DIFFERENTIAL OPERATORS
ON A POLARIZED ABELIAN VARIETY

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ABSTRACT. Let $L$ be an ample line bundle over a complex abelian variety $A$. We show that the space of all global sections over $A$ of $\text{Diff}^n_A(L, L)$ and $S^n(\text{Diff}^1_A(L, L))$ are both of dimension one. Using this it is shown that the moduli space $M_X$ of rank one holomorphic connections on a compact Riemann surface $X$ does not admit any nonconstant algebraic function. On the other hand, $M_X$ is biholomorphic to the moduli space of characters of $X$, which is an affine variety. So $M_X$ is algebraically distinct from the character variety if $X$ is of genus at least one.

1. Introduction

Let $X$ be a compact connected Riemann surface of genus at least one. A holomorphic connection on a holomorphic line bundle $L$ over $X$ is a first-order differential operator

$$D : L \rightarrow K_X \otimes L$$

satisfying the Leibniz rule, which says $D(fs) = fD(s) + \partial f \otimes s$, where $f$ is a locally defined holomorphic function and $s$ is a local holomorphic section of $L$. Let $M_X$ denote the moduli space of all rank one holomorphic connections on $X$. In other words, $M_X$ parametrizes isomorphism classes of pairs of the form $(L, D)$, where $D$ is a holomorphic connection on $L$. The space $M_X$ is a smooth quasi-projective variety of dimension $2g$, where $g$ is the genus of $X$.

Since any holomorphic connection on a Riemann surface is flat, the monodromy map identifies $M_X$ with the character variety $\mathcal{R} := \text{Hom}(\pi_1(X), \mathbb{C}^*)$. This identification is in fact a biholomorphism between $M_X$ and $\mathcal{R}$. We show that $\mathcal{R}$ is not algebraically isomorphic to $M_X$. More precisely, while $\mathcal{R}$ is an affine variety, $M_X$ does not have any nonconstant function (Theorem 3.2).

Let $A$ be a complex abelian variety and $L$ an ample line bundle over $A$. By $\text{Diff}^1_A(L, L)$ we denote the sheaf of differential operators of order one on $L$.

In Theorem 2.3 we prove that

$$\dim H^0(A, S^n(\text{Diff}^1_A(L, L))) = 1$$

for all $n \geq 1$. As a corollary we have (Corollary 2.10)

$$\dim H^0(A, \text{Diff}^n_A(L, L)) = 1.$$
In [11], using the method of the present paper, we prove a Torelli theorem for the moduli space of $\tau$-connections on a compact Riemann surface.

2. Differential operators and connections

Let $A$ be a complex abelian variety. Fix an ample line bundle $L$ over $A$.

For $n > 0$, let $\text{Diff}_A^n(L, L)$ denote the vector bundle over $A$ defined by the sheaf of differential operators of order $n$ on $L$. So,

$$\text{Diff}_A^n(L, L) = \text{Hom}_O(L, L) = O$$

and $\text{Diff}^{n-1}_A(L, L)$ is a subbundle of $\text{Diff}_A^n(L, L)$ in an obvious way. More precisely, there is an exact sequence

$$0 \rightarrow \text{Diff}^{n-1}_A(L, L) \rightarrow \text{Diff}_A^n(L, L) \xrightarrow{\sigma_n} S^n(TA) \rightarrow 0,$$

where $\sigma_n$ is the symbol homomorphism and $S^n(TA)$ is the $n$-th symmetric power of the tangent bundle. By convention, the 0-th symmetric power of a vector bundle is the trivial line bundle.

The $n$-th symmetric power of the homomorphism $\sigma_1$ in (2.1) gives an exact sequence

$$(2.2) \quad 0 \rightarrow S^{n-1}(\text{Diff}^1_A(L, L)) \rightarrow S^n(\text{Diff}^1_A(L, L)) \xrightarrow{\sigma_1^n} S^n(TA) \rightarrow 0,$$

of vector bundles. The vector bundle $S^{n-1}(\text{Diff}^1_A(L, L))$ is realized as a subbundle using the composition

$$S^{n-1}(\text{Diff}^1_A(L, L)) \xrightarrow{\alpha} S^{n-1}(\text{Diff}^1_A(L, L)) \otimes \text{Diff}^1_A(L, L) \xrightarrow{\beta} S^n(\text{Diff}^1_A(L, L)),$$

where $\alpha$ is defined using the inclusion of $O$ in $\text{Diff}^1_A(L, L)$ in the exact sequence (2.1) and $\beta$ is the symmetrization.

