POSITIVITY, SUMS OF SQUARES
AND THE MULTI-DIMENSIONAL MOMENT PROBLEM

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Abstract. Let \( K \) be the basic closed semi-algebraic set in \( \mathbb{R}^n \) defined by some finite set of polynomials \( S \) and \( T \), the preordering generated by \( S \). For \( K \) compact, \( f \) a polynomial in \( n \) variables nonnegative on \( K \) and real \( \epsilon > 0 \), we have that \( f + \epsilon \in T \). In particular, the \( K \)-Moment Problem has a positive solution. In the present paper, we study the problem when \( K \) is not compact. For \( n = 1 \), we show that the \( K \)-Moment Problem has a positive solution if and only if \( S \) is the natural description of \( K \) (see Section 1). For \( n \geq 2 \), we show that the \( K \)-Moment Problem fails if \( K \) contains a cone of dimension 2. On the other hand, we show that if \( K \) is a cylinder with compact base, then the following property holds:

\[
\forall f \in \mathbb{R}[X], f \geq 0 \text{ on } K \Rightarrow \exists q \in T \text{ such that } \forall \text{ real } \epsilon > 0, f + \epsilon q \in T.
\]

This property is strictly weaker than the one given in Schm"udgen (1991), but in turn it implies a positive solution to the \( K \)-Moment Problem. Using results of Marshall (2001), we provide many (noncompact) examples in hypersurfaces for which \((†)\) holds. Finally, we provide a list of 8 open problems.

Let \( K \) be the basic closed semi-algebraic set in \( \mathbb{R}^n \) defined by some finite set of polynomial inequalities \( g_1 \geq 0, \ldots, g_s \geq 0 \) and let \( T \) be the preordering in the polynomial ring \( \mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n] \) generated by \( g_1, \ldots, g_s \). For \( K \) compact, Schm"udgen proves in [15] that:

\((*)\) The \( K \)-Moment Problem has a positive solution.

\((†)\) \( \forall f \in \mathbb{R}[X], f \geq 0 \text{ on } K \Rightarrow \forall \text{ real } \epsilon > 0, f + \epsilon \in T \).

In the present paper, we consider the status of \((*)\) and \((†)\) when \( K \) is not compact. At the same time, we consider a third property:

\((‡)\) \( \forall f \in \mathbb{R}[X], f \geq 0 \text{ on } K \Rightarrow \exists q \in T \text{ such that } \forall \text{ real } \epsilon > 0, f + \epsilon q \in T \).

which we prove is strictly weaker than \((†)\) and, at the same time, which implies \((*)\). Many (non-compact) examples are given where \((‡)\) holds. Many examples are given where \((*)\) fails. The question of whether or not \((‡)\) and \((*)\) are equivalent is an open problem.

In Section 1 we introduce the problem and the notation. In Section 2 we settle the case \( n = 1 \). In Section 3 we study property \((*)\) using the method developed by Berg, Christensen and Jensen in [2]. We combine this with a result of Scheiderer in [13] to prove that \((*)\) fails whenever \( K \) contains a cone of dimension 2. In Section 4 we mention applications of results of the second author in [9]. In particular, we construct a large number of noncompact examples where \((‡)\) holds. In Section 5 we
prove (‡) holds for cylinders with compact cross section. In Section 6 we provide a list of open problems.

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1. Introduction

Fix an integer \( n \geq 1 \) and denote the polynomial ring \( \mathbb{R}[X_1,\ldots,X_n] \) by \( \mathbb{R}[X] \) for short. \( \sum \mathbb{R}[X]^2 \) denotes the set of finite sums \( \sum f_i^2, \ f_i \in \mathbb{R}[X]. \) For \( S = \{g_1,\ldots,g_s\}, \) a finite subset of \( \mathbb{R}[X], \) let \( KS \) denote the basic closed semi-algebraic set in \( \mathbb{R}^n \) determined by \( S; \) i.e.,

\[
KS = \{ x \in \mathbb{R}^n \mid g_i(x) \geq 0, \ i = 1,\ldots,s \}.
\]

Let \( TS \) denote the preordering in \( \mathbb{R}[X] \) generated by \( S, \) i.e., the set of all sums \( \sum \sigma_\epsilon g_\epsilon^e, \ e = (e_1,\ldots,e_s) \) running through the finite set \( \{0,1\}^s, \) \( \sigma_\epsilon \in \sum \mathbb{R}[X]^2, \) where \( g_\epsilon^e := g_1^{e_1} \cdots g_s^{e_s}. \)

Denote by \( \mathcal{T}_{S}^{\text{alg}} \) the set \( \{ f \in \mathbb{R}[X] \mid f \geq 0 \text{ on } KS \}. \) Points \( x \in \mathbb{R}^n \) are in one-to-one correspondence with algebra homomorphisms \( L: \mathbb{R}[X] \to \mathbb{R} \) via

\[
x = (L(X_1),\ldots,L(X_n)), \ L(f) = f(x).
\]

Under this correspondence, points \( x \in KS \) correspond to algebra homomorphisms \( L \) satisfying \( L(g_i) \geq 0, \ i = 1,\ldots,s \) or, equivalently, \( L(TS) \geq 0. \)

One can also consider the dual cone

\[
KS^{\text{lin}} = \{ L: \mathbb{R}[X] \to \mathbb{R} \mid L \text{ is linear (}\neq 0) \text{ and } L(TS) \geq 0 \},
\]

and the double dual cone \( \mathcal{T}_{S}^{\text{lin}} = \{ f \in \mathbb{R}[X] \mid L(f) \geq 0 \text{ for all } L \in \mathcal{T}_{S}^{\text{lin}} \}; \) see Section 3 for a more thorough discussion of the double dual cone. Since every algebra homomorphism is, in particular, a linear map, we have that

\[
KS \hookrightarrow KS^{\text{lin}} \text{ and } \mathcal{T}_{S}^{\text{alg}} \supseteq \mathcal{T}_{S}^{\text{lin}} \supseteq \mathcal{T}_{S}.
\]

Note: \( T_S, T_{S}^{\text{lin}} \) generally depend on \( S, \) \( T_{S}^{\text{alg}} \) depends only on the basic closed set \( KS. \)

Note: \( L \in KS^{\text{lin}} \Rightarrow L(1) > 0, \) since \( 1 \in TS \) implies \( L(1) \geq 0. \) If \( L(1) = 0 \) then, for any \( f \in \mathbb{R}[X], \) the identity \( (k + f)^2 = k^2 + 2kf + f^2 \) implies \( 2kL(f) + L(f^2) \geq 0 \) for all real \( k, \) and so \( L(f) = 0. \)

We are interested here in the relationship between \( T_{S}^{\text{alg}}, T_{S}^{\text{lin}} \) and \( TS \) for all the various choices of (finite) \( S \) in \( \mathbb{R}[X], n \geq 1. \) In particular, we are interested in the relationship between the condition

\[ (*) \quad T_{S}^{\text{alg}} = T_{S}^{\text{lin}} \]

and the condition

\[ (\dagger) \quad \forall f \in \mathbb{R}[X], f \in T_{S}^{\text{alg}} \Rightarrow f + \epsilon \in T_{S} \text{ for any real } \epsilon > 0. \]

We are also interested in a weak version of (\dagger) referred to as (\ddagger) (see below) and a strong version of (\dagger); namely, \( T_{S}^{\text{alg}} = TS. \) The condition \( T_{S}^{\text{alg}} = TS \) is examined in a recent paper by Scheiderer [13].
The study of the relationship between $T^\text{alg}_S$ and $T_S$ goes back at least to Hilbert and is a cornerstone of modern semi-algebraic geometry. The Positivstellensatz proved by Stengle in 1974 implies, in particular, that

$$\forall f \in \mathbb{R}[X], f \in T^\text{alg}_S \iff \exists p, q \in T_S \text{ and } m \geq 0 \text{ such that } pf = f^{2m} + q.$$  

The interest in $T^\text{lin}_S$ comes from functional analysis. For a linear functional $L$ on $\mathbb{R}[X]$, one is interested in when there exists a positive Borel measure $\mu$ on $\mathbb{R}^n$ supported by some given closed set $K = K_S$ in $\mathbb{R}^n$ such that

$$\forall f \in \mathbb{R}[X], L(f) = \int_{\mathbb{R}^n} f d\mu.$$ 

According to a result of Haviland [5], [6], this will be the case iff for all $f \in T^\text{alg}_S$. The Moment Problem (or at least a version of it) is the following: When is it true that every $\mu \in \mathcal{M}(\mathbb{R}^n)$ supported by some given closed set $K_S$? In view of the result of Haviland, this is equivalent to asking: When does (*) hold?

