A NOTE ON MEYERS’ THEOREM IN $W^{k,1}$

IRENE FONSECA, GIOVANNI LEONI, JAN MALÝ, AND ROBERTO PARONI

Abstract. Lower semicontinuity properties of multiple integrals
$$u \in W^{k,1}(\Omega; \mathbb{R}^d) \mapsto \int_{\Omega} f(x, u(x), \cdots, \nabla^k u(x)) \, dx$$
are studied when $f$ may grow linearly with respect to the highest-order de-
rivative, $\nabla^k u$, and admissible $W^{k,1}(\Omega; \mathbb{R}^d)$ sequences converge strongly in
$W^{k-1,1}(\Omega; \mathbb{R}^d)$. It is shown that under certain continuity assumptions on $f$, convexity, 1-quasiconvexity or $k$-polyconvexity of
$$\xi \mapsto f(x_0, u(x_0), \cdots, \nabla^{k-1} u(x_0), \xi)$$
enures lower semicontinuity. The case where $f(x_0, u(x_0), \cdots, \nabla^{k-1} u(x_0), \cdots)$ is $k$-quasiconvex remains open except in some very particular cases, such as when $f(x, u(x), \cdots, \nabla^k u(x)) = h(x)g(\nabla^k u(x))$.

1. Introduction

In a classical paper Meyers [26] proved that $k$-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of a functional
$$u \mapsto \int_{\Omega} f(x, u(x), \cdots, \nabla^k u(x)) \, dx$$
with respect to weak convergence (weak$^*$ convergence if $p = \infty$) in the Sobolev space $W^{k,p}(\Omega; \mathbb{R}^d)$ and under appropriate growth and continuity conditions on the integrand $f$, thus extending to the case $k > 1$ the notion of quasi-convexity introduced by Morrey when $k = 1$. Here $\Omega$ is an open, bounded subset of $\mathbb{R}^N$, with $N \geq 1$, and $k, d \in \mathbb{N}$, $1 \leq p \leq \infty$. Meyers’ theorem uses results of Agmon, Douglis and Nirenberg [11] concerning Poisson kernels for elliptic equations. Fusco [22] later gave a simpler proof using De Giorgi’s Slicing Lemma. He also extended the result

Received by the editors April 1, 2001.

2000 Mathematics Subject Classification. Primary 49J45, 49Q20.

Key words and phrases. $k$-quasiconvexity, higher-order lower semicontinuity, gradient
truncation.

The research of I. Fonseca was partially supported by the National Science Foundation under
Grant No. DMS–9731957.

The research of G. Leoni was partially supported by MURST, Project “Metodi Variazionali ed Equazioni Differenziali Non Lineari”, by the Italian CNR, through the strategic project “Metodi e modelli per la Matematica e l’Ingegneria”, and by GNAFA.

The research of J. Malý was supported by CEZ MSM 113200007, grants GA ČR 201/00/0768 and GA UK 170/99.

The authors wish to thank Guy Bouchitté for stimulating discussions on the subject of this
work, and the Center for Nonlinear Analysis (NSF Grant No. DMS–9803791) for its support
during the preparation of this paper.

©2002 American Mathematical Society
to Carathéodory integrands when \( p = 1 \), while the case \( p > 1 \) has been recently established by Guidorzi and Poggiolini [24] under the Lipschitz condition
\[
|f(x, v, \xi) - f(x, v, \xi_1)| \leq C(1 + |\xi|^{p-1} + |\xi_1|^{p-1})|\xi - \xi_1|
\]
(note that this condition is automatically satisfied for \( k = 1 \) and \( k = 2 \), see [25] and [24]), and by Braides, Fonseca and Leoni in [8], who obtained a general relaxation result in \( W^{k,p}(\Omega; \mathbb{R}^d) \) with respect to weak convergence.

However, when \( k > 1 \) and \( p > 1 \), due to loss of reflexivity of the space \( W^{k,1}(\Omega; \mathbb{R}^d) \) one can only conclude that an energy bounded sequence \( \{u_n\} \subset W^{k,1}(\Omega; \mathbb{R}^d) \) with
\[
\sup_n\|u_n\|_{W^{k,1}} < \infty
\]
admits a subsequence (not relabelled) such that
\[
(1.1) \quad u_n \rightharpoonup u \quad \text{in} \quad W^{k-1,1}(\Omega; \mathbb{R}^d),
\]
where \( u \in W^{k-1,1}(\Omega; \mathbb{R}^d) \) and \( \nabla^{k-1}u \) is a vector-valued function of bounded variation. In this paper we seek to establish lower semicontinuity in the space \( W^{k,1}(\Omega; \mathbb{R}^d) \) under this natural notion of convergence.

When \( k = 1 \) the scalar case \( d = 1 \) has been extensively treated, while the vectorial case \( d > 1 \) was first studied by Fonseca and Müller in [19], who proved (sequential) lower semicontinuity in \( W^{1,1}(\Omega; \mathbb{R}^d) \) of a functional
\[
u \mapsto \int_{\Omega} f(x, u(x), \nabla u(x)) \, dx
\]
with respect to strong convergence in \( L^1(\Omega; \mathbb{R}^d) \) (see also [3], [20], [17], [18] and the references contained therein). The approach in [19] is based on blow-up and truncation methods.

Similar truncation techniques have been used quite successfully in the study of existence and qualitative properties of solutions of second-order elliptic equations and systems (see e.g. [2] and the references contained within). Their main drawback lies in the fact that they cannot be easily extended to truncated gradients or higher-order derivatives. This may explain in part why several important results for second-order elliptic equations have no analog for higher-order equations.

The main result of this paper extends Meyers’ Theorem to the case where weak convergence in \( W^{k,1}(\Omega; \mathbb{R}^d) \) is replaced by (1.1) together with a weak form of coercivity of the convex, 1-quasiconvex or \( k \)-polyconvex density \( f \) (see Theorems 1.2, 1.3 and 1.4 below). We start with the case where \( f \) depends essentially only on \( x \) and on the highest-order derivatives, that is, \( \nabla^k u(x) \). This situation is significantly simpler than the general case, since it does not require one to truncate the initial sequence \( \{u_n\} \subset W^{k,1}(\Omega; \mathbb{R}^d) \). Using the notation and terminology introduced in Section 2, we state the following:

**Theorem 1.1.** Let \( f : \Omega \times E^{d}_{[k-1]} \times E^d_k \to [0, \infty) \) be a Borel integrand. Suppose that for all \( (x_0, v_0) \in \Omega \times E^{d}_{[k-1]} \) and \( \varepsilon > 0 \) there exist \( \delta_0 > 0 \) and a modulus of continuity \( \rho \), with \( \rho(s) \leq C_0(1 + s) \) for \( s > 0 \) and for some \( C_0 > 0 \), such that
\[
(1.2) \quad f(x_0, v_0, \xi) - f(x, v, \xi) \leq \varepsilon(1 + f(x, v, \xi)) + \rho(|v - v_0|)
\]
for all $x \in \Omega$ with $|x - x_0| \leq \delta_0$, and for all $(v, \xi) \in \mathcal{E}^1 \times E^d_k$ of Fonseca and Leoni (Theorem 1.7 in [17]) to higher-order derivatives, where the condition (a) was proved by Amar and De Cicco [2]. Theorem 1.1 extends a result of Fonseca and Müller [19], [20]). Even in the simple case where $k = 1$ were obtained previously by Serrin [28] in the scalar case $d = 1$ and by Ambrosio and Dal Maso [4] in the vectorial case $d > 1$ (see also Fonseca and Müller [19], [20]). Even in the simple case where $f = f(\xi)$ it is not known if Theorem 1.1(a) still holds without the coercivity condition

$$f(\xi) \geq \frac{1}{C_1} |\xi| - C_1.$$  

The main tool in the proof of Theorem 1.1 used also in an essential way in subsequent results, is the blow–up method introduced by Fonseca and Müller [19], [20], which reduces the domain $\Omega$ to a ball and the target function $u$ to a polynomial.