Theorem 2.3. For $n \geq 1$, the homomorphism

$$H^0(A, S^n(\text{Diff}^1_A(L, L))) = H^0(A, O) \rightarrow H^0(A, S^n(\text{Diff}^1_A(L, L)))$$

obtained using (2.2) repeatedly is an isomorphism.

Proof. Consider the long exact sequence of cohomologies

$$(2.4) \quad H^0(A, S^{n-1}(\text{Diff}^1_A(L, L))) \rightarrow H^0(A, S^n(\text{Diff}^1_A(L, L)))$$

obtained from (2.2). To prove the theorem, it suffices to show that the above homomorphism $h_n$ is injective for all $n \geq 1$. Indeed, if $h_n$ is injective, then the injective map

$$H^0(A, S^{n-1}(\text{Diff}^1_A(L, L))) \rightarrow H^0(A, S^n(\text{Diff}^1_A(L, L)))$$

is also surjective.

A connected homomorphism, like $h_n$ in (2.4), is the cup product by the extension class for the corresponding short exact sequence. So we need to understand the extension class

$$\overline{C}_n \in H^1(A, \text{Hom}(S^n(TA), S^{n-1}(\text{Diff}^1_A(L, L))))$$
for the exact sequence (2.2). Using the homomorphism $S^{n-1}(\sigma_1)$ in (2.2), the cohomology class $C_n$ gives

$$C_n \in H^1(A, \operatorname{Hom}(S^n(TA), S^{n-1}(TA))). \tag{2.5}$$

This cohomology class $C_n$ is clearly the extension class for the exact sequence

$$0 \longrightarrow S^{n-1}(\operatorname{Diff}_A^1(L, L))/S^{n-2}(\operatorname{Diff}_A^1(L, L)) = S^{n-1}(TA) \tag{2.6}$$

obtained from (2.2). Before we describe $C_n$, we need to identify the extension class for the exact sequence from which (2.2) is built, namely, the one obtained by setting $n = 1$ in (2.1).

The first step would be to show that the extension class for the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \operatorname{Diff}_A^1(L, L) \xrightarrow{\sigma_1} TA \longrightarrow 0 \tag{2.7}$$

is the first Chern class of $L$. Although this fact is well known, we give brief details of the argument.

We fix a convention. For our convenience, the first Chern class will always denote $2\pi\sqrt{-1}$ times the standard rational class. So, for example, $c_1(L) \in H^2(J, 2\pi\sqrt{-1}\mathbb{Z})$.

Let $\{U_i\}_{i \in I}$ be a covering of $A$ by analytic open sets and

$$\phi_i : L|_{U_i} \longrightarrow \mathcal{O}_{U_i}$$

be local trivializations of $L$. The composition $\phi_j \circ (\phi_i)^{-1}$ is a multiplication by a function on $U_i \cap U_j$. This function will be denoted by $\phi_{i,j}$.

Using $\phi_i$ and the differentiation action of $T U_i$ on $\mathcal{O}_{U_i}$, we have a splitting

$$\psi_i : T U_i \longrightarrow \operatorname{Diff}_i^1(L|_{U_i}, L|_{U_i})$$

of the symbol map. The difference $\psi_i - \psi_j$ on $U_i \cap U_j$ factors as a composition homomorphism

$$T(U_i \cap U_j) \xrightarrow{\gamma} \mathcal{O}_{U_i \cap U_j} \hookrightarrow \operatorname{Diff}_{U_i \cap U_j}^1(L|_{U_i \cap U_j}, L|_{U_i \cap U_j}),$$

and the one-form $\gamma$ on $U_i \cap U_j$ coincides with $d\phi_{i,j}/\phi_{i,j}$. Therefore, the one-cocycle $\{d\phi_{i,j}/\phi_{i,j}\}_{i,j \in I}$ represents the extension class in $H^1(A, \Omega^1_A)$ for the exact sequence (2.7). On the other hand, $\{d\phi_{i,j}/\phi_{i,j}\}$ represents the Chern class $c_1(L)$.