1.1 Remark. For each $n$-tuple of nonnegative integers $k = (k_1, \ldots, k_n)$, denote the monomial $X_1^{k_1} \cdots X_n^{k_n}$ by $X^k$ for short. The monomials form a basis for $\mathbb{R}[X]$; so each linear functional $L$ on $\mathbb{R}[X]$ corresponds to a function $p : (\mathbb{Z}^+)^n \to \mathbb{R}$ via $p(k) = L(X^k)$. We say $p$ is positive definite if

$$\sum_{i,j=1}^m p(k_i + k_j)c_i c_j \geq 0$$

for arbitrary (distinct) $k_1, \ldots, k_m \in (\mathbb{Z}^+)^n$, $c_1, \ldots, c_m \in \mathbb{R}$, and $m \geq 1$. For $g \in \mathbb{R}[X]$, $g = \sum a_k X^k$, $g(E)p : (\mathbb{Z}^+)^n \to \mathbb{R}$ is defined by $g(E)p(\ell) = \sum \phi_k a_k p(k + \ell)$.

The condition that $L \in K^\text{lin}_S$ corresponds exactly to the condition that the functions $g^e(E)p$, $e \in \{0,1\}^n$ are positive definite; see [4], [11] or [15]. Consequently, (*) is equivalent to the assertion that every nonzero function $p : (\mathbb{Z}^+)^n \to \mathbb{R}$ with $g^e(E)p$ positive definite for all $e \in \{0,1\}^n$ comes from a positive Borel measure on $\mathbb{R}^n$ supported by $K_S$.

Schmüdgen’s 1991 paper [15] settles the Moment Problem in the compact case. In [15], Schmüdgen proves if $K_S$ is compact, then (*) and (†) both hold. In particular, this settles the problem posed by Berg and Maserick in [11], [4]. In [17], Wörmann proves $K_S$ is compact iff $T_S$ is archimedean and uses this to obtain Schmüdgen’s result as a corollary of the Kadison-Dubois Theorem.

Denote by $P_d$ the (finite-dimensional) vector space consisting of all polynomials in $\mathbb{R}[X]$ of degree $\leq 2d$, and let $T_d = T_S \cap P_d$. $T_d$ is obviously a cone in $P_d$, i.e., $T_d + T_d \subseteq T_d$ and $\mathbb{R}^+ T_d \subseteq T_d$. Denote by $\overline{T_d}$ (resp., $\text{int}(T_d)$) the closure (resp., interior) of $T_d$ in $P_d$. The identity $pq = \frac{1}{2}((p + q)^2 - p^2 - q^2)$ applied to monomials $p, q$ of degree $\leq d$ implies $P_d = T_d - T_d$. This means that $T_d$ contains a basis of $P_d$; so $\text{int}(T_d) \neq \emptyset$.

1.2 Remark. $T_S$ is archimedean iff $1 \in \text{int}(T_d)$ for all $d \geq 0$. This is just a matter of sorting out the definitions: If $k \pm f \in T_S$ for sufficiently large $k \in \mathbb{Z}$, then $1 \pm \epsilon f \in T_S$ for sufficiently small $\epsilon > 0$. In fact, one even knows that $T_S$ is archimedean iff $k - \sum_{i=1}^n X_i^2 \in T_S$ for $k$ sufficiently large [17]; so $T_S$ is archimedean iff $1 \in \text{int}(T_1)$.
We are also interested in the condition
\[(\ddagger) \quad \forall f \in \mathbb{R}[X], f \in T_S^{\text{alg}} \Rightarrow \exists q \in T_S \text{ such that } \forall \text{ real } \epsilon > 0, f + \epsilon q \in T_S.\]

Obviously, \((\ddagger) \Rightarrow (\ddagger)\) (taking \(q = 1\)). The significance of \((\ddagger)\) is clear from the following:

1.3 Proposition. The following are equivalent:

1. \((\ddagger)\) holds;
2. \(T_S^{\text{alg}} = T_S^{\text{lin}} = \bigcup_{d \geq 0} \overline{T_d}\).

Proof. Clearly \(\overline{T_d} \subseteq T_S^{\text{lin}}\). For \(f \in P_d, q \in T_d\), the condition that \(f + \epsilon q \in T_S\) for all real \(\epsilon > 0\) is equivalent to

\[
\lambda f + (1 - \lambda)q \in T_d \text{ for every } 0 < \lambda < 1,
\]

i.e., that every point on the open line segment joining \(q\) and \(f\) belongs to \(T_d\). In particular, it implies that \(f \in \overline{T_d}\). Thus we see that \((1) \Rightarrow (2)\). Now assume \((2)\), and suppose \(f \in T_S^{\text{alg}}\). Thus, for \(d\) sufficiently large, \(f \in \overline{T_d}\). Pick \(q\) to be any point in the interior of \(T_d\). (Recall \(\text{int}(T_d) \neq \emptyset\).) Then obviously the open line segment joining \(f\) with \(q\) belongs to \(T_d\), and so \((1)\) holds. \(\square\)

Thus, condition \((\ddagger)\) implies not only that \((*)\) holds, but also that \(T_S^{\text{lin}} = \bigcup_{d \geq 0} \overline{T_d}\).

Note: The requirement in \((\ddagger)\) that \(q\) belongs to \(T_S\) is unnecessary: Using

\[
q = \left(\frac{q + 1}{2}\right)^2 - \left(\frac{q - 1}{2}\right)^2,
\]

we see that if \(f + \epsilon q \in T_S, q \in \mathbb{R}[X]\), then \(f + \epsilon \left(\frac{q + 1}{2}\right)^2 \in T_S\).

See Section 4 for additional discussion of \((\ddagger)\) and for examples where \((\ddagger)\) holds. See Section 5 for examples where \((\ddagger)\) holds but \((\ddagger)\) does not hold. No examples are known where \((*)\) holds but \((\ddagger)\) does not hold. No examples are known where \(T_S^{\text{lin}} \neq \bigcup_{d \geq 0} \overline{T_d}\). In fact, there is a general shortage of examples.

Instead of working with \(T_S\), one can work with the \(\sum \mathbb{R}[X]^2\)-module \(M_S\) generated by \(S\), i.e., the set of all sums \(s_0 + \sum_{i=1}^s s_i g_i, \sigma_i \in \sum \mathbb{R}[X]^2, i = 0, \ldots, s, \) and \(M_S^{\text{lin}}\), the double dual cone of \(M_S\). One is interested in the analog of \((*)\):

\[(*)_M\]

\[T_S^{\text{alg}} = M_S^{\text{lin}},\]

the analog to \((\ddagger)\):

\[(\ddagger)_M\]

\[\forall f \in \mathbb{R}[X], f \geq 0 \text{ on } K_S \Rightarrow \forall \text{ real } \epsilon > 0, f + \epsilon \in M_S,\]

and the analog to \((\ddagger)\):

\[(\ddagger)_M\]

\[\forall f \in \mathbb{R}[X], f \in T_S^{\text{alg}} \Rightarrow \exists q \in M_S \text{ such that } \forall \text{ real } \epsilon > 0, f + \epsilon q \in M_S.\]

Again, \(T_S^{\text{alg}} = M_S\) implies \((\ddagger)_M\) which, in turn, implies \((\ddagger)_M\), and \((\ddagger)_M\) is equivalent to \(T_S^{\text{alg}} = M_S^{\text{lin}} = \bigcup_{d \geq 0} \overline{M_d}\) where \(\overline{M_d}\) denotes the closure in \(P_d\) of \(M_d := M_S \cap P_d\). \((\ddagger)_M\) is known to hold in a few cases, for example, if \(K_S\) is compact and either \(n = 1\) \([7]\) or the polynomials \(g_1, \ldots, g_s\) are linear \([7], [10]\) or \(s \leq 2\) \([7]\). In \([7]\), Jacobi and Prestel develop a powerful valuation-theoretic method for testing the validity of \((\ddagger)_M\) when \(K_S\) is compact.
2. The case \( n = 1 \)

In this section we consider the case \( n = 1 \), extending the work of Berg and Maserick in [4].

**2.1 Theorem.** Suppose \( n = 1 \). If \( K_S \) is compact, then (†) and (†\(_M\)) hold. If \( K_S \) is not compact, then the conditions (†), (‡), (§) and \( T^{alg}_S = T_S \) are equivalent and the conditions (†\(_M\)), (‡\(_M\)), (§\(_M\)) and \( T^{alg}_S = M_S \) are equivalent.

**Proof.** If \( K_S \) is compact, then (†) holds by Schmüdgen’s result and (‡\(_M\)) holds by [7, Remark 4.7]. We defer the proof of the rest of Theorem 2.1 to Section 3 where it is an immediate consequence of Theorem 3.5.

