TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 11, Pages 4681–4702
S 0002-9947(02)03020-9
Article electronically published on May 7, 2002

SCALING COUPLING OF REFLECTING BROWNIAN MOTIONS AND THE HOT SPOTS PROBLEM

MIHAI N. PASCU

Abstract. We introduce a new type of coupling of reflecting Brownian motions in smooth planar domains, called *scaling coupling*. We apply this to obtain monotonicity properties of antisymmetric second Neumann eigenfunctions of convex planar domains with one line of symmetry. In particular, this gives the proof of the hot spots conjecture for some known types of domains and some new ones.

1. Introduction

The technique of coupling of diffusions is a useful tool for various estimates in probability and analysis. In particular, coupling of reflecting Brownian motions can be used for obtaining various estimates related to eigenfunctions and eigenvalues of the Laplacian in smooth bounded domains.

There are two main known types of coupling of reflecting planar Brownian motions in the literature: synchronous and mirror couplings (see [BK] for a discussion of these couplings). We introduce a new type of coupling of reflecting planar Brownian motions in smooth $C^{1,\alpha}$ domains, called *scaling coupling*. We first give the construction in the case of the unit disk; we show that given a reflecting Brownian motion Z_t in the unit disk U, starting at $z_0 \in \overline{U} - \{0\}$, the formula

$$\frac{1}{\sup\limits_{s \le t} |Z_s|} Z_s$$

defines a time change of a reflecting Brownian motion $\widetilde{Z}_t$ in U, starting at $\widetilde{z}_0 = \frac{1}{|z_0|} z_0 \in \partial U$. We define $(Z, \widetilde{Z})$ as a coupling of reflecting Brownian motions starting at z_0 and $\widetilde{z}_0 = \frac{1}{|z_0|} z_0$, respectively. By means of automorphisms of the unit disk, the construction is then extended to any pair of starting points $z_0, \widetilde{z}_0 \in \overline{U}$, not both on the boundary of U. The coupling is uniquely defined by the choice of an additional point lying on the hyperbolic line in U determined by z_0 and $\widetilde{z}_0$, not separating them, which can be viewed as the parameter of the coupling.

Finally, the construction is extended to $C^{1,\alpha}$ domains ($0 < \alpha < 1$) by means of conformal maps. The restriction to the class of $C^{1,\alpha}$ domains is necessary in order to insure the conformality at the boundary of U of a mapping from the unit disk

Received by the editors September 9, 2001 and, in revised form, January 12, 2002.
2000 *Mathematics Subject Classification.* Primary 60J65; Secondary 60J45, 35B05, 35K05.
Key words and phrases. Coupling of diffusions, reflecting Brownian motion, hot spots conjecture, eigenfunctions, Neumann problem, Laplacian.
I would like to thank Richard Bass, Krzysztof Burdzy, Nicolae N. Pascu and Nicolae R. Pascu for several helpful discussions.

onto a $C^{1,\alpha}$ domain, hence to insure that the image of a reflecting Brownian motion in U under a conformal map is a reflecting Brownian motion in the image domain.

Scaling coupling of reflecting Brownian motions in $C^{1,\alpha}$ domains is a coupling in the following generalized sense: there exist a.s. finite stopping times τ and $\widetilde{\tau}$ (with respect to the filtration $(\mathcal{F}_t^Z)$, respectively $(\mathcal{F}_t^{\widetilde{Z}})$) such that

$$Z_{t+\tau} = \widetilde{Z}_{t+\widetilde{\tau}}$$

for all $t \geq 0$. The usual coupling of diffusions can be viewed as a particular case of the above, namely the case when $\tau = \widetilde{\tau}$ a.s. and $\mathcal{F}_t^Z = \mathcal{F}_t^{\widetilde{Z}}$ for all $t \geq 0$.

We show that in the case of convex domains, we can choose the coupling so that we have $\tau \leq \widetilde{\tau}$ a.s., which shows that for $t \geq \widetilde{\tau}$, $\widetilde{Z}_t$ "follows" the path of $Z_{t+\tau-\widetilde{\tau}}$. Moreover, we show that the inequality $\tau \leq \widetilde{\tau}$ is characteristic to the class of convex domains.

The above property of scaling coupling is used to prove strict monotonicity properties of antisymmetric second Neumann eigenfunctions of the Laplacian of a convex $C^{1,\alpha}$ domain D $(0 < \alpha < 1)$ having a line of symmetry, along the family of hyperbolic lines in D which intersect this line of symmetry.

The proof uses the expansion of the function

$$u(t,x) = P(\tau^x > t)$$

in terms of the mixed Dirichlet-Neumann eigenfunctions for the Laplacian on $D^+ = D \cap \{(x,y) : y > 0\}$, with Dirichlet conditions on the part of ∂D^+ lying on the horizontal axis and Neumann conditions on the remaining part of the boundary of D^+ (τ^x denotes the lifetime of reflecting Brownian motion in D^+ starting at $x \in \overline{D^+}$, killed on hitting the horizontal axis).

Using the properties of the scaling coupling, we show a monotonicity property of τ^x (Proposition 3.3) along a certain family of curves (hyperbolic lines in D, defined as conformal images of diameters in the unit disk), which gives the monotonicity of $u(t,x) = P(\tau^x > t)$ as a function of x on the indicated family of curves in D^+.

Using the fact that an antisymmetric second Neumann eigenfunction φ for D is a first mixed Dirichlet-Neumann eigenfunction for D^+ (the nodal domain of φ, since the eigenfunction is assumed to be antisymmetric with respect to the horizontal axis), and using the eigenfunction expansion of $u(t,x)$, the monotonicity of $u(t,x)$ translates into the monotonicity of the second Neumann eigenfunction φ for D, along the same family of curves. In particular, this shows that an antisymmetric second Neumann eigenfunction of a convex $C^{1,\alpha}$ domain attains its maximum only at the boundary of the domain.

The main result of our paper is the following:

Theorem 1.1. *Let D be a convex $C^{1,\alpha}$ domain $(0 < \alpha < 1)$ which is symmetric with respect to the horizontal axis. If φ is a second Neumann eigenfunction for D which is antisymmetric with respect to the horizontal axis, then φ is* ***strictly*** *monotone on the family of hyperbolic lines in D which intersect nontrivially the horizontal axis.*

In particular, φ must attain its maximum and minimum over D solely at the boundary of D.

In our particular setting, this is exactly the object of the hot spots conjecture (due to Jeffrey Rauch, 1974), which states that a second Neumann eigenfunction of a bounded simply connected domain (later modified to bounded convex domain) satisfies a strong maximum/minimum principle, that is, it attains its maximum and minimum over the domain only at the boundary of the domain.

The structure of the paper is the following: in Section 2.1 we review some basic facts from complex analysis needed in the paper, and we introduce our notation.

Next we construct the scaling coupling of reflecting Brownian motions, first for the case of the unit disk (Section 2.2), and then we generalize it to smooth bounded convex domains. We believe this construction is of independent interest. The construction and the properties of the scaling coupling are detailed in Section 2.3.

In Section 3.1 we give some basic results about eigenfunctions and eigenvalues needed in our paper, and we give the formal statement of the hot spots conjecture.

In Section 3.2 we introduce the notion of hyperbolic lines in smooth domains, and we apply the properties of scaling coupling to obtain the strict monotonicity properties of antisymmetric second Neumann eigenfunctions for smooth convex domains with one axis of symmetry along this family of curves.

In Section 3.3 we identify classes of domains for which our main theorem in Section 3.2 applies, and we compare our results with recent advances in the proof of the hot spots conjecture. We show that our main result applies to convex domains with two axes of symmetry, obtaining the known results of Bañuelos and Burdzy [BB] and Jerison and Nadirashvili [JN]. We show that for convex domains with just one axis of symmetry having diameter-to-width ratio larger than 3.06, the second Neumann eigenspace is 1-dimensional and it is given by an antisymmetric eigenfunction. This shows that our main result in Theorem 3.6 applies, and therefore we obtain the proof of the hot spots conjecture for a new class of domains.

The techniques developed in the present paper can be easily modified to give the proof of the hot spots conjecture for second Neumann eigenfunctions of new classes of domains. In Section 3.4 we discuss some developments, showing how the hypotheses used in this paper (i.e., the symmetry and the convexity of the domain) can be weakened in order to obtain new results. It can be shown that the hot spots conjecture holds for second Neumann eigenfunction of smooth convex domains for which the nodal line is a hyperbolic line (Theorem 3.14), and for certain types of second Neumann eigenfunctions of some doubly connected domains (Theorem 3.16).

2. Scaling coupling of reflecting Brownian motions in $C^{1,\alpha}$ domains

2.1. **Preliminaries.** We denote the unit disk in $\mathbb{R}^2$ by $U = \{z : |z| < 1\}$.

A curve $\Gamma \subset \mathbb{R}^2$ is said to be of class $C^{1,\alpha}$ ($0 < \alpha < 1$) if it has a parametrization $w(t)$ that is continuously differentiable, $w' \neq 0$ and w' is Lipschitz of order α; that is, for some $M > 0$ and for all t, t' we have

$$|w'(t) - w'(t')| \leq M\, |t - t'|^{\alpha} .$$

A domain $D \subset \mathbb{R}^2$ is said to be a $C^{1,\alpha}$ domain $(0 < \alpha < 1)$ if its boundary is a Jordan curve Γ of class $C^{1,\alpha}$.

It is known ([P], pp. 48–49) that if $f : U \to D$ is a conformal map of the unit disk onto the $C^{1,\alpha}$ domain D $(0 < \alpha < 1)$, then f and f' have continuous extensions to $\overline{U}$, f' is Lipschitz of order α on $\overline{U}$, and $f' \neq 0$ on $\overline{U}$.