When the integrand $f$ depends on the full set of variables in an essential way, the situation becomes significantly more complicated, since one needs to truncate gradients and higher-order derivatives in order to localize lower-order terms.

The following theorem was proved for $k = 1$ by Fonseca and Leoni in [17] (Theorem 1.8). Here we extend the result to the higher-order case.

**Theorem 1.2.** Let $f : \Omega \times \mathcal{E}^1 \times E^d_k \to [0, \infty)$ be a Borel integrand, with $f(x, v, \cdot)$ 1-quasiconvex in $E^d_k$. Suppose that for all $(x_0, v_0) \in \Omega \times \mathcal{E}^1$ either $f(x_0, v_0, \cdot) \equiv 0$, or for every $\varepsilon > 0$ there exist $C, \delta_0 > 0$ such that

$$f(x_0, v_0, \xi) - f(x, v, \xi) \leq \varepsilon(1 + f(x, v, \xi)),$$

$$(1.5)$$

$$(1.6)$$

$$C|\xi| - \frac{1}{C} \leq f(x_0, v_0, \xi) \leq C(1 + |\xi|)$$
for all \((x, v) \in \Omega \times E_{[k-1]}^d\) with \(|x - x_0| + |v - v_0| \leq \delta_0\) and for all \(\xi \in E_k^d\).

Let \(u \in BV^k(\Omega; \mathbb{R}^d)\), and let \(\{u_n\}\) be a sequence of functions in \(W^{k,1}(\Omega; \mathbb{R}^d)\) converging to \(u\) in \(W^{k-1,1}(\Omega; \mathbb{R}^d)\). Then
\[
\int_{\Omega} f(x, u, \ldots, \nabla^ku) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} f(x, u_n, \ldots, \nabla^ku_n) \, dx.
\]

A standing open problem is to decide whether Theorem 1.2 continues to hold under the weaker assumption that \(f(x, v, \cdot)\) is \(k\)-quasiconvex, which is the natural assumption in this context.

In the scalar case \(d = 1\) (that is, when \(u\) is an \(\mathbb{R}\)-valued function), and for first-order gradients, i.e., \(k = 1\), condition (1.6) can be eliminated; see Theorem 1.1 in \(\Omega\text{-valued function})\), and for first-order gradients, i.e., \(k = 1\), condition (1.6) can be eliminated; see Theorem 1.1 in [17]. In particular, in [17] Fonseca and Leoni have shown the following result:

**Proposition 1.3** (cf. [17], Corollary 1.2). Let \(g : \mathbb{R}^N \to [0, \infty)\) be a convex function, and let \(h : \Omega \times \mathbb{R} \to [0, \infty)\) be a lower semicontinuous function. If \(u \in BV(\Omega; \mathbb{R})\) and \(\{u_n\}\subset W^{1,1}(\Omega; \mathbb{R})\) converges to \(u\) in \(L^1(\Omega; \mathbb{R})\), then
\[
\int_{\Omega} h(x, u)g(\nabla u) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} h(x, u_n)g(\nabla u_n) \, dx.
\]

It is interesting to observe that the analogue of this result is false when \(k \geq 2\).

**Theorem 1.4.** Let \(\Omega := (0,1)^N\), \(N \geq 3\), and let \(h\) be a smooth cut-off function on \(\mathbb{R}\) with \(0 \leq h \leq 1\), \(h(u) = 1\) for \(u \leq \frac{1}{2}\), \(h(u) = 0\) for \(u \geq 1\). There exists a sequence of functions \(\{u_n\}\subset W^{1,1}(\Omega; \mathbb{R})\) converging to zero in \(W^{1,1}(\Omega; \mathbb{R})\) such that \(\|\Delta u_n\|_{L^1(\Omega; \mathbb{R})}\) is uniformly bounded and
\[
\limsup_{n \to \infty} \int_{\Omega} h(u_n)(1 - \Delta u_n)^+ \, dx < \int_{\Omega} h(0) \, dx.
\]

As in Theorem 1.1, conditions (1.5) and (1.6) can be considerably weakened if we assume that \(f(x, v, \cdot)\) is convex rather than 1-quasiconvex. Indeed, we have the following result:

**Theorem 1.5.** Let \(f : \Omega \times E_{[k-1]}^d \times E_k^d \to [0, \infty]\) be a lower semicontinuous function, with \(f(x, v, \cdot)\) convex in \(E_k^d\). Suppose that for all \((x_0, v_0) \in \Omega \times E_{[k-1]}^d\) either \(f(x_0, v_0, \cdot) \equiv 0\), or there exist \(C_1, \delta_0 > 0\), and a continuous function \(g : B(x_0, \delta_0) \times B(v_0, \delta_0) \to E_k^d\) such that
\[
(1.7) \quad f(x, v, g(x, v)) \in L^\infty (B(x_0, \delta_0) \times B(v_0, \delta_0); \mathbb{R}),
\]
\[
(1.8) \quad f(x, v, \xi) \geq C_1 |\xi| - \frac{1}{C_1},
\]
for all \((x, v) \in \Omega \times E_{[k-1]}^d\) with \(|x - x_0| + |v - v_0| \leq \delta_0\) and for all \(\xi \in E_k^d\). Let \(u \in BV^k(\Omega; \mathbb{R}^d)\), and let \(\{u_n\}\subset W^{k,1}(\Omega; \mathbb{R}^d)\) converging to \(u\) in \(W^{k-1,1}(\Omega; \mathbb{R}^d)\). Then
\[
\int_{\Omega} f(x, u, \ldots, \nabla^ku) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} f(x, u_n, \ldots, \nabla^ku_n) \, dx.
\]

Theorem 1.5 was obtained by Fonseca and Leoni for the case \(k = 1\) in Theorem 1.1 of [18]. It is interesting to observe that without a condition of the type (1.7), Theorem 1.5 is false in general. This has been recently proved by Černý and Malý in [12].
The proofs of Theorems 1.4(b) and (c), 1.5 and 1.6 can be deduced easily from the corresponding ones in [17] and [18], where \( k = 1 \). It suffices to write

\[
\int_{\Omega} f(x,u(x),\ldots,\nabla^{k}u(x)) \, dx =: \int_{\Omega} F(x,v(x),\nabla v(x)) \, dx
\]

with \( v := (u,\ldots,\nabla^{k-1}u) \), and then to perturb the new integrand \( F \) in order to recover the full coercivity conditions necessary to apply the results in [17], [18]. This approach cannot be used for \( k \)-polyconvex integrands, and a new proof is needed to treat this case. Thus Theorem 1.1(a) and Theorem 1.6 below are the only truly genuine higher-order results, in that they cannot be reduced in a trivial way to a first-order problem.

For each \( \xi \in E_{k}^{d} \) let \( \mathcal{M}(\xi) \in \mathbb{R}^{r} \) be the vector whose components are all the minors of \( \xi \).