Note: Already in the case \( n = 1 \), we see the dichotomy between the compact case and the noncompact case. If \( K_S \) is compact, then (†) (resp., (‡\(_M\))) holds but, unlike what happens in the noncompact case, this does not imply that \( T^{alg}_S = T_S \) (resp., \( T^{alg}_S = M_S \)) holds. For example, if \( S = \{(1 - X^2)^2\} \), then \( T^{alg}_S \neq T_S \); see [10, Example 4.3.3].

The second part of Theorem 2.1 is only useful if we know when \( T^{alg}_S = T_S \) (resp., \( T^{alg}_S = M_S \)) holds. For the rest of the section we concentrate our attention on answering this question.

**2.2 Theorem.** Suppose \( n = 1 \). If \( K_S \) is not compact and \( T^{alg}_S = T_S \), then:

(i) If \( K_S \) has a smallest element, call it \( a \), then \( r(X - a) \in S \) for some real \( r > 0 \).

(ii) If \( K_S \) has a largest element, call it \( a \), then \( r(a - X) \in S \) for some real \( r > 0 \).

(iii) For every \( a, b \in K_S \) with \( a < b \) and \( (a, b) \cap K_S = \emptyset \), \( r(X - a)(X - b) \in S \) for some real \( r > 0 \).

Conversely, for any \( K_S \), if conditions (i), (ii) and (iii) hold, then \( T^{alg}_S = T_S \).

**2.3 Notes.** (1) Since \( K_S \) is a closed semi-algebraic set in \( \mathbb{R} \), it is the union of finitely many closed intervals and points (including possibly closed intervals of the form \((-\infty, a]\) or \([a, \infty)\)).

(2) If \( K \subseteq \mathbb{R} \) is any closed semi-algebraic set, then one checks easily that \( K = K_S \) for \( S \) the set of polynomials defined as follows:

- If \( a \in K \) and \((-\infty, a] \cap K = \emptyset \), then \( X - a \in S \).
- If \( a \in K \) and \([a, \infty) \cap K = \emptyset \), then \( a - X \in S \).
- If \( a, b \in K \), \( a < b \), \((a, b) \cap K = \emptyset \), then \((X - a)(X - b) \in S \).

- \( S \) has no other elements except these.

We call \( S \) the natural choice of generators for \( K \).

(3) Theorem 2.2 proves that, for \( K = K_S \) not compact, \( T^{alg}_S = T_S \) holds iff \( S \) contains the natural choice of generators (up to scalings by positive reals).

(4) If \( K_S \) is compact, then \( T^{alg}_S = T_S \) can hold without conditions (i), (ii) and (iii) holding. For example, if \( S = \{1 - X^2\} \), then \( 1 \pm X = \frac{1}{2}(1 \pm X)^2 \pm \frac{1}{2}(1 - X^2) \in T_S \) and so \( T^{alg}_S = T_S \).

**Proof.** Suppose \( T^{alg}_S = T_S \) and \( K_S \) is not compact. We can assume that the elements of \( S \) have degree \( \geq 1 \). Since \( K_S \) is not compact, it either contains an interval of the form \([a, \infty)\) or it contains an interval of the form \((-\infty, a]\). Replacing \( X \) by \(-X\) if necessary, we can assume we are in the first case. Thus each \( g_i \in S \) is nonnegative on some interval \([a, \infty)\), so has positive leading coefficient. Thus, for any element
p = \sum \sigma_{x}g_{x}^{e_{1}}\ldots g_{x}^{e_{r}} \text{ of } T_{S}, \text{ the degree of } p \text{ is equal to the maximum of the degrees of the terms } \sigma_{x}g_{x}^{e_{1}}\ldots g_{x}^{e_{r}}. \text{ Also, since each } \sigma_{x} \text{ is a sum of squares, it has even degree.}

Suppose that } K_{S} \text{ has a smallest element, say } a. \text{ Let } p = X - a. \text{ Then } p \geq 0 \text{ on } K_{S} \text{ and so } p \in T_{S}. \text{ Since } p \text{ has degree 1, } p \text{ is a sum of terms of the form } \sigma, \sigma \in \mathbb{R}^{+}, \text{ and } \sigma_{g_{i}} \text{ with } \sigma \in \mathbb{R}^{+} \text{ and } g_{i} \in S \text{ linear. Each such } g_{i} \text{ is } \geq 0 \text{ at } (a \in K_{S}). \text{ Consequently, at least one linear } g_{i} \in S \text{ is equal to zero at } a; \text{ so } g_{i} = r(X - a) \text{ as required.}

Suppose now that } a, b \in K_{S} \text{ are such that } a < b \text{ and } (a, b) \cap K_{S} = \emptyset. \text{ Let } p = (X - a)(X - b). \text{ Then } p \geq 0 \text{ on } K_{S} \text{ and so } p \in T_{S}. \text{ Since } p \text{ has degree 2, it is a sum of terms of the form } \sigma \in \sum \mathbb{R}[X]^{2} \text{ of degree 0 or 2, } \sigma_{g_{i}} \text{ with } \sigma \in \mathbb{R}^{+} \text{ and } g_{i} \in S \text{ linear or quadratic, and } \sigma_{g_{i}g_{j}} \text{ with } \sigma \in \mathbb{R}^{+} \text{ and } g_{i}, g_{j} \in S \text{ linear. Since any linear } g_{i} \in S \text{ is increasing and } g_{i}(a) \geq 0, g_{i} \text{ is positive on the interval } (a, b). \text{ Thus } p \geq \sigma_{g_{1}g_{2} \ldots + \sigma_{g_{t}}} \text{ on } (a, b) \text{ where } g_{1}, \ldots, g_{t} \text{ are quadratics in } S \text{ which assume at least one negative value on } (a, b), \text{ and } \sigma_{1}, \ldots, \sigma_{t} \text{ are positive reals. Define the width of a quadratic } g \text{ to be } r_{2} - r_{1} \text{ where } r_{1} \leq r_{2} \text{ are the roots of } g \text{ (or 0 if } g \text{ has no roots). Each } g_{i} \text{ opens upward and is nonnegative at } a, b \text{ (since } a, b \in K_{S}), \text{ has its roots between } a \text{ and } b \text{ and consequently has width at most } b - a, i = 1, \ldots, t. \text{ It suffices to show that } g_{i} \text{ has width exactly } b - a \text{ for some } i \in \{1, \ldots, t\}, \text{ for then } g_{i} \text{ necessarily has the form } r(X - a)(X - b) \text{ for some real } r > 0. \text{ Since the width of } p \text{ is equal to } b - a \text{ and } \sigma_{g_{i}} \text{ has the same width as } g_{i}, \text{ this is a consequence of the following elementary result:}

2.4 Lemma. If } f_{1}, f_{2} \text{ are quadratics with positive leading coefficients, then}

\text{width}(f_{1} + f_{2}) \leq \max\{\text{width}(f_{1}), \text{width}(f_{2})\}.

Proof. Without loss of generality, we can assume width}(f_{1}) \geq \text{width}(f_{2}) \text{ and that } f_{1} \text{ has positive width. Shifting and scaling, we can assume } f_{1} = X^{2} - X, f_{2} = c(X - a)(X - (a + b)), c > 0, 0 \leq b \leq 1. \text{ Thus } f_{1} + f_{2} = (c + 1)X^{2} - (2ac + bc + 1)X + ca(a + b) \text{ and}

\text{width}(f_{1} + f_{2}) = \sqrt{(2ac + bc + 1)^{2} - 4(c + 1)ca(a + b)}/c + 1.

Thus we are reduced to showing that

\begin{equation}
(2ac + bc + 1)^{2} - 4(c + 1)ca(a + b) \leq (c + 1)^{2},
\end{equation}

i.e., that

\begin{equation}
(2ac + bc + 1)^{2} \leq (c + 1)^{2} + 4(c + 1)ca(a + b).
\end{equation}

Expanding and canceling, this reduces to showing that

\begin{equation}
b^{2}c + 4a + 2b \leq c + 2 + 4a^{2} + 4ab
\end{equation}

or, equivalently, that

\begin{equation}
(1 - b^{2})(c + 1) + (2a + b - 1)^{2} \geq 0.
\end{equation}

Since } 0 \leq b \leq 1 \text{ and } c > 0, \text{ this is clear. \hfill \Box}

Note: Equality holds iff } b = 1 \text{ and } a = 0, \text{ i.e., iff } f_{2} = cf_{1}.

