WEAK AMENABILITY OF MODULE EXTENSIONS
OF BANACH ALGEBRAS

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Abstract. We start by discussing general necessary and sufficient conditions for a module extension Banach algebra to be \( n \)-weakly amenable, for \( n = 0, 1, 2, \ldots \). Then we investigate various special cases. All these case studies finally provide us with a way to construct an example of a weakly amenable Banach algebra which is not \( 3 \)-weakly amenable. This answers an open question raised by H. G. Dales, F. Ghahramani and N. Grønbæk.

Introduction

Suppose that \( \mathfrak{A} \) is a Banach algebra, and that \( X \) is a Banach \( \mathfrak{A} \)-bimodule. A derivation from \( \mathfrak{A} \) into \( X \) is a linear operator \( D: \mathfrak{A} \to X \) satisfying

\[
D(ab) = D(a)b + aD(b) \quad (a, b \in \mathfrak{A}).
\]

A derivation \( D \) is inner if there is \( x_0 \in X \) such that \( D(a) = ax_0 - x_0a \) for \( a \in \mathfrak{A} \).

The quotient space \( \mathcal{H}^1(\mathfrak{A}, X) \) of all continuous derivations from \( \mathfrak{A} \) into \( X \) modulo the subspace of inner derivations is called the first cohomology group of \( \mathfrak{A} \) with coefficients in \( X \). A Banach algebra \( \mathfrak{A} \) is said to be amenable if \( \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\} \) for every Banach \( \mathfrak{A} \)-bimodule \( X \); here \( \mathfrak{A}^* \) denotes the Banach dual module of \( X \).

The algebra \( \mathfrak{A} \) is said to be weakly amenable if \( \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = \{0\} \), and is called \( n \)-weakly amenable, for an integer \( n \geq 0 \), if \( \mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = \{0\} \), where \( \mathfrak{A}^{(n)} \) is the \( n \)-th dual module of \( \mathfrak{A} \) when \( n \geq 1 \), and is \( \mathfrak{A} \) itself when \( n = 0 \). The algebra \( \mathfrak{A} \) is said to be permanently weakly amenable if it is \( n \)-weakly amenable for all \( n \geq 1 \).

The concept of weak amenability was first introduced by Bade, Curtis and Dales in [1] for commutative Banach algebras, and was extended to the noncommutative case by Johnson in [22] (see also [7], [9], [11]–[16], [21] and [24]). Dales, Ghahramani and Grønbæk initiated the study of \( n \)-weak amenability of Banach algebras in their recent paper [10], where they revealed many important properties of this sort of Banach algebra. An interesting problem concerning this class of Banach algebras is the relation between \( n \)-weak amenability and \( m \)-weak amenability for different integers \( n \) and \( m \). For instance, if \( \mathfrak{A} \) is a commutative Banach algebra, then the assertion that \( \mathfrak{A} \) is weakly amenable is equivalent to saying that it is permanently weakly amenable ([1] Theorem 1.5); but, for noncommutative Banach algebras, things are different—we only know that \((n + 2)\)-weak amenability implies \( n \)-weak amenability.
amenability for \( n \geq 1 \) ([10, Proposition 1.2]), and weak amenability does not imply 2-weak amenability ([10, Theorems 5.1 and 5.2]). After investigating varieties of classical Banach algebras, Dales, Ghahramani and Grønbæk raised and left open the following question in [10]: Does weak amenability imply 3-weak amenability?

This paper is designed to answer the preceding question. We will construct a counterexample to the question. For this purpose, we study \( n \)-weak amenability of the module extension Banach algebra \( A \oplus X \), the \( l_1 \)-direct sum of a Banach algebra \( A \) and a nonzero Banach \( A \)-module \( X \) with the algebra product defined as follows:

\[
(a, x) \cdot (b, y) = (ab, ay + xb) \quad (a, b \in A, \; x, y \in X).
\]

Some aspects of algebras of this form have been discussed in [2] and [10]. We choose this class of Banach algebras to investigate for the preceding question because this class is neither too small nor is it too large; it contains permanently weakly amenable Banach algebras (see Section 6), and it contains no amenable Banach algebras due to [8, Lemma 2.7], since \( X \) is a complemented nilpotent ideal in the algebra. If \( A \) has both left and right approximate identities and they are also, respectively, left and right approximate identities for \( X \), then \( A \oplus X \) cannot be pointwise approximately biprojective (see [30]). The class of module extension Banach algebras also includes the natural triangular Banach algebra whose amenability has been investigated in [12]. We will give some comment on the latter algebra in Section 2.

This paper is organized as follows: in Section 1 we study the construction of module actions of \( 2^{-m} \)-th dual algebras on \( 2^{-m} \)-th dual modules. This extends the corresponding discussion in [10]. In Section 2 we give the main theorems which deal with the necessary and sufficient conditions for \( A \oplus X \) to be \( n \)-weakly amenable. Section 3 discusses various techniques for lifting derivations. These will be applied in Section 4 to give the proofs of the main theorems. Sections 5 and 6 deal with the special cases of \( X = A, A^* \) and \( X_0 \), where \( X_0 \) denotes an \( A \)-bimodule with the right module action trivial. In Section 7 we first discuss the condition for \( A \oplus (X_1 \oplus X_2) \) to be weakly amenable, where \( \oplus \) denotes the \( l_1 \) direct sum (of modules). Then, we give an example of a weakly amenable Banach algebra of this form and prove that it is not 3-weakly amenable. This finally answers the preceding open question in the negative.

Since \( (A \oplus X)^* = (0 \oplus X)^\perp + (A \oplus 0)^\perp \), where \( \perp \) denotes the direct \( A \)-module \( l_\infty \)-sum, and \( (0 \oplus X)^\perp \) (respectively, \( (A \oplus 0)^\perp \)) is isometrically isomorphic to \( A^* \) (respectively, \( X^* \)) as \( A \)-bimodules, for convenience, in this paper we simply identify the corresponding terms and write:

\[
(A \oplus X)^* = A^* + X^*.
\]

Similarly, we will identify the underlying space of the \( n \)-th conjugate \( (A \oplus X)^{(n)} \) with \( A^{(n)} + X^{(n)} \). The sum is an \( l_1 \)-sum when \( n \) is even and is an \( l_\infty \)-sum when \( n \) is odd.

1. Bimodule actions of \( A^{(2m)} \) on \( X^{(2m)} \)

Suppose that \( A \) is a Banach algebra, and \( X \) is a Banach \( A \)-bimodule. According to [10], pp. 27 and 28, \( X^{**} \) is a Banach \( A^{**} \)-bimodule, where \( A^{**} \) is equipped with the first Arens product. The module actions are successively defined as follows.
First, for \( x \in X, f \in X^*, \phi \in X^{**} \) and \( u \in \mathfrak{A}^{**}, \) define \( \phi f, fx \in \mathfrak{A}^* \) and \( uf \in X^* \) by
\[
\langle a, \phi f \rangle = \langle fa, \phi \rangle, \quad \langle a, fx \rangle = \langle xa, f \rangle \quad (a \in \mathfrak{A}), \\
\langle x, uf \rangle = \langle fx, u \rangle \quad (x \in X).
\]
Then, for \( \phi \in X^{**} \) and \( u \in \mathfrak{A}^{**}, \) define \( u\phi, \phi u \in X^{**} \) by
\[
\langle f, u\phi \rangle = \langle \phi f, u \rangle, \quad \langle f, \phi u \rangle = \langle uf, \phi \rangle \quad (f \in X^*).
\]
These give the left and right \( \mathfrak{A}^{**} \)-module actions on \( X^{**}. \) Also, the definition for \( uf \) with \( u \in \mathfrak{A}^{**} \) and \( f \in X^* \) gives a left Banach \( \mathfrak{A}^{**} \)-module action on \( X^*. \) When \( u = a \in \mathfrak{A}, \) all the above \( \mathfrak{A}^{**} \)-module actions agree with the \( \mathfrak{A} \)-module actions on the corresponding dual modules \( X^* \) and \( X^{**}. \) Moreover, it is readily seen that, with these module actions, the first Arens product on \( (\mathfrak{A} \oplus X)^{**} \) may be represented by
\[
(u, \phi) \cdot (v, \psi) = (uv, u\psi + \phi v) \quad (u, v \in \mathfrak{A}^{**}, \phi, \psi \in X^{**}).
\]

Viewing \( \mathfrak{A}^{(2m)} \) as a new \( \mathfrak{A} \) and \( X^{(2m)} \) as a new \( X, \) the preceding procedure will successively define \( X^{(2m+2)} \) as a Banach \( \mathfrak{A}^{(2m+2)} \)-bimodule. Here, and throughout the paper, the first Arens product is consistently assumed on each \( \mathfrak{A}^{(2n)} \). Since some relations arising from the procedure are important for later use, we now give the definition in detail as follows.

Suppose that the bimodule action of \( \mathfrak{A}^{(2m)} \) on \( X^{(2m)} \) has been defined, where \( m \geq 1 \). Then in a natural way, \( X^{(2m+k)}, \; k \geq 1, \) is a Banach \( \mathfrak{A}^{(2m)} \)-bimodule with the module multiplications \( uA \) and \( Au \in X^{(2m+k)}, \) for \( A \in X^{(2m+k)} \) and \( u \in \mathfrak{A}^{(2m)}, \) defined by
\[
\langle \gamma, uA \rangle = \langle \gamma u, A \rangle, \quad \langle \gamma, Au \rangle = \langle u\gamma, A \rangle \quad (\gamma \in X^{(2m+k-1)}).
\]
If \( u = a \in \mathfrak{A}, \) these module actions coincide with \( \mathfrak{A} \)-module actions on \( X^{(2m+k)} \).