**Theorem 1.6.** Let \( h : \Omega \times E_{k-1}^{d} \times \mathbb{R}^{r} \to [0,\infty] \) be a lower semicontinuous function, with \( h(x,v,\cdot) \) convex in \( \mathbb{R}^{r} \). Suppose that for all \( (x_{0},v_{0}) \in \Omega \times E_{k-1}^{d} \) either

\[
h(x_{0},v_{0},\cdot) \equiv 0, \quad \text{or there exist } C, \delta_{0} > 0, \quad \text{and a continuous function } g : B(x_{0},\delta_{0}) \times B(v_{0},\delta_{0}) \to \mathbb{R}^{r} \quad \text{such that}
\]

\[
h(x,v,g(x,v)) \in L^{\infty}(B(x_{0},\delta_{0}) \times B(v_{0},\delta_{0}); \mathbb{R}),
\]

\[
h(x,v,v) \geq C|v| - \frac{1}{C}
\]

for all \( (x,v) \in \Omega \times E_{k-1}^{d} \) with \( |x-x_{0}| + |v-v_{0}| \leq \delta_{0} \) and for all \( v \in \mathbb{R}^{r} \). Let \( u \in BV^{k}(\Omega;\mathbb{R}^{d}) \), and let \( \{u_{n}\} \) be a sequence of functions in \( W^{k,p}(\Omega;\mathbb{R}^{d}) \) that converges to \( u \) in \( W^{k-1,1}(\Omega;\mathbb{R}^{d}) \), where \( p \) is the minimum between \( N \) and the dimension of \( E_{k-1}^{d} \). Then

\[
\int_{\Omega} h(x,u,\ldots,\nabla^{k-1}u,\mathcal{M}(\nabla^{k}u)) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} h(x,u_{n},\ldots,\nabla^{k-1}u_{n},\mathcal{M}(\nabla^{k}u_{n})) \, dx.
\]

Theorem 1.6 is closely related to a result of Ball, Currie and Olver [6], where it was assumed that

\[
h(x,v,v) \geq \gamma(|v|) - \frac{1}{C},
\]

with

\[
\frac{\gamma(s)}{s} \to \infty \quad \text{as } s \to \infty.
\]

Also, as stated above and with \( k = 1 \), Theorem 1.6 was proved by Fonseca and Leoni in [18], Theorem 1.4.

## 2. Preliminaries

We start with some notation. Here \( \Omega \subset \mathbb{R}^{N} \) is an open, bounded subset; \( \mathcal{L}^{N} \) and \( \mathcal{H}^{N-1} \) are, respectively, the \( N \)-dimensional Lebesgue measure and the \( (N-1) \)-dimensional Hausdorff measure in \( \mathbb{R}^{N} \). Let \( Q \) be the the unit cube \((-1/2,1/2)^{N}\) and set \( Q(x_{0},\varepsilon) := x_{0} + \varepsilon Q \).

For each \( j \in \mathbb{N} \) the symbol \( \nabla^{j}u \) stands for the vector-valued function whose components are all derivatives of order \( j \) of \( u \). If \( u \) is \( C^{\infty} \), then for \( j \geq 2 \) we have...
that $\nabla^j u(x) \in E^d_j$, where $E^d_j$ stands for the space of symmetric $j$-linear maps from $\mathbb{R}^N$ into $\mathbb{R}^d$. We set $E^d_0 := \mathbb{R}^d$, $E^d_1 := \mathbb{R}^{d \times N}$ and

$$E^d_{[j-1]} := E^d_0 \times \cdots \times E^d_{j-1}, \quad E^d_{[0]} := E^d_0.$$ 

For any integer $k \geq 2$ we define

$$BV^k(\Omega; \mathbb{R}^d) := \{ u \in W^{k-1,1}(\Omega; \mathbb{R}^d) : \nabla^{k-1} u \in BV(\Omega; E^d_{k-1}) \},$$

where $\nabla^j u$ is the Radon–Nikodým derivative of the distributional derivative $D^j u$ of $\nabla^{j-1} u$, with respect to the $N$–dimensional Lebesgue measure $\mathcal{L}^N$.

We recall that a function $f : E^d_k \to \mathbb{R}$ is said to be $k$-quasiconvex if

$$f(\xi) \leq \int_Q f(\xi + \nabla^k w(y)) \, dy$$

for all $\xi \in E^d_k$ and all $w \in C^\infty_0(Q; \mathbb{R}^d)$.

The following theorem was proved in the case $k = 1$ by Ambrosio and Dal Maso \[4\], while Fonseca and Müller \[19\] treated general integrands of the form $f = f(x, u, \nabla u)$, but their argument requires coercivity. The case $k \geq 2$ is due to Amar and De Cicco \[2\]. For completeness we give a proof for all $k \geq 1$.

**Proposition 2.1.** Let $f : E^d_k \to [0, \infty)$ be a $k$-quasiconvex function such that

$$0 \leq f(\xi) \leq C (1 + |\xi|)$$

for all $\xi \in E^d_k$. Moreover, when $k \geq 2$ assume that

$$f(\xi) \geq C_1 |\xi| \quad \text{for } |\xi| \text{ large.}$$

If $\{u_n\}$ is a sequence of functions in $W^{k,1}(Q; \mathbb{R}^d)$ converging to $0$ in $W^{k-1,1}(Q; \mathbb{R}^d)$, then

$$f(0) \leq \liminf_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx.$$

**Proof.** We start with the case $k \geq 2$. Without loss of generality, we may assume that

$$\liminf_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx = \lim_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx < \infty,$$

so that by condition (2.2),

$$K := \sup_n \int_Q |\nabla^k u_n| \, dx < \infty.$$

Let $\varepsilon > 0$, $M \in \mathbb{N}$, and decompose $L := Q \setminus (1 - \varepsilon) Q$ into $M$ layers with mutually disjoint interiors, $L_i := \alpha_{i+1} Q \setminus \alpha_i Q$, so that

$$1 - \varepsilon = \alpha_1 < \alpha_2 < \ldots < \alpha_M < 1 =: \alpha_{M+1}.$$

Since

$$\sum_{i=1}^M \int_{L_i} (1 + |\nabla^k u_n|) \, dx \leq 1 + K$$

for all $n \in \mathbb{N}$, there exist $i_\varepsilon \in \{1, \ldots, M\}$ and a subsequence of $\{u_n\}$ (not relabelled) such that

$$\int_{L_{i_\varepsilon}} (1 + |\nabla^k u_n|) \, dx \leq \frac{1 + K}{M} \quad \text{for all } n \in \mathbb{N}. \quad (2.3)$$
Let $\varphi \in C^\infty_c(Q; [0, 1])$ with $\varphi(x) = 1$ in $\alpha_i Q$, $\varphi(x) = 0$ if $x \notin \alpha_{i+1} Q$. Since $f$ is $k$-quasiconvex,

$$f(0) \leq \liminf_{n \to \infty} \int_Q f(\nabla^k (\varphi u_n)) \, dx$$

$$\leq \liminf_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx + \int_{Q \setminus \alpha_{i+1} Q} f(0) \, dx$$

$$+ C \limsup_{n \to \infty} \int_{L_{i\varepsilon}} (1 + |\nabla^k (\varphi u_n)|) \, dx,$$

where we have used (2.1). As $u_n \to 0$ in $W^{k-1,1}(Q; \mathbb{R}^d)$ strongly, we have

$$\limsup_{n \to \infty} \int_{L_{i\varepsilon}} (1 + |\nabla^k (\varphi u_n)|) \, dx \leq \limsup_{n \to \infty} \int_{L_{i\varepsilon}} (1 + |\nabla^k u_n|) \, dx \leq \frac{1 + K}{M}$$

by (2.3). We conclude that

$$(1 - \varepsilon)^N f(0) \leq \alpha_{i+1}^N f(0) \leq \liminf_{n \to \infty} \int_Q f(\nabla^k u_n) \, dx + \frac{1 + K}{M},$$

and the result now follows by letting first $\varepsilon \to 0^+$ and then $M \to \infty$.