Suppose now that (i), (ii), and (iii) hold. \text{ If } K_{S} \text{ has a smallest element } a, \text{ then } X - a \in T_{S}; \text{ so } X - d = (X - a) + (a - d) \in T_{S} \text{ for any } d \leq a. \text{ Similarly, if } K_{S} \text{ has a largest element } a, \text{ then } a - X \in T_{S}; \text{ so } d - X = (a - X) + (d - a) \in T_{S} \text{ for any } d \geq a. \text{ Also, if } a, b \in K_{S} \text{ are such that } a < b \text{ and } (a, b) \cap K_{S} = \emptyset, \text{ then } (X - a)(X - b) \in T_{S}. \text{ \hfill \Box}
Moreover, by the proof of [4] Lemma 4, if \( a \leq c \leq d \leq b \), then \((X - c)(X - d)\) is in the preordering generated by \((X - a)(X - b)\); so \((X - c)(X - d) \in T_S\).

Suppose \( f \in \mathbb{R}[X] \), \( f \geq 0 \) on \( K_S \). We prove \( f \in T_S \) by induction on the degree. If \( f \) has degree zero, it is clear. If \( f \geq 0 \) on \( 
\), then \( f \in \sum \mathbb{R}[X]^2 \); so, in particular, \( f \in T_S \). Thus we can assume that \( f(c) < 0 \) for some \( c \). There are three possibilities: either \( K_S \) has a least element \( a \) and \( c < a \) or \( K_S \) has a largest element \( a \) and \( c > a \) or there exist \( a, b \in K_S \), \( a < b \) with \((a, b) \cap K_S = \emptyset \), and \( a < c < b \). In the first case, \( f \) has a least root \( d \) in the interval \((c, a] \), \( X - d \in T_S \), \( f = (X - d)g \) for some \( g \in \mathbb{R}[X] \) and one checks that \( g \geq 0 \) on \( K_S \). In the second case, \( f \) has a greatest root \( d \) in the interval \([a, c) \) and a least root \( e \) in the interval \((c, b) \), \( (X - d)(X - e) \in T_S \), \( f = (X - d)(X - e)g \) and \( g \geq 0 \) on \( K_S \). Thus, in any case, the result follows by induction on the degree.

The question of when \( T_S^{\text{alg}} = M_S \) holds for \( K = K_S \) not compact is more complicated. Theorem 2.2 provides an obvious necessary condition: \( S \) must contain the natural choice of generators (up to scalings by positive reals). But this necessary condition is sufficient only in very special cases:

**2.5 Theorem.** Suppose \( K \) is not compact and \( S \) is the natural choice of generators for \( K \). Then \( T_S^{\text{alg}} = M_S \) holds iff either \( |S| \leq 1 \) or \( |S| = 2 \) and \( K \) has an isolated point.

**Proof.** If \( |S| \leq 1 \), then \( M_S = T_S \), and so \( T_S^{\text{alg}} = M_S \) by Theorem 2.2; also see [4, Theorem 1]. Suppose \( |S| = 2 \) and \( K \) has an isolated point. If this isolated point is the least element of \( K \), then \( S = \{g_1, g_2\} \), \( g_1 = X - b \), \( g_2 = (X - b)(X - c) \), \( b < c \). Then one checks that

\[
g_1g_2 = (X - c)^2g_1 + (c - b)g_2 \in M_S;
\]

so \( T_S = M_S \). The case where the isolated point is the greatest element of \( K \) is similar. The remaining case is where \( S = \{g_1, g_2\} \), \( g_1 = (X - a)(X - b) \), \( g_2 = (X - b)(X - c) \), \( a < b < c \). In this last case, one checks that

\[
g_1g_2 = \frac{b - a}{c - a}(X - c)^2g_1 + \frac{c - b}{c - a}(X - a)^2g_2 \in M_S.
\]

Thus, in all cases, \( M_S = T_S \); so \( T_S^{\text{alg}} = M_S \) by Theorem 2.2.

Suppose now that either \(|S| = 2 \) and \( K \) has no isolated points, or \(|S| \geq 3 \). Say \( S = \{g_1, \ldots, g_s\} \). Replacing \( X \) by \(-X\) if necessary, we can assume that \( K \) contains an interval of the form \([a, \infty)\). Reindexing, we can suppose \( g_1 \) is either linear of the form \( g_1 = X - b \) (corresponding to the case where \( K \) has a least element \( b \)) or \( g_1 = (X - a)(X - b) \), \( a < b \) where \((a, b)\) is the left-most gap of \( K \) (corresponding to the case where \( K \) has no least element) and that \( g_2 = (X - c)(X - d) \) where \((c, d)\) is the right-most gap of \( K \). The remaining elements \( g_3, \ldots, g_s \) of \( S \) then correspond to the intermediate gaps, if any. By our hypothesis, \( b < c \). We claim that \( g_1g_2 \notin M_S \). Since \( g_1g_2 \geq 0 \) on \( K \), this will complete the proof. Suppose, to the contrary, that

\[
g_1g_2 = \sigma_0 + \sigma_1g_1 + \cdots + \sigma_sg_s, \quad \sigma_i \in \sum \mathbb{R}[X]^2.
\]

Comparing degrees, we see that \( \sigma_1 \) has degree \( \leq 2 \). Note: Since \( g_1(x)g_2(x) \) is strictly negative for \( x \) close to \( b \), \( b < c \), \( \sigma_1 \) is not identically zero. Let \( g = g_2 - \sigma_1 \). Thus \( g_1g = \sigma_0 + \sigma_2g_2 + \cdots + \sigma_sg_s \). Thus \( g_1(c)g(c) \geq 0 \), \( g_1(d)g(d) \geq 0 \); so \( g(c) \geq 0 \), \( g(d) \geq 0 \). On the other hand, from \( g = g_2 - \sigma_1 \), it follows that \( g(c) \leq 0 \), \( g(d) \leq 0 \).
This implies $g(c) = g(d) = 0$, which in turn implies that $\sigma_1(c) = \sigma_1(d) = 0$. Since $\sigma_1$ is a sum of squares of degree $\leq 2$ which are not identically zero, this is a contradiction.

3. Examples where (*) fails

In this section, we study condition (*) using an extension of the method introduced by Berg, Christensen and Jensen in [2]. A major tool here is the recent work of Scheiderer in [13].

As we have seen in Section 1,
\[ T_{\text{alg}} S = T_S \Rightarrow (\dagger) \Rightarrow (\ddagger) \Rightarrow (*) . \]

If $T_{\text{lin}} S = T_S$, we can obviously say more:

3.1 Proposition. If $T_{\text{lin}} S = T_S$, then the conditions $(\dagger)$, $(\ddagger)$, $(*)$ and $T_{\text{alg}} S = T_S$ are equivalent.

Similarly, for $M_S$:

3.2 Proposition. If $M_{\text{lin}} S = M_S$, then the conditions $(\dagger_M)$, $(\ddagger_M)$, $(* M)$ and $T_{\text{alg}} S = M_S$ are equivalent.

Proof. This is clear.

We recall results from the theory of locally convex vector spaces. Any vector space $V$ over $\mathbb{R}$ comes equipped with a unique finest locally convex topology [3, Sect. 1.1.9]. For a cone $C$ in $V$, $C^{\text{lin}}$ denotes the double dual cone of $C$, i.e., the set of all $x \in V$ such that $L(x) \geq 0$ holds for all linear functionals $L$ on $V$ with $L(C) \geq 0$. We need the following two results:

3.3 Lemma. If $C$ is a cone in a vector space $V$ over $\mathbb{R}$, then
\begin{enumerate}
\item $C^{\text{lin}}$ is the closure of $C$;
\item $C^{\text{lin}} = C \iff C$ is closed.
\end{enumerate}

Proof. According to the Bipolar Theorem [3, Theorem 1.3.6], $C^{\text{lin}}$ is the smallest closed convex set in $V$ containing $C$. Since the closure of a cone is a cone (so, in particular, is convex), (1) is clear. (2) is immediate from (1).

3.4 Lemma. Suppose $C$ is a cone in a vector space $V$ over $\mathbb{R}$ and $C_i = C \cap V_i$ where $V_i, i \geq 0$ are finite-dimensional subspaces of $V$ with $V_i \subseteq V_{i+1}$ and $V = \bigcup_{i \geq 0} V_i$. Then the following are equivalent:
\begin{enumerate}
\item $C$ is closed in $V$;
\item $C_i$ is closed in $V_i$ for each $i \geq 0$.
\end{enumerate}

Proof. See [3, Lemma 6.3.3].

Our next result provides examples where $T_S$ and $M_S$ are closed.

3.5 Theorem. If $K_S$ contains a cone of dimension $n$, then $T_S$ and $M_S$ are closed.


Note: Theorem 3.5 applies, in particular, if $n = 1$ and $K_S$ is not compact. Combining this with Lemma 3.3(2) and Propositions 3.1 and 3.2 completes the proof of Theorem 2.1.
Proof. The first assertion follows from the second, replacing $S$ by the set of all products $g^e$, $e \in \{0,1\}^s$, $e \neq (0,\ldots,0)$. By Lemma 3.4, to prove the second assertion, it suffices to show that $M_i$ is closed in $P_i$, for each $i \geq 0$. Let $S = \{g_1,\ldots,g_s\}$. We can assume each $g_i$ is not zero. By our hypothesis, $K_S$ contains a cone $C$ with nonempty interior. Making a linear change in coordinates if necessary, we can assume the vertex of $C$ is at the origin.