Then, for \( F \in X^{(2m+1)} \) and \( \Phi \in X^{(2m+2)}, \) define \( F\Phi, \Phi F \in \mathfrak{A}^{(2m+1)} \) by
\[
\langle u, F\Phi \rangle = \langle F, \Phi u \rangle = \langle uF, \Phi \rangle \quad (u \in \mathfrak{A}^{(2m)}).
\]
and
\[
\langle u, \Phi F \rangle = \langle Fu, \Phi \rangle = \langle uF, \Phi \rangle \quad (u \in \mathfrak{A}^{(2m)}).
\]

Throughout this paper, for a Banach space \( Y \) and an element \( y \in Y, \) \( \hat{y} \) always denotes the image of \( y \) in \( Y^{**} \) under the canonical mapping. When \( F \in X^{(2m+1)} \) and \( \phi \in X^{(2m)} \), we denote \( F\hat{\phi} \) by \( F\phi \) and \( \hat{\phi}F \) by \( \phi F. \) It is easy to check that
\[
\langle u, F\phi \rangle = \langle \phi u, F \rangle, \quad \langle u, \phi F \rangle = \langle u\phi, F \rangle \quad \text{for } u \in \mathfrak{A}^{(2m)}.
\]
By using the canonical image of \( F \) or \( \Phi \) in the appropriate \( 2l \)-th dual space of the space that it belongs to, we can then signify a meaning for \( F\Phi \) and \( \Phi F \) for every \( F \in X^{(2m+1)} \) and \( \Phi \in X^{(2m)}; \) they are elements of \( \mathfrak{A}^{(2k+1)}, \) where \( k = \max\{m, n - 1\}. \)

Now for \( \mu \in \mathfrak{A}^{(2m+2)} \) and \( F \in X^{(2m+1)} \), we define \( \mu F \in X^{(2m+1)} \) by
\[
\langle \phi, \mu F \rangle = \langle F\phi, \mu \rangle \quad (\phi \in X^{(2m)}).
\]
This actually defines a left Banach \( \mathfrak{A}^{(2m+2)} \)-module action on \( X^{(2m+1)}. \)

Finally, for \( \mu \in \mathfrak{A}^{(2m+2)} \) and \( \Phi \in X^{(2m+2)}, \) define \( \Phi\mu, \mu\Phi \in X^{(2m+2)} \) by
\[
\langle F, \Phi\mu \rangle = \langle \Phi F, \mu \rangle, \quad \langle F, \mu\Phi \rangle = \langle \mu F, \Phi \rangle \quad (F \in X^{(2m+1)}).
\]
These finally define the \( \mathfrak{A}^{(2m+2)} \)-module actions on \( X^{(2m+2)} \) and, therefore, complete our definition.
If \( \lim u_\alpha = \mu \) in \( \sigma(\mathcal{A}^{(2m+2)}) \), \( \mathcal{A}^{(2m+1)} \) and \( \lim \phi_\beta = \Phi \) in \( \sigma(X^{(2m+2)}, X^{(2m+1)}) \), where \( (u_\alpha) \subset \mathcal{A}^{(2m)} \) and \( (\phi_\beta) \subset X^{(2m)} \), and \( \sigma(Y^*, Y) \) denotes the weak* topology on \( Y^* \), then
\[
\mu \Phi = \lim_{\alpha} u_\alpha \phi_\beta, \quad \Phi \mu = \lim_{\beta} \phi_\beta u_\alpha \quad \text{in} \quad \sigma(X^{(2m+2)}, X^{(2m+1)}).
\]
For \( \mu \in \mathcal{A}^{(2m+2)} \) and \( \phi \in X^{(2m)} \), since \( \mu \phi = \mu \hat{\phi} \), \( \phi \mu = \hat{\phi} \mu \), we have
\[
(\Phi, \mu \phi) = (\phi F, \mu), \quad (\Phi, \phi \mu) = (F \phi, \mu) \quad (F \in X^{(2m+1)}).
\]
One can also easily check the relations
\[
\begin{align*}
uf = \hat{u} f &= (uf)^*, \\
\hat{f} \phi &= (f \phi)^*, \\
\hat{u} \hat{\phi} &= (u \phi)^*,
\end{align*}
\]
where \( f \in X^{(2m-1)} \), \( \phi \in X^{(2m)} \) and \( u \in \mathcal{A}^{(2m)} \) \( m \geq 1 \). Therefore, each product agrees with those previously defined.

Concerning dual module morphisms, we have the following.

**Lemma 1.1.** Suppose that \( X \) and \( Y \) are Banach \( \mathcal{A} \)-bimodules. Then, for every continuous \( \mathcal{A} \)-bimodule morphism \( \tau: X \to Y \) and for each \( m \geq 1 \), \( \tau^{(m)}: X^{(2m)} \to Y^{(2m)} \), the \( 2m \)-th dual operator of \( \tau \) is an \( \mathcal{A}^{(2m)} \)-bimodule morphism.

**Proof.** It suffices to prove the lemma in the case where \( m = 1 \). However, for this simple case, the proof is straightforward if we note that \( \tau^{**} \) is weak*-weak* continuous.

In the following, to avoid involving unnecessarily complicated notation, for an element \( y \) in a Banach space \( Y \), we will use the same notation \( y \) to represent its canonical image in any of the \( 2m \)-th dual spaces \( Y^{(2m)} \).

Take \( \mathcal{A}^{(n)} \hat{\oplus} X^{(n)} \) as the underlying space of \( (\mathcal{A} \oplus X)^{(n)} \). From induction, by using the relations in (1.1) and (1.2), one can verify that the \( (\mathcal{A} \oplus X) \)-bimodule actions on \( (\mathcal{A} \oplus X)^{(n)} \) are formulated as follows:

\[
(1.3) \quad (a, x) \cdot (a^{(n)}, x^{(n)}) = \begin{cases}
(aa^{(n)} + x^{(n)} a, a x^{(n)}), & \text{if } n \text{ is odd}; \\
(aa^{(n)}, a x^{(n)} + x a^{(n)}), & \text{if } n \text{ is even},
\end{cases}
\]
and
\[
(1.4) \quad (a^{(n)}, x^{(n)}) \cdot (a, x) = \begin{cases}
(a^{(n)} a + x^{(n)} x, x^{(n)} a), & \text{if } n \text{ is odd}; \\
(a^{(n)} a, a^{(n)} x + x^{(n)} a), & \text{if } n \text{ is even},
\end{cases}
\]
where \( (a, x) \in \mathcal{A} \oplus X \) and \( (a^{(n)}, x^{(n)}) \in \mathcal{A}^{(n)} \hat{\oplus} X^{(n)} = (\mathcal{A} \oplus X)^{(n)} \).

2. **Main Theorems**

Suppose that \( \mathcal{A} \) is a Banach algebra, and \( X \) is a Banach \( \mathcal{A} \)-bimodule. For \( n \)-weak amenability of the Banach algebra \( \mathcal{A} \oplus X \), we have the following main results, whose proofs will be given in Section 4.

**Theorem 2.1.** For \( m \geq 0 \), \( \mathcal{A} \oplus X \) is \((2m+1)\)-weakly amenable if and only if the following conditions hold:
1. \( \mathcal{A} \) is \((2m+1)\)-weakly amenable;
2. \( H^1(\mathcal{A}, X^{(2m+1)}) = \{0\} \);
Proof. Assume, towards a contradiction, that span(\mathbb{A}) \neq X^{(2m+1)} such that \mathbb{A}F - F\mathbb{A} = 0 for \mathbb{A} in \mathbb{A} and \Gamma(x) = xF - FX for x \in X.

Proposition 2.4. Since \mathbb{A} \oplus X is 2m-weakly amenable if and only if the only continuous \mathbb{A}-bimodule morphism \Gamma: X \rightarrow \mathbb{A}^{(2m)} for which x\Gamma(y) + T(x)y = 0 (x, y \in X) in \mathbb{A}^{(2m+1)} is T = 0.

Theorem 2.2. For m \geq 0, \mathbb{A} \oplus X is 2m-weakly amenable if and only if the following conditions hold:

1. the only continuous derivations D: \mathbb{A} \rightarrow \mathbb{A}^{(2m)} for which there is a continuous operator T: X \rightarrow X^{(2m)} such that T(ax) = D(a)x + aT(x) and T(ax) = aD(x) + T(x)a (a \in \mathbb{A}, x \in X) are the inner derivations;
2. \mathcal{H}^1(\mathbb{A}, X^{(2m)}) = \{0\};
3. the only continuous \mathbb{A}-bimodule morphism \Gamma: X \rightarrow \mathbb{A}^{(2m)} for which x\Gamma(y) + \Gamma(x)y = 0 (x, y \in X) in X^{(2m+1)} is zero;
4. for every continuous \mathbb{A}-bimodule morphism T: X \rightarrow X^{(2m)}, there exists u \in \mathbb{A}^{(2m)} for which au = ua for a \in \mathbb{A} and T(x) = xu - ux for x \in X.

Remark 2.3. A simple calculation shows that, when m = 0, condition 3 in Theorem 2.1 is equivalent to the following:

3'. there is no nonzero continuous \mathbb{A}-bimodule morphism \Gamma: X \rightarrow \mathbb{A}^*.

For the general case, condition 3 in Theorem 2.1 is equivalent to the following:

3^m. if \Gamma: X \rightarrow \mathbb{A}^{(2m+1)} is a continuous \mathbb{A}-bimodule morphism, then \Gamma(X) \subset \mathbb{A}^\perp and there is G \in X^{(2m+1)} \cap X^\perp for which aG - Ga = 0 in X^{(2m+1)} (a \in \mathbb{A}) and \Gamma(x) = xG - GX (x \in X).

Proposition 2.4. Suppose that condition 4 of Theorem 2.1 holds for an m \geq 0. Then, span(\mathbb{A}X + X\mathbb{A}) is dense in X.

Proof. Assume, towards a contradiction, that span(\mathbb{A}X + X\mathbb{A}) is not dense in X. Take a nonzero element F \in X^* \cap (\mathbb{A}X + X\mathbb{A})^\perp, and define T: X \rightarrow X^* by

T(x) = F(x)F.

Since F|_{\mathbb{A}X + X\mathbb{A}} = 0, it is easy to see that T is a nonzero, continuous \mathbb{A}-bimodule morphism and that \mathbb{A}T(X) = T(X)\mathbb{A} = \{0\}. Also, for x, y \in X, we have xT(y) = T(x)y = 0 in \mathbb{A}^* since T(X) \subset (\mathbb{A}X)^\perp \cap (X\mathbb{A})^\perp. This shows that condition 4 of Theorem 2.1 does not hold for \mathbb{A}X + X\mathbb{A}. So it does not hold for all m \geq 0. This is a contradiction.

Corollary 2.5. For m = 0, condition 4 in Theorem 2.1 is equivalent to the following:

4^0. span(\mathbb{A}X + X\mathbb{A}) is dense in X and there is no nonzero \mathbb{A}-bimodule morphism T: X \rightarrow X^* satisfying \langle x, T(y) \rangle + \langle y, T(x) \rangle = 0 for x, y \in X.

Proof. Suppose that condition 4 in Theorem 2.1 holds. From the preceding proposition, span(\mathbb{A}X + X\mathbb{A}) is dense in X. If the \mathbb{A}-bimodule morphism T: X \rightarrow X^* satisfies

\langle x, T(y) \rangle + \langle y, T(x) \rangle = 0 \quad for \ x, y \in X,

then, for every a \in \mathbb{A},

\langle a, xT(y) + T(x)y \rangle = \langle ax, T(y) \rangle + \langle y, T(ax) \rangle = 0.