We recall that an analytic function f is called convex in U if it maps U conformally onto a convex domain.

For a $C^{1,\alpha}$ domain D $(0 < \alpha < 1)$ we denote by ν_D the inward unit normal vector field on ∂D, the boundary of D.

By reflecting Brownian motion in D we mean reflecting Brownian motion with respect to the normal vector field ν_D.

Consider an arbitrarily fixed probability space $(\Omega, \mathcal{F}, P)$. Whenever a (reflecting) Brownian motion B_t on $(\Omega, \mathcal{F}, P)$ is considered, we denote by $(\mathcal{F}_t^B)$ the filtration (satisfying the usual conditions) with respect to which B_t is adapted.

We define reflecting Brownian motion in D as a solution of the stochastic differential equation

$$X_t = X_0 + B_t + \frac{1}{2}\int_0^t \nu_D(X_s)\,dL_s, \qquad t \geq 0. \tag{2.1}$$

Formally, we have

Definition 2.1. X_t is a reflecting Brownian motion in D starting at $x_0 \in \overline{D}$ if it satisfies (2.1), where:

(a) B_t is a 2-dimensional Brownian motion started at 0,
(b) L_t is a continuous nondecreasing process which increases only when $X_t \in \partial D$,
(c) X_t is $(\mathcal{F}_t^B)$-adapted, and almost surely $X_0 = x_0$ and $X_t \in \overline{D}$ for all $t \geq 0$.

Remark 2.2. Bass and Hsu [BaH2] showed that in $C^{1,\alpha}$ domains there exists exactly one solution to (2.1) for a given Brownian motion B_t. Also, by [BaH1] the process X_t corresponds to the Dirichlet form $\mathcal{E}(f,f) = \frac{1}{2}\int_{\overline{D}} |\nabla f|^2$.

2.2. **Scaling coupling of reflecting Brownian motions in the unit disk.** We will first construct the scaling coupling in the case of the unit disk, and then extend it to bounded $C^{1,\alpha}$ domains by means of conformal maps.

The key to our construction is the following result:

Theorem 2.3. *Let Z_t be a reflecting Brownian motion in U starting at $z_0 \in \overline{U} - \{0\}$ and let $|z_0| \leq a \leq 1$ be arbitrarily fixed. The process $\tilde{Z}_t$, defined by*

$$\tilde{Z}_t = \frac{1}{M_{\gamma_t}} Z_{\gamma_t}, \qquad t \geq 0, \tag{2.2}$$

where

$$M_t = a \vee \sup_{s \leq t} |Z_s|,$$

$$C_t = \int_0^t \frac{1}{M_s^2}\,ds, \tag{2.3}$$

$$\gamma_t = \inf\{s > 0 : C_s \geq t\}, \tag{2.4}$$

is an $(\mathcal{F}_{\gamma_t}^Z)$-adapted reflecting Brownian motion in U starting at $\widetilde{z}_0 = \frac{z_0}{a}$.

Proof. We apply Itô's formula to the semimartingale Z_t and the nondecreasing process M_t with $f(x,y) = \frac{x}{y}$. If

$$Z_t = Z_0 + B_t + \frac{1}{2}\int_0^t \nu_U(Z_s)\,dL_s$$

is the semimartingale representation of Z_t given by Definition 2.1, we have

$$\begin{aligned}\frac{Z_t}{M_t} &= \frac{Z_0}{M_0} + \int_0^t \frac{1}{M_s} dZ_s - \int_0^t \frac{1}{M_s^2} Z_s dM_s \\ &= \frac{Z_0}{M_0} + \int_0^t \frac{1}{M_s} dB_s + \frac{1}{2}\int_0^t \frac{1}{M_s}\nu_U(Z_s) dL_s - \int_0^t \frac{1}{M_s^2} Z_s dM_s.\end{aligned} \tag{2.5}$$

If $\tau = \inf\{s : |Z_s| = 1\}$, note that L_s is constant on $[0, \tau]$ and $M_s \equiv 1$ on $[\tau, \infty)$; further, when M_s is increasing, $\frac{Z_s}{M_s}$ is on ∂U, and therefore $-\frac{Z_s}{M_s} = \nu_U(\frac{Z_s}{M_s})$. The difference of the last two integrals in the last equality above can thus be written

$$\begin{aligned}&\frac{1}{2}\int_{t\wedge\tau}^t \frac{1}{M_s}\nu_U(Z_s) dL_s - \int_0^{t\wedge\tau} \frac{1}{M_s^2} Z_s dM_s \\ &= \frac{1}{2}\int_{t\wedge\tau}^t \nu_U(\frac{Z_s}{M_s}) dL_s + \frac{1}{2}\int_0^{t\wedge\tau} \nu_U(\frac{Z_s}{M_s}) d\log M_s^2 \\ &= \frac{1}{2}\int_0^t \nu_U(\frac{Z_s}{M_s}) dL_s + \frac{1}{2}\int_0^t \nu_U(\frac{Z_s}{M_s}) d\log M_s^2 \\ &= \frac{1}{2}\int_0^t \nu_U(\frac{Z_s}{M_s}) d\tilde{L}_s,\end{aligned}$$

where $\tilde{L}_s = L_s + \log M_s^2$ is readily seen to be a nondecreasing process which increases only if either L_s or M_s does; this happens only when $\frac{Z_s}{M_s}$ is on the boundary of U.

Note that since $0 < |z_0| \le M_t \le 1$ for all $t \ge 0$, C_t defined by (2.3) is strictly increasing and $C_t \to \infty$ a.s. It follows that γ_t defined by (2.4) is continuous and increasing, and on substituting in (2.5) we obtain

$$\tilde{Z}_t = \frac{Z_{\gamma_t}}{M_{\gamma_t}} = \frac{Z_0}{M_0} + \int_0^{\gamma_t} \frac{1}{M_s} dB_s + \frac{1}{2}\int_0^{\gamma_t} \nu_U(\frac{Z_s}{M_s}) d\tilde{L}_s.$$

Setting $\widetilde{B}_t = \int_0^{\gamma_t} \frac{1}{M_s} dB_s$, we obtain

$$\begin{aligned}\langle \widetilde{B}^i, \widetilde{B}^j \rangle_t &= \int_0^{\gamma_t} \frac{1}{M_s^2} d\langle \widetilde{B}^i, \widetilde{B}^j \rangle_s \\ &= \delta_{ij} \int_0^{\gamma_t} \frac{1}{M_s^2} ds \\ &= \delta_{ij} C_{\gamma_t} \\ &= \delta_{ij} t,\end{aligned}$$

$i, j \in \{1, 2\}$, and thus $\widetilde{B}_t$ is a 2-dimensional Brownian motion starting at 0.

We have thus shown that

$$\begin{aligned}\tilde{Z}_t &= \tilde{Z}_0 + \widetilde{B}_t + \frac{1}{2}\int_0^{\gamma_t} \nu_D(\frac{Z_s}{M_s})d\tilde{L}_s \\ &= \tilde{Z}_0 + \widetilde{B}_t + \frac{1}{2}\int_0^{t} \nu_D(\frac{Z_{\gamma_u}}{M_{\gamma_u}})d\tilde{L}_{\gamma_u} \\ &= \tilde{Z}_0 + \widetilde{B}_t + \frac{1}{2}\int_0^{t} \nu_D(\tilde{Z}_u)d\,\bar{L}_u,\end{aligned}$$

where $\overline{L}_u = \tilde{L}_{\gamma_u}$ is a nondecreasing process which increases only when $\tilde{Z}_u = \frac{Z_{\gamma_u}}{M_{\gamma_u}}$ is at the boundary of U. This proves the theorem. □

Remark 2.4. In $\mathbb{R}$, essentially the same proof shows that if Z_t is a 1-dimensional Brownian motion starting at $z_0 > 0$, and $M_t = a \vee \sup_{s\le t} Z_s$ $(a \geq z_0)$, then $\frac{1}{M_t}Z_t$ is a time change of a reflecting Brownian motion on $(-\infty, 1]$, the time change being given by (2.3)–(2.4) above.

The above construction also applies to higher dimensions, to give a scaling coupling of reflecting Brownian motions in the unit sphere in $\mathbb{R}^n$, $n \geq 3$. However, since the conformal images of the unit sphere in $\mathbb{R}^n$ $(n \geq 3)$ are again spheres, we cannot use the conformal mapping arguments above in order to extend the construction to more general domains.

Definition 2.5. We call the pair $(Z_t, \tilde{Z}_t)$ constructed above a *scaling coupling* of reflecting Brownian motions in U, starting at $z_0 \in \overline{U} - \{0\}$, respectively at $\tilde{z}_0 = \frac{1}{a}z_0 \in \overline{U}$.