Since the exact sequence (2.2) is simply the $n$-th symmetric power of (2.7), the extension class $C_n$ in (2.5) is also $c_1(L)$. To explain this, first note that the cup product of $c_1(L) \in H^1(A, \Omega^1_A)$ with the identity automorphism of $S^n(TA)$ is a cohomology class

$$c \in H^1(A, \operatorname{Hom}(S^n(TA), \Omega^1_A \otimes S^n(TA))).$$

Using the contraction $\Omega^1_A \otimes S^n(TA) \longrightarrow S^{n-1}(TA)$, the cohomology class $c$ gives

$$C'_n \in H^1(A, \operatorname{Hom}(S^n(TA), S^{n-1}(TA))).$$

The extension class $C_n$ in (2.5) coincides with $C'_n$. Indeed, since the extension class for (2.7) is $c_1(L)$, this is an immediate consequence of the fact that (2.2) is the symmetric power of (2.7).

Take a translation invariant $(1,1)$-form $\omega$ on the abelian variety $A$ such that $\omega$ represents the first Chern class $c_1(L)$. It is easy to see that there is exactly one
such form. Since $L$ is ample, the form $\omega$ must be positive. In other words, the homomorphism

$$
(2.8) \quad \hat{\omega} : TA \rightarrow \Omega^0_{A,1}
$$

that sends any $v \in T_pA$ to the contraction of $\omega(p)$ with $v$ is an isomorphism.

Since $TA$ is trivial, any section of $S^n(TA)$ is invariant under translations in $A$. Take a nonzero section

$$
0 \neq \xi \in H^0(A, S^n(TA)).
$$

Using the contraction map $\hat{\omega}$ in (2.8), the section $\xi$ gives a $(0,1)$-form

$$
\xi \in \Omega^{0,1}(S^{n-1}(TA))
$$

with values in $S^{n-1}(TA)$. We noted earlier that $C_n$ in (2.5) coincides with $C'_n$. Therefore, the $S^{n-1}(TA)$-valued $(0,1)$-form $\xi$ represents the cohomology class

$$
S^{n-1}(\sigma_1) \circ h_n(\xi) \in H^1(A, S^{n-1}(TA))
$$

in Dolbeault cohomology, where $h_n$ is the connecting homomorphism in (2.4) and

$$
S^{n-1}(\sigma_1) : H^1(A, S^{n-1}(\text{Diff}_A^1(L, L))) \rightarrow H^1(A, S^{n-1}(TA))
$$

is the homomorphism obtained, in an obvious fashion, from $S^{n-1}(\sigma_1)$ in (2.2).

Since both $\omega$ and $\xi$ are invariant under the translations in $A$, the form $\xi$ is also invariant under the translations. Furthermore, since $\hat{\omega}$ in (2.8) is an isomorphism and $\xi \neq 0$, we have $\xi \neq 0$. From this it follows that the cohomology class in $H^1(A, S^{n-1}(TA))$ represented by $\xi$ is nonzero. To see this, note that $\omega$ being positive defines a Kähler structure on $A$. In order to prove that the cohomology class in $H^1(A, S^{n-1}(TA))$ represented by $\xi$ is nonzero, it suffices to show that the form $\xi$ is harmonic for the Dolbeault complex for $S^{n-1}(TA)$. However, since the Kähler form is translation invariant, $\xi$ being translation invariant must be harmonic.

We already noted that the Dolbeault cohomology class represented by $\xi$ coincides with $S^{n-1}(\sigma_1) \circ h_n(\xi)$. Since this class is nonzero, $h_n(\xi)$ must be nonzero. In other words, the homomorphism $h_n$ in (2.4) is injective. We noted earlier that the injectivity of $h_n$ proves the theorem. Therefore, the proof of the theorem is complete.