This shows that xT(y) + T(x)y = 0 for x, y \in X. Therefore, T = 0 and so 4^0 holds.
Conversely, if 40 holds, and $T: X \rightarrow X^*$ is a continuous $\mathfrak{A}$-bimodule morphism satisfying $xT(y) + T(x)y = 0$ in $\mathfrak{A}^*$, then, for every $x = ax_1 + x_2b \in \mathfrak{A}X + X\mathfrak{A}$ and $y \in X$, we have

$$\langle x, T(y) \rangle + \langle y, T(x) \rangle = \langle ax_1T(y) + T(x_1)y + b, T(y)x_2 + yT(x_2) \rangle = 0.$$ 

Since $\text{span}(\mathfrak{A}X + X\mathfrak{A})$ is dense in $X$, this implies that $\langle x, T(y) \rangle + \langle y, T(x) \rangle = 0$ for all $x, y \in X$. Hence $T = 0$, and so condition 4 of Theorem 2.1 holds for $m = 0$. □

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras, and let $\mathcal{M}$ be a Banach $\mathfrak{A}, \mathfrak{B}$-module. The algebra $\mathcal{T}$ with the triangular matrix structure

$$\mathcal{T} = \begin{pmatrix} \mathfrak{A} & \mathcal{M} \\ 0 & \mathfrak{B} \end{pmatrix}$$

is called a triangular Banach algebra. The sum and product on $\mathcal{T}$ are given by the usual $2 \times 2$ matrix operations and obvious internal module actions. The norm on $\mathcal{T}$ is

$$\left\| \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \right\| = \|a\|_{\mathfrak{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathfrak{B}}.$$ 

Denote by $\mathfrak{A} \oplus \mathfrak{B}$ the direct $l_1$-sum Banach algebra of $\mathfrak{A}$ and $\mathfrak{B}$, and view $\mathcal{M}$ as an $(\mathfrak{A} \oplus \mathfrak{B})$-bimodule with the module actions given by

$$(a, b) \cdot m = am, \quad m \cdot (a, b) = mb, \quad a \in \mathfrak{A}, \quad b \in \mathfrak{B}, \quad m \in \mathcal{M}.$$ 

Then $\mathcal{T}$ is isometrically isomorphic to the module extension Banach algebra $(\mathfrak{A} \oplus \mathfrak{B}) \oplus \mathcal{M}$. With this setting and some calculations, one sees that Theorems 2.1 and 2.2 imply some main results in [12]. For instance, if $\mathfrak{A}$ and $\mathfrak{B}$ are unital and $\mathcal{M}$ is a unital $\mathfrak{A}, \mathfrak{B}$-module, then $\mathcal{T}$ is weakly amenable if and only if both $\mathfrak{A}$ and $\mathfrak{B}$ are weakly amenable. In fact, the condition can be weakened further to the following: there exist a bounded approximate identity of $\mathfrak{A}$ and a bounded approximate identity of $\mathfrak{B}$ that are also, respectively, left and right approximate identities for $\mathcal{M}$.

3. LIFTING DERIVATIONS

In this section we give several lemmas concerning the lifting of derivations (and module morphisms) from $\mathfrak{A}$ (or $X$) into $\mathfrak{A}^{(n)}$ or $X^{(n)}$ to derivations from $\mathfrak{A} \oplus X$ into $(\mathfrak{A} \oplus X)^{(n)}$.

**Lemma 3.1.** Suppose that $\Gamma: X \rightarrow \mathfrak{A}^{(2m+1)}$ is a continuous $\mathfrak{A}$-bimodule morphism. Then $\overline{\Gamma}: \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m+1)}$, defined by

$$\overline{\Gamma}((a, x)) = (\Gamma(x), 0),$$

is a continuous derivation. The derivation $\overline{\Gamma}$ is inner if and only if there exists $F \in X^{(2m+1)}$ such that $aF - Fa = 0$ and $\Gamma(x) = xF - Fx$ for $a \in \mathfrak{A}$ and $x \in X$.

**Proof.** It is straightforward to check that $\overline{\Gamma}$ is a continuous derivation. Noting that $(\Gamma(x), 0) = \overline{\Gamma}((0, x))$ and $\overline{\Gamma}((a, 0)) = (0, 0)$, one can also see easily that the element $F \in \mathfrak{A}^{(2m+1)}$ described in the lemma exists if $\overline{\Gamma}$ is inner.

Conversely, if such an element $F$ exists, then

$$\overline{\Gamma}((a, x)) = (\Gamma(x), 0) = (xF - Fx, aF - Fa) = (a, x) \cdot (0, F) - (0, F) \cdot (a, x),$$

showing that $\overline{\Gamma}$ is inner. □
A similar proof gives the following lemma.

**Lemma 3.2.** Suppose that \( T: X \to X^{(2m)} \) is a (continuous) \( \mathfrak{A} \)-bimodule morphism. Then \( \overline{T}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)} \), defined by

\[
\overline{T}((a, x)) = (0, T(x)),
\]

is a continuous derivation. The derivation \( \overline{T} \) is inner if and only if there exists \( u \in \mathfrak{A}^{(2m)} \) such that \( ua = au \) for \( a \in \mathfrak{A} \), and \( T(x) = xu - ux \) for all \( x \in X \).

Concerning dual operators we have the following.

**Lemma 3.3.** Suppose that \( k > 0 \) is an integer, and that \( D: \mathfrak{A} \to X^{(k)} \) is a (continuous) derivation. Then, for every integer \( m \geq 0 \), \( D^{(2m+1)}: X^{(k+2m+1)} \to \mathfrak{A}^{(2m+1)} \), the \((2m+1)\)-th dual operator of \( D \), satisfies

\[
D^{(2m+1)}(aF) = aD^{(2m+1)}(F) - (D(a)F)|_{\mathfrak{A}^{(2m)}},
\]

and

\[
D^{(2m+1)}(Fa) = D^{(2m+1)}(F)a - (FD(a))|_{\mathfrak{A}^{(2m)}},
\]

for \( a \in \mathfrak{A} \) and \( F \in X^{(k+2m+1)} \).

**Proof.** The lemma is true for \( m = 0 \) because

\[
\langle b, D^* (aF) \rangle = \langle D(b)a, F \rangle = \langle D(ba) - bD(a), F \rangle = \langle b, aD^*(F) - D(a)F \rangle
\]

and

\[
\langle b, D^* (Fa) \rangle = \langle aD(b), F \rangle = \langle D(ab) - D(a)b, F \rangle = \langle b, D^*(F)a - FD(a) \rangle,
\]

for \( a, b \in \mathfrak{A} \) and \( F \in X^{(k+1)} \).

For \( m > 0 \), from Proposition 1.7 of [10], \( D^{(2m)}: \mathfrak{A}^{(2m)} \to X^{(k+2m)} \) is a (continuous) derivation; here we take the first Arens product in each \( \mathfrak{A}^{(2m)} \). Then, the above shows that \( D^{(2m+1)} = (D^{(2m)})^* : X^{(k+2m+1)} \to (\mathfrak{A}^{(2m)})^* \) satisfies

\[
D^{(2m+1)}(uF) = uD^{(2m+1)}(F) - (D^{(2m)}(u)F)|_{\mathfrak{A}^{(2m)}},
\]

and

\[
D^{(2m+1)}(Fu) = D^{(2m+1)}(F)u - (FD^{(2m)}(u))|_{\mathfrak{A}^{(2m)}},
\]

for \( u \in \mathfrak{A}^{(2m)} \) and \( F \in X^{(k+2m+1)} \). In particular, when \( u = a \in \mathfrak{A} \), these give the formulæ in the lemma.

**Lemma 3.4.** Let \( m \) be an integer. Suppose that \( D: \mathfrak{A} \to X^{(2m+1)} \) is a (continuous) derivation. Then \( \overline{D}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)} \), defined by

\[
\overline{D}((a, x)) = (-D^{(2m+1)}(x), D(a)) \quad \text{for} \quad (a, x) \in \mathfrak{A} \oplus X,
\]

is also a (continuous) derivation. Moreover,

1. if \( \overline{D} \) is inner, then so is \( D \);
2. if \( D \) is inner, then there exists a (continuous) derivation \( \overline{D}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)} \) satisfying \( \overline{D}((a, 0)) = 0 \) (\( a \in \mathfrak{A} \)) and for which \( \overline{D} - D \) is inner.
Proof. For $a, b \in \mathfrak{A}$ and $x, y \in X$, we have, from Lemma \ref{lem.3.3},
\[
\overline{D}((a, x) \cdot (b, y)) = \overline{D}((ab, ay + xb)) = \left(-D^{(2m+1)}(ay + xb), D(ab)\right) = (\left[-aD^{(2m+1)}(y) - (D(a)y)\mathfrak{A}^{(2m)} + D^{(2m+1)}(x)b - (xD(b))\mathfrak{A}^{(2m)}\right], D(a)b + aD(b) = (\left[-aD^{(2m+1)}(y) - D(a)y + D^{(2m+1)}(x)b - xD(b), D(a)b + aD(b)\right) = (\left[-aD^{(2m+1)}(y) + xD(b), aD(b)\right] + \left(-D^{(2m+1)}(x)b + D(a)y, D(a)b\right) = (a, x) \cdot \left(-D^{(2m+1)}(y), D(b)\right) + \left(-D^{(2m+1)}(x), D(a)\right) \cdot (b, y) = (a, x) \cdot \overline{D}((b, y)) + \overline{D}((a, x)) \cdot (b, y).
\]
Therefore, $\overline{D}$ is a (continuous) derivation.

If $\overline{D}$ is inner, then, for some $u \in \mathfrak{A}^{(2m+1)}$ and $F \in X^{(2m+1)}$, we have
\[
\overline{D}((a, x)) = (a, x) \cdot (u, F) - (u, F) \cdot (a, x).
\]

Thus,
\[
(0, D(a)) = \overline{D}((a, 0)) = (a, 0) \cdot (u, F) - (u, F) \cdot (a, 0) = (au - ua, aF - Fa).
\]

This shows that $D(a) = aF - Fa$ for all $a \in \mathfrak{A}$, and hence $D$ is inner.