We will need the following proposition, showing that the conformal image of a reflecting Brownian motion in the unit disk is a time change of a reflecting Brownian motion in the image domain. More precisely, we have

Proposition 2.6. *Let Z_t be a reflecting Brownian motion in U starting at $z_0 \in \overline{U}$. If f is a conformal map of U onto the $C^{1,\alpha}$ domain D $(0 < \alpha < 1)$, then $W_t = f(Z_{\alpha_t})$ is an $(\mathcal{F}^Z_{\alpha_t})$-adapted reflecting Brownian motion in D starting at $f(z_0)$, where:*

$$\alpha_t = \inf\{s : A_s \geq t\}$$

and

$$A_t = \int_0^t |f'(Z_s)|^2\, ds.$$

Proof. Recall that since D is a $C^{1,\alpha}$ domain, $f \in C^1(\overline{D})$. If

$$Z_t = Z_0 + B_t + \frac{1}{2}\int_0^t \nu_U(Z_s)dL_s$$

is the semimartingale representation of Z_t given by Definition 2.1, by applying Itô's formula with $f = (u, v)$ we have

$$f(Z_t) = f(Z_0) + \int_0^t (\nabla u, \nabla v)(Z_s) dB_s + \frac{1}{2} \int_0^t (\nabla u \cdot \nu_U, \nabla v \cdot \nu_U)(Z_s) dL_s.$$

By Lévy's theorem ([Ba], p. 311), the stochastic integral above is a time change of a 2-dimensional Brownian motion, the increasing process being given by $A_t = \int_0^t |f'(Z_s)|^2 \, ds$. Replacing t by $\alpha_t = \inf\{s : A_s \geq t\}$, the stochastic integral above becomes an $(\mathcal{F}^Z_{\alpha_t})$-adapted Brownian motion (note that by Definition 2.1 we have $\mathcal{F}^Z_t = \mathcal{F}^B_t$ for all $t \geq 0$).

So it suffices to show that when L_s increases, $(\nabla u \cdot \nu_U, \nabla v \cdot \nu_U)(Z_s)$ has no tangential component to the boundary of D.

However this follows since, by preliminary remarks, f has a conformal extension to $\overline{U}$; one can use Cauchy-Riemann equations and the geometric interpretation of the argument of f' to show that whenever L_s increases we have

$$(\nabla u \cdot \nu_U, \nabla v \cdot \nu_U)(Z_s) = |f'(Z_s)| \, \nu_D(f(Z_s)),$$

which concludes the proof. □

2.3. **Scaling coupling of reflecting Brownian motions in $C^{1,\alpha}$ domains.** Before carrying out the general construction of scaling coupling, we introduce the notion of hyperbolic line in $C^{1,\alpha}$ domains $(0 < \alpha < 1)$, as follows:

Definition 2.7. i) We define a hyperbolic line in U as being a line segment or an arc of a circle contained in $\overline{U}$ which meets the boundary of U orthogonally. We denote by $\mathcal{H}_U$ the family of all hyperbolic lines in U. If z_1, z_2 are two distinct points on a hyperbolic line $l \in \mathcal{H}_U$, we define the hyperbolic segment with endpoints z_1 and z_2 (denoted by $[z_1, z_2]$) as the part of l between (and including) z_1 and z_2.

ii) For a $C^{1,\alpha}$ $(0 < \alpha < 1)$ domain D, we define a hyperbolic line/segment in D as the conformal image of a hyperbolic line/segment in U. We denote by $\mathcal{H}_D$ the family of all hyperbolic lines in D.

Simple geometric considerations show the following:

Proposition 2.8. *Let D be a $C^{1,\alpha}$ $(0 < \alpha < 1)$ domain.*

i) Given two distinct points in $\overline{D}$, there exists a unique hyperbolic line in D passing through them.

ii) For an arbitrarily chosen diameter d of U, we have

$$\mathcal{H}_U = \{\varphi(d) : \varphi \in \mathcal{A}\},$$

where $\mathcal{A}$ is the family of all automorphisms of U. If f is an arbitrarily chosen conformal map of U onto D, then

$$\mathcal{H}_D = \{f \circ \varphi(d) : \varphi \in \mathcal{A}\}.$$

Remark 2.9. Part i) of the proposition shows that, given any two distinct points z_1 and z_2 in a $C^{1,\alpha}$ domain D $(0 < \alpha < 1)$, the hyperbolic segment $[z_1, z_2]$ in D is uniquely determined. Hence the notion of hyperbolic segment is well defined in the above definition. We will denote by $z_1 z_2$ the unique hyperbolic line passing through z_1 and z_2.

Now we give the construction of scaling coupling for general $C^{1,\alpha}$ domains. Let D be a $C^{1,\alpha}$ domain ($0 < \alpha < 1$) and let $w_0, \widetilde{w}_0 \in \overline{D}$ be distinct, not both on ∂D, and arbitrarily fixed. By Proposition 2.8, there is a unique hyperbolic line $w_0\widetilde{w}_0$ in D, passing through w_0 and $\widetilde{w}_0$. Consider a point w_1 on $w_0\widetilde{w}_0 - [w_0, \widetilde{w}_0]$, $w_1 \notin \partial D$. Let $f : U \to D$ be the unique conformal map of U onto D (given by the Riemann mapping theorem) with $f(0) = w_1$ and $\arg f'(0) = 0$. Let $z_0 = f^{-1}(w_0)$ and $\widetilde{z}_0 = f^{-1}(\widetilde{w}_0)$. Note that by definition, $f^{-1}(w_0\widetilde{w}_0)$ is a hyperbolic line in U, and since $0 = f^{-1}(w_1) \in f^{-1}(w_0\widetilde{w}_0)$, it follows that $f^{-1}(w_0\widetilde{w}_0)$ is in fact a diameter of U. Note that by the choice of w_1, we have $0 \notin [z_0, \widetilde{z}_0]$, and therefore $|z_0| \neq |\widetilde{z}_0|$. Without loss of generality we can assume that $|z_0| < |\widetilde{z}_0|$.

Let Z_t be a reflecting Brownian motion in U starting at z_0. Define processes W_t, $\widetilde{W}_t$ by

$$W_t = f(Z_{\alpha_t}), \qquad t \geq 0, \tag{2.6}$$

$$\widetilde{W}_t = f\Big(\frac{1}{M_{\beta_t}} Z_{\beta_t}\Big), \qquad t \geq 0, \tag{2.7}$$

where $M_t = \left|\frac{z_0}{\widetilde{z}_0}\right| \vee \sup\limits_{s \leq t} |Z_s|$, $t \geq 0$, and

$$A_t = \int_0^t |f'(Z_s)|^2 \, ds, \qquad \alpha_t = \inf\{s : A_s \geq t\}, \qquad t \geq 0, \tag{2.8}$$

$$B_t = \int_0^t \frac{1}{M_s^2} \left| f'\Big(\frac{Z_s}{M_s}\Big)\right|^2 ds, \qquad \beta_t = \inf\{s : B_s \geq t\}, \qquad t \geq 0. \tag{2.9}$$

Theorem 2.10. *W_t and $\widetilde{W}_t$ defined by (2.6)–(2.9) are $(\mathcal{F}^Z_{\alpha_t})$, respectively $(\mathcal{F}^Z_{\beta_t})$-adapted reflecting Brownian motions in D, starting at w_0, respectively $\widetilde{w}_0$.*

Proof. That W_t is an $(\mathcal{F}^Z_{\alpha_t})$-adapted reflecting Brownian motion in D follows from Proposition 2.6.

To prove that $\widetilde{W}_t$ is an $(\mathcal{F}^Z_{\beta_t})$-adapted reflecting Brownian motion, note that by Lemma 2.3, $\frac{Z_t}{M_t}$ is a time change γ_t (given by (2.4)) of a reflecting Brownian motion in $\overline{U}$, starting at $\widetilde{z}_0$.

By Proposition 2.6, $f(\frac{Z_{\gamma_t}}{M_{\gamma_t}})$ is a time change $\tilde{\alpha}_t$ of a reflecting Brownian motion in D, starting at $f(\widetilde{z}_0) = \widetilde{w}_0$, where

$$\tilde{\alpha}_t = \inf\{s : \tilde{A}_s \geq t\} \text{ and } \tilde{A}_t = \int_0^t \left| f'\Big(\frac{Z_{\gamma_s}}{M_{\gamma_s}}\Big)\right|^2 ds, \qquad t \geq 0. \tag{2.10}$$

In order to prove the claim it suffices to show that the combined effect of the two time changes γ_t and $\tilde{\alpha}_t$ is the time change β_t given by (2.9); that is, $\gamma_{\tilde{\alpha}_t} = \beta_t$ for all $t \geq 0$.

C_u given by (2.4) is a bijection on $[0,\infty)$, with inverse $C^{-1}=\gamma$. With the substitution $s=C_u$ in the definition of $\tilde{A}_t$, we obtain

$$\begin{aligned}\tilde{A}_{C_t} &= \int_0^{C_t} \left| f'(\frac{Z_{\gamma_s}}{M_{\gamma_s}})\right|^2 ds \\ &= \int_0^t \left| f'(\frac{Z_u}{M_u})\right|^2 \frac{dC_u}{du} du \\ &= \int_0^t \frac{1}{M_u^2}\left| f'(\frac{Z_u}{M_u})\right|^2 du \\ &= B_t,\end{aligned}$$

for all $t\geq 0$. This shows that $\tilde{A}_{C_t}=B_t$ for all $t\geq 0$, or equivalently, by taking inverses, $\gamma_{\tilde{\alpha}_t}=\beta_t$ for all $t\geq 0$, as needed. □

Definition 2.11. For a $C^{1,\alpha}$ domain D $(0<\alpha<1)$ and arbitrarily fixed distinct points $w_0,\widetilde{w}_0\in\overline{D}$ (not both on ∂D), $w_1\in w_0\widetilde{w}_0-[w_0,\widetilde{w}_0]$ (not on ∂D), the pair $(W_t,\widetilde{W}_t)$ defined by (2.6)-(2.9) is called a *scaling coupling* of reflecting Brownian motions in D starting at $w_0\in\overline{D}$ and $\widetilde{w}_0\in\overline{D}$, respectively.