For $n \geq 1$, consider the exact sequence

$$
0 \rightarrow \text{Diff}_A^{n-1}(L, L)/\text{Diff}_A^{n-2}(L, L) = S^{n-1}(TA)
$$

(2.9)

$$
\rightarrow \text{Diff}_A^1(L, L)/\text{Diff}_A^0(L, L) \xrightarrow{\alpha_n} S^n(TA) \rightarrow 0
$$

obtained from (2.1), where $\text{Diff}_A^{n-1}(L, L)$ denotes 0. It is known that the exact sequence (2.8) is isomorphic to the exact sequence (2.6). Therefore, the injectivity of the homomorphism $h_n$ in (2.4) implies that the connecting homomorphism

$$
H^0(A, S^n(TA)) \rightarrow H^1(A, S^{n-1}(TA))
$$

in the long exact sequence of cohomologies for (2.9) is also injective. Consequently, the injective homomorphism

$$
H^0(A, \text{Diff}_A^{n-1}(L, L)) \rightarrow H^0(A, \text{Diff}_A^1(L, L))
$$

obtained from (2.1) is also surjective. Therefore, we have the following corollary of Theorem 2.3.
Corollary 2.10. The inclusion
\[ H^0(A, \mathcal{O}) \rightarrow H^0(A, \text{Diff}^1_A(L, L)) \]
onobtained from (2.1) is an isomorphism for all \( n \geq 0 \).

Consider the exact sequence
\[ (2.11) \quad 0 \rightarrow \Omega^1_A \rightarrow \text{Diff}^1_A(L, L)^* \rightarrow \mathcal{O} \rightarrow 0, \]
which is the dual of (2.7). We will denote by \( \Gamma \) the image of the section of \( \mathcal{O} \) defined by the constant function 1. The subset of the total space of the vector bundle \( \text{Diff}^1_A(L, L)^* \) defined by the inverse image \( \tau^{-1}(\Gamma) \) will be denoted by \( \mathcal{C}(L) \).

Let
\[ (2.12) \quad p : \mathcal{C}(L) \rightarrow A \]
be the obvious projection. The exact sequence (2.11) shows that for any point \( x \in A \), the inverse image \( p^{-1}(x) \) is an affine space for the holomorphic cotangent space \( (\Omega^1_A)_x \).

Let \( U \subset A \) be an open subset and \( \theta \) a holomorphic section over \( U \) of the fiber bundle \( \mathcal{C}(L) \). Such a section \( \theta \) defines a holomorphic connection on \( L|_U \) \([1]\). The exact sequence (2.7) for a holomorphic line bundle over a complex manifold is known as the Atiyah exact sequence. A splitting of the Atiyah exact sequence is a holomorphic connection \([1]\). A section \( \theta \) of \( \mathcal{C}(L) \) over \( U \) clearly gives a splitting over \( U \) of the exact sequence (2.7).

The subset \( \mathcal{C}(L) \subset \text{Diff}^1_A(L, L)^* \) being a Zariski open set has a natural algebraic structure. By \( \mathcal{O}_{\mathcal{C}(L)} \) we will denote the structure sheaf of this algebraic variety.

Proposition 2.13. For the variety \( \mathcal{C}(L) \),
\[ H^0(\mathcal{C}(L), \mathcal{O}_{\mathcal{C}(L)}) = \mathbb{C} \]
or, in other words, there is no nonconstant algebraic function on \( \mathcal{C}(L) \).

Proof. Let \( P := P\text{Diff}^1_A(L, L)^* \) be the projective bundle over \( A \) consisting of lines in \( \text{Diff}^1_A(L, L)^* \). Similarly, \( P' := P\Omega^1_A \) denotes the projective bundle defined by the lines in \( \Omega^1_A \). Using the inclusion of \( \Omega^1_A \) in \( \text{Diff}^1_A(L, L)^* \) in (2.11), we have \( P' \) as a subbundle of the projective bundle \( P \). Let
\[ P_0 := P - P' \]
be the complement. It is easy to see that \( P_0 \) is naturally identified with \( \mathcal{C}(L) \). The identification is defined by the obvious projection to \( P \) of the complement of the zero section in \( \text{Diff}^1_A(L, L)^* \).