Conversely, if $D$ is inner, then there exists $F \in X^{(2m+1)}$ such that $D(a) = aF - Fa$ for $a \in \mathfrak{A}$. Let $T: X \rightarrow \mathfrak{A}^{(2m+1)}$ be defined by
\[
T(x) = -D^{(2m+1)}(x) - (xF - Fx) \quad (x \in X),
\]
and let $\overline{T}: \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m+1)}$ be defined by
\[
\overline{T}((a, x)) = (T(x), 0) \quad ((a, x) \in \mathfrak{A} \oplus X).
\]

Then
\[
(\overline{D} - \overline{T})((a, x)) = (xF - Fx, aF - Fa) = (a, x) \cdot (0, F) - (0, F) \cdot (a, x)
\]
for $(a, x) \in \mathfrak{A} \oplus X$. Therefore, $\overline{D} - \overline{T}$ is an inner derivation. This in turn implies that $\overline{T}$ is a (continuous) derivation. So $\overline{D} = \overline{T}$ satisfies all the requirements. This completes the proof. □

If $D$ is a (continuous) derivation from $\mathfrak{A}$ into $\mathfrak{A}^{(2m+1)}$, $m \geq 0$, we define $\overline{D}: \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m+1)}$ by
\[
\overline{D}((a, x)) = (D(a), 0).
\]

If $D$ is a (continuous) derivation from $\mathfrak{A}$ into $X^{(2m)}$, $m \geq 0$, we define $\overline{D}: \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m)}$ by
\[
\overline{D}((a, x)) = (0, D(a)).
\]

If $T: X \rightarrow \mathfrak{A}^{(2m)}$ is a (continuous) $\mathfrak{A}$-bimodule morphism, satisfying $xT(y) + T(x)y = 0$ in $X^{(2m)}$ for $x, y \in X$, we define $\overline{T}: \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m)}$ by
\[
\overline{T}((a, x)) = (T(x), 0).
\]
Finally, if $T: X \rightarrow X^{(2m+1)}$ is a (continuous) $\mathfrak{A}$-bimodule morphism, satisfying $xT(y) + T(x)y = 0$ for $x, y \in X$, we define $\overline{T}: \mathfrak{A} \oplus X \rightarrow (\mathfrak{A} \oplus X)^{(2m+1)}$ by
\[
\overline{T}((a, x)) = (0, T(x)).
\]
Then, straightforward calculations yield the following result.

**Lemma 3.5.** The operators $\overline{D}$ and $\overline{T}$ defined above are (continuous) derivations. Furthermore, the derivation $\overline{D}$ is inner if and only if $D$ is inner, and $\overline{T}$ is inner if and only if $T = 0$.

### 4. Proofs of the main theorems

We first prove Theorem 2.1.

**Proof.** Denote by $\Delta_1$ the projection from $(\mathfrak{A} \oplus X)^{(2m+1)}$ onto $\mathfrak{A}^{(2m+1)}$ with kernel $X^{(2m+1)}$. Let $\Delta_2$ be the projection $id - \Delta_1$: $(\mathfrak{A} \oplus X)^{(2m+1)} \to X^{(2m+1)}$, and let $\tau_1$: $\mathfrak{A} \to \mathfrak{A} \oplus X$ be the inclusion mapping (i.e., $\tau_1(a) = (a, 0)$). Then $\Delta_1$ and $\Delta_2$ are $\mathfrak{A}$-bimodule morphisms, and $\tau_1$ is an algebra homomorphism.

We now prove the sufficiency in Theorem 2.1. Suppose that conditions 1–4 hold. We first prove Theorem 2.1. Suppose also that $D$: $\mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous derivation. Then $D \circ \tau_1$: $\mathfrak{A} \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous derivation. This implies that $\Delta_1 \circ D \circ \tau_1$: $\mathfrak{A} \to \mathfrak{A}^{(2m+1)}$ and $\Delta_2 \circ D \circ \tau_1$: $\mathfrak{A} \to X^{(2m+1)}$ are continuous derivations. By conditions 1 and 2, they are inner. Therefore, $D \circ \tau_1$ is inner. From Lemmas 3.5 and 3.6,

$$D \circ \tau_1 = \Delta_1 \circ D \circ \tau_1 + \Delta_2 \circ D \circ \tau_1 : \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$$

is a continuous derivation, and there is a continuous derivation $\tilde{D}$: $\mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ satisfying $\tilde{D}((a, 0)) = 0$ for $a \in \mathfrak{A}$ and such that $D \circ \tau_1 - \tilde{D}$ is inner.

On the other hand,

$$(D - D \circ \tau_1)((a, 0)) = D((a, 0)) - D \circ \tau_1((a, 0))$$

$$= D \circ \tau_1(a) - D \circ \tau_1(a) = 0 \quad (a \in \mathfrak{A}).$$

Let $\tilde{D} = D - D \circ \tau_1 + \tilde{D}$. Then $\tilde{D}$ is a continuous derivation from $\mathfrak{A} \oplus X$ into $(\mathfrak{A} \oplus X)^{(2m+1)}$ satisfying $\tilde{D}((a, 0)) = 0$ for $a \in \mathfrak{A}$. Moreover,

$$\tilde{D}((0, ax)) = \tilde{D}((0, a) \cdot (0, x)) = (a, 0) \cdot \tilde{D}((0, x)) = a \tilde{D}((0, x)) \quad (a \in \mathfrak{A}, x \in X),$$

and

$$\tilde{D}((0, xa)) = \tilde{D}((0, x) \cdot (a, 0)) = \tilde{D}((0, x))a \quad (a \in \mathfrak{A}, x \in X).$$

Denote by $\tau_2$: $X \to \mathfrak{A} \oplus X$ the inclusion mapping given by $\tau_2(x) = (0, x)$ ($x \in X$). Then $\tilde{D} \circ \tau_2$: $X \to (\mathfrak{A} \oplus X)^{(2m+1)}$ is a continuous $\mathfrak{A}$-bimodule morphism. From condition 3, there exists $F \in X^{(2m+1)}$ for which $\Delta_1 \circ \tilde{D} \circ \tau_2 = xF - Fx$, and $aF - Fa = 0$ for $x \in X$ and $a \in \mathfrak{A}$. Since

$$(0, 0) = \tilde{D}((0, 0)) = \tilde{D}((0, x) \cdot (0, y))$$

$$= \tilde{D}((0, x)) \cdot (0, y) + (0, x) \cdot \tilde{D}((0, y))$$

$$= ([\Delta_2 \circ \tilde{D}((0, x))]y, 0) + (x[\Delta_2 \circ \tilde{D}((0, y))], 0)$$

$$= ([\Delta_2 \circ \tilde{D} \circ \tau_2(x)]y + x[\Delta_2 \circ \tilde{D} \circ \tau_2(y)], 0),$$

we have

$$([\Delta_2 \circ \tilde{D} \circ \tau_2(x)]y + x(\Delta_2 \circ \tilde{D} \circ \tau_2(y)) = 0 \quad (x, y \in X).$$
From condition 4,  \( \Delta_2 \circ \hat{D} \circ \tau_2 = 0 \). Thus,
\[
\hat{D}((a, x)) = \hat{D}((0, x)) = \hat{D} \circ \tau_2(x)
\]
\[
= (\Delta_1 \circ \hat{D} \circ \tau_2(x), \Delta_2 \circ \hat{D} \circ \tau_2(x))
\]
\[
= (xF - Fx, 0) = (a, x) \cdot (0, F) - (0, F) \cdot (a, x).
\]
We have that \( \hat{D} \) is inner. Thus \( D = \hat{D} + (\hat{D} \circ \tau_1 - \hat{D}) \) is inner. This proves that \( \mathfrak{A} \oplus X \) is \((2m + 1)\)-weakly amenable.

Necessity: Suppose that \( \mathfrak{A} \oplus X \) is \((2m + 1)\)-weakly amenable. Then from Lemmas 4.2 and 3.4, \( \mathcal{H}^i(\mathfrak{A}, \mathfrak{A}(2m+1)) = \{0\} \) and \( \mathcal{H}^4(\mathfrak{A}, X(2m+1)) = \{0\} \). Therefore, conditions 1 and 2 hold. Furthermore, Lemma 3.1 gives condition 3, and Lemma 3.5 shows that condition 4 holds. This completes the proof of Theorem 2.1.

We now turn to the proof of Theorem 2.2.

**Proof.** Denote by \( \tau_1 \) and \( \tau_2 \) the inclusion mappings described in the preceding proof from, respectively, \( \mathfrak{A} \) and \( X \) into \( \mathfrak{A} \oplus X \), and denote by \( \Delta_1 \) and \( \Delta_2 \) the natural projections from \( (\mathfrak{A} \oplus X)(2m) \) onto \( \mathfrak{A}(2m) \) and \( X(2m) \), respectively. These are \( \mathfrak{A} \)-bimodule morphisms.

To prove the sufficiency we assume that conditions 1–4 in Theorem 2.2 hold. Let \( D: (\mathfrak{A} \oplus X) \to (\mathfrak{A} \oplus X)(2m) \) be a continuous derivation. Then \( \Delta_1 \circ D \circ \tau_1: \mathfrak{A} \to \mathfrak{A}(2m) \) and \( \Delta_2 \circ D \circ \tau_2: \mathfrak{A} \to X(2m) \) are continuous derivations.

**Claim 1:** \( \Delta_1 \circ D \circ \tau_2: X \to \mathfrak{A}(2m) \) is trivial.

Let \( \Gamma = \Delta_1 \circ D \circ \tau_2 \). To prove claim 1, by condition 3 it suffices to show that \( \Gamma \) is an \( \mathfrak{A} \)-bimodule morphism satisfying \( x \Gamma(y) + \Gamma(x)y = 0 \) in \( X(2m) \) for \( x, y \in X \). In fact,
\[
0 = D((0, 0)) = D((0, x) \cdot (0, y)) = D((0, x)) \cdot (0, y) + (0, x) \cdot D((0, y))
\]
\[
= (0, x \Gamma(y)) + (0, x \Gamma(y)).
\]
Thus, \( x \Gamma(y) + \Gamma(x)y = 0 \). On the other hand,
\[
\Gamma(ax) = \Delta_1 \circ D((0, ax)) = \Delta_1 \circ D((a, 0) \cdot (0, x))
\]
\[
= \Delta_1 ((a, 0) \cdot (0, x) + (a, 0) \cdot D((0, x)))
\]
\[
= \Delta_1 ((a, 0) \cdot D((0, x))) = \Delta_1 (aD \circ \tau_2(x)) = a \Gamma(x).
\]
Similarly, \( \Gamma(xa) = \Gamma(x)a \) and so \( \Gamma \) is an \( \mathfrak{A} \)-bimodule morphism. Therefore, claim 1 is true.

Now let \( T = \Delta_2 \circ D \circ \tau_2: X \to X(2m) \), and set \( D_1 = \Delta_1 \circ D \circ \tau_1: \mathfrak{A} \to \mathfrak{A}(2m) \).

**Claim 2:** \( T(ax) = D_1(a)x + aT(x) \) and \( T(xa) = xD_1(a) + T(x)a \) for \( a \in \mathfrak{A} \) and \( x \in X \).