Next we consider the case where $k = 1$. Let $\varepsilon > 0$, fix $n \in \mathbb{N}$, set

$$M_n := \left[ n \int_Q (1 + |\nabla u_n|) \, dx \right] + 1,$$

where $[\cdot]$ denotes the integer part, and decompose $L := Q \setminus (1 - \varepsilon) Q$ into $M_n$ layers with mutually disjoint interiors, $L_i^{(n)} := \alpha_{i+1}^{(n)} Q \setminus \alpha_i^{(n)} Q$, so that

$$1 - \varepsilon = \alpha_1^{(n)} < \alpha_2^{(n)} < \ldots < \alpha_{M_n}^{(n)} < 1 =: \alpha_{M+1}^{(n)}$$

and, in addition, $\alpha_{i+1}^{(n)} - \alpha_i^{(n)} = \frac{\varepsilon}{M_n}$, $i = 1, \ldots, M_n$. Let $\varphi_i^{(n)} \in C^\infty_c(Q; [0, 1])$ with $\varphi_i^{(n)}(x) = 1$ in $\alpha_i^{(n)} Q$, $\varphi_i^{(n)}(x) = 0$ if $x \notin \alpha_{i+1}^{(n)} Q$, $||\nabla \varphi_i^{(n)}|| \leq \frac{2M_n}{\varepsilon}$, $i = 1, \ldots, M_n$. We have

$$\int_Q f \left( \nabla \left( \varphi_i^{(n)} u_n \right) \right) \, dx \leq \int_Q f(\nabla u_n) \, dx + \int_{Q \setminus \alpha_i^{(n)} Q} f(0) \, dx$$

$$+ C \int_{L_i^{(n)}} (1 + |\nabla u_n|) \, dx + C \frac{M_n}{\varepsilon} \int_{L_i^{(n)}} |u_n| \, dx.$$

Thus

$$\frac{1}{M_n} \sum_{i=1}^{M_n} \int_Q f \left( \nabla \left( \varphi_i^{(n)} u_n \right) \right) \, dx \leq \int_Q f(\nabla u_n) \, dx + \int_{Q \setminus \alpha_1^{(n)} Q} f(0) \, dx$$

$$+ C \frac{M_n}{\varepsilon} \int_{Q \setminus \alpha_i^{(n)} Q} (1 + |\nabla u_n|) \, dx + C \frac{M_n}{\varepsilon} \int_{Q \setminus \alpha_1^{(n)} Q} |u_n| \, dx$$

$$\leq \int_Q f(\nabla u_n) \, dx + O(\varepsilon) + C \frac{M_n}{\varepsilon} \int_{Q \setminus \alpha_1^{(n)} Q} |u_n| \, dx.$$

We may, therefore, find $i = i(n, \varepsilon) \in \{1, \ldots, M\}$ such that, in view of the quasicontvexity of $f$,

$$f(0) \leq \int_Q f \left( \nabla \left( \varphi_i^{(n)} u_n \right) \right) \, dx \leq \int_Q f(\nabla u_n) \, dx + O(\varepsilon) + C \frac{M_n}{\varepsilon} \int_Q |u_n| \, dx,$$

and the conclusion follows by letting $n \to \infty$ and then $\varepsilon \to 0^+$. \(\square\)
Proposition 2.2. Let \( h : \mathbb{R}^+ \to [0, \infty) \) be a convex function such that
\[
h(v) \to \infty \quad \text{as } |v| \to \infty.
\]
Let \( u \in W^{1,p}(\Omega; \mathbb{R}^d) \), and let \( \{ u_n \} \) be a sequence of functions in \( W^{1,p}(\Omega; \mathbb{R}^d) \) that converges to \( u \) in \( L^q(\Omega; \mathbb{R}^d) \), where \( p = \min\{d, N\} \). Then
\[
\int_{\Omega} h(\mathcal{M}(\nabla u)) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} h(\mathcal{M}(\nabla u_n)) \, dx.
\]

Proposition 2.2 has been proved by Dal Maso and Sbordone (cf. Theorem 2.2 in [14]) using Cartesian currents, and by Fusco and Hutchinson (cf. Theorem 2.6 in [23]).

Next we present an approximation result for convex functions.

Proposition 2.3. Let \( M \) be a closed set of \( \mathbb{R}^p \), and let \( V \) be an reflexive and separable Banach space. Let \( f : M \times V \to (0, +\infty] \) be an \( M \times \) (weak-\( V \)) sequentially lower semicontinuous function, convex in the last variable and such that there exists a continuous function \( v_0 : M \to V \) with
\[
(f(\cdot, v_0(\cdot)))^+ \in L^\infty_{\mathrm{loc}}(M; \mathbb{R}).
\]
Then there exist two sequences of continuous functions
\[
a_j : M \to \mathbb{R}, \quad b_j : M \to V^*,
\]
where \( V^* \) is the dual space of \( V \), such that
\[
f(t, v) = \sup_j (a_j(t) + \langle b_j(t), v \rangle)^+
\]
for all \( t \in M \) and \( v \in V \).

Proposition 2.3 was proved by Fonseca and Leoni in [18], following closely the argument of Ambrosio in [3], who studied the case where (2.4) is replaced by the assumption that \( f(\cdot, v_0(\cdot)) \) is continuous.

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Without loss of generality, we may assume that
\[
\liminf_{n \to \infty} \int_{\Omega} f(x, u_n(x), \ldots, \nabla^k u_n(x)) \, dx = \lim_{n \to \infty} \int_{\Omega} f(x, u_n(x), \ldots, \nabla^k u_n(x)) \, dx < \infty.
\]
Passing to a subsequence, if necessary, there exists a nonnegative Radon measure \( \mu \) such that
\[
f(x, u_n(x), \ldots, \nabla^k u_n(x)) \mathcal{L}^N | \Omega \rightharpoonup \mu
\]
as \( n \to \infty \), weakly* in the sense of measures. We claim that
\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q(x_0, \varepsilon))}{\varepsilon^N} \geq f(x_0, u(x_0), \ldots, \nabla^k u(x_0))
\]
for \( \mathcal{L}^N \) a.e. \( x_0 \in \Omega \). If (3.1) holds, then the conclusion of the theorem follows immediately. Indeed, let \( \varphi \in C_c(\Omega; \mathbb{R}), 0 \leq \varphi \leq 1 \). We have
\[
\lim_{n \to \infty} \int_{\Omega} f(x, u_n, \ldots, \nabla^k u_n) \, dx \geq \liminf_{n \to \infty} \int_{\Omega} \varphi f(x, u_n, \ldots, \nabla^k u_n) \, dx
\]
\[
= \int_{\Omega} \varphi \, d\mu \geq \int_{\Omega} \varphi \frac{d\mu}{d\mathcal{L}^N} \, dx \geq \int_{\Omega} \varphi f(x, u, \ldots, \nabla^k u) \, dx.
\]
By letting $\varphi \to 1$, and using the Lebesgue Monotone Convergence Theorem, we obtain the desired result. Thus, to conclude the proof of the theorem it suffices to show (3.1).