Since the quotient bundle \( \text{Diff}^1_A(L, L)^*/\Omega^1_A \) is trivial, the divisor \( P' \) on \( P \) is the divisor of the tautological line bundle \( \mathcal{O}_P(1) \) over \( P \). So a meromorphic function on \( P \) with pole of order \( d \) along \( P' \) is a section of \( \mathcal{O}_P(d) \). Therefore, it suffices to prove that
\[ \dim H^0(P, \mathcal{O}_P(d)) = 1 \]
for all \( d \geq 0 \).

Let \( \gamma \) denote the projection of \( P \) to \( A \). Taking direct image to \( A \), we have the identification
\[ H^0(P, \mathcal{O}_P(d)) = H^0(A, \gamma_* \mathcal{O}_P(d)) = H^0(A, S^d(\text{Diff}^1_A(L, L))). \]
Now Theorem 2.3 implies that \( \dim H^0(P, \mathcal{O}_P(d)) = 1 \) for \( d \geq 0 \). This completes the proof of the proposition.
In the next section we will specialize to Jacobians of curves.

3. Rank one connections on a curve

Let $X$ be a connected smooth projective curve over $\mathbb{C}$ or, equivalently, a compact connected Riemann surface. The genus $g$ of $X$ is assumed to be positive. Fix once and for all a point $x_0 \in X$. Let $J := \text{Pic}^0(X)$ be the Jacobian of $X$. We will denote by $\Theta$ the line bundle over $J$ defined by the divisor that consists of all $L$ with

$$H^0(X, \mathcal{O}_X((g-1)x_0) \otimes L) \neq 0.$$ 

It is known that $\Theta$ is ample. More precisely, it defines a principal polarization on $J$.

Let $M_X$ denote the moduli space of rank one holomorphic connections on $X$. In other words, $M_X$ parametrizes pairs of the form $(L, D)$, where $L$ is a holomorphic line bundle over $X$ and $D$ is a holomorphic connection on $L$. Since $\dim X = 1$, any holomorphic connection on $X$ is flat. The moduli space of holomorphic connections on a smooth projective variety has been constructed in [5]. In particular, $M_X$ is a quasi-projective variety.

Let $\phi : M_X \to J$ be the forgetful morphism. So $\phi$ sends a pair $(L, D)$ to $L$.

Let $\mathcal{R}$ denote the character variety $\text{Hom}(\pi_1(X), \mathbb{C}^*)$ of the fundamental group. If we fix generators of the fundamental group $\pi_1(X)$, then $\mathcal{R}$ gets identified with the $2g$-fold self-product $\mathbb{C}^* \times \mathbb{C}^* \times \cdots \times \mathbb{C}^*$. By associating its monodromy to a flat connection, the space $\mathcal{R}$ gets identified with $M_X$. More precisely, for $D \in M_X$, this identification associates $D$ with the element in $\mathcal{R}$ that sends any $g \in \pi_1(X)$ to the holonomy of $D$ around $g$. This identification of $M_X$ with $\mathcal{R}$ is biholomorphic but not necessarily algebraic [5]. In fact, we will see that $M_X$ is not algebraically isomorphic to $\mathcal{R}$.

Since $\mathcal{R}$ is a product of copies of $\mathbb{C}^*$, it is an affine variety. In particular, there are many nonconstant functions on $\mathcal{R}$. In view of that, the following theorem shows that $M_X$ is not algebraically isomorphic to $\mathcal{R}$.

**Theorem 3.2.** For the variety $M_X$,

$$\dim H^0(M_X, \mathcal{O}_{M_X}) = 1,$$

where $\mathcal{O}_{M_X}$ denotes the structure sheaf.

**Proof.** Set the pair $(A, L)$ is Section 2 to be $(J, \Theta)$. Consider the fiber bundle

$$p : \mathcal{C}(\Theta) \to J$$

constructed in (2.12). In view of Proposition 2.13, the theorem follows immediately from the following proposition.

**Proposition 3.3.** The fiber bundle $\mathcal{C}(\Theta)$ over $J$ defined by $p$ is algebraically isomorphic to $M_X$ defined in (3.1).

**Proof.** We already remarked that $\mathcal{C}(\Theta)$ is an affine bundle over $J$ for the cotangent bundle, that is, any fiber of $p$ is an affine space for the cotangent space at that point. Now note that $M_X$ is also an affine bundle for the cotangent bundle. Indeed, the space of holomorphic connections on a degree zero line bundle over $X$ is an affine
space for $H^0(X, K_X)$, where $K_X$ denotes the holomorphic cotangent bundle. On the other hand, $H^0(X, K_X)$ are the fibers $\Omega^1_J$.