Remark 2.12. The above construction of scaling coupling for a $C^{1,\alpha}$ domain with starting points $(w_0,\widetilde{w}_0)$ relied on the choice of a conformal map from the unit disk U onto D. As it is known, the choice is uniquely determined by the values of $f(0)$ and $\arg f'(0)$. However, the choice of just $w_1=f(0)$ uniquely determines the conformal map, up to a rotation of the unit disk. By the angular symmetry of the construction of scaling coupling in the case of the unit disk, it follows that the construction is invariant under rotations of the unit disk, and therefore the construction does not depend of the choice of $\arg f'(0)$ (we chose $\arg f'(0)=0$ for simplicity).

It follows that, given two distinct point in $\overline{D}$ (not both on the boundary of D), the scaling coupling with these starting points is uniquely determined once a choice for $f(0)$ (lying on the hyperbolic line passing through them, and not separating them) has been made. We will therefore refer to $w_1=f(0)$ as the parameter of the scaling coupling.

In order to derive the main property of the scaling coupling in the case of convex domains, we need the following characterization of convexity:

Proposition 2.13. *Let $f:U\to D$ be a conformal map of U onto the simply connected domain D. The following are equivalent:*

$$D \textit{ is convex;} \tag{2.11}$$

$$\left|rf'(re^{i\theta})\right| \textit{ is an increasing function of } r\in[0,1), \textit{ for all } 0\leq\theta<2\pi. \tag{2.12}$$

Proof. Since $f'(0)\neq 0$, without loss of generality we can assume that $f(0)=f'(0)-1=0$.

Note that the domain D is convex iff the function f is convex, which (under the condition $f(0)=f'(0)-1=0$) is equivalent (see [D], p. 42) to

$$\operatorname{Re}\left(1+z\frac{f''(z)}{f'(z)}\right)>0,\quad z\in U.$$

In polar coordinates, $z = re^{i\theta}$, we obtain equivalently that

$$\operatorname{Re}\left(\frac{1}{r} + e^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) > 0, \quad 0 \le \theta < 2\pi, \quad 0 < r < 1.$$

Note that since

$$\begin{aligned}\operatorname{Re}\left(\frac{1}{r} + e^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) &= \frac{1}{r} + \operatorname{Re}\left(\frac{\partial}{\partial r}\log f'(re^{i\theta})\right) \\ &= \frac{1}{r} + \frac{\partial}{\partial r}\ln\left|f'(re^{i\theta})\right| \\ &= \frac{\partial}{\partial r}\ln\left|rf'(re^{i\theta})\right|,\end{aligned}$$

the previous statement is equivalent to (2.12), as needed. □

The main feature of the scaling coupling is given by

Proposition 2.14. *With the notation of Theorem 2.10, there exist almost surely finite stopping times τ_1 and τ_2 with respect to the filtration $(\mathcal{F}^Z_{\alpha_t})$, respectively $(\mathcal{F}^Z_{\beta_t})$, such that for all $t \ge 0$ we have*

$$\alpha_{t+\tau_1} = \beta_{t+\tau_2}$$

Moreover, if the domain D is convex, with probability one we have $\beta_t \le \alpha_t$ for all $t \ge 0$.

Proof. Set $\tau = \inf\{s : |Z_s| = 1\}$, $\tau_1 = A_\tau$ and $\tau_2 = B_\tau$. Obviously τ is an a.s. finite stopping time with respect to the filtration $(\mathcal{F}^Z_t)$, and we have $M_s \equiv 1$ for all $s \ge \tau$.

It follows that τ_1 and τ_2 are also a.s. finite stopping times with respect to the filtration $(\mathcal{F}^Z_{\alpha_t})$, respectively $(\mathcal{F}^Z_{\beta_t})$, and that for all $t \ge \tau$ we have

$$A_t - \tau_1 = \int_\tau^t \left|f'(Z_s)\right|^2 ds = B_t - \tau_2.$$

Since $\alpha_t = A_t^{-1}$, $\beta_t = B_t^{-1}$, this implies the first part of the proposition.

For the second part, note that since D is convex, Proposition 2.13 shows that

$$|Z_s f'(Z_s)| \le \left|\frac{Z_s}{M_s} f'(\frac{Z_s}{M_s})\right|.$$

Hence we obtain

$$A_t = \int_0^t \left|f'(Z_s)\right|^2 ds \le \int_0^t \frac{1}{M_s^2}\left|f'(\frac{Z_s}{M_s})\right|^2 ds = B_t,$$

and therefore we have $\alpha_t \ge \beta_t$, for all $t \ge 0$. □

Remark 2.15. The pair $(W_t, \widetilde{W}_t)$ in Definition 2.11 is a coupling in the following extended sense: there exist a.s. finite stopping times τ_1 and τ_2 with respect to the filtration $(\mathcal{F}^Z_{\alpha_t})$, respectively $(\mathcal{F}^Z_{\beta_t})$, such that:

$$W_{t+\tau_1} = \widetilde{W}_{t+\tau_2}, \qquad \text{for all } t \ge 0. \tag{2.13}$$

The usual coupling of diffusions can be thought as a particular case of the above, namely the case when $\tau_1 = \tau_2$ a.s. (and the two filtrations coincide).

The scaling coupling is readily seen to satisfy (2.13) by using

$$\tau_1 = \int_0^\tau |f'(Z_s)|^2 \, ds,$$

$$\tau_2 = \int_0^\tau \frac{1}{M_s^2} \left| f'(\frac{Z_s}{M_s}) \right|^2 ds,$$

where $\tau = \inf\{s > 0 : |Z_s| = 1\}$.

Moreover, in the particular case of convex domains, we have $\tau_1 \leq \tau_2$ a.s., which shows that for $t \geq \tau_2$, $\widetilde{W}_t$ "follows" the path of $W_{t+\tau_1-\tau_2}$.

Remark 2.16. Note that by the equivalence in Proposition 2.13, and using the support theorem for Brownian motion, it follows that the class of $C^{1,\alpha}$ domains ($0 < \alpha < 1$) for which the inequality $\beta_t \leq \alpha_t$ holds almost surely for all $t \geq 0$ (and all starting points w_0, $\widetilde{w}_0$) coincides with the class of convex $C^{1,\alpha}$ domains.

3. Hot spots problem

3.1. **Results on eigenvalues and eigenfunctions.** We will review first some basic facts about eigenfunctions and eigenvalues. We make the remark that for the convenience of arguments involving Brownian motion, we will be using $\frac{1}{2}\Delta$ instead of the Laplace operator Δ. The results hold for the Laplacian Δ by scaling.

Fix an arbitrarily $C^{1,\alpha}$ domain D ($0 < \alpha < 1$).

We say that λ is an eigenfunction for $\frac{1}{2}\Delta$ in D, if there exists a nontrivial solution $\varphi \in C^2(D) \cap C^1(\overline{D})$ to

$$\frac{1}{2}\Delta\varphi + \lambda\varphi = 0. \tag{3.1}$$

φ is then called an eigenfunction corresponding to the eigenvalue λ.

If $\frac{\partial\varphi}{\partial\nu_D} = 0$ on ∂D, the we will refer to λ and φ as being a *Neumann eigenvalue* and *Neumann eigenfunction*, respectively.

If $\frac{\partial\varphi}{\partial\nu_D} = 0$ only on a nonempty proper open subset of ∂D, and $\varphi = 0$ on the remaining part of ∂D, we refer to λ and φ as being a *mixed Dirichlet-Neumann eigenvalue* and *eigenfunction*, respectively.

It is known (see [C], p. 46) that the set of Neumann/mixed Dirichlet-Neumann eigenvalues forms an unbounded sequence

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots \nearrow \infty.$$

We will refer to λ_i as the i^{th} Neumann/mixed Dirichlet-Neumann eigenvalue for D, and to the eigenfunctions belonging to the eigenspace corresponding to λ_i as the i^{th} Neumann/mixed Dirichlet-Neumann eigenfunctions for D.

Recall that the nodal set of an eigenfunction φ is the set $\varphi^{-1}(\{0\}) = \{x \in \overline{D} : \varphi(x) = 0\}$, and a nodal domain of φ is a component of $\overline{D} - \varphi^{-1}(\{0\})$.

The Courant nodal domain theorem ([C], p. 19) asserts that for each $k \geq 1$, the number of nodal domains of a k^{th} eigenfunction of a simply connected domain is less than or equal to k.

As an immediate consequence we have that a first eigenfunction has constant sign, and a second eigenfunction has precisely 2 nodal domains. Moreover, λ_1 is characterized as being the only eigenvalue with eigenfunction of constant sign ([C], p. 20).

It is also known that in the case of a second Neumann eigenfunction the nodal set is a smooth (C^∞) curve, called the nodal line, and that there are no closed nodal lines ([B], p. 128).

The hot spots conjecture (due to Jeffrey Rauch, 1974) is a strong maximum principle for second Neumann eigenfunctions of a simply connected domain D, and it can be formulated as follows:

Conjecture 3.1. For every second Neumann eigenfunction φ_2 of D, and for all $y \in D$, we have

$$\min_{x\in\partial D} \varphi_2(x) < \varphi_2(y) < \max_{x\in\partial D} \varphi_2(x). \tag{3.2}$$

By an abuse of language, if (3.2) holds for a second Neumann eigenfunction φ_2 of D, we say that the hot spots conjecture holds for φ_2.

According to Kawohl [K], Conjecture 3.1 holds for balls, annuli and rectangles in $\mathbb{R}^d$. Burdzy and Werner constructed a counterexample to Conjecture 3.1, in which the maximum of φ_2 is attained only in the interior of D and the minimum at the boundary of D. More recently, Bass and Burdzy constructed a stronger counterexample, in which both the maximum and the minimum of φ_2 are attained only in the interior of D. In both counterexamples the domain D was not convex.