In fact, from claim 1,
\[
(0, T(ax)) = D((0, ax)) = D((a, 0) \cdot (0, x)) = D((a, 0)) \cdot (0, x) + (a, 0) \cdot D((0, x))
\]
\[
= (0, D_1(a)x) + a(0, T(x)) = (0, D_1(a)x + aT(x)).
\]
Similarly, \( (0, T(xa)) = (0, xD_1(a) + T(x)a) \) for \( a \in \mathfrak{A} \) and \( x \in X \). Thus, claim 2 is true.

Therefore, by condition 1, \( D_1 = \Delta_1 \circ D \circ \tau_1 \) is inner. Suppose that \( a \in \mathfrak{A}(2m) \) satisfies \( D_1(a) = au - ua \) for \( a \in \mathfrak{A} \). Let \( T_1: X \to X(2m) \) be defined by \( T_1(x) = xu - ux \).
for \( x \in X \). Then \( T - T_1: X \to X^{(2m)} \) is a continuous \( \mathfrak{A} \)-bimodule morphism. In fact, from claim 2, for \( a \in \mathfrak{A} \) and \( x \in X \),
\[
(T - T_1)(ax) = T(ax) - T_1(ax) = (D_1(a)x + aT(x)) - (axu - uax) = (au - ua)x + aT(x) - (axu - uax) = a(ux - xu) + aT(x) = a(T - T_1)(x).
\]

Similarly, \( T - T_1 \) is a right \( \mathfrak{A} \)-module morphism. From condition 4, there is a \( v \in \mathfrak{A}^{(2m)} \) such that \( av = va \) for \( a \in \mathfrak{A} \), and \( (T - T_1)(x) = xv - vx \) for \( x \in X \). From Lemma 3.2, we have that
\[
\Delta \text{ is inner. This shows that under conditions 1–4 of Theorem 2.2, for a \( \mathfrak{A} \)-bimodule morphism, it is inner by condition 2.}
\]

From Lemma 3.5,
\[
\Delta_2 \circ D \circ \tau_1: (a, x) \mapsto (0, \Delta_2 \circ D \circ \tau_1(a)), \quad \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)}
\]
is also inner. Using claim 1, we now have
\[
D((a, x)) = (D_1(a), \Delta_2 \circ D \circ \tau_1(a) + T(x)) = \Delta_2 \circ D \circ \tau_1((a, x)) + (T - T_1)((a, x)) + (D_1(a), T_1(x)).
\]

Since
\[
(D_1(a), T_1(x)) = (au - ua, xu - ux) = (a, x) \cdot (u, 0) - (u, 0) \cdot (a, x),
\]
for \( a \in \mathfrak{A} \) and \( x \in X \), it gives an inner derivation from \( \mathfrak{A} \oplus X \) into \((\mathfrak{A} \oplus X)^{(2m)}\). Hence as a sum of three inner derivations, \( D \) is inner. This shows that under conditions 1–4 of Theorem 2.2, \( \mathfrak{A} \oplus X \) is \( 2m \)-weakly amenable.

Now we prove the necessity. Suppose that \( \mathfrak{A} \oplus X \) is \( 2m \)-weakly amenable. Let \( D: \mathfrak{A} \to \mathfrak{A}^{(2m)} \) be a continuous derivation with the property given in condition 1. Then \( \overline{D}: \mathfrak{A} \oplus X \to (\mathfrak{A} \oplus X)^{(2m)} \) defined by
\[
\overline{D}((a, x)) = (D(a), T(x)), \quad (a, x) \in \mathfrak{A} \oplus X,
\]
is a continuous derivation and hence is inner. This implies that \( D \) is inner, and so condition 1 holds. The other conditions are consequences of Lemma 3.3 and Lemma 3.2.

The proof is complete. \( \square \)

5. The Algebras \( \mathfrak{A} \oplus \mathfrak{A} \) and \( \mathfrak{A} \oplus \mathfrak{A}^* \)

In this and the following section we consider several concrete cases. This section deals mainly with the two cases \( X = \mathfrak{A} \) and \( X = \mathfrak{A}^* \) as Banach \( \mathfrak{A} \)-bimodules.

We first note that, if \( \mathfrak{A} \) is not amenable, then there is a Banach \( \mathfrak{A} \)-bimodule \( X \) such that \( \mathcal{H}^1(\mathfrak{A}, X^*) \neq \{0\} \). From Theorem 2.1 for this \( X \), \( \mathfrak{A} \oplus X \) is not weakly amenable. In fact, the Banach algebra \( \mathfrak{A} \oplus X \) is never weakly amenable when \( X = \mathfrak{A}^* \), as implied in the following proposition.

**Proposition 5.1.** Suppose that \( \mathfrak{A} \) is a Banach algebra. Then \( \mathfrak{A} \oplus \mathfrak{A}^* \) is not \( n \)-weakly amenable for every \( n \geq 0 \).
Proof. From Proposition 1.2 of [11], it suffices to prove the cases of \( n = 0, \ n = 1 \) and \( n = 2 \). Note that condition 3 does not hold, because the identity mapping from \( X (= \mathfrak{A}^*) \) onto \( \mathfrak{A}^* \) is a nonzero, continuous \( \mathfrak{A} \)-bimodule morphism. So the proposition is true for \( n = 1 \).

For \( n = 2m \) with \( m = 0 \) or \( m = 1 \), if condition 4 in Theorem 2.2 holds for \( X = \mathfrak{A}^* \), then the operator \( T \) described in this condition has the property that \( T(f) \in \mathfrak{A}^\perp \) for \( f \in X \). In fact, for \( a \in \mathfrak{A} \), we have
\[
(a, T(f)) = (a, f \cdot u - u \cdot f) = (af - fa, u) = (f, ua - au) = 0.
\]

But \( X = \mathfrak{A}^* \) certainly does not annihilate \( \mathfrak{A} \). So, as \( \mathfrak{A} \)-bimodule morphisms, the identity mapping (in the case \( m = 0 \)) from \( X \) onto \( X \) and the inclusion mapping (in the case \( m = 1 \)) from \( X \) into \( X^{**} \) do not satisfy condition 4. Consequently, \( \mathfrak{A} \oplus \mathfrak{A}^* \) is not \( 2m \)-weakly amenable for \( m = 0 \) and 1.

Now we consider the case that \( X = \mathfrak{A} \). To avoid any confusion, from now on, when we regard \( \mathfrak{A} \) as an \( \mathfrak{A} \)-bimodule, we will use the notation \( A \) instead of \( \mathfrak{A} \). If \( X = A \), condition 4 in Theorem 2.2 never holds for any integer \( m \) (the canonical embedding is a nonzero morphism). It turns out that \( \mathfrak{A} \oplus A \) is never \( 2m \)-weakly amenable for any \( m \geq 0 \). If \( A \) is commutative, for the same reason we can conclude more as in the next proposition. Recall that an \( \mathfrak{A} \)-bimodule \( X \) is symmetric if \( ax = xa \) for \( a \in \mathfrak{A} \) and \( x \in X \).

**Proposition 5.2.** Suppose that \( \mathfrak{A} \) is a commutative Banach algebra. Then for every nonzero, symmetric \( \mathfrak{A} \)-bimodule \( X \), \( \mathfrak{A} \oplus X \) is not \( 2m \)-weakly amenable.

**Proof.** Let \( X \) be symmetric. Then \( xu = ux \) for \( u \in \mathfrak{A}^{(2m)} \) and \( x \in X \). Since the canonical embedding from \( X \) into \( X^{(2m)} \) is a nontrivial \( \mathfrak{A} \)-bimodule morphism, condition 4 in Theorem 2.2 does not hold for such a module \( X \).

But \( \mathfrak{A} \oplus A \) can be weakly amenable. Before giving an example let us go through some relation identities for corresponding elements of \( A^{(n)} \) and \( \mathfrak{A}^{(n)} \). Suppose that \( \phi \in A^{(n)} \). We denote the same element in \( \mathfrak{A}^{(n)} \) by \( \phi \).

**Lemma 5.3.** Suppose that \( \mathfrak{A} \) is a Banach algebra, and let \( m \geq 0 \). Then, for \( \phi, \psi \in A^{(2m)} \) and \( F \in A^{(2m+1)} \), we have
\[
(\tilde{\phi} \tilde{\psi})^\sim = \tilde{\phi} \tilde{\psi} = (\phi \psi)^\sim, \quad \phi F = (\tilde{\phi} F)^\sim = \tilde{\phi} F, \quad F \phi = (F \tilde{\phi})^\sim = F \tilde{\phi}.
\]

**Proof.** It is straightforward to check the identities for the case \( m = 0 \). Then, an induction on \( m \) completes the proof for the general case.

A special case of Lemma 5.3 is the following group of identities which will be used in the proof of the next theorem:
\[
(a \phi)^\sim = a \tilde{\phi}, \quad (\phi a)^\sim = \tilde{\phi} a, \quad xF = (\tilde{x} F)^\sim = \tilde{x} F, \quad F x = (F \tilde{x})^\sim = F \tilde{x},
\]
where \( a \in \mathfrak{A}, x \in A, \phi \in A^{(2m)} \) and \( F \in A^{(2m+1)} \). From these identities, we also see that, for \( X = A \) and \( m \geq 0 \), condition 3 in Theorem 2.1 holds if and only if there is no nonzero \( \mathfrak{A} \)-bimodule morphism \( T \) from \( A \) into \( A^{(2m+1)} \), and that, if this is the case, then condition 4 holds automatically. Moreover, with \( X = A \), conditions 1 and 2 of Theorem 2.1 are the same.
Theorem 5.4. For a Banach algebra $\mathfrak{A}$:

1. if span{$ab - ba; a, b \in \mathfrak{A}$} is not dense in $\mathfrak{A}$, then $\mathfrak{A} \oplus A$ is not weakly amenable;
2. if span{$ab - ba; a, b \in \mathfrak{A}$} is dense in $\mathfrak{A}$, then $\mathfrak{A} \oplus A$ is weakly amenable, provided that $\mathfrak{A}$ is weakly amenable and has a bounded approximate identity.

Proof. By condition 1 of Theorem 2.1 without loss of generality, we can assume that $\mathfrak{A}$ is weakly amenable for both cases. If span{$ab - ba; a, b \in \mathfrak{A}$} is not dense in $\mathfrak{A}$, then there exists $f \in \mathfrak{A}^*$ such that $f \neq 0$ and $\langle ab - ba, f \rangle = 0$ for $a, b \in \mathfrak{A}$. So $af = fa$ for $a \in \mathfrak{A}$. Then $T: A \to \mathfrak{A}^*$, defined by

$$T(x) = \hat{x}f = f\hat{x},$$

is an $\mathfrak{A}$-bimodule morphism. According to Proposition 1.3 of [10], $\mathfrak{A}^2$, the linear span of all product elements $ab, a, b \in \mathfrak{A}$, is dense in $\mathfrak{A}$. So there are $a, b \in \mathfrak{A}$ such that $\langle ab, f \rangle \neq 0$. This implies that $T \neq 0$. Therefore, condition 3 does not hold. As a consequence, $\mathfrak{A} \oplus A$ is not weakly amenable.