Take $x_0 \in \Omega$ such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q(x_0, \varepsilon))}{\varepsilon^N} < \infty,$$

(3.2)

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0, \varepsilon)} \frac{|u(x) - T_k(x)|}{|x - x_0|^k} dx = 0,$$

where

$$T_k(x) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \nabla^\alpha u(x_0)(x - x_0)^\alpha,$$

and set

$$v_0 := (u(x_0), \ldots, \nabla^{k-1} u(x_0)).$$

Choosing $\varepsilon_m \downarrow 0$ such that $\mu(\partial Q(x_0, \varepsilon_m)) = 0$, then

$$\lim_{m \to \infty} \frac{\mu(Q(x_0, \varepsilon_m))}{\varepsilon_m^N} = \lim_{m \to \infty} \lim_{n \to \infty} \frac{1}{\varepsilon_m^N} \int_{Q(x_0, \varepsilon_m)} f(x, u_n, \ldots, \nabla^k u_n) dx$$

$$= \lim_{m \to \infty} \lim_{n \to \infty} \int_Q f(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k w_{n,m}(y), \nabla T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k \nabla^2 w_{n,m}(y), \ldots, \nabla^k w_{n,m}(y)) dy,$$

where

$$w_{n,m}(y) := u_n(x_0 + \varepsilon_m y) - T_{k-1}(x_0 + \varepsilon_m y).$$

Clearly $w_{n,m} \in W^{k,1}(Q; \mathbb{R}^d)$, and, by (3.2), $\lim_{m \to \infty} \lim_{n \to \infty} \|w_{n,m} - w_0\|_{W^{k-1,1}(Q; \mathbb{R}^d)} = 0$, where

$$w_0(y) := \sum_{|\alpha| = k} \frac{1}{\alpha!} \nabla^\alpha u(x_0) y^\alpha.$$

By a standard diagonalization argument, we may extract a subsequence $w_m := w_{n,m}$ that converges to $w_0$ in $W^{k-1,1}(Q; \mathbb{R}^d)$, such that $\nabla^j w_m \to \nabla^j w_0$ pointwise a.e. for $j = 0, \ldots, k - 1$, and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{m \to \infty} \int_Q f(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k w_m(y), \ldots, \nabla^k w_m(y)) dy.$$

By condition (3.2), for all $\varepsilon > 0$ and for $m$ large enough,

$$(1 + \varepsilon) \frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon$$

$$\geq \lim_{m \to \infty} \left( \int_Q f(x_0, u(x_0), \ldots, \nabla^{k-1} u(x_0), \nabla^k w_m(y)) dy - \int_Q \rho(|z_m(y)|) dy \right),$$
where

\[
  z_m(y) := (T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k w_m(y), \ldots, \nabla^{k-1} T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m \nabla^{k-1} w_m(y)) - v_0.
\]

By Fatou’s Lemma, and since $\rho$ is continuous with $\rho(0) = 0$, we have

\[
  C_0 - \limsup_{m \to \infty} \int_Q \rho(|z_m(y)|) \, dy = \liminf_{m \to \infty} \int_Q [C_0(1 + |z_m(y)|) - \rho(|z_m(y)|)] \, dy \geq \int_Q \liminf_{m \to \infty} [C_0(1 + |z_m(y)|) - \rho(|z_m(y)|)] \, dy = C_0,
\]

and so

\[
  \int_Q \rho(|z_m(y)|) \, dy \to 0 \quad \text{as} \quad m \to \infty.
\]

Thus

\[
  (1 + \varepsilon) \frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq \lim_{m \to \infty} \int_Q f(x_0, v_0, \nabla^k w_m(y)) \, dy.
\]

If $g(\xi) := f(x_0, v_0, \xi)$ satisfies condition (a), then use Proposition 2.1, and if either condition (b) or (c) holds, then apply Theorem 1.7 in [17] to conclude that

\[
  (1 + \varepsilon) \frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq f(x_0, u(x_0), \ldots, \nabla^k u(x_0)),
\]

and it suffices to let $\varepsilon \to 0^+$. \qed

**Proof of Theorem 1.2.** Theorem 1.2 can be easily deduced from Theorem 1.8 in [17]. It suffices to write

\[
  \int_{\Omega} f(x, u(x), \ldots, \nabla^k u(x)) \, dx =: \int_{\Omega} F(x, \mathbf{v}(x), \mathbf{\nabla v}(x)) \, dx
\]

with $\mathbf{v} := (u, \ldots, \nabla^{k-1} u)$. Note, however, that the coercivity condition (1.6) for $F$ now reads

\[
  F(x_0, \mathbf{v}_0, \eta) \geq C|\eta_k| - \frac{1}{C},
\]

where

\[
  \eta = (\eta_1, \ldots, \eta_k) \in E_1^d \times \cdots \times E_k^d \quad \text{and} \quad F(x, \mathbf{v}, \eta) := f(x, \mathbf{v}, \eta_k).
\]

In order to be in position to apply Theorem 1.8 we need to ensure full coercivity. Due to the strong convergence of admissible sequences $\{u_n\}$ in $W^{k-1,1}(\Omega; \mathbb{R}^d)$, and therefore of $\{v_n\}$ in $L^1 \left( \Omega; E^d_{[k-1]} \right)$, it suffices to consider

\[
  F_\varepsilon(x, \mathbf{v}, \eta) := F(x, \mathbf{v}, \eta) + \varepsilon \chi_A(x, \mathbf{v}) |(\eta_1, \cdots, \eta_{k-1})|,
\]

where $\chi_A(x, \mathbf{v}) := 1$ if $x \in A$ and $\eta \in E^d_{[k-1]}$.
where $A := \{(x,v) \in \Omega \times E^d_{[k-1]} : f(x,v,\cdot) \neq 0\}$. Theorem 1.8 in [17] now yields

$$\liminf_{n \to \infty} \int_{\Omega} f(x,u_n,\ldots,\nabla^k u_n) \, dx = \liminf_{n \to \infty} \int_{\Omega} F(x,v_n,\nabla v_n) \, dx \geq \int_{\Omega} F(x,v,\nabla v) \, dx - \varepsilon \int_{\Omega} \left| (\nabla u,\ldots,\nabla^{k-1} u) \right| \, dx,$$

where $\varepsilon \to 0^+$. □

4. PROOF OF THEOREM 1.4

Throughout this section we assume that $N \geq 3$.

**Lemma 4.1.** Let $D$ be a cube with $|D| \leq 1$. Then there exist constants $C > 0$ and $\lambda \in (0,1)$, depending only on $N$, a function $u \in W^{2,\infty}(D;\mathbb{R})$ with compact support in $D$, and sets $A$, $E$, $G \subset D$, with $A \cup E \cup G = D$ and $|E| \leq \lambda |D|$, such that

1. $\|\Delta u\|_{L^1(D;\mathbb{R})} \leq C |D|$, $\|u\|_{W^{1,1}(D;\mathbb{R})} \leq C |D|^{1+\frac{1}{N}}$,
2. $\Delta u = 1$ on $A$,
3. $u = 0$ on $E$, $u \geq 1$ on $G$.

**Proof.** After a translation we may assume that there exists $B(0,R) \subset D$ such that

$$C^{-1} R^N \leq |D| \leq C R^N, \quad R \in (0,1/2),$$

for some $C > 0$. We search for a radial function of type

$$u(x) := \varphi(|x|),$$

where $\varphi$ is a $C^2$-function on $(0,\infty)$ such that

1. $\varphi(t) = 0$ for $t \geq R$,
2. $\varphi'(0+) = 0$.