Recent advances in the hot spots problem identified classes of domains for which Conjecture 3.1 holds. It is known that Conjecture 3.1 holds for bounded convex domains with two orthogonal axes of symmetry (see [BB], [JN]), or just one axis of symmetry and additional hypotheses on the domain (see [BB]). The question whether Conjecture 3.1 holds for bounded convex domains in $\mathbb{R}^d$ is still open.

3.2. **Main results.** The main result in this section is Theorem 3.6, which shows that the hot spots conjecture holds for antisymmetric second Neumann eigenfunctions of smooth convex domains having a line of symmetry.

Let $D \subset \mathbb{R}^2$ be a convex $C^{1,\alpha}$ domain ($0 < \alpha < 1$) having a line of symmetry. Without loss of generality we will assume that D is symmetric with respect to the horizontal axis.

Set $D^+ = D \cap \{(x,y) : y > 0\}$, $\Gamma^+ = \partial D^+ \cap \partial D$, $\Gamma_0 = \overline{D} \cap \{(x,y) : y = 0\}$. For $w_0 \in \overline{D^+}$, denote $\tau^{w_0} = \inf\{s : \operatorname{Im} W_s = 0\}$, where W_s is a reflecting Brownian motion in D starting at w_0.

Remark 3.2. Let f_1 be a conformal map of U^+ onto D^+, such that the parts of the boundaries of U^+ and D^+ lying on the horizontal axis correspond to each other. By the symmetry principle, f_1 extends to a conformal map of U onto D. Since $\Gamma_0 = f_1([-1,1])$, it follows that Γ_0 is a hyperbolic line in $\overline{D}$.

The key to Theorem 3.4 is the following lemma, showing a monotonicity property of τ^{w_0} (as a function of w_0) on the family of hyperbolic lines in D intersecting the horizontal axis:

Lemma 3.3. *Given $\widetilde{w}_0 \in \overline{D}$, $w_1 \in D \cap \Gamma_0$ and $w_0 \in [w_1, \widetilde{w}_0]$, there exist filtrations $(\mathcal{F}_t)$, $(\widetilde{\mathcal{F}}_t)$ and $(\mathcal{F}_t)$, respectively $(\widetilde{\mathcal{F}}_t)$-adapted reflecting Brownian motions W_t, $\widetilde{W}_t$ in D, starting at w_0, respectively $\widetilde{w}_0$, such that if τ^{w_0}, $\tau^{\widetilde{w}_0}$ are the hitting times to the horizontal axis of W_t, respectively $\widetilde{W}_t$, then with probability one we have*

$$\tau^{w_0} \le \tau^{\widetilde{w}_0}.$$

Proof. The proof is trivial if $w_0 = \widetilde{w}_0$ or $w_0 = w_1$, so we can assume that $w_0, \widetilde{w}_0$ and w_1 are distinct.

Let $(W_t, \widetilde{W}_t)$ be a scaling coupling of reflecting Brownian motions in D starting at $(w_0, \widetilde{w}_0)$, with parameter w_1, that is, a scaling coupling obtained as the image under a conformal map $f : U \to D$ with $f(0) = w_1$ of the scaling coupling $(Z_t, \tilde{Z}_t)$ in U. The filtrations $(\mathcal{F}_t)$, $(\widetilde{\mathcal{F}}_t)$ are the corresponding "time changes" of the filtrations $(\mathcal{F}_t^Z)$, respectively $(\mathcal{F}_t^{\tilde{Z}})$, as indicated in Proposition 2.6.

Remark 3.2 above shows that Γ_0 is a hyperbolic line in U; hence $f^{-1}(\Gamma_0)$ is a hyperbolic line in U. Since $0 = f^{-1}(w_1) \in f^{-1}(\Gamma_0)$, it follows that $f^{-1}(\Gamma_0)$ is in fact a diameter of U. By Remark 2.12, without loss of generality we can assume that $f^{-1}(\Gamma_0) = [-1, 1]$.

If $s > 0$ is such that $\operatorname{Im} \widetilde{W}_s = 0$, by construction of the coupling we have $\operatorname{Im} f(\frac{Z_{\beta_s}}{M_{\beta_s}}) = 0$. Because under f the parts of the boundaries of U^+ and D^+ lying on the horizontal axis correspond to each other, $\operatorname{Im} \frac{Z_{\beta_s}}{M_{\beta_s}} = 0$. Since M_{β_s} is real, it follows that $\operatorname{Im} Z_{\beta_s} = 0$.

Since D is convex, by Proposition 2.14 it follows that $\beta_s \le \alpha_s$. Since α_s is increasing (and a bijection) on $[0, \infty)$, there exists $s' \le s$ such that $\alpha_{s'} = \beta_s$.

It follows that $\operatorname{Im} Z_{\alpha_{s'}} = \operatorname{Im} Z_{\beta_s} = 0$, and hence $W_{s'} = f(Z_{\alpha_{s'}})$ is on the horizontal axis.

We have shown that if $\operatorname{Im} \widetilde{W}_s = 0$, then there exists $s' \le s$ such that $\operatorname{Im} W_{s'} = 0$, which implies that $\tau^{w_0} \le \tau^{\widetilde{w}_0}$ a.s., as needed. $\square$

We can now prove a first version of our main result:

Theorem 3.4. *Let D be a convex $C^{1,\alpha}$ domain ($0 < \alpha < 1$) which is symmetric with respect to the horizontal axis. If φ is a second Neumann eigenfunction for D which is antisymmetric with respect to the horizontal axis, then φ is monotone on the family of hyperbolic lines in D which intersect the horizontal axis.*

In particular, φ must attain its maximum and minimum over $\overline{D}$ on the boundary of D.

Proof. By the assumption, φ must be identically zero on the horizontal axis, and therefore the nodal line for φ is Γ_0 (the part of the horizontal axis contained in $\overline{D}$). It follows that D^+ and D^- are the nodal domains of φ.

Since φ has constant sign on each nodal domain, without loss of generality we will assume that φ is positive on D^+. Since φ is antisymmetric with respect to the horizontal axis, it suffices to prove the monotonicity of φ in D^+ along the indicated family of curves.

Consider an arbitrarily fixed hyperbolic line in D, which intersects the horizontal axis, and denote by w_1 the point of intersection. If w_0, $\widetilde{w}_0 \in \overline{D^+}$ are arbitrarily chosen points lying on this hyperbolic line, such that $w_0 \in [w_1, \widetilde{w}_0]$, we will show that $\varphi(w_0) \le \varphi(\widetilde{w}_0)$.

Since the restriction of a second Neumann eigenfunction for D to one of its nodal domains has constant sign, by preliminary remarks it follows that it is a first mixed Dirichlet-Neumann eigenfunction for the corresponding nodal domain. Therefore, the restriction of φ to $\overline{D^+}$ is a first mixed Dirichlet-Neumann eigenfunction for D^+, with Neumann conditions on Γ^+ and Dirichlet conditions on Γ_0.

It can be shown ([Pa], p. 20) that the transition density $p_{D^+}(t,x,y)$ of reflecting Brownian motion in D^+, killed on hitting the horizontal axis, has an eigenfunction expansion in terms of the mixed Dirichlet-Neumann eigenfunctions for D^+, with Dirichlet boundary conditions on Γ_0 and Neumann conditions on Γ^+. More precisely, it can be shown that

$$p_{D^+}(t,x,y) = \sum_{i\geq 1} e^{-\mu_i t}\varphi_i(x)\varphi_i(y),$$

where $0 < \mu_1 < \mu_2 \leq ...$ are the mixed Dirichlet-Neumann eigenvalues for D^+ repeated according to multiplicity, and $\{\varphi_i\}_{i\geq 1}$ is an orthonormal sequence of eigenfunctions corresponding to the eigenvalues $\{\mu_i\}_{i\geq 1}$. Moreover, the convergence is uniform and absolute on $\overline{D^+}$.

Note that since μ_1 is simple, the corresponding eigenspace is 1-dimensional, and therefore $\varphi_1 = c\varphi$, for some nonzero constant c. Also note that since $\mu_1 < \mu_i$ for all $i \geq 2$, we can write

$$\begin{aligned} p_{D^+}(t,x,y) &= e^{-\mu_1 t}\varphi_1(x)\varphi_1(y) + R(t,x,y) \\ &= c^2 e^{-\mu_1 t}\varphi(x)\varphi(y) + R(t,x,y), \end{aligned} \tag{3.3}$$

where $\lim\limits_{t\to\infty} e^{\mu_1 t} R(t,x,y) = 0$, uniformly in $x, y \in \overline{D^+}$.

Consider the function $u : (0,\infty)\times D \to \mathbb{R}$ given by $u(t,x) = E[1; \tau^x > t]$, where τ^x is the lifetime of reflecting Brownian motion in D^+ starting at x, killed on hitting Γ_0. Integrating the eigenfunction expansion (3.3), we obtain

$$\begin{aligned} u(t,x) &= \int\limits_{D^+} p_{D^+}(t,x,y)dy \\ &= c^2 e^{-\mu_1 t}\varphi(x)\int\limits_{D^+}\varphi(y)dy + \int\limits_{D^+} R(t,x,y)dy \\ &= a e^{-\mu_1 t}\varphi(x) + R_1(t,x), \end{aligned} \tag{3.4}$$

where, by assumption, $a = c^2 \int\limits_{D^+}\varphi(y)dy > 0$ and $R_1(t,x)$ approaches zero faster than $e^{-\mu_1 t}$ as $t \to \infty$.