If span{$ab - ba; a, b \in \mathfrak{A}$} is dense in $\mathfrak{A}$, and $\mathfrak{A}$ has a bounded approximate identity $(e_i)$, then, for every given continuous $\mathfrak{A}$-bimodule morphism $T: A \to \mathfrak{A}^*$, we have $T(a) = af = fa$, where $f$ is a weak* cluster point of $(T(e_i))$. Therefore, $\langle ab - ba, f \rangle = 0$ for all $a, b \in \mathfrak{A}$. This shows that $f = 0$ and hence $T = 0$. Thus conditions 3 and 4 in Theorem 2.1 hold for $m = 0$. The other two conditions hold automatically for $m = 0$. So, from Theorem 2.1 the second statement of the theorem is true.

From case 1 of Theorem 5.4 we immediately have the following corollary.

Corollary 5.5. If $\mathfrak{A}$ is a commutative Banach algebra, then $\mathfrak{A} \oplus A$ is not weakly amenable.

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. According to a classical result due to Halmos (Theorem 8 of [18]), every element in $B(\mathcal{H})$ can be written as a sum of two commutators (see also [4] and [31]). Together with the fact that $B(\mathcal{H})$ has an identity and, as a $C^*$-algebra, is weakly amenable [17], from Theorem 5.3 we see that $B(\mathcal{H}) \oplus B(\mathcal{H})$ is weakly amenable. Later in this section we will see that it is in fact $(2m + 1)$-weakly amenable.

Proposition 5.6. Suppose that $V = \text{span}\{au - ua; u \in \mathfrak{A}^*, a \in \mathfrak{A}\}$ is not dense in $\mathfrak{A}^* \mathfrak{A} + \mathfrak{A} \mathfrak{A}^*$ (if $\mathfrak{A}$ has an identity, this is equivalent to saying that $V$ is not dense in $\mathfrak{A}^*$). Then $\mathfrak{A} \oplus A$ is not 3-weakly amenable.

Proof. In fact, in this case $\mathfrak{A}^* \mathfrak{A} \not\subset cl(V)$, since otherwise it would follow that both $\mathfrak{A} \mathfrak{A}^*$ and $\mathfrak{A} \mathfrak{A}^*$ were in $cl(V)$, and then $cl(V) \supset eqn\mathfrak{A}^* + \mathfrak{A}^* \mathfrak{A}$, which contradicts the assumption that $V$ is not dense in $\mathfrak{A}^*$. Then $T: A \to \mathfrak{A}^*$ is continuous $\mathfrak{A}$-bimodule morphism from $A$ into $\mathfrak{A}^*$. Therefore, condition 3 in Theorem 2.1 does not hold for $m = 1$. This shows that $\mathfrak{A} \oplus A$ is not 3-weakly amenable.

Regarding the open question of whether weak amenability implies 3-weak amenability, Theorem 5.4 and Proposition 5.6 suggest that one might find a counterexample in the Banach algebras of the form $\mathfrak{A} \oplus A$. Unfortunately, $B(\mathcal{H})$ cannot be
Let $H$ be an infinite-dimensional Hilbert space. Then there exist two elements $Q_0$ and $S_0$ in $B(H)$ such that, for each $B \in B(H)$, there exist $P_B \in B(H)$ and $R_B \in B(H)$ with $\|P_B\| \leq \|B\|$ and $\|R_B\| \leq \|B\|$ for which

$$B = (P_B \circ Q_0 - Q_0 \circ P_B) + (R_B \circ S_0 - S_0 \circ R_B).$$

**Proof.** For an infinite-dimensional Hilbert space $H$, there exists an isometry $\eta$: $H \rightarrow \sum_{i=1}^{\infty} \hat{H}_i$, where $\sum_{i=1}^{\infty} \hat{H}_i$ denotes an $l_2$ direct sum and each $\hat{H}_i$ is a copy of $H$.

Let $Q$: $H \rightarrow \sum_{i=1}^{\infty} \hat{H}_i$ and $S$: $\sum_{i=1}^{\infty} \hat{H}_i \rightarrow \sum_{i=1}^{\infty} \hat{H}_i$ be the bounded operators given by the infinite matrices

$$Q = \begin{pmatrix} I \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & 0 & \ddots \end{pmatrix}.$$  

Let $Q_0 = \eta^{-1} \circ Q$ and $S_0 = \eta^{-1} \circ S \circ \eta$. Then $Q_0, S_0 \in B(H)$. Given an element $B \in B(H)$, let $P$: $\sum_{i=1}^{\infty} \hat{H}_i \rightarrow H$ and $R$: $\sum_{i=1}^{\infty} \hat{H}_i \rightarrow \sum_{i=1}^{\infty} \hat{H}_i$ be the bounded operators given by the infinite matrices

$$P = (B \ 0 \ 0 \ \cdots) \quad \text{and} \quad R = \begin{pmatrix} 0 & B & 0 & \cdots \\ 0 & 0 & B & \cdots \\ 0 & 0 & 0 & B \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$  

Take $P_B = P \circ \eta$ and $R_B = \eta^{-1} \circ R \circ \eta$. Then $P_B, R_B \in B(H)$ and $\|P_B\| \leq \|B\|$, $\|R_B\| \leq \|B\|$. We have that $B = (P_B \circ Q_0 - Q_0 \circ P_B) + (R_B \circ S_0 - S_0 \circ R_B)$. \hfill $\square$

The following result on the $2n$-th dual of $B(H)$ seems not to be known.

**Lemma 5.8.** For every integer $n \geq 0$,

$$B(H)^{(2n)} = \text{span}\{au - ua; \ a \in B(H), \ u \in B(H)^{(2n)}\}.$$  

**Proof.** By taking weak* limits and using induction, one can show the result immediately from Lemma 5.7. \hfill $\square$

**Proposition 5.9.** For each integer $m \geq 0$, $B(H) \oplus B(H)$ is $(2m + 1)$-weakly amenable but is not $2m$-weakly amenable.

**Proof.** First, as a C*-algebra, $B(H)$ is permanently weakly amenable. So conditions 1 and 2 of Theorem 2.1 hold for $X = \mathfrak{A} = B(H)$ and $m \geq 0$. To show that conditions 3 and 4 also hold, it suffices to prove that every continuous $B(H)$-bimodule morphism $T$ from $B(H)$ into $B(H)^{(2m+1)}$ is trivial.

In fact, letting $e$ be the identity of $B(H)$ and $F = T(e)$, we have $T(a) = aF = Fa$ for all $a \in B(H)$. Therefore, for all $u \in B(H)^{(2m)}$, we have $\langle au - ua, F \rangle = 0$. From Lemma 5.8, this implies that $F = 0$. Hence $T = 0$. Therefore, $B(H) \oplus B(H)$ is $(2m + 1)$-weakly amenable for $m \geq 0$. \hfill $\square$
On the other hand, we have indicated in the paragraph before Proposition 5.2 that \( A \oplus A \) is never \( 2m \)-weakly amenable. So \( B(H) \oplus B(H) \) is not \( 2m \)-weakly amenable for \( m \geq 0 \). This completes the proof.

Remark 5.10. Denote by \( K(H) \) the algebra of compact operators on \( H \). Using Theorem 1 of [29] one can also prove that \( K(H) \oplus K(H) \) and \( B(H) \oplus K(H) \) are \((2m + 1)\)- (but not \( 2m \)-) weakly amenable. On the other hand, it is interesting to recall Proposition 2.4 which implies that \( K(H) \oplus B(H) \) is not weakly amenable.

6. The algebra \( A \oplus X_0 \)

In this section we consider the case that the module action on one side of \( X \) is trivial. We denote by \( X_0 \) (respectively, \( \mathcal{Y} \)) specifically the \( A \)-bimodules with right (respectively, left) module action trivial. We observe that, when \( X = X_0 \), conditions 3 and 4 in Theorem 2.1 are reduced, respectively, to the following:

- 3\(_0\)' for each continuous \( A \)-bimodule morphism \( \Gamma: X_0 \to A(2m+1) \), there is \( F \in X_0^{(2m+1)} \) such that \( Fa = 0 \) for \( a \in A \) and \( \Gamma(x) = xF \) for \( x \in X_0 \);
- 4\(_0\)' \( A \) is amenable for \( X_0 \).

Also, conditions 1, 3 and 4 in Theorem 2.2 are reduced, respectively, to the following:

- 1\(_0\)' every continuous derivation \( D: A \to A(2m) \) with the property that there is a continuous operator \( T: X_0 \to X_0^{(2m)} \) such that \( T(ax) = D(a)x + aT(x) \) for \( a \in A \) and \( x \in X_0 \) is inner;
- 3\(_0\)' the only continuous \( A \)-bimodule morphism \( \Gamma: X_0 \to A(2m) \) satisfying \( \Gamma(x) = 0 \) \( (x, y \in X_0) \) in \( X_0^{(2m)} \) is zero;
- 4\(_0\)' for every continuous \( A \)-bimodule morphism \( T: X_0 \to X_0^{(2m)} \), there exists \( u \in \mathcal{A}(2m) \) such that \( uu = uu \) for \( a \in A \) and \( T(x) = ux \) for \( x \in X_0 \).

Proposition 6.1. Suppose that \( A \) is a \((2m + 1)\)-weakly amenable Banach algebra with a bounded approximate identity and satisfying that \( A \mathcal{A}(2m) = \mathcal{A}(2m) \). Then, \( A \oplus X_0 \) is \((2m + 1)\)-weakly amenable if and only if \( AX_0 \) is dense in \( X_0 \).

Proof. Since \( A \) has a bounded approximate identity, from Proposition 1.5 of [29], condition 2 in Theorem 2.1 always holds for \( X = X_0 \). If \( A \mathcal{A}(2m) = \mathcal{A}(2m) \), then there is no nonzero, continuous \( A \)-bimodule morphism \( T: X_0 \to \mathcal{A}(2m+1) \), since such a morphism must satisfy \( \langle au, T(x) \rangle = \langle u, T(xa) \rangle = 0 \) \( (a \in A, u \in \mathcal{A}(2m)) \). So condition 3\(_0\)' holds automatically.

For \( m = 0 \), the above proposition yields the following.

Corollary 6.2. Suppose that \( A \) is a weakly amenable Banach algebra with a bounded approximate identity. Then \( A \oplus X_0 \) is weakly amenable if and only if \( AX_0 \) is dense in \( X_0 \).

A dual result to Corollary 6.2 is as follows.