For $f$ any nonzero element of $M_S$, let $f = f_0 + \cdots + f_d$ be its homogeneous decomposition. Observe that, for any $a \in C$ and any $\lambda > 0$, 
\[ f(\lambda a) = f_0 + \lambda f_1(a) + \cdots + \lambda^d f_d(a). \]

Since $\lambda a \in C \subseteq K_S$ for all positive $\lambda$, it follows that, if $f_d(a) \neq 0$, then $f_d(a) > 0$.

Now suppose 
\[ p = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s, \]
\[ \sigma_i \in \sum \mathbb{R}[X]^2. \]

Let $d$ denote the maximum of the degrees of $\sigma_0, \sigma_1 g_1, \ldots, \sigma_s g_s$. Using the fact that $C$ has nonempty interior, we can pick a point $a \in C$ such that the homogeneous parts of highest degree of the various polynomials in question do not vanish; so the homogeneous part of $p$ having degree $d$ is positive when evaluated at $a$ (so, in particular, it is not zero). To summarize: if $p$ has degree $d$, then $\sigma_0$ has degree $\leq d$ and each $\sigma_i$, $i \geq 1$ has degree $\leq d - \deg(g_i)$. Thus any $p \in M_m$ is expressible as above with $\sigma_i = \sum f_{ij}^2$, $\deg(f_{0ij}) \leq m$, $\deg(f_{ij}) \leq m - \frac{1}{2} \deg(g_i)$, $i \geq 1$. Also, as in [2], we can assume $\sigma_i = \sum_{j=1}^N f_{ij}^2$, where $N$ is the dimension of $P_m$.

Making an additional linear change of coordinates, we can also assume that the point $(1,\ldots,1)$ lies in the interior of $C$ and none of the $g_i$ vanish at this point. Thus, for real distinct $a_1,\ldots,a_{2m+1}$ sufficiently close to 1, the set 
\[ H = \{(a_{j_1},\ldots,a_{j_{2m+1}}) \mid 1 \leq j_k \leq 2m+1, k = 1,\ldots,2m+1 \} \]
is contained in $C$ and $g_i(a) > 0$ for each $a \in H$. As explained in [2], two elements of $P_m$ are equal iff they are equal on $H$, and the topology on $P_m$ coincides with the topology of pointwise convergence on $H$. Also, for any $a \in H$, $f_{0ij}(a)^2 \leq \sigma_0(a) \leq p(a)$, and $f_{ij}(a)^2 \leq \sigma_i(a) \leq p(a)/D_i$ where $D_i$ is the minimum of the $g_i(a), a \in H$, if $i \geq 1$. Thus, if $p_t$ is a sequence in $M_m$ converging to some $p \in P_m$, then, as in [2], there exists a subsequence $p_{t_k}$ with the associated $f_{t_k,ij}$ converging pointwise on $H$ to some $f_{ij} \in P_m$ of the appropriate degree; so $p \in M_m$. 

Theorem 3.5 does not cover the case where $K_S$ is a curve. At the same time, $T_S$ is closed for many curves, at least, for the “right” choice of $S$. We give two examples to illustrate.

3.6 Example. Consider the curve 
\[ C = \{(x,y) \in \mathbb{R}^2 \mid y = q(x)\} \]
in $\mathbb{R}^2$, $q \in \mathbb{R}[X]$. Then $C = K_S$ where $S = \{-Y - q\}$. We claim that (i) holds for this choice of $S$. Suppose $f \geq 0$ on $C$. Then $f(X,Y) = f(X,q(X)) + h(X,Y)$ with $h$ vanishing on $C$. Also $f(X,q(X)) \geq 0$ on $\mathbb{R}$. So $f(X,q(X)) \in \sum \mathbb{R}[X]^2$. Thus we are reduced to showing that $\epsilon + h \in T_S$ for each real $\epsilon > 0$. Since $Y - q$ is irreducible, $h = (Y - q)h_1$ for some $h_1 \in \mathbb{R}[X,Y]$. So 
\[ \epsilon + h = \epsilon(1 + \frac{h_1}{2\epsilon}(Y - q))^2 - \frac{h_1^2}{4\epsilon}(Y - q)^2 \in T_S. \]
Note: $Y - q \notin T_S$. If $Y - q = \sigma_0 - \sigma_1(Y - q)^2$, $\sigma_0, \sigma_1 \in \sum \mathbb{R}[X,Y]^2$, then $\sigma_0$ vanishes on $C$; so $(Y - q)^2$ divides $\sigma_0$. Dividing through by $Y - q$, this implies $Y - q$ divides 1, a contradiction. Thus $T_S^{alg} \neq T_S$. So $T_S$ is not closed for this choice of $S$. On the other hand, the curve $C$ is described more naturally as $C = K_S$ where $S = \{Y - q, -(Y - q)\}$, and, for this choice of $S$, $T_S^{alg} = T_S$ (so, in particular, $T_S$ is closed). To prove this, just write $h_1 = r^2 - s^2$, $r = (h_1 + 1)/2$, $s = (h_1 - 1)/2$. So $f = f(X, q(X)) + r^2(Y - q) - s^2(Y - q) \in T_S$.

3.7 Example. Consider the curve

$$C = \{(x, y) \in \mathbb{R}^2 \mid y^2 = q(x)\}$$

where $q \in \mathbb{R}[X]$ is not a square. Thus $C = K_S$ where $S = \{Y^2 - q, -(Y^2 - q)\}$. Assume that $C$ is not compact. Replacing $X$ by $-X$ if necessary, we can assume that $q(x) > 0$ for $x$ sufficiently large (so the leading coefficient of $q$ is positive). We claim that, under this assumption, $T_S$ is closed. Since $T_S = \sum \mathbb{R}[X,Y]^2 + (Y^2 - q)$, where $(Y^2 - q)$ denotes the ideal generated by $Y^2 - q$, it suffices to prove that $\sum A^2$ is closed in $A$, where $A$ denotes the coordinate ring of $C$, i.e.,

$$A = \frac{\mathbb{R}[X,Y]}{(Y^2 - q)}.$$

Every $f \in A$ is expressible uniquely as $f = g + h\sqrt{q}$, $g, h \in \mathbb{R}[X]$. Let $Q_d$ denote the subspace of $A$ consisting of all $f = g + h\sqrt{q} \in A$ with $\deg(g), \deg(h) \leq 2d$. According to Lemma 3.4, it suffices to show that $(\sum A^2) \cap Q_d$ is closed in $Q_d$ for each $d \geq 0$. We need certain degree estimates. If $f = g + h\sqrt{q}$, $g, h \in \mathbb{R}[X]$, then $f^2 = (g^2 + h^2q) + (2gh)\sqrt{q}$. Thus every element of $\sum A^2$ has the form

$$p = \sum_{i=1}^k (g_i + h_i\sqrt{q})^2 = \sum_{i=1}^k (g_i^2 + h_i^2q) + 2 \sum_{i=1}^k g_i h_i \sqrt{q}.$$

Thus, if $p = p_1 + p_2\sqrt{q}$, $p_1, p_2 \in \mathbb{R}[X]$, then

$$\deg(p_1) = \max\{2\deg(g_i), 2\deg(h_i) + \deg(q) \mid 1 \leq i \leq k\}.$$

In particular, if $p \in Q_d$, then $\deg(g_i) \leq d$ and $\deg(h_i) \leq d - \deg(q)/2$, $i = 1, \ldots, k$. With these estimates in hand, the proof that $(\sum A^2) \cap Q_d$ is closed in $Q_d$ follows along the same lines as in the proof of Theorem 3.5 and can safely be left to the reader.