Affine bundles for the cotangent bundle are classified by $H^1(J, \Omega^1_J)$. We will quickly recall how a cohomology class is associated to an affine bundle.

Let $q : Z \rightarrow J$ be an affine bundle for $\Omega^1_J$. Let $\{U_i\}_{i \in I}$ be a covering of $J$ by analytic open subsets and

$$(3.4) \quad \psi_i : U_i \rightarrow Z|_{U_i}$$

holomorphic sections. Since the fibers of $Z$ are affine spaces, $\psi_j - \psi_i$ is a holomorphic section of $\Omega^1_{U_i \cap U_j}$. These one-forms $\{\psi_j - \psi_i\}_{i,j \in I}$ define a cocycle. Let $\beta_Z \in H^1(J, \Omega^1_J)$ be the corresponding cohomology class. It is easy to see that another affine bundle $Z'$ will be holomorphically isomorphic to $Z$ if $\beta_Z$ coincides with the corresponding cohomology class $\beta_{Z'}$ for $Z'$. If these two affine bundles are analytically isomorphic, then from the GAGA principle of [4], it follows that they must be algebraically isomorphic.

If $\beta_Z \neq 0$ and $\beta_{Z'} = \lambda \beta_Z$, where $\lambda \in \mathbb{C}^*$, then also the two fiber bundles $Z$ and $Z'$ are algebraically isomorphic. However, if $\lambda \neq 1$, then there will be no isomorphism preserving the affine space structures. Nevertheless, there will be an isomorphism $h : Z' \rightarrow Z$ of fiber bundles satisfying the identity $h(z + \theta) = h(z) + \lambda \theta$, where $\theta \in \Omega^1_J$.

Let $\beta_p$ (respectively, $\beta_q$) be the cohomology class in $H^1(J, \Omega^1_J)$ associated to $\mathcal{C}(\Theta)$ (respectively, $M_X$). We will show that both $\beta_p$ and $2\beta_q$ coincide with $c_1(\Theta)$.

In the proof of Theorem 2.3, we have seen that the extension class for the Atiyah exact sequence (2.7) for $\Theta$ coincides with $c_1(\Theta)$. We already noted that any section $\psi : U \rightarrow \mathcal{C}(\Theta)|_U$ as in (3.4) gives a splitting over $U$ of the Atiyah exact sequence for $\Theta$. Consequently, $\beta_p$ coincides with $c_1(\Theta)$.

Let

$$f : J \rightarrow M_X$$

be a $C^\infty$ section of the map $\phi$ in (3.1). The obstruction to the holomorphicity of the map $f$ gives a form $\omega_f$ on $J$ of type $(1,1)$. This form $\omega_f$ can be described as follows. For any point $z \in J$, let

$$df(z) : T^R_z J \rightarrow T^R_{f(z)} M_X$$

be the homomorphism of real tangent spaces given by the differential of $f$. Let

$$J_z : T^R_z J \rightarrow T^R_z J$$

be the almost complex structure of $J$ at $z$. Similarly, the almost complex structure of $M_X$ at $f(z)$ will be denoted by $J_{f(z)}$. Now, for any $v \in T^R_z J$, the difference

$$J_{f(z)} \circ df(z)(v) - df(z) \circ J_z(v)$$

is an element of $(T^R_z J)^*$. Indeed, this is an immediate consequence of the fact that the kernel of the differential homomorphism

$$d\phi(f(z)) : T^R_{f(z)} M_X \rightarrow T^R_z J$$

is identified with $(T^R_z J)^*$ using the affine space structure of the fibers of $\phi$. The resulting homomorphism $T^R_z J \rightarrow (T^R_z J)^*$ that sends any $v$ to $J_{f(z)} \circ df(z)(v) - df(z) \circ J_z(v)$ defines the $(1,1)$-form $\omega_f$.