Corollary 6.3. Suppose that \( A \) is a weakly amenable Banach algebra with a bounded approximate identity. Let \( \mathcal{Y} \) be a Banach \( A \)-bimodule with left module action trivial. Then, \( A \oplus \mathcal{Y} \) is weakly amenable if and only if \( AX_0 \) is dense in \( \mathcal{Y} \).

View \( A \) as a left \( A \)-module and then impose a trivial right \( A \)-module action on it. This results in a Banach \( A \)-bimodule. We denote it by \( A_0 \). Suppose that \( \phi \in A_0^{(n)} \).
We denote the same element in $\mathfrak{A}^{(n)}$ by $\tilde{\phi}$. Similarly to Lemma 5.3 one can check that the following equalities hold:

\[(u\phi)^\sim = u\tilde{\phi}, \quad \phi u = 0, \quad \phi F = \tilde{\phi}F,\]

\[F\phi = 0, \quad uF = 0, \quad (Fu)^\sim = F\tilde{u},\]

where $u \in \mathfrak{A}^{(2m)}$, $\phi \in A_0^{(2m)}$, $F \in A_0^{(2m+1)}$ ($m \geq 0$).

**Proposition 6.4.** Suppose that $\mathfrak{A}$ is a $(2m + 1)$-weakly amenable Banach algebra with a bounded approximate identity. Then $\mathfrak{A} \oplus A_0$ is $(2m + 1)$-weakly amenable.

**Proof.** As in the proof of Proposition 6.1 it suffices to verify conditions 3′ and 4′. Condition 4′ holds since $\mathfrak{A}$ has a bounded approximate identity for $A_0$. Let $(x_\alpha) \subset A_0$ be a net such that $(\tilde{x}_\alpha)$ is a bounded approximate identity for $\mathfrak{A}$. If $\Gamma: A_0 \to \mathfrak{A}^{(2m+1)}$ is a continuous $\mathfrak{A}$-bimodule morphism, we let $\tilde{F}$ be a weak* cluster point of $(\Gamma(x_\alpha))$. Let the element in $A_0^{(2m+1)}$ corresponding to $F$ be $F$. Then $F$ satisfies the requirement in condition 3′.

Concerning $2m$-weak amenability, we have the following.

**Proposition 6.5.** Let $m \geq 1$, and suppose that $\mathfrak{A}$ is a commutative $2m$-weakly amenable Banach algebra with a bounded approximate identity. Then $\mathfrak{A} \oplus A_0$ is $2m$-weakly amenable.

**Proof.** It suffices to show that conditions 3′′ and 4′′ hold. Suppose that an $\mathfrak{A}$-bimodule morphism $\Gamma: A_0 \to \mathfrak{A}^{(2m)}$ satisfies $\Gamma(x)y = 0$ in $A_0^{(2m)}$ ($x, y \in A_0$). Then

\[0 = (\Gamma(x)y)^\sim = \Gamma(x)\tilde{y} = \tilde{y}\Gamma(x) = \Gamma(\tilde{yx}) \quad (x, y \in A_0).\]

This implies that $\Gamma(ax) = 0$ for $a \in \mathfrak{A}$ and $x \in A_0$. So $\Gamma(x) = 0$ for all $x \in A_0$. Therefore, condition 3′′ holds.

Assume that $T: A_0 \to A_0^{(2m)}$ is a continuous $\mathfrak{A}$-bimodule morphism. Let $v$ be a weak* cluster point of $(T(x_i))$, where $(\tilde{x}_i)$ is a bounded approximate identity for $\mathfrak{A}$. Let $u = \tilde{v}$. Then, $T(x) = \lim T(\tilde{xx}_i) = \tilde{x}v$. However, $(\tilde{x}v)^\sim = \tilde{x}\tilde{v} = \tilde{x}u = u\tilde{x} = (ux)^\sim$. Hence $T(x) = ux$. On the other hand, $ua = au$ since $\mathfrak{A}$ is commutative. Condition 4′′ holds.

Although we have already had an example of a Banach algebra which is $(2m+1)$-weakly amenable but not $2m$-weakly amenable (see Proposition 5.4 another known example is the nuclear algebra $\mathcal{N}(E)$ with $E$ a reflexive Banach space having the approximation property [10 Corollary 5.4]), we end this section by giving one more example of a weakly amenable Banach algebra which is not 2-weakly amenable.

Suppose that $\mathfrak{A}$ is a weakly amenable Banach algebra with a bounded approximate identity and satisfying that $\mathfrak{A}\mathfrak{A}^* \neq \mathfrak{A}^*\mathfrak{A}$ (an example is $\mathfrak{A} = L^1(G)$ with $G$ a non-SIN locally compact group; see [28] and [25] for the reference of SIN groups, and Theorem 32.44 of [20] as well as [26] for the property we need here). Without loss of generality, we assume that $\mathfrak{A}\mathfrak{A}^* \not\subseteq \mathfrak{A}^*\mathfrak{A}$.

**Example 6.6.** For the above Banach algebra $\mathfrak{A}$, $\mathfrak{A} \oplus A_0$ is weakly amenable but is not 2-weakly amenable.
Proof. From Proposition 6.4, \( A \oplus A_0 \) is weakly amenable. We show that condition 3\( _0 \) does not hold for \( m = 1 \). Take a \( \varphi \in \mathfrak{A}^{**} \) for which \( \varphi|_{\mathfrak{A}^r} = 0 \) but \( \varphi|_{\mathfrak{A} \cdot A} \neq 0 \) (notice that by Cohen’s factorization theorem, \( \mathfrak{A}^{r} \) is closed in \( \mathfrak{A}^{*} \)). Then \( \varphi a = 0 \) for all \( a \in \mathfrak{A} \) and \( a \varphi \neq 0 \) for some \( a \in \mathfrak{A} \). Let \( T: A_0 \to \mathfrak{A}^{**} \) be defined by \( T(x) = \tilde{x} \varphi \). Then \( T \) is a continuous \( \mathfrak{A} \)-bimodule morphism and \( T \neq 0 \).

Thus, we have \( T(x)y = 0 \) for all \( x, y \in A_0 \). Therefore, condition 3\( _0 \) is not satisfied. \( \square \)

7. **Weak amenability does not imply 3-weak amenability**

Suppose that \( X_1 \) and \( X_2 \) are two Banach \( \mathfrak{A} \)-bimodules. We denote by \( X_1 \oplus X_2 \) the direct module sum of \( X_1 \) and \( X_2 \), i.e., the \( l_1 \) direct sum of \( X_1 \) and \( X_2 \) with the module actions given by \( a(x_1, x_2) = (ax_1, ax_2), \) \( (x_1, x_2)a = (x_1a, x_2a) \).

For this module we have the following equality:

\[
(x_1, x_2) \cdot (f_1^* , f_2^*) = x_1 f_1^* + x_2 f_2^* \quad ((x_1, x_2) \in X_1 \oplus X_2, \ (f_1^* , f_2^*) \in (X_1 \oplus X_2)^*) .
\]

In this section we shall first study the weak amenability of the Banach algebra \( \mathfrak{A} \oplus (X_1 \oplus X_2) \). Then we shall give an example of a weakly amenable Banach algebra of this form which is not 3-weakly amenable.

**Lemma 7.1.** Suppose that \( \mathfrak{A} \oplus X_1 \) and \( \mathfrak{A} \oplus X_2 \) are weakly amenable. Then the following are equivalent:

(i) \( \mathfrak{A} \oplus (X_1 \oplus X_2) \) is weakly amenable;

(ii) there is no nonzero, continuous \( \mathfrak{A} \)-bimodule morphism \( \gamma: X_1 \to X_2^* \);

(iii) there is no nonzero, continuous \( \mathfrak{A} \)-bimodule morphism \( \eta: X_2 \to X_1^* \).

**Proof.** Suppose that (i) holds. We show that (ii) also holds. Indeed, suppose that \( \gamma: X_1 \to X_2^* \) is a continuous \( \mathfrak{A} \)-bimodule morphism. Consider the continuous \( \mathfrak{A} \)-bimodule morphism \( T: X_1 \oplus X_2 \to (X_1 \oplus X_2)^* \) defined by

\[
T((x_1, x_2)) = (-\gamma^*(x_2), \gamma(x_1)), \quad (x_1, x_2) \in X_1 \oplus X_2.
\]

For \( (x_1, x_2), (y_1, y_2) \in X_1 \oplus X_2 \), and \( a \in \mathfrak{A} \), we have

\[
\langle a, (x_1, x_2) \cdot T((y_1, y_2)) \rangle = (a, -x_1 \gamma^*(y_2) + x_2 \gamma(y_1)) + (a, -\gamma^*(x_2)y_1 + \gamma(x_1)y_2)
\]

\[
= \langle a, -\gamma(x_1)y_2 + x_2 \gamma(y_1) \rangle + \langle a, -x_2 \gamma(y_1) + \gamma(x_1)y_2 \rangle = 0.
\]

So \( (x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) = 0 \). Then, from condition 4 of Theorem [21], \( T = 0 \). Thus \( \gamma = 0 \). As a consequence, (ii) holds.

To prove that (ii) implies (iii), we suppose that \( \eta: X_2 \to X_1^* \) is a continuous \( \mathfrak{A} \)-bimodule morphism. Then \( \gamma: X_1 \to X_2^* \) defined by \( \gamma = \eta^*|_{X_1} \) is a continuous \( \mathfrak{A} \)-bimodule morphism. Therefore, \( \gamma = 0 \). This implies that \( \eta^* = 0 \) since \( \eta^* \) is weak*-weak* continuous and \( X_1 \) is weak* dense in \( X_1^{**} \). Thus, \( \eta = 0 \), showing that (iii) holds. Similarly, one can prove that (iii) implies (ii).

Finally, we prove that (ii) + (iii) implies (i). Because \( \mathfrak{A} \oplus X_1 \) and \( \mathfrak{A} \oplus X_2 \) are weakly amenable, conditions 1–3 of Theorem [21] hold automatically for \( X = X_1 \oplus X_2 \) and \( m = 0 \). We show that condition 4 also holds. Suppose that \( T: X \to X^* \) is a continuous \( \mathfrak{A} \)-bimodule morphism satisfying

\[
(x_1, x_2) \cdot T((y_1, y_2)) + T((x_1, x_2)) \cdot (y_1, y_2) = 0 \quad ((x_1, x_2), (y_1, y_2) \in X).
\]
Let $P_1: X^* \to X_i^*$ be the natural projections and let $\tau_i: X_i \to X$ be the natural embeddings, $i = 1, 2$. Then, by taking $x_2 = y_2 = 0$ and $x_1 = y_1 = 0$ separately, we have

\[
x_1 \cdot P_1 \circ T \circ \tau_1(y_1) + P_1 \circ T \circ \tau_1(x_1) \cdot y_1 = 0,
\]
\[
x_2 \cdot P_2 \circ T \circ \tau_2(y_2) + P_2 \circ T \circ \tau_2(x_2) \cdot y_2 = 0,
\]

for all $x_i, y_i \in X_i$, $i = 1, 2$. So we have $P_i \circ T \circ \tau_i = 0$ by applying condition 4 of Theorem 2.1 to the weakly amenable Banach algebras $\mathfrak{A} \oplus X_i$, $i = 1, 2$. Furthermore, (ii) and (iii) imply that $P_1 \circ T \circ \tau_2: X_2 \to X_1^*$ and $P_2 \circ T \circ \tau_1: X_1 \to X_2^*$ are trivial. Therefore, we have $T = 0$. Condition 4 of Theorem 2.1 holds for $X = X_1 + X_2$. From Theorem 2.1, $\mathfrak{A} \oplus (X_1 + X_2)$ is weakly amenable. This completes the proof.