Since $T_S$ is closed, the various conditions (i), (ii), (xii) and $T_S^{alg} = T_S$ are equivalent. We claim that these conditions hold if $\deg(q) = 1$ or 2 and that they fail if $\deg(q) \geq 3$. The first assertion is elementary; e.g., see [13, Proposition 2.17]. Suppose now that $\deg(q) \geq 3$. If $q$ is not in $\sum \mathbb{R}[X]^2$, then $q$ takes on negative values. So there exists a quadratic $p \in \mathbb{R}[X]$ that takes on negative values with $p \geq 0$ on $C$. If $p \in T_S$, then $p = \sum_{i=1}^k (g_i^2 + h_i^2q)$ for some $g_i, h_i \in \mathbb{R}[X]$. Then $\sum_{i=1}^k h_i^2 \neq 0$; so $\deg(p) \geq \deg(q)$, a contradiction. If $q \in \sum \mathbb{R}[X]^2$, then $q = r^2 + s^2$, $r, s \in \mathbb{R}[X]$. Take $p = p_1 + Y$ where $p_1 \in \mathbb{R}[X]$ has degree $\leq \max\{\deg(r), \deg(s)\} + 1$ and is such that $p_1 \geq |r| + |s|$ on $\mathbb{R}$. Then

$$p(x, \pm \sqrt{q(x)}) = p_1(x) \pm \sqrt{q(x)} \geq |r(x)| + |s(x)| \pm \sqrt{r(x)^2 + s(x)^2} \geq 0$$

for each $x \in \mathbb{R}$; so $p \geq 0$ on $C$. If $p \in T_S$, then $p_1 = \sum_{i=1}^k (g_i^2 + h_i^2q)$, $1 = 2 \sum_{i=1}^k g_i h_i$ for some $g_i, h_i \in \mathbb{R}[X]$. Then $\sum h_i^2 \neq 0$; so $\deg(p_1) \geq \deg(q)$. Since $\deg(q) = \max\{2\deg(r), 2\deg(s)\}$, this contradicts the assumption that $\deg(q) \geq 3$.  


Note: According to [13] Theorem 3.4, if $C$ is nonsingular and $\deg(q) \geq 3$, then the preordering $T_{S}^{\text{alg}}$ is not even finitely generated.

We turn now to the case where $\dim(K_{S}) \geq 2$. In [2] Theorem 4, Berg, Christensen and Jensen use their weak version of Theorem 3.5 to show that $(\ast)$ fails for $S = \emptyset$ (so $K_{S} = \mathbb{R}^{n}$, $T_{S} = \sum \mathbb{R}[X]^{2}$) if $n \geq 2$. In [3] Theorem 6.3.9, this same method is used to show that $(\ast)$ fails for $S = \{X_{1}, \ldots, X_{n}\}$ if $n \geq 2$. We proceed to generalize these results. In general, for $T_{S}$ closed, showing that $(\ast)$ fails is equivalent to showing $\exists \ p \in T_{S}^{\text{alg}}, \ p \notin T_{S}$. Such a polynomial always exists when $\dim(K_{S})$ is large enough, e.g., the Motzkin polynomial if $S = \emptyset$, $n \geq 2$. The following general result is proved in [13].

3.8 Theorem. If $\dim(K_{S}) \geq 3$, then there exists a polynomial $p \geq 0$ on $\mathbb{R}^{n}$ such that $p \notin T_{S}^{\text{lin}}$.


Theorem 3.8 implies, in particular, that $(\ast)$ fails whenever $n \geq 3$ and $K_{S}$ contains a cone of dimension $n$. Another result in [13] allows us to greatly improve on this:

3.9 Theorem. If $n = 2$ and $K_{S}$ contains a 2-dimensional cone, then there exists a polynomial $p \geq 0$ on $\mathbb{R}^{2}$ such that $p \notin T_{S}$.


Note: The proof of Theorem 3.9 given in [13] is highly nontrivial. It uses deep results on the existence of “enough” psd regular functions on noncompact nonrational curves [13] Section 3] in conjunction with an extension theorem for extending such functions to the ambient space [13] Section 5].

3.10 Corollary. $(\ast)$ fails whenever $n \geq 2$ and $K_{S}$ contains a cone of dimension 2.

Note: We are not claiming now that $T_{S}$ is closed.

Proof. Changing coordinates, we may assume the cone is given by

$$X_{1} \geq 0, \ X_{2} \geq 0, \ X_{i} = 0, \ i \geq 3.$$ 

Define $S' \subseteq \mathbb{R}[X_{1}, X_{2}]$ by $S' = \{g'_{1}, \ldots, g'_{i}\}$ where $g'_{i} = g_{i}(X_{1}, X_{2}, 0, \ldots, 0)$. Thus $K_{S'}$ contains the cone defined by $X_{1} \geq 0, \ X_{2} \geq 0$. So, by Theorem 3.5, $T_{S'}$ is closed. Use Theorem 3.9 to pick $p = p(X_{1}, X_{2})$ such that $p \geq 0$ on $\mathbb{R}^{2}$ and $p \notin T_{S'}$. Then $p \in T_{S}^{\text{alg}}$. We claim that $p \notin T_{S}^{\text{lin}}$. For suppose $p \in T_{S}^{\text{lin}}$. Any linear functional $L$ on $\mathbb{R}[X_{1}, X_{2}]$ with $L(T_{S'}) \geq 0$ extends to a linear functional $L'$ on $\mathbb{R}[X]$ with $L'(T_{S}) \geq 0$ if $L'(f) = L(f(X_{1}, X_{2}, 0, \ldots, 0))$. This is clear. Thus, for any such $L$, $L(p) = L'(p) \geq 0$; so $p \in T_{S}^{\text{lin}}$. Since $T_{S}^{\text{lin}} = T_{S}$, this contradicts the choice of $p$.

The method of Corollary 3.10 applies in other cases as well:

3.11 Example. Take $n = 2$, $S = \{X, 1 - X, Y^{3} - X^{2}\}$. $K_{S}$ consists of the points $(x, y) \in \mathbb{R}^{2}$ on the vertical strip $0 \leq x \leq 1$ with $y \geq x^{2/3}$. We claim that $(\ast)$ fails for $S$. Let $S' \subseteq \mathbb{R}[Y]$ be defined by $S' = \{Y^{3}\}$. Then $K_{S'} = \{0, \infty\}$; so, by Theorem 3.5, $T_{S'}$ is closed. One checks that $Y \notin T_{S'}$. (Degree considerations show that

\footnote{Apparently this is not true for $\dim(K_{S}) = 2$. In the “Added to proof” at the end of [13], Scheiderer reports the discovery of a smooth compact surface $K_{S}$ with $T_{S}^{\text{alg}} = T_{S}$.}
\(Y = \sigma_0 + \sigma_1 Y^3, \sigma_0, \sigma_1 \in \mathbb{R}[Y]^2\) is not possible.) Clearly \(Y \geq 0\) on \(K_S\). We claim \(Y \notin T_{S}^{\text{fin}}\). We argue as in the proof of Corollary 3.10: Every linear functional \(L\) on \(\mathbb{R}[Y]\) with \(L(T_{S}) \geq 0\) extends to a linear functional \(L'\) on \(\mathbb{R}[X, Y]\) with \(L'(T_S) \geq 0\) via \(L'(f) = L(f(0, Y))\). Thus, if \(Y \in T_{S}^{\text{fin}}\), then \(L(Y) = L'(Y) \geq 0\) for every such \(L\); so \(Y \in T_{S}^{\text{fin}} = T_S\).

4. Consequences of the results in \([9]\)

In this section we recall results from \([9]\), and mention applications of these results to the problem at hand. The results in \([9]\) extend Schmüdgen’s result in the compact case, replacing the assumption that \(K_S\) is compact with the assumption that \(p \in 1 + T_S\) is chosen so that the coordinate functions \(X_1, \ldots, X_n\) are bounded on \(K_S\) by \(kp^\ell\) for \(k, \ell\) sufficiently large. The method of proof in \([9]\) is a generalization of Wörmann’s method in \([17]\).

Note: Such a polynomial \(p\) always exists, e.g., \(p = 1 + \sum_{i=1}^n X_i^2\). The “best” choice of \(p\) may depend on \(K_S\). For example:

— If \(K_S\) is compact, we can take \(p = 1\).
— If \(n = 2\) and \(S = \{X, 1 - X\}\), we can take \(p = 1 + Y^2\).

Fixing such a polynomial \(p\), we have:

4.1 Theorem. For any \(f \in \mathbb{R}[X]\) there exist integers \(k, \ell\) such that \(kp^\ell \pm f \in T_S\).

\(\)Proof. See \([9]\) Corollary 1.4. \(\square\)

This gives better control over the element \(q\) appearing in condition (‡). We can choose \(q\) to be a power of \(p\) if we want. In more detail, we have:

4.2 Corollary. The following are equivalent:

1. (‡) holds;
2. \(\forall f \in \mathbb{R}[X], f \in T_S^{\text{alg}} \Rightarrow \exists \ell \geq 0 \text{ such that } \forall \text{ real } \epsilon > 0, f + \epsilon p^\ell \in T_S\).

\(\)Proof. (2) \(\Rightarrow\) (1) is clear. For (1) \(\Rightarrow\) (2), use Theorem 4.1 to pick \(k, \ell\) so that \(kp^\ell - q \in T_S\) and use the identity \(f + \epsilon p^\ell = (f + \frac{q}{\epsilon} \ell) + \epsilon (kp^\ell - q)\). \(\square\)

The second main result in \([9]\) is:

4.3 Theorem. For any \(f \in \mathbb{R}[X]\), the following are equivalent:

1. \(f \in T_S^{\text{alg}}\);
2. \(\text{for any sufficiently large integer } m \text{ and all rational } \epsilon > 0, \text{ there exists an integer } \ell \geq 0 \text{ such that } p^\ell(f + \epsilon p^m) \in T_S\).