Further, we want that for some $a > 0$,

$$\Delta u(x) = \begin{cases} -a & \text{if } |x| < r, \\ 1 & \text{if } r \leq |x| < R, \end{cases}$$

where $r$ is determined by the equation

$$r^{2-N} R^N = 2N(N-2).$$

Note that $r \in (0, R)$, because $R < 1$ and $N \geq 3$. In order to find $a$ and $\varphi$ satisfying (4.1), (4.4) and (4.5), we note that

$$\Delta u(x) = \varphi''(|x|) + |x|^{-1} (N-1) \varphi'(|x|), \quad \text{for } |x| \neq 0,$$

or, equivalently,

$$\Delta u(x) = t^{1-N} (t^{N-1} \varphi'(t))', \quad \text{where } t = |x|.$$

On the interval $(r,R)$, (4.6) now yields

$$(t^{N-1} \varphi'(t))' = t^{N-1},$$
and thus, by (4.3),
\[
\varphi'(t) = \frac{t}{N} \left( 1 - \frac{R^N}{t^N} \right).
\]
(4.8)

On the interval \((0, r)\), and in view of (4.6), we have
\[
(t^{N-1} \varphi'(t))' = -at^{N-1},
\]
which, together with (4.5), implies that
\[
\varphi'(t) = -\frac{at}{N}.
\]
(4.9)

We have
\[
-\frac{ar}{N} = \varphi'(r-) = \varphi'(r+) = \frac{r}{N} \left( 1 - \frac{R^N}{r^N} \right),
\]
and thus
\[
(4.10)
\]
Now the function \(u\) is uniquely determined by its properties. Obviously we have (4.10) by setting
\[
A := B(0, R) \setminus B(0, r), \quad E := D \setminus B(0, R), \quad G := B(0, r),
\]
with \(|E| \leq \lambda |D|\) and \(\lambda = \lambda(N)\). In light of (4.6) and (4.10) we have
\[
\|\Delta u\|_{L^1(D; B)} \leq |B(0, R) \setminus B(0, r)| + a |B(0, r)|
\]
\[
= \omega_N \left( R^N - r^N + \left( \frac{R^N}{r^N} - 1 \right) r^N \right) \leq 2\omega_N R^N,
\]
where \(\omega_N := |B(0, 1)|\). If \(x \in G\), we have by (4.10), (4.3) and (4.7),
\[
u(x) \geq \varphi(r) = -\int_r^R \varphi'(t) \, dt = \frac{1}{N} \int_r^R (R^N t^{1-N} - t) \, dt
\]
\[
\geq \frac{1}{N(N-2)} (r^{2-N} R^N - R^2) - \frac{R^2}{2N} = 2 - \frac{R^2}{2(N-2)} \geq 1,
\]
as \(R \leq \frac{1}{2}, B(0, R) \subset D\) and the side length of \(D\) does not exceed 1. This proves (4.11). By (4.8) and (4.9),
\[
\int_D |\nabla u| \, dx \leq C \int_0^R t^{N-1} |\varphi'(t)| \, dt \leq C \left( \int_0^R r^{-N} R^N t^N \, dt + \int_r^R R^N \, dt \right) \leq CR^{N+1},
\]
which, with the aid of the Poincaré inequality for zero boundary values, proves (4.11). \qed

Proof of Theorem 1.4. We set \(\Omega = (0, 1)^N\), and we construct the \(\frac{1}{n}\) periodic sequence \(\{u_n\}\) as follows: divide \(\Omega\) into small cubes \(D_\alpha\) of measure \(\frac{1}{n^N}\), \(\alpha \in I_n\), where the set of indices \(I_n\) has cardinality \(n^N\). On each \(D_\alpha\) we construct \(u_n\) as indicated in Lemma 4.1 and denote by \(A_\alpha, E_\alpha, G_\alpha\) the corresponding sets. Then \(u_n \to 0\) in \(W^{1,1}(\Omega; \mathbb{R})\), because
\[
\|u_n\|_{W^{1,1}(\Omega; \mathbb{R})} = \sum_{\alpha \in I_n} \|u_n\|_{W^{1,1}(D_\alpha; \mathbb{R})} \leq n^N C \left( \frac{1}{n^N} \right)^{1+\frac{1}{N}} \to 0 \quad \text{as} \quad n \to \infty.
\]
and \( \{\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})}\} \) is uniformly bounded since
\[
\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})} = \sum_{\alpha \in I_n} \|\Delta u_n\|_{L^1(D_\alpha;\mathbb{R})} \leq N C \frac{1}{n^N} = C.
\]
Consider the functional
\[
F(v) := \int_{\Omega} h(v)(1 - \Delta v)^+ \, dx.
\]
For \( \alpha \in I_n \) we have by (4.1)–(4.3),
\[
h(u_n) = 1 \quad \text{and} \quad \Delta u_n = 0 \quad \text{on } E_\alpha,
\]
\[
\Delta u_n = 1 \quad \text{on } A_\alpha,
\]
\[
h(u_n) = 0 \quad \text{on } G_\alpha,
\]
and thus
\[
\int_{D_\alpha} h(u_n)(1 - \Delta u_n)^+ \, dx = |E_\alpha| \leq \lambda |D_\alpha|.
\]
Summing up over \( \alpha \in I_n \), we conclude that
\[
\int_{\Omega} h(u_n)(1 - \Delta u_n)^+ \, dx \leq \lambda < 1 = F(0).
\]

**Remark 4.2.** We cannot obtain an a priori bound on \( \|u_n\|_{W^{2,1}(\Omega;\mathbb{R})} \), because the function
\[
h(v)(1 - \text{trace } \xi)^+
\]
is convex in the last variable, and, after adding a small multiple of \( |\xi| \), the corresponding functional is lower semicontinuous on \( W^{1,1}(\Omega;\mathbb{R}) \) according to Theorem 1.5. A direct heuristic computation using the notation of Lemma 4.1 yields
\[
\int_Q |\nabla^2 u| \, dx \sim \int_0^R \left( t^{N-1} |\varphi''(t)| + t^{N-2} |\varphi'(t)| \right) \, dt
\]
\[
\sim \left( \int_0^R t^{N-1} \frac{R^N}{t^N} \, dt + \int_R^R \frac{R^N}{t} \, dt \right)
\]
\[
\sim R^N \log R,
\]
and so an inequality of the type
\[
\|\nabla^2 u\|_{L^1(Q;E^1_d)} \leq C|Q|
\]
will not hold.

5. Proof of Theorems 1.3 and 1.6

**Proof of Theorem 1.3.** As in the proof of Theorem 1.2 it is easy to obtain Theorem 1.3 from Theorem 1.1 in [18] by considering \( v := (u,...,\nabla^{k-1} u) \) and the reformulated functionals
\[
\int_{\Omega} \left( F(x,v(x),\nabla v(x)) + \varepsilon \chi_{A}(x,v(x)) \right) \|((\nabla v)_{1},\cdots,(\nabla v)_{k-1})\| \, dx,
\]
with
\[
\nabla v = ((\nabla v)_{1},\cdots,(\nabla v)_{k}) \in E^d_1 \times \cdots \times E^d_k.
\]
Observe that, as opposed to Theorems 1.1(b) and (c), 1.2 and 1.5 Theorem 1.6 cannot be deduced easily from the analogous result already obtained in the case where \( k = 1 \), i.e., Theorem 1.4 in [13]. Indeed, there is no obvious way of perturbing the new integrand \( H(x, v, M(\nabla v)) := h(x, u, \ldots, \nabla^{k-1} u, M(\nabla^k u)) \), with \( v := (u, \ldots, \nabla^{k-1} u) \), in such a way that (1.10) is satisfied for the perturbed integrand, i.e.,

\[
H_{\varepsilon}(x, v, M(\nabla v)) \geq C_{\varepsilon} |M(\nabla v)| - \frac{1}{C_{\varepsilon}}
\]

and

\[
\lim_{\varepsilon \to 0^+} \liminf_{n \to \infty} \int_{\Omega} H_{\varepsilon}(x, v_n, M(\nabla v_n)) \, dx \leq \liminf_{n \to \infty} \int_{\Omega} H(x, v_n, M(\nabla v_n)) \, dx.
\]

This is due to the fact that \( M(\nabla v) \) involves terms of the form \( \nabla u \nabla^k u \) for which we have no bounds.