**Proposition 7.2.** The algebra $\mathfrak{A} \oplus (X_1 + X_2)$ is weakly amenable if and only if both $\mathfrak{A} \oplus X_1$ and $\mathfrak{A} \oplus X_2$ are weakly amenable and condition (ii) or condition (iii) in Lemma 7.1 holds.

**Proof.** If $\mathfrak{A} \oplus (X_1 + X_2)$ is weakly amenable, then conditions 1–4 of Theorem 2.1 hold for this algebra. It follows that these conditions also hold for the algebras $\mathfrak{A} \oplus X_1$ and $\mathfrak{A} \oplus X_2$. So the latter two are also weakly amenable. The rest has been given in Lemma 7.1.

In the remainder of the paper we focus on constructing an example of a weakly amenable Banach algebra which is not 3-weakly amenable. Recall that we always equip $\mathfrak{A}^{(2m)}$ with the first Arens product. The following lemma has been proved in [31].

**Lemma 7.3.** Suppose that $\mathfrak{A}$ is a left (right) ideal in $\mathfrak{A}^{**}$. Then it is also a left (respectively, right) ideal in $\mathfrak{A}^{(2m)}$ for all $m \geq 1$.

Suppose that $\mathfrak{B}$ is a Banach algebra and $\mathfrak{A} = \mathfrak{B}^{**}$. If $\mathfrak{B}$ is an ideal in $\mathfrak{B}^{**}$, then a natural way to make $\mathfrak{B}$ an $\mathfrak{A}$-bimodule is using (the first) Arens product to give the module actions. In this way $\mathfrak{B}^{**}$ is an $\mathfrak{A}^{**}$-bimodule. For $b \in \mathfrak{B} \subseteq \mathfrak{B}^{**}$ and $u \in \mathfrak{A}^{**}$, the module coupling $u \cdot b$ and $b \cdot u$ result in elements of $\mathfrak{B}^{**}$. Since $\mathfrak{B} \subseteq \mathfrak{B}^{(4)} (= \mathfrak{A}^{**})$, we can also consider the products $ub$ and $bu$ in the sense of Arens in $\mathfrak{B}^{(4)}$. But, from the above lemma, $ub, bu \in \mathfrak{B} \subseteq \mathfrak{B}^{**}$. It is routine to check that, as elements in $\mathfrak{B}^{**}$, $u \cdot b = ub$ and $b \cdot u = bu$.

From this point on, $\mathcal{H}$ will denote an infinite-dimensional, separable Hilbert space, $B(\mathcal{H})$ will denote the Banach algebra of all bounded operators on $\mathcal{H}$, and $K(\mathcal{H})$ the ideal of all compact operators on $\mathcal{H}$. It is well known that, with any Arens product, $K(\mathcal{H})^{**} = B(\mathcal{H})$ (see [27] p. 103 for details).

**Lemma 7.4.** There is an element $a_0 \in B(\mathcal{H})$ such that $a_0 \notin K(\mathcal{H})$, $a_0$ is not right invertible in $B(\mathcal{H})$ and $a_0 K(\mathcal{H})$ is dense in $K(\mathcal{H})$.

**Proof.** Let $(e_i)_{i=1}^\infty$ be an orthonormal basis of $\mathcal{H}$. Let $a_0 \in B(\mathcal{H})$ be defined by

\[
a_0(e_i) = \begin{cases}
  e_i & \text{if } i \text{ is even}; \\
  -e_i & \text{if } i \text{ is odd}.
\end{cases}
\]

Clearly, $a_0 \notin K(\mathcal{H})$. Also, $a_0$ is neither right nor left invertible because any one-sided inverse of $a_0$ must satisfy

\[
a_0^{-1}(e_i) = \begin{cases}
  ie_i & \text{if } i \text{ is even}; \\
  e_i & \text{if } i \text{ is odd}.
\end{cases}
\]
which cannot be a bounded operator.

For each $n \geq 1$, denote by $V_n$ the subspace of $\mathcal{H}$ generated by $\{e_1, e_2, \ldots, e_n\}$, and let $P_n$ be the orthogonal projection from $\mathcal{H}$ onto $V_n$. Then, from Corollary II.4.5 of [B], for every $k \in K(\mathcal{H})$ and $\varepsilon > 0$, there is $n = n(k, \varepsilon)$, such that $\|P_n \circ k - k\| < \varepsilon$.

For this $n = n(k, \varepsilon)$, let $b_n \in B(\mathcal{H})$ be defined by

$$b_n(e_i) = \begin{cases} ie_i & \text{if } i \leq n \text{ and } i \text{ is even;} \\ \varepsilon_i & \text{if } i \leq n \text{ and } i \text{ is odd;} \\ 0 & \text{for } i \geq n. \end{cases}$$

Then $a_0 \circ b_n = P_n$ and $a_0 \circ b_n \circ P_n = P_n^2 = P_n$. Let $k_n = b_n \circ P_n \circ k$. Then $k_n \in K(\mathcal{H})$, and $a_0 \circ k_n = P_n \circ k$. Also, $\|a_0 \circ k_n - k\| = \|P_n \circ k - k\| < \varepsilon$. Since $k \in K(\mathcal{H})$ and $\varepsilon \geq 0$ are arbitrarily given, this shows that $a_0 K(\mathcal{H})$ is dense in $K(\mathcal{H})$.

For the element $a_0$ in the above lemma, $a_0 B(\mathcal{H})$ is a proper right ideal of $B(\mathcal{H})$ since the identity $1 \notin a_0 B(\mathcal{H})$. The closure of $a_0 B(\mathcal{H})$ is also a proper right ideal of $B(\mathcal{H})$ (see p. 46). So there is $F \in B(\mathcal{H})^*$ such that $F \neq 0$ but $F a_0 = 0$. Then, $F B(\mathcal{H}) \neq \{0\}$ is a right $B(\mathcal{H})$-submodule of $B(\mathcal{H})^*$. Take

$$X_0 = (K(\mathcal{H}))_0, \quad \mathcal{Y} = \mathcal{O}(cl(F B(\mathcal{H}))).$$

Then we have the following example.

**Example 7.5.** $B(\mathcal{H}) \oplus (X_0 + \mathcal{Y})$ is weakly amenable but not 3-weakly amenable.

**Proof.** Clearly, we have $B(\mathcal{H}) X_0 = X_0$ and $\mathcal{Y} B(\mathcal{H}) = \mathcal{O}_2$. By Corollaries 6.2 and 6.3, the Banach algebras $B(\mathcal{H}) X_0$ and $B(\mathcal{H}) \oplus \mathcal{Y}$ are weakly amenable.

Suppose that $T: \mathcal{Y} \to X_0^0$ is a continuous $B(\mathcal{H})$-bimodule morphism. We prove that $T$ is trivial. Let $f = T(F)$. Then $f a_0 = T(F a_0) = 0$, and so $(a_0 K(\mathcal{H}), f) = \{0\}$. We then have $f = 0$ since $a_0 K(\mathcal{H})$ is dense in $K(\mathcal{H})$. This shows that $T(F) = 0$ and hence $T(F B(\mathcal{H})) = \{0\}$. Thus, $T = 0$. From Proposition 7.2, $B(\mathcal{H}) \oplus (X_0 + \mathcal{Y})$ is weakly amenable.

To prove that $B(\mathcal{H}) \oplus (X_0 + \mathcal{Y})$ is not 3-weakly amenable, we show that it fails condition 4 of Theorem 2.1 for $m = 1$. Since

$$(X_0)^{**} = \mathcal{O}(K(\mathcal{H})^*) = \mathcal{O}(B(\mathcal{H})^*) \supset \mathcal{Y},$$

there exists a nonzero $B(\mathcal{H})$-bimodule morphism from $\mathcal{Y}$ into $(X_0)^{**}$ (e.g., the inclusion mapping). Let $\tau: \mathcal{Y} \to (X_0)^{**}$ be such a morphism, and let $\Delta: (X_0)^{**} \to (X_0)^*$ be the projection with the kernel $X_0^1$. Take $T = \Delta \circ \tau: \mathcal{Y} \to X_0^0$. From the preceding paragraph, we have that $T = 0$.

So

$$\langle x, \tau(y) \rangle = \langle x, T(y) \rangle = 0 \quad (y \in \mathcal{Y}, x \in X_0).$$

Now let $\Gamma: X_0 + \mathcal{Y} \to (X_0 + \mathcal{Y})^{**}$ be the continuous $B(\mathcal{H})$-bimodule morphism defined by

$$\Gamma((x, y)) = (\tau(y), 0).$$

Then, for $(x_1, y_1), (x_2, y_2) \in X_0 + \mathcal{Y}$, and $u \in B(\mathcal{H})^{**}$, we have

$$\langle u, (x_1, y_1) \cdot \Gamma((x_2, y_2)) + \Gamma((x_1, y_1)) \cdot (x_2, y_2) \rangle = \langle u \cdot x_1, 0 \rangle, \quad \langle 0, y_2 \cdot u \rangle, \quad \tau(y_1), \quad 0 \rangle$$

$$= \langle u \cdot x_1, \tau(y_2) \rangle = (ux_1, \tau(y_2)) = 0.$$
Here we used the fact that \( u \cdot x_1 = ux_1 \in X_0 \) (see the paragraph following Lemma 7.3). So
\[
(x_1, y_1) \cdot \Gamma((x_2, y_2)) + \Gamma((x_1, y_1)) \cdot (x_2, y_2) = 0
\]
for all \((x_1, y_1), (x_2, y_2) \in X_0 + \partial Y\). But \( \Gamma \neq 0 \); so condition 4 of Theorem 2.1 does not hold for \( m = 1 \) and \( X = X_0 + \partial Y \). As a consequence, \( B(H) \oplus (X_0 + \partial Y) \) is not 3-weakly amenable.

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