
= \xi(x_N)(-C\Delta e^{-\sqrt{\lambda_0}|x|} + Ce^{-\sqrt{\lambda_0}|x|} \frac{-\xi''(x_N) - (f'(v(t)) + \lambda_0)\xi(x_N)}{\xi(x_N)})
= \xi(x_N)(-C\Delta e^{-\sqrt{\lambda_0}|x|} + 2\lambda_0 Ce^{-\sqrt{\lambda_0}|x|}) > 0,
\]
if \( |x'| \geq R_1 > 0 \) is large enough.

Since \( u \in H^1_0(R^N_+) \), we see that \( u(x) \to 0 \) as \( |x| \to +\infty \). Thus we may choose \( R_1 > 0 \) large enough, such that \( \frac{u(x,x_N)}{u(x)} \leq \lambda_0 \) for \( |x'| \geq R_1 \). As a result,
\[
-\Delta u - (f'(u(x_N)) + \lambda_0)u < 0, \quad \text{if } |x'| \geq R_1.
\]
Thus we see
\[
-\Delta (u - v) - (f'(u(t)) + \lambda_0)(u - v) < 0, \quad \text{if } |x'| \geq R_1 \text{ and } x_N \in [0, R].
\]
In view of (A.1), we see that we can choose \( C > 0 \) large enough, such that \( u(x) \leq v(x) \) if \( |x'| \leq R_1 \) and \( x_N \in [0, R] \), or \( x_N = 0 \), or \( x_N = R \) (we can always choose \( \lambda_0 < \mu_0 \)). Let \( \eta = (u - v)^+ \text{ if } x_N \in [0, R] \), and \( \eta = 0 \text{ if } x_N \geq R \). Because \( u \leq v \) for \( x_N = R \), we see that \( \eta \in H^1_{0}(R^N_+) \).

Thus
\[
\int_{R^N_+} \left( |D(u - v)^+|^2 - (f'(u(t)) + \lambda_0)|(u - v)^+|^2 \right)
= \int_{|x'| \geq R_1 \cap \{x_N \in [0, R]\}} \left( D(u - v)^+ D\eta - (f'(u(t)) + \lambda_0)(u - v)^+ \eta \right) \leq 0,
\]
which implies \( (u - v)^+ = 0 \). That is, \( u \leq v \). Here we have used the natural generalization of Lemma 2.1 to the half space.

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