The cohomology class in $H^1(J, \Omega^1_J)$ represented by $\omega_f$ coincides with $\beta_p$. In fact, this is the Dolbeault analog of the earlier construction of the cohomology class $\beta_Z$. 
Any holomorphic line bundle over $X$ of degree zero admits a unique unitary flat connection. Let $f$ be the map that associates to any $L$ in $J$ the unitary flat connection on $L$. From \cite{2} Theorem 2.11 we know that $2\omega_f$ coincides with the pullback, using $f$, of a certain natural symplectic form on $M_X$. The symplectic form on $M_X$ in question is the one defined in \cite{3} on the representation space $R$. On the other hand, the pullback of this symplectic form coincides with $c_1(\Theta)$. This is well known; the details can be found in \cite{2}.

Therefore, both $\beta_f$ and $2\beta_f$ coincide with $c_1(\Theta)$. This completes the proof of the proposition.

We already noted that Proposition 3.3 completes the proof of Theorem 3.2. Therefore, the proof of Theorem 3.2 is complete.

Let $Y$ be another compact connected Riemann surface. Let $M_Y$ denote the moduli space of rank one holomorphic connections on $Y$. Let $\Omega(X)$ (respectively, $\Omega(Y)$) denote the natural symplectic form on $M_X$ (respectively, $M_Y$) constructed in \cite{3}.

**Proposition 3.5.** If there is an algebraic isomorphism of $M_X$ with $M_Y$ that takes the symplectic form $\Omega(X)$ to $\Omega(Y)$, then the Riemann surface $X$ is isomorphic to $Y$.

*Proof.* The Torelli theorem says that if the Jacobian of $X$ is isomorphic to the Jacobian of $Y$ as a principally polarized abelian variety, then $X$ is isomorphic to $Y$. The principal polarization in question is the one given by theta. The proposition will be proved by recovering the Jacobian of $X$, along with its polarization, from the symplectic variety $(M_X, \Omega(X))$.

Let $\phi_Y : M_Y \rightarrow \text{Pic}^0(Y)$ be the projection defined in (3.1) for $Y$.

There is no nonconstant algebraic map from the affine line to an abelian variety. This is an immediate consequence of the fact that there is no nonzero holomorphic one-form on the projective line. Therefore, any algebraic isomorphism $\psi : M_X \rightarrow M_Y$ induces an isomorphism $\overline{\psi} : \text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$ which is determined by the identity $\phi_Y \circ \psi = \overline{\psi} \circ \phi$, where $\phi$ is as in (3.1). Consequently, $M_X$ determines both $\text{Pic}^0(X)$ and the projection $\phi$.

Take a $C^\infty$ section $f : \text{Pic}^0(X) \rightarrow M_X$ (as in the proof of Proposition 3.3) of the projection $\psi$. As in the proof of Proposition 3.3, let $\omega_f$ denote the $(1,1)$-form on $\text{Pic}^0(X)$ given by the obstruction to the holomorphicity of $f$. If $f_0 : \text{Pic}^0(X) \rightarrow M_X$ is another section of $\psi$, then it is easy to check that

$\omega_f - \omega_{f_0} = \overline{\partial}(f - f_0).$

Note that using the affine bundle structure of $M_X$, the difference $f - f_0$ defines a $(1,0)$-form on $\text{Pic}^0(X)$.

Set $f_0$ to be the section that sends any line bundle $L$ in $\text{Pic}^0(X)$ to the (unique) unitary flat connection on $L$. In the proof of Proposition 3.3 we saw that $\omega_{f_0}$...
represents $c_1(\Theta)/2$. Therefore, the identity (3.6) implies that the cohomology class in $H^1(\text{Pic}^0(X), \Omega^1_{\text{Pic}^0(X)})$ represented by the form $2\omega_f$ coincides with $c_1(\Theta)$.

Therefore, the algebraic variety $M_X$ equipped with the symplectic form $\Omega(X)$ determines the principally polarized abelian variety $(\text{Pic}^0(X), c_1(\Theta))$. This completes the proof of the proposition.

Since only the cohomology class represented by the symplectic form is used, $X$ is isomorphic to $Y$ if there is an isomorphism of $M_X$ with $M_Y$ that takes the cohomology class for the symplectic form $\Omega(Y)$ to that for $\Omega(X)$.

References


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