**Proof of Theorem 1.6.** Let \( f(x, v, \xi) := h(x, v, M(\xi)) \). We proceed as in the proof of Theorem 1.1 until we reach (3.3); precisely,

\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{m \to \infty} \int_Q f(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k w_m(y), \ldots, \nabla^k w_m(y)) \, dy,
\]

where now \( w_m \in W^{k,p}(Q; \mathbb{R}^d) \), and \( ||w_m - w_0||_{W^{k-1,1}(Q; \mathbb{R}^d)} \to 0 \) as \( m \to \infty \), where

\[
w_0(y) := \sum_{|\alpha| = k} \frac{1}{\alpha!} \nabla^\alpha u(x_0) y^\alpha.
\]

If \( f(x_0, v_0, \cdot) \equiv 0 \), with \( v_0 := (u(x_0), \ldots, \nabla^{k-1} u(x_0)) \), then there is nothing to prove. Otherwise, let \( \delta_0 > 0 \) be given by (1.9) and (1.10). Setting

\[
Q_m := \{ y \in Q : \left| (w_m(y), \ldots, \nabla^{k-1} w_m(y)) \right| \leq \delta_0/(2\varepsilon_m) \},
\]

by (5.1) and (1.10) we have

\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \limsup_{m \to \infty} \int_{Q_m} f(x_0 + \varepsilon_m y, T_{k-1}(x_0 + \varepsilon_m y) + \varepsilon_m^k w_m(y), \ldots, \nabla^k w_m(y)) \, dy
\]

\[
\geq C \limsup_{m \to \infty} \int_{Q_m} |M(\nabla^k w_m(y))| \, dy - 1/C,
\]

and so there exists a constant \( K > 0 \) such that

\[
\int_{Q_m} |M(\nabla^k w_m(y))| \, dy \leq K \quad \text{for all} \quad m \in \mathbb{N}.
\]

By Proposition 2.3 with \( M = (x_0 + \varepsilon_1 Q) \times B(v_0, \delta_0/2) \) and \( V = \mathbb{R}^r \), in view of (1.9) there exist two sequences of continuous functions

\[
a_j : M \to \mathbb{R}, \quad b_j : M \to \mathbb{R}^r
\]

such that

\[
h_j(x, v, \eta) = \sup_j (a_j(x, v) + b_j(x, v) \cdot \eta)^+
\]

for all \((x, v) \in M \) and \( \eta \in \mathbb{R}^r \). Define

\[
h_j(x, v, \eta) := (a_j(x, v) + b_j(x, v) \cdot \eta)^+, \quad f_j(x, v, \xi) := h_j(x, v, M(\xi)).
\]
Clearly $h_j$ is continuous, convex in $\eta$, and

\begin{equation}
0 \leq h_j(x, v, \eta) \leq C_j(|\eta| + 1),
\end{equation}

for all $(x, v) \in M$ and $\eta \in \mathbb{R}^q$, where

$$C_j := \max \{|a_j(x, v)| + |b_j(x, v)| : (x, v) \in M\}.$$

Fix $\varepsilon > 0$ and find $0 < \delta_j \leq \delta_0/2$ such that

$$|a_j(x, v) - a_j(x_0, v_0)| + |b_j(x, v) - b_j(x_0, v_0)| \leq \varepsilon$$

for all $(x, v) \in (x_0 + \delta_jQ) \times B(v_0, \delta_j)$. Since the function $s \mapsto s^+$ is Lipschitz continuous with Lipschitz constant 1, we have

$$|f_j(x, v, \xi) - f_j(x_0, v_0, \xi)| \leq |a_j(x, v) - a_j(x_0, v_0)| + |b_j(x, v) - b_j(x_0, v_0)| |M(\xi)|$$

\begin{equation}
\leq \varepsilon(1 + |M(\xi)|)
\end{equation}

for all $(x, v) \in (x_0 + \delta_jQ) \times B(v_0, \delta_j)$ and all $\xi \in E^k$. By (5.1) and for any $j \in \mathbb{N}$ we obtain

$$\frac{d\mu}{d\mathcal{L}^q}(x_0) \geq \liminf_{m \to \infty} \int_{Q_m} f(x_0 + \varepsilon_my, T_{k-1}(x_0 + \varepsilon_my) + \varepsilon_m^k w_m(y), \ldots, \nabla^k w_m(y)) dy$$

\begin{equation}
\geq \liminf_{m \to \infty} \int_{Q_m} f_j(x_0 + \varepsilon_my, T_{k-1}(x_0 + \varepsilon_my) + \varepsilon_m^k w_m(y), \ldots, \nabla^k w_m(y)) dy
\end{equation}

\begin{equation}
\geq \liminf_{m \to \infty} \left( \int_{Q_m} f_j(x_0, v_0, \nabla^k w_m(y)) dy - \varepsilon - \varepsilon \int_{Q_m} |M(\nabla^k w_m(y))| dy \right)
\end{equation}

\begin{equation}
\geq \liminf_{m \to \infty} \int_{Q_m} f_j(x_0, v_0, \nabla^k w_m(y)) dy - \varepsilon - \varepsilon K,
\end{equation}

where we have used (5.3) and (5.2). Define

$$z_m(y) := (\varepsilon_m^{-1} w_m(y), \ldots, \varepsilon_m^{k-2} w_m(y)), \quad u_m(y) := \nabla^{k-1} w_m(y).$$

Fix an integer $P \in \mathbb{N}$ such that $e^P > 1 + \|\nabla^{k-1} w_0\|_{\infty}$. For $m$ sufficiently large, say $m \geq m_P$, we have $e^{2P+1} \leq \delta_0/(2\varepsilon_m)$; so in view of (5.2) we may find $i_m \in \{P+1, \ldots, 2P\}$ such that

$$\{y \in Q : \varepsilon_m \leq |(z_m(y), u_m(y))| \leq e^{i_m+1} \} \subset Q_m$$

and

$$\int_{\{y \in Q : \varepsilon_m \leq |(z_m(y), u_m(y))| \leq e^{i_m+1} \}} (1 + |M(\nabla^k w_m(x))|) dx \leq \frac{1 + K}{P}.$$

Since $\{P+1, \ldots, 2P\}$ is a finite set, we may find $i_P \in \{P+1, \ldots, 2P\}$ such that

\begin{equation}
\int_{\{y \in Q : e^{i_P} \leq |(z_m(y), u_m(y))| \leq e^{i_P+1} \}} (1 + |M(\nabla^k w_m(x))|) dx \leq \frac{1 + K}{P}
\end{equation}

for infinitely many indices $m \in \mathbb{N}$. From now until the end of the proof we assume without loss of generality that the whole sequence satisfies (5.7).