Since $u(t,x) = P(\tau^x > t)$, and using the monotonicity property in Lemma 3.3, it follows that for any $t \geq 0$ we have

$$\begin{aligned} u(t,w_0) &= P(\tau^{w_0} > t) \\ &\leq P(\tau^{\widetilde{w}_0} > t) \\ &= u(t,\widetilde{w}_0). \end{aligned}$$

Using the analytic representation (3.4) of $u(t,\cdot)$, we therefore obtain

$$a\varphi(w_0) + e^{\mu_1 t}R_1(t,w_0) \leq a\varphi(\widetilde{w}_0) + e^{\mu_1 t}R_1(t,\widetilde{w}_0)$$

for all $t \geq 0$. Letting $t \to \infty$, it follows that $a\varphi(w_0) \leq a\varphi(\widetilde{w}_0)$, and since $a > 0$, we obtain $\varphi(w_0) \leq \varphi(\widetilde{w}_0)$, as needed. $\square$

The above theorem leaves open the question of whether φ can also attain its maximum/minimum over $\overline{D}$ inside the domain. We will show that under the hypotheses of the previous theorem this cannot happen; more generally, we will show that φ is strictly monotone on the family of hyperbolic lines which intersect nontrivially the axis of symmetry of D. We have:

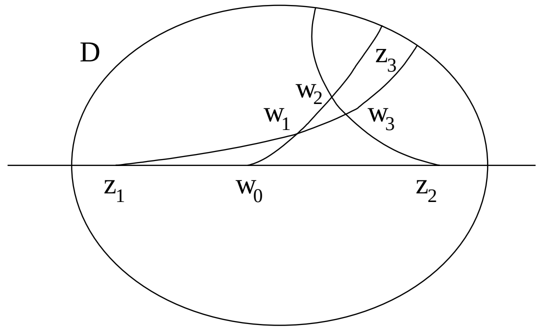

FIGURE 1. Hyperbolic lines in D

Theorem 3.5. *Under the hypotheses of the previous theorem, for any $a \neq 0$, the intersection between a level set $\varphi^{-1}(\{a\}) = \{x \in \overline{D} : \varphi(x) = a\}$ of φ and a hyperbolic line of D which intersects the horizontal axis consists of at most one point.*

Proof. Assume there exist distinct points w_1, w_2 such that $\varphi(w_1) = \varphi(w_2) \neq 0$ and the hyperbolic line $w_1 w_2$ intersects the horizontal axis, say at w_0. Without loss of generality we can assume $\varphi(w_1) = \varphi(w_2) > 0$, and therefore $w_1, w_2 \in \overline{D^+}$. Also, we may assume that $w_1 \in [w_0, w_2]$.

Consider points $z_1, z_2 \in \Gamma_0$ such that $z_1 < w_0 < z_2$.

We will show that $[w_1, z_3] \cap [z_2, w_2] \neq \emptyset$, where z_3 is the endpoint of the hyperbolic line $z_1 w_1$ lying on $\partial D^+ - \Gamma_0$.

Note that by Proposition 2.8, the intersection of any two hyperbolic lines consists of at most one point.

Since $[w_1, z_3] \cap w_1 w_2 = \{w_1\}$, it follows that the hyperbolic segment $[w_1, z_3]$ is contained in the right (hyperbolic) half plane determined by the hyperbolic line $w_1 w_2$ (see Figure 1). If $[w_1, z_3] \cap [z_2, w_2] = \emptyset$, then $[w_1, z_3]$ is also contained in the left (hyperbolic) half plane determined by the hyperbolic line $z_2 w_2$. It follows that $z_3 \in [w_1, z_3]$ must be contained in their intersection; however, since $z_3 \in \overline{D^+} - \Gamma_0$, this is possible only if $z_3 = w_2$, contradicting $[w_1, z_3] \cap [z_2, w_2] = \emptyset$.

Therefore we must have $[w_1, z_3] \cap [z_2, w_2] \neq \emptyset$, and we denote by w_3 the point of intersection.

By applying the previous theorem to the hyperbolic lines $w_1 w_2, w_1 w_3$ and respectively $w_3 w_2$, we obtain

$$\begin{aligned}
\varphi(w_1) &\leq \varphi(z) \leq \varphi(w_2), \\
\varphi(w_1) &\leq \varphi(z') \leq \varphi(w_3), \\
\varphi(w_3) &\leq \varphi(z'') \leq \varphi(w_2),
\end{aligned}$$

for all $z \in [w_1, w_2]$, $z' \in [w_1, w_3]$, $z'' \in [w_3, w_2]$. Since by hypothesis $\varphi(w_1) = \varphi(w_2)$, we obtain that $\varphi(z) = \varphi(w_1) = \varphi(w_2) = \varphi(w_3)$, for all $z \in$. $[w_1, w_2] \cup [w_1, w_3] \cup [w_3, w_2]$.

Choosing now an arbitrarily fixed $z_2 \in [w_1, w_3]$ and applying again the previous theorem to the hyperbolic line $z w_2$ (which intersects the horizontal axis between w_0 and z_2), we obtain

$$\varphi(w_2) = \varphi(z) \leq \varphi(z') \leq \varphi(w_2),$$

for all $z' \in [z, w_2]$. Therefore we have that $\varphi(z') = \varphi(w_2)$ for all $z' \in [z, w_2]$, where $z \in [w_1, w_3]$.

It follows that φ is constant on the interior of the (hyperbolic) triangle with vertices w_1, w_2 and w_3. Since the points w_1, w_2, w_3 do not lie on the same hyperbolic line, the interior of this triangle is not empty.

However, φ, being a nonconstant real analytic function, cannot be constant on a nonempty open set. The contradiction shows that φ is injective on every hyperbolic line (of course, except Γ_0) intersecting the horizontal axis, thus proving the claim. $\square$

Using the above theorem, we can strengthen the result in Theorem 3.4, and we state:

Theorem 3.6. *Let D be a convex $C^{1,\alpha}$ domain $(0 < \alpha < 1)$ which is symmetric with respect to the horizontal axis. If φ is a second Neumann eigenfunction for D which is antisymmetric with respect to the horizontal axis, then φ is **strictly** monotone on the family of hyperbolic lines in D which nontrivially intersect the horizontal axis.*

In particular, φ must attain its maximum and minimum over $\overline{D}$ solely at the boundary of D.

As an immediate consequence, we have:

Corollary 3.7. *Let D be a convex $C^{1,\alpha}$ domain $(0 < \alpha < 1)$ which is symmetric with respect to the horizontal axis. If φ is a second Neumann eigenfunction for D which is antisymmetric with respect to the horizontal axis, then the hot spots conjecture holds for φ.*

As another consequence of Theorem 3.6, we obtain the following properties of the level sets of antisymmetric second Neumann eigenfunctions of convex $C^{1,\alpha}$ domains with a line of symmetry:

Corollary 3.8. *Let D be a convex $C^{1,\alpha}$ domain $(0 < \alpha < 1)$ which is symmetric with respect to the horizontal axis. If φ is a second Neumann eigenfunction for D which is antisymmetric with respect to the horizontal axis, then the level curves of φ are unions of disjoint simple curves in $\overline{D}$ (possibly reduced to a single point). Moreover, no level curve of φ can terminate in D.*

Proof. Since φ is antisymmetric, it suffices to consider the case of the level curves $\varphi^{-1}(\{a\})$, with $a > 0$.

Let z_t, $0 \le t \le 1$, be a $C^{1,\alpha}$ parametrization of the part of the boundary of D lying in the upper half-plane, and let $w_0 \in D$ be an arbitrarily chosen point on the horizontal axis.

By Theorem 3.5, $\varphi^{-1}(\{a\}) \cap [w_0, z_t]$ consists of at most one point, for any $t \ge 0$.

Define $\gamma(t) = \varphi^{-1}(\{a\}) \cap [w_0, z_t]$, for all values of t for which the intersection is nonempty, and let I be the domain of γ.

Consider $t \in I$ such that $\gamma(t) \in D$. By Theorem 3.6, we have that $\varphi(z_t) > \varphi(\gamma(t)) = a$. By the continuity of φ, we have that $\varphi > a$ on a whole neighborhood of z_t, and therefore, for t' sufficiently close t, we have $\varphi(z_{t'}) > a$. Therefore, for all t' close to t, we have $\varphi^{-1}(\{a\}) \cap [w_0, z_t] \neq \emptyset$, showing that $t' \in I$; thus t belongs to the interior of I. In particular this shows that the level set $\varphi^{-1}(\{a\})$ cannot terminate in D. Also, by the continuity of φ and z_t, γ is continuous at t.

If I_t is the largest connected component of I which contains t, it follows by the continuity of φ that I_t is a closed interval, and therefore $\gamma(I_t)$ is a simple curve terminating at the boundary of D. The fact that $\gamma(I_t)$ cannot form a closed loop follows by Theorem 3.6.

Moreover, $\gamma(I_t)$ is a connected component of the level curve $\varphi^{-1}(\{a\})$, and conversely, any connected component of $\varphi^{-1}(\{a\})$ is of the form $\gamma(I_t)$ for some $t \in [0,1]$. $\square$

3.3. **Comparisons with known results.** The hot spots conjecture is known to be true for relatively small classes of domains (parallelepipeds, balls and annuli in $\mathbb{R}^d$, obtuse triangles). According to the knowledge of the author, the only papers in the literature which contain the proof of the hot spots conjecture for general classes of domains are [BB] and [JN]. We will refer to these papers for a comparison of our results.

For a bounded planar domain D, we will denote the diameter of D by d_D, the length of the projection of D on the vertical axis by w_D, the length of the projection of D on the horizontal axis by l_D, and will call them the diameter, the width and respectively the length of D.