\(\)Proof. See \([9]\) Corollary 3.1. \(\square\)

Let 
\[
T'_S = \{f \in \mathbb{R}[X] \mid p^\ell f \in T_S \text{ for some } \ell \geq 0\}.
\]

For \(d \geq 0\), let \(T'_d = T'_S \cap P_d\).

4.4 Corollary. \(T_S^{\text{alg}} = (T'_S)^{\text{fin}} = \bigcup_{d \geq 0} T'_d\).

\(\)Proof. Suppose \(f \in T_S^{\text{alg}}\). By Theorem 4.3 there exists \(m \geq 0\) such that, for all \(\epsilon > 0\), \(f + \epsilon p^m \in T'_S\). This implies \(f \in T'_d\) for \(d\) sufficiently large. \(\square\)
Note: In view of the result of Haviland in [5], Corollary 4.4 implies, in particular, that a linear functional \( L \) on \( \mathbb{R}[X] \) comes from a positive Borel measure on \( \mathbb{R}^n \) supported by \( K_S \) if \( L(T^1_{alg}) \geq 0 \); also see [10].

Note: The preordering \( T^1_{alg} \) is not finitely generated in general. This is the case, for example, if \( K_S \) contains a cone of dimension 2; see Corollary 3.10.

Theorem 4.3 also yields a large number of noncompact examples where (†) holds. Denote by \( \mathbb{R}[X,Y] \) the polynomial ring in \( n+1 \) variables \( X_1, \ldots, X_n, Y \) and consider the finite set \( S' = S \cup \{ 1 - pY, -(1 - pY) \} \) in \( \mathbb{R}[X,Y] \). \( K_{S'} \) consists of those points on the hypersurface

\[
H = \{(x,y) \in \mathbb{R}^{n+1} \mid p(x)y = 1 \}
\]

in \( \mathbb{R}^{n+1} \) which map to \( K_S \) under the projection \((x,y) \mapsto x\).

4.5 Corollary. (†) holds for \( S' \).

Proof. Suppose \( f \in T^1_{alg} \). Thus \( f(X, \frac{1}{p^m}) \geq 0 \) on \( K_S \). Writing \( f(X, \frac{1}{p^m}) = \frac{h}{p^k} \), \( g \in \mathbb{R}[X] \), we see that \( g \geq 0 \) on \( K_S \). So there exists \( m \geq 0 \) such that for each \( \epsilon > 0 \), \( p\epsilon (g + \epsilon p^m) \in T_S \) for some \( \ell \geq 0 \). Since \( p - 1 \in T_S \), there is no harm in assuming \( m \geq k \).

Thus

\[
\frac{g}{p^k} + \epsilon p^{m-k} = \frac{h}{p^k}
\]

for some \( h \in T \). This implies that the polynomial \( f + \epsilon p^{m-k} - hp^{k+\ell}Y^{2(k+\ell)} \) vanishes on \( H \). So

\[
f + \epsilon p^{m-k} = hp^{k+\ell}Y^{2(k+\ell)} + j(1-pY)
\]

for some \( j \in \mathbb{R}[X,Y] \). Using the identity \( j = (\frac{i+j}{2})^2 - (\frac{i-j}{2})^2 \), this implies \( f + \epsilon p^{m-k} \in T_{S'} \) for all real \( \epsilon > 0 \).

Thus, for example, although (⋆) fails for the plane (Corollary 3.10), (†) holds for the surface \( Z = \frac{1}{1+X+Y^2} \) in \( \mathbb{R}^3 \).

4.6 Corollary. For a nonzero linear functional \( L \) on \( \mathbb{R}[X] \), the following are equivalent:

(1) \( L \) comes from a positive Borel measure on \( \mathbb{R}^n \) supported by \( K_S \);

(2) \( L \) extends to a linear functional \( L' \) on \( \mathbb{R}[X,Y] \) satisfying \( L'(T_{S'}) \geq 0 \).

Proof. (1) \( \Rightarrow \) (2). Define

\[
L'(f) = \int_{\mathbb{R}^n} f(x, \frac{1}{p(x)})d\mu(x)
\]

where \( \mu \) is the measure. Since \( p \geq 1 \) on \( K_S \), \( L' \) is well-defined. (2) \( \Rightarrow \) (1). If \( f \in T^1_{alg} \), then \( f \in T^1_{alg} = T^1_{lin} \), so \( L(f) = L'(f) \geq 0 \). (1) follows now, by Haviland’s result.

Remark. \( T^2_{alg} \) and \( T_{S'} \) are related by \( T^2_{alg} = T_{S'} \cap \mathbb{R}[X] \).

4.7 Example. We use Corollary 4.5 to construct examples where \( \dim(K_S) = 1 \), \( K_S \) is not compact, (†) holds and \( T_{S'} \) is not closed.

(1) Take \( n = 1 \), \( S = \{ X^3 \} \), \( p = 1 + X^2 \). Thus \( K_{S'} \) is defined by \( Y = 1/(1+X^2) \), \( X \geq 0 \). We claim \( T_{S'} \) is not closed. Otherwise, by Corollary 4.5 and Proposition 3.1, \( T^1_{alg} = T_{S'} \); so, intersecting with \( \mathbb{R}[X] \), \( T^1_{alg} = T^2_{alg} \). Since \( X \geq 0 \) on \( K_S \), this implies \( (1+X^2)^\ell X \in T_S \) for some \( \ell \geq 0 \), i.e., \( (1+X^2)^\ell X = \sigma_0 + \sigma_1 X^3 \), \( \sigma_i \in \mathbb{R}[X]^2 \).
\(i = 0, 1\). Evaluating at \(X = 0\), we see that \(X^2\) divides \(\sigma_0\). Consequently, \(X\) divides \((1 + X^2)^i\), a contradiction.

(2) Take \(n = 2, S = \{Y^2 - X^3, -(Y^2 - X^3)\}\). Since \(Y^2 = X^3\) on \(K_S\), we can take \(p = 1 + X^2\). Thus \(K_S\) is defined by \(Z = 1/(1 + X^2), Y^2 = X^3\). We claim that \(T_S\) is not closed. Otherwise, as in (1), \((1 + X^2)^\ell X \in T_S\) for some \(\ell \geq 0\); so, by the method of Example 3.7, \((1 + X^2)^\ell X = \sigma_0 + \sigma_1 X^3\) for some \(\sigma_i \in \sum \mathbb{R}[X]^2, i = 0, 1\).

5. CYLINDERS WITH COMPACT CROSS SECTION

We denote by \(\mathbb{R}[X,Y]\) the polynomial ring in \(n + 1\) variables \(X_1, \ldots, X_n, Y\). In this section we consider a subset \(S = \{g_1, \ldots, g_s\}\) of \(\mathbb{R}[X,Y]\) where the polynomials \(g_1, \ldots, g_s\) involve only the variables \(X_1, \ldots, X_n\); so \(K_S\) has the form \(K \times \mathbb{R}, K \subseteq \mathbb{R}^n\). We further assume that \(K\) is compact. We describe this situation by saying that \(K_S\) is a cylinder with compact cross section. We aim to prove:

5.1 Theorem. If \(K_S\) is a cylinder with compact cross section, then (1) holds.

Proof. Suppose \(f \in \mathbb{R}[X,Y]\) is such that \(f \geq 0\) on \(K_S\). Fix \(d \geq 1\) so that the degree of \(f\) as a polynomial in \(Y\) is \(\leq 2d\). Let \(\epsilon > 0\) be given. Let \(f_1 = f + \epsilon + \epsilon Y^{2d}, f_1 = a_0 + a_1 Y + \cdots + a_{2d} Y^{2d}, a_i \in \mathbb{R}[X]\). \(f_1\) is strictly positive on \(K \times \mathbb{R}\) and its leading coefficient is strictly positive on \(K\). Denote by \(A\) the ring of all continuous functions from \(K\) to \(\mathbb{R}\) and consider \(\phi = \hat{a_0} + \hat{a_1} Y + \cdots + \hat{a_{2d}} Y^{2d} \in A[Y]\) where \(\hat{\alpha}\) is the continuous function corresponding to \(\alpha\). Since \(a_{2d}\) is strictly positive, the roots of \(\phi\) depend continuously on the coefficients \(\square\) page 3) and, since \(\phi\) has no real roots, \(\phi\) factors as \(\phi = \hat{a}_{2d} Y^d\) where the coefficients of \(\psi\) are continuous complex-valued functions on \(K\) and the “bar” denotes complex conjugation. (At each point of \(K\), take \(\psi = \prod_{i=1}^d Y - z_i\) where the \(z_1, \ldots, z_d\) are the roots with positive imaginary part, counting multiplicities. The coefficients are elementary symmetric functions of \(z_1, \ldots, z_d\), so are continuous.) Decomposing \(\psi\) as \(\psi_1 + i\psi_2, \psi_1, \psi_2 \in A[Y]\) yields

\[
(1) \quad \phi = \hat{a}_{2d}(\psi_1^2 + \psi_2^2).
\]