Set

$$v_m(y) := G(|(z_m(y), u_m(y))|) u_m(y)$$
and

\[ D_m := \{ y \in Q : |(z_m(y), u_m(y))| < e^{i\rho} \}, \]
\[ D_m^+ := \{ y \in Q : e^{i\rho} \leq |(z_m(y), u_m(y))| \leq e^{i\rho + 1} \}, \]
\[ D_m^- := \{ y \in Q : |(z_m(y), u_m(y))| > e^{i\rho + 1} \}, \]

where

\[ G(s) := \begin{cases} 
1 & \text{if } s < e^{i\rho}, \\
\frac{e^{i\rho + 1} - s}{e^{i\rho + 1} - e^{i\rho}} & \text{if } e^{i\rho} \leq s \leq e^{i\rho + 1}, \\
0 & \text{if } s > e^{i\rho + 1}.
\end{cases} \]

Note that

\[ |D_m^- \cup D_m^+| = |\{ y \in Q : |(z_m(y), u_m(y))| \geq e^{i\rho} \}| \]
\[ \leq |\{ y \in Q : |(z_m(y), u_m(y)) - (0, \nabla^k w_0(y))| \geq 1 \}| \]
\[ \leq ||z_m||_{L^1(Q)} + ||u_m - \nabla^k w_0||_{L^1(Q)} \rightarrow 0 \quad \text{as } m \rightarrow \infty, \]

where we have used the fact that \( e^{i\rho} > 1 + ||\nabla^k w_0||_{\infty}. \) Also,

\[ |\nabla z_m| = \left| (\varepsilon^{-k+1}\nabla w_m, \ldots, \varepsilon^m \nabla^k w_m) \right| \]
\[ \leq \varepsilon_m \left| (\varepsilon^{-k+1}w_m, \varepsilon^{-k+2}\nabla w_m, \ldots, \varepsilon^m \nabla^k w_m) \right| = \varepsilon_m \left| (z_m, u_m) \right|. \]

We claim that

\[ |\mathcal{M}(\nabla v_m(y))| \leq C \left( 1 + \varepsilon_m e^{i\rho + 1} \right) |\mathcal{M}(\nabla^k w_m(y))|. \]

In view of the definition of \( g \) this is immediate for \( x \in D_m \cup D_m^+ \). Thus it remains to assert (5.10) in \( D_m^- \). We have

\[ \nabla v_m = (G(|(z_m, u_m)|) I + G'(|(z_m, u_m)|) \frac{u_m \otimes u_m}{|(z_m, u_m)|}) \nabla u_m \]
\[ + G'(|(z_m, u_m)|) \frac{u_m \otimes z_m}{|(z_m, u_m)|} \nabla z_m, \]

where \( I \) is the identity matrix. Since in \( D_m^- \),

\[ |GI + G' \frac{u_m \otimes u_m}{|(z_m, u_m)|}| + |G' \frac{u_m \otimes z_m}{|(z_m, u_m)|}| \leq C, \]

we have

\[ |\mathcal{M}_l \left( GI + G' \frac{u_m \otimes u_m}{|(z_m, u_m)|} \nabla u_m \right)| \]
\[ \leq |\mathcal{M}_l \left( GI + G' \frac{u_m \otimes u_m}{|(z_m, u_m)|} \right)| |\mathcal{M}_l (\nabla u_m)| \leq C |\mathcal{M}_l (\nabla u_m)|, \]

and

\[ |\mathcal{M}_l \left( G' \frac{u_m \otimes z_m}{|(z_m, u_m)|} \nabla z_m \right)| \leq |\mathcal{M}_l \left( G' \frac{u_m \otimes z_m}{|(z_m, u_m)|} \right)| |\mathcal{M}_l (\nabla z_m)| \]
\[ \leq \begin{cases} 
0 & l > 1, \\
C |\mathcal{M}_1 (\nabla z_m)| & l = 1,
\end{cases} \]
where $\mathcal{M}_l(X)$ is the vector whose components are all the minors of $X$ of order $l$. Here we have used the facts that

$$|\mathcal{M}_l(X + Y)| \leq C \sum_{i=0}^{l} |\mathcal{M}_i(X)||\mathcal{M}_{l-i}(Y)|,$$

that

$$|\mathcal{M}_l(XY)| \leq |\mathcal{M}_l(X)||\mathcal{M}_l(Y)|,$$

and that $u_m \otimes z_m$ is a rank-one matrix. Then, in view of (5.11), (5.12) and (5.13),

$$\lim_{m \to \infty} \mathcal{M}_j(x_0, \nabla v_m) dy = C_j \int_{D_m^+} (1 + |\mathcal{M}(\nabla v_m)|) dy = C_j |D_m^+| \to 0,$$

while from (5.4), (5.10) and (5.8), and taking $m > m_P$ so that $\varepsilon_m e^{P+1} < 1$,

$$\int_{D_m} f_j(x_0, v_0, \nabla v_m) dy \leq C_j \int_{D_m} (1 + |\mathcal{M}(\nabla v_m)|) dy$$

$$\leq CC_j \int_{D_m} (1 + |\mathcal{M}(\nabla^k w_m(y))|) dy$$

$$\leq CC_j \frac{1 + K}{P}.$$

Consequently, in view of (5.11), (5.12) and (5.13),

$$\frac{d\mu}{dL^N}(x_0) \geq \lim_{m \to \infty} \mu \int_{Q} f_j(x_0, v_0, \nabla v_m) dy - \varepsilon - \varepsilon K - CC_j \frac{1 + K}{P},$$

and by (5.2), (5.7), and in view of the fact that $v_m \equiv 0$ in $D_m^+$,

$$\sup_m \int_{Q} |\mathcal{M}(\nabla v_m)| dy \leq K_1 < \infty,$$

where $K_1$ is independent of $m$ and $j$. Define

$$h_{j,\varepsilon}(v) := h_j(x_0, v_0, v) + \varepsilon |v|,$$

$$f_{j,\varepsilon}(\xi) := h_{j,\varepsilon}(\mathcal{M}(\xi)).$$
Then by (5.14), (5.15), and Proposition 2.2,
\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \liminf_{m \to \infty} \int_{Q} h_{j,\varepsilon}(\mathcal{M}(\nabla v_m(y))) \, dy - \varepsilon - \varepsilon (K + K_1) - CC_j \frac{1 + K}{P}
\]
\[
\geq h_{j,\varepsilon}(\mathcal{M}(\nabla^k w_0(x_0))) - \varepsilon - \varepsilon (K + K_1) - CC_j \frac{1 + K}{P}
\]
\[
= f_j(x_0, u(x_0), \ldots, \nabla^k u(x_0)) + \varepsilon \left| \mathcal{M}(\nabla^k w_0(x_0)) \right|
\]
\[
- \varepsilon - \varepsilon (K + K_1) - CC_j \frac{1 + K}{P}.
\]
Letting first \( P \to \infty \), then taking the supremum in \( j \) yields
\[
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(x_0, u(x_0), \ldots, \nabla^k u(x_0)) + \varepsilon \left| \mathcal{M}(\nabla^k w_0(x_0)) \right| - \varepsilon - \varepsilon (K + K_1),
\]
by (5.13). To complete the proof it suffices to let \( \varepsilon \to 0^+ \).

\[\square\]

References


A NOTE ON MEYERS’ THEOREM IN $W^{1,1}$


[22] N. Fusco, Quasiconvessità e semicontinuità per integrali multipli di ordine superiore, Ricerche Mat. 29 (1980), 93–134. MR 83h:49022


Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213
E-mail address: fonseca@cmu.edu

Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale, Alessandria, Italy 15100
E-mail address: leoni@unipmn.it

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83,186 75 Praha 8, Czech Republic
E-mail address: maly@karlin.mff.cuni.cz

Dipartimento di Ingegneria Civile, Università degli Studi di Udine, Udine, Italy 33100
E-mail address: roberto.paroni@dic.univud.it

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use