In [BB], Bañuelos and Burdzy used probabilistic techniques (based on synchronous and mirror couplings of reflecting Brownian motions in polygonal domains) to prove the following:

Theorem 3.9. *Suppose that a convex polygonal domain D is symmetric with respect to the horizontal axis S and the ratio d_D/w_D is greater than 1.54. Let x and y be the intersection points of S with ∂D. Make at least one of the following two assumptions:*

a) D has another line of symmetry S_1 which is perpendicular to S;

b) For every $r > 0$, the intersection of the circle $\partial B(x,r)$ with D is either empty or a connected arc, and the same holds for $\partial B(y,r)$.

Then the hot spots conjecture holds for D.

More recently, Jerison and Nadirashvili used deformation of the domain techniques (see [JN]) in order to refine the above result in the case of domains having two orthogonal axes of symmetry. They showed the following:

Theorem 3.10. *Let Ω be a bounded convex domain in the plane that is symmetric with respect to both coordinate axes. Let u be any Neumann eigenfunction with lowest nonzero eigenvalue. Then, except in the case of a rectangle, u achieves its maximum over $\overline{\Omega}$ on the boundary at exactly one point, and likewise for its minimum. Furthermore, if $x^0 \in \partial\Omega$ and $-x^0 \in \partial\Omega$ denote the places where u achieves its maximum and minimum, then u is monotone along the two arcs of the boundary from $-x^0$ to x^0. Let ν be any outer normal to $\partial\Omega$ at x^0; that is, $\nu \cdot (x - x^0) < 0$ for all $x \in \Omega$. Then $\nu \cdot \nabla u(x) > 0$ for all $x \in \Omega$.*

As indicated in [JN], the uniqueness of the location of the extrema of u may fail for domains not having two orthogonal axis of symmetry (for example an equilateral triangle has a second Neumann eigenfunction which attains its maximum at two of its vertices and the minimum at the third vertex). We make the remark that in the present paper we are concerned with the hot spots conjecture in the form presented in Conjecture 3.1, and we will not attempt to prove that the extrema are attained at a single point (this may fail for general domains, as shown in the example above).

Theorem 3.9 provides additional hypotheses under which the hot spots conjecture holds, for the case of convex domains having just one axis of symmetry (additional hypothesis b)).

We will show that our main result in Theorem 3.6 gives the hot spots conjecture for smooth convex domains with two orthogonal axes of symmetry (without any restriction on their diameter to width ratio), thus obtaining the cited results (Theorem 3.9 with the additional hypothesis a) and Theorem 3.10).

Also, we will show that we can apply Theorem 3.6 to the case of domains having just one axis of symmetry and satisfying an additional restriction on their diameter to width ratio, obtaining a result similar to Theorem 3.9 (with additional hypothesis b)), but which is complimentary to it.

We will consider first the case of convex domains having two orthogonal axes of symmetry. We have:

Theorem 3.11. *If D is a convex $C^{1,\alpha}$ domain ($0<\alpha<1$), symmetric with respect to both coordinate axes, then the hot spots conjecture holds for D.*

Proof. It is known (see [N]) that the eigenspace corresponding to the second Neumann eigenvalue of a simply connected planar domain is at most 2-dimensional.

We will consider first the case when the eigenspace corresponding to the second Neumann eigenvalue for D is 1-dimensional. We will show that in this case we can find a second Neumann eigenfunction $\widetilde{\varphi}$ for D which is antisymmetric with respect to one of the coordinate axes. From this and Theorem 3.6, the result follows.

Let φ be a second Neumann eigenfunction for D. If φ is antisymmetric with respect to one of the coordinate axes, we are done. Otherwise we consider the functions $\varphi_1(x,y)=\varphi(x,y)-\varphi(x,-y)$ and $\varphi_2(x,y)=\varphi(x,y)-\varphi(-x,y)$, and note that they are antisymmetric with respect to the horizontal, respectively vertical axis and that both of them are eigenfunctions (possibly identically zero) corresponding to the second eigenvalue of D.

Note that φ_1 and φ_2 cannot both be identically zero. In fact, φ would otherwise be symmetric with respect to both coordinate axes. However, this cannot happen, since the nodal line of φ cannot form a closed loop and since φ has exactly two nodal domains.

It follows that we can always find a (not identically zero) second eigenfunction $\widetilde{\varphi}$ for D which is antisymmetric with respect to one of the coordinate axes. It follows that Theorem 3.6 applies to $\widetilde{\varphi}$, and therefore the hot spots conjecture holds for $\widetilde{\varphi}$. Since the eigenspace corresponding to the second Neumann eigenvalue for D is 1-dimensional, it follows that the hot spots conjecture holds for D, ending the proof in this case.

We will consider now the case when the eigenspace corresponding to the second Neumann eigenvalue is 2-dimensional. We will show that we can find two independent second Neumann eigenfunctions $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ which are antisymmetric with respect to the horizontal, respectively vertical axis.

Consider φ_1 and φ_2, two linearly independent second Neumann eigenfunctions for D.

Note that if one of the eigenfunctions is symmetric with respect to one of the axes, then it has to be antisymmetric with respect to the other axis; conversely, if one of the eigenfunctions is antisymmetric with respect to one of the axes, then it has to be symmetric with respect to the other axis. To see this, if for example φ_1 is antisymmetric with respect to the horizontal axis, note that $\overline{\varphi}_1(x,y)=\widetilde{\varphi}_1(x,y)-$

$\widetilde{\varphi}_1(-x,y)$ is a second Neumann eigenfunction for D, antisymmetric with respect to both coordinate axes; unless $\overline{\varphi}_1$ is identically zero, $\overline{\varphi}_1$ would have at least four nodal domains, which is impossible by the Courant nodal domain theorem. Hence $\overline{\varphi}_1$ is identically zero, or equivalently φ_1 is symmetric with respect to the horizontal axis. Similar reasoning applies to the other cases.

It follows that φ_1 and φ_2 cannot be symmetric with respect to the same symmetry axis, for they are independent. Also, neither φ_1 nor φ_2 cannot be symmetric with respect to both symmetry axes.

Without loss of generality we can thus assume that φ_1 is not symmetric with respect to the horizontal axis, and that φ_2 is not symmetric with respect to the vertical axis. We can therefore choose $\widetilde{\varphi}_1(x,y) = \varphi_1(x,y) - \varphi_1(x,-y)$ and $\widetilde{\varphi}_2(x,y) = \varphi_2(x,y) - \varphi_2(-x,y)$ and note that they are independent, not identically zero second Neumann eigenfunctions for D, antisymmetric with respect to the horizontal, respectively vertical axis. Moreover, from the previous part of the proof it follows that $\widetilde{\varphi}_1$ is symmetric with respect to the vertical axis, and $\widetilde{\varphi}_2$ is symmetric with respect to the horizontal axis.

It follows that Theorem 3.6 applies to $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$, and therefore the hot spots conjecture holds for them. Moreover, $\widetilde{\varphi}_1$ is strictly monotone on the family of hyperbolic lines intersecting the horizontal axis, and $\widetilde{\varphi}_2$ is strictly monotone on the family of hyperbolic lines intersecting the vertical axis. In particular, both $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ are strictly monotone on all hyperbolic lines passing through the origin.

Consider now an arbitrarily chosen second Neumann eigenfunction φ for D. Since $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ are independent and the eigenspace corresponding to the second Neumann eigenvalue is 2-dimensional, we can find constants a, b such that $\varphi = a\widetilde{\varphi}_1 + b\widetilde{\varphi}_2$.

Consider a point $z_0 = (x_0, y_0) \in \overline{D}$ where φ attains its maximum over $\overline{D}$. We will show that we must have $z_0 \in \partial D$. If $a\widetilde{\varphi}_1(z_0) < 0$, since $\widetilde{\varphi}_1$ is antisymmetric with respect to the horizontal axis and $\widetilde{\varphi}_2$ is symmetric with respect to the horizontal axis, we obtain

$$\begin{aligned} \varphi(z_0) &= a\widetilde{\varphi}_1(x_0,y_0) + b\widetilde{\varphi}_2(x_0,y_0) \\ &< a\widetilde{\varphi}_1(x_0,-y_0) + b\widetilde{\varphi}_2(x_0,-y_0) \\ &= \varphi(x_0,-y_0), \end{aligned}$$

a contradiction. It follows that we must have $a\widetilde{\varphi}_1(z_0) \geq 0$, and similarly $b\widetilde{\varphi}_2(z_0) \geq 0$. Since both $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ are strictly monotone on the hyperbolic line through 0 and z_0, it follows that φ is strictly increasing on the hyperbolic half line with the endpoint 0 and passing through z_0. Since φ attains its maximum at z_0, it follows that $z_0 \in \partial D$. Similar reasoning shows that if φ attains its minimum at a point, then the point must belong to the boundary ∂D, and therefore the hot spots conjecture holds for φ.

Since φ was arbitrarily chosen, it follows that the hot spots conjecture holds for D. □

In [BB] it is shown that if D is a convex domain symmetric with respect to the horizontal axis, and the ratio d_D/w_D is larger than 1.54, then the eigenspace corresponding to the second eigenvalue is 1-dimensional, and there is no second Neumann eigenfunction for D which is antisymmetric with respect to the horizontal axis, and hence our result in Theorem 3.6 does not apply. Rephrasing this, we can say that if a domain D having the horizontal axis as line of symmetry is "long enough", then the second Neumann eigenfunctions for D have to be symmetric with respect to the horizontal axis. One might expect that if the domain is "wide enough", then the second Neumann eigenfunctions have to be antisymmetric with respect to the horizontal axis. This is true, and we state:

Proposition 3.12. *Suppose that D is a convex $C^{1,\alpha}$ domain $(0 < \alpha < 1)$ which is symmetric with respect to the horizontal axis, and the diameter-to-length ratio d_D/l_D is larger than $\frac{4j_0}{\pi} \approx 3.06$. Then the eigenspace corresponding to the second Neumann eigenvalue for D is 1-dimensional, and it is given by a function which is antisymmetric with respect to the horizontal axis.*

Proof. The first part follows from Proposition 2.4i) in [BB]. To prove the second part of the statement, we will use a modification of an argument found in [BB].

Assume there exists a second Neumann eigenfunction φ for D which is symmetric with respect to the horizontal axis. Let D_1 be one of the nodal domains of φ, with the property that the horizontal component of the inner pointing normal at the common boundary of D_1 and D always points in the same direction (i.e. either left or right). The existence of D_1 with the above property follows from the convexity of the domain and the fact that the nodal domains of φ are symmetric with respect to the horizontal axis.

We will estimate the first mixed Dirichlet-Neumann eigenvalue for D_1 (denoted λ_1), which is the same as the second eigenvalue for D (denoted μ_2).

For this, consider a reflecting Brownian motion (X_t, Y_t) in D_1, killed upon hitting the nodal line of φ. Note that the horizontal component X_t is a Brownian motion plus (or minus) a nondecreasing process (we are using here the fact that the horizontal component of the inner pointing normal to the common boundary of D_1 and D is always pointing in the same direction).

By a comparison argument, it can be shown that the distribution of the lifetime of X_t is majorized by the distribution of the lifetime of a 1-dimensional Brownian motion on an interval of length l_D, killed at one end and reflected at the other. This latter is majorized by $ce^{-\lambda t}$, where λ is the first Dirichlet-Neumann eigenvalue for the given interval, i.e. $\lambda = \frac{\pi^2}{8l_D^2}$ (for it is the second Neumann eigenvalue of an interval of double length, i.e. $\frac{\pi^2}{2(2l_D)^2}$). It follows that we have $\mu_2 = \lambda_1 \geq \lambda = \frac{\pi^2}{8l_D^2}$.

It is also known (see [BB]) that for convex domains, we have $\mu_2 \leq \frac{2j_0^2}{l_D^2}$; thus we must have $\frac{\pi^2}{8l_D^2} \leq \frac{2j_0^2}{d_D^2}$, or equivalently $\frac{d_D}{l_D} \leq \frac{4j_0}{\pi}$.

It follows that if this inequality doesn't hold (i.e. for sufficiently "wide" domains, symmetric with respect to the horizontal axis), there are no second Neumann eigenfunctions for D which are symmetric with respect to the horizontal axis.

To conclude the proof, consider an arbitrarily fixed second Neumann eigenfunction φ_1 for D, and define $\widetilde{\varphi}(x, y) = \varphi_1(x, y) - \varphi_1(x, -y)$. From the previous part of the proof it follows that $\widetilde{\varphi}$ is not identically zero, and hence it is a second Neumann eigenfunction for D, antisymmetric with respect to the horizontal axis. □

Using the above proposition and Corollary 3.7, we obtain the proof of the hot spots conjecture for a new class of domains, as follows:

Corollary 3.13. *Suppose that D is a $C^{1,\alpha}$ convex domain $(0 < \alpha < 1)$, symmetric with respect to the horizontal axis and with diameter-to-width ratio d_D/l_D larger than $\frac{4j_0}{\pi} \approx 3.06$. Then the hot spots conjecture holds for D.*

3.4. **Further developments.** We conclude with three remarks.

First, note that because of the diameter-to-width ratio restriction in Theorem 3.9, essentially the results of Burdzy and Bañuelos apply to convex domains with one axis of symmetry, for which the eigenspace corresponding to the second Neumann eigenvalue is 1-dimensional, being given by an eigenfunction which is symmetric with respect to the axis of symmetry. Our result in Corollary 3.13 is complementary to this, giving the proof of the hot spots conjecture for (smooth) convex domains with one axis of symmetry, for which the eigenspace corresponding to the second Neumann eigenvalue is 1-dimensional, being given by an eigenfunction which is antisymmetric with respect to the axis of symmetry.

In our view, these two results together should give a resolution to the hot spots conjecture in the case of convex domains with a line of symmetry (and no restrictions on their diameter-to-width ratio), but we were not able to implement it.

Second, we will discuss how the hypotheses of our main result in Theorem 3.6 can be weakened. There are two main ingredients in the proof: the symmetry and the convexity.

Even though the symmetry of the domain and the antisymmetry of the Neumann eigenfunction hypotheses are needed in the proof, we can carry out the proof with weaker assumptions. A careful examination of the proof shows that we can replace these hypotheses as follows:

Theorem 3.14. *Let D be a convex $C^{1,\alpha}$ $(0<\alpha<1)$ domain and let φ be a second Neumann eigenfunction for D. If the nodal line of φ is a hyperbolic line in D, then φ is strictly monotone on the family of hyperbolic lines in D which intersect the nodal line of φ nontrivially.*

In particular, the hot spots conjecture holds for φ.

The hypothesis on the nodal line φ in the above theorem can still be weakened, by requiring instead that the nodal domains of φ are *hyperbolically starlike* in D (i.e. starlike with respect to the hyperbolic lines in D). Geometrically this means that the nodal line of φ is "completely visible" from w along the hyperbolic lines in D. We state:

Theorem 3.15. *Let D be a convex $C^{1,\alpha}$ $(0<\alpha<1)$ domain and let φ be a second Neumann eigenfunction for D, with nodal domains $D^+=\{x\in D:\varphi(x)>0\}$ and $D^-=\{x\in D:\varphi(x)<0\}$. Assume there exists a point $w\in D^-$ such that D^- is hyperbolically starlike with respect to w. Then φ is monotone in D^+ along the family of hyperbolic lines in D which pass through w. In particular, φ must attain its maximum over $\overline{D}$ at the boundary of D. A similar statement holds for D^-.*

The convexity of the domain is a key element in our construction of scaling coupling of reflecting Brownian motions (see Proposition 2.14), needed in order to prove the hot spots conjecture, and therefore we cannot dispense with it. However, the scaling coupling can be defined in certain annuli-like domains, and it has the same properties outlined in our construction for convex $C^{1,\alpha}$ domains $(0<\alpha<1)$. This in turn leads to a proof of the hot spots conjecture for certain types of second Neumann eigenfunctions of doubly connected domains, where almost nothing is known in the literature. We have:

Theorem 3.16. *Let $D=\{f(z):a<|z|<1\}$, where $f:U\to\mathbb{C}$ is a conformal map of U onto a convex $C^{1,\alpha}$ domain $(0<\alpha<1)$ and $a\in(0,1)$ is arbitrarily fixed. If φ is a second Neumann eigenfunction for D, for which the nodal line is* *a part of a hyperbolic line l in $f(U)$, then φ is monotone in D on the family of hyperbolic lines in $f(U)$ passing through $f(0)$. In particular, φ attains its maximum and minimum over $\overline{D}$ at the boundary of D.*

References

[B] C. Bandle, Isoperimetric Inequalities and Applications, Pitman, London (1980). MR **81e:**35095

[Ba] R. Bass, Probabilistic Techniques in Analysis, Springer, New York (1995). MR **96e:**60001

[BaB] R. Bass and K. Burdzy, Fiber Brownian motion and the "hot spots" problem, Duke Math. J. **105** (2000), pp. 25 – 58. MR **2001g:**60190

[BaH1] R. Bass and P. Hsu, Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains, Ann. Probab. **19** (1991), pp. 486 – 508. MR **92i:**60142

[BaH2] R. F. Bass and E. P. Hsu, Pathwise uniqueness for reflecting Brownian motion in Euclidean domains, Probab. Theory and Rel. Fields **117** (2000), pp. 183 – 200. MR **2001e:**60164

[BB] R. Bañuelos and K. Burdzy, On the "Hot Spots" conjecture of J. Rauch, J. Funct. Anal. **164** (1999), pp. 1 – 33. MR **2000m:**35085

[BK] K. Burdzy and W. Kendall, Efficient Markovian couplings: Examples and counterexamples, Ann. Appl. Probab. **10** (2000), pp. 362 – 409. MR **2002b:**60129

[BW] K. Burdzy and W. Werner, A counterexample to the "hot spots" conjecture, Ann. of Math. **149** (1999), pp. 309 – 317. MR **2000b:**35044

[C] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, Orlando (1984). MR **86g:**58140

[D] Peter L. Duren, Univalent Functions, New York, Springer (1983). MR **85j:**30034

[JN] D. Jerison and N. Nadirashvili, The "hot spots" conjecture for domains with two axes of symmetry, J. Amer. Math. Soc. **13** (2000), pp. 741 – 772. MR **2001f:**35110

[K] B. Kawohl, Rearrangements and Convexity of Level Sets in PDE, Lecture Notes in Mathematics, Vol. 1150, Springer-Verlag, Berlin (1985). MR **87a:**35601

[N] N. S. Nadirashvili, On the multiplicity of the eigenvalues of the Neumann problem, Soviet. Math. Dokl. **33** (1986), pp. 281 – 282. MR **88a:**35068

[Pa] M. N. Pascu, Probabilistic approaches to eigenvalue problems, Ph. D. thesis, University of Connecticut (2001).

[P] Ch. Pommerenke, Boundary behaviour of conformal maps, New York, Springer (1991). MR **95b:**30008

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269

Current address: Department of Mathematics, Purdue Uniwersity, West Lafayette, Indiana 47907-1395

E-mail address: pascu@math.purdue.edu