Now use the compactness of \(K\) to approximate the coefficients of \(\psi_1, \psi_2\) closely by polynomials in \(\mathbb{R}[X]\). In this way we obtain polynomials \(h_1, h_2 \in \mathbb{R}[X][Y] = \mathbb{R}[X, Y]\) of degree \(\leq 2d\) in \(Y\) whose coefficients approximate closely the corresponding coefficients of \(\psi_1, \psi_2\). Then

\[
(2) \quad f + \epsilon + \epsilon Y^{2d} = a_{2d}(h_1^2 + h_2^2) + \sum_{i=0}^{2d} b_i Y^i,
\]

where \(b_0, \ldots, b_{2d}\) are (obviously) polynomials in \(X\) with \(|b_i| < \epsilon\) on \(K, i = 0, \ldots, 2d\). By Schmüdgen’s Theorem, \(a_{2d} \in T\) and \(\epsilon \pm b_i \in T\) where \(T\) is the preordering in \(\mathbb{R}[X]\) generated by \(g_1, \ldots, g_s\). This yields

\[
(3) \quad \epsilon Y^{2i} + b_i Y^i \in T_S, \text{ for } i \text{ even.}
\]

For \(i\) odd, say \(i = 2k + 1\), use the identity \(Y^{2k+1} = \frac{1}{2} Y^{2k}((Y + 1)^2 - Y^2 - 1)\) plus the fact that \(\epsilon Y^{2k}(Y + 1)^2 + b_i Y^{2k}(Y + 1)^2, \epsilon Y^{2k}Y^2 - b_i Y^{2k}Y^2\) and \(\epsilon Y^{2k} - b_i Y^{2k}\) all belong to \(T_S\) to obtain

\[
(4) \quad \epsilon(Y^{2i+1} + Y^i + Y^{i-1}) + b_i Y^i \in T_S, \text{ for } i \text{ odd.}
\]
Thus, adding \( \epsilon (2 + Y + 3Y^2 + Y^3 + 3Y^4 + \cdots + 2Y^{2d}) \) to each side of (2), we see finally that

\[
f + \epsilon (3 + Y + 3Y^2 + Y^3 + \cdots + 3Y^{2d}) \in T_\mathcal{S}.
\]

Since the polynomial \( q = 3 + Y + 3Y^2 + Y^3 + \cdots + 3Y^{2d} \) depends only on \( d \) (not on \( \epsilon \)), the proof is complete.

Cylinders with compact cross section also provide examples where (‡) holds but (†) does not hold.

**5.2 Example.** Take \( n = 2 \). Take \( S \subseteq \mathbb{R}[X,Y] \) defined by \( S = \{ X^3(1-X)^3 \} \). Then \( K_S \) is the strip \([0,1] \times \mathbb{R} \) and (†) holds by Theorem 5.1. Consider \( f = XY^2 \). Clearly \( f \geq 0 \) on \( K_S \). We claim that \( \forall \epsilon > 0, f + \epsilon \notin T_\mathcal{S} \). This will show, in particular, that (†) does not hold. For suppose

\[
f + \epsilon = \sigma_0 + \sigma_1 X^3(1-X)^3, \quad \sigma_0, \sigma_1 \in \sum \mathbb{R}[X,Y]^2.
\]

Comparing degrees in \( Y \), we see that \( \sigma_0, \sigma_1 \) have the form

\[
\sigma_0 = \sum (a_i + b_i Y)^2, \quad \sigma_1 = \sum (c_j + d_j Y)^2,
\]

with \( a_i, b_i, c_j, d_j \in \mathbb{R}[X] \). Equating coefficients of \( Y^2 \), this gives

\[
X = \sum b_i^2 + \sum d_j^2 X^3(1-X)^3.
\]

Evaluating at \( X = 0 \), we see that \( X \) divides each \( b_i \). Dividing through by \( X \), we see that \( X \) divides 1, a contradiction.

**5.3 Notes.** (1) Example 5.2 generalizes to arbitrary dimension, taking \( f = pY^2 \) with \( p \in \mathbb{R}[X] \), \( p \geq 0 \) on \( K \) and \( p \notin T \). Such a polynomial \( p \) will exist (regardless of the description of \( K \)) if \( \dim(K) \) is sufficiently large; see Theorem 3.8.

(2) In Example 5.2 we are not using the “natural” description of the strip \( 0 \leq X \leq 1 \). The question of whether or not (†) holds for the natural description (i.e., \( S = \{ X, 1-X \} \)) is open.

Theorem 5.1 also extends to sets of the form \( K \times [0, \infty) \) with \( K \) compact:

**5.4 Corollary.** Suppose \( S' = S \cup \{Y\} \), \( S \) as in Theorem 5.1 (so \( K_{S'} = K \times [0, \infty) \)). Then (†) holds for \( S' \).

Note: The result is false if we take the “wrong” description of \( K \times [0, \infty) \), e.g., for \( S' = S \cup \{Y^3\} \), (*) fails. (Just modify the proof of Example 3.11.)

**Proof.** Suppose \( f = f(X,Y) \) is nonnegative on \( K_{S'} \). Then \( f(X,Y^2) \) is nonnegative on \( K_S \). By the proof of Theorem 5.1, for \( q = 3 + Y + 3Y^2 + \cdots + 3Y^{2d} \), \( d \) the degree of \( f \) in \( Y \), we have, for all real \( \epsilon > 0 \),

\[
f(X,Y^2) + \epsilon q = \sum \sigma_\epsilon g^\epsilon,
\]

with \( \sigma_\epsilon \in \sum \mathbb{R}[X,Y]^2 \). Replacing \( Y \) by \(-Y\), adding, and dividing by 2, using the standard identity

\[
\frac{1}{2}(g(Y)^2 + g(-Y)^2) = h(Y^2)^2 + k(Y^2)^2 Y^2
\]
What about the presentation \( K \)? If \( n \geq 2 \) and \( K \) contains a cone of dimension 2, then \( \{1 - X^2\} \) holds. What if \( K \) contains a cone of dimension 2, and \( \{1 - Y^2\} \)?

5. It should be possible to use the method of Corollary 4.5 to construct a curve where \( \dagger \) holds but \( \ddagger \) fails. We were not able to do this. In general, it would be nice to know more about the curve case.

6. Find examples where \( K \) is not compact, \( \dim(K_S) \geq 2 \), and \( \dagger \) holds. So far, we only know examples where \( \dagger \) holds. Does \( \ddagger \) hold for the strip \( 0 \leq X \leq 1 \) in \( \mathbb{R}^2 \) with the presentation \( S = \{X, 1 - X\} \)?

7. For \( n = 1 \) and \( K_S \) compact, whether \( T_S \) is closed or depends on the presentation. For example, for \( S = \{1 + X, 1 - X\} \), \( T_S \) is closed, but for \( S = \{(1 - X^2)^3\} \), \( T_S \) is not closed. In both cases, \( K_S = [-1, 1] \). If \( K_S = [-1, 1]^n \), \( n \geq 3 \), then \( T_S \) is never closed, regardless of the presentation (Theorem 3.8). What if \( n = 2 \)? We know \( T_S \) is not closed for the presentation \( S = \{(1 - X^2)^3, (1 - Y^2)^3\} \).

8. Certain results are independent of the presentation, i.e., depend only on \( K_S \): if \( K_S \) is compact, then \( \dagger \) holds, regardless of the presentation; if \( n \geq 2 \) and \( K_S \) contains a cone of dimension 2, then \( * \) fails, regardless of the presentation; if \( K_S \) contains a cone of dimension \( n \), then \( T_S \) is closed, regardless of the presentation. Other results depend on the particular presentation. In general, one would like to know to what extent results are independent of the presentation.

Update, March 2002:

When the present paper was submitted, Powers and Scheiderer informed us that they were also working on similar questions. Their paper has since appeared [11]. At the end of [11], the authors announce that they can answer open problems 4, 6 and 7 above, based on unpublished work of Scheiderer [13]. However, in a
recent private communication, Scheiderer has withdrawn the claim about problem 6, which therefore appears still to be open. The above open problems have also led to recent work by Schwartz [16]. It seems that the methods developed in [16] might provide an answer to problem 1, and possibly to problem 2 as well.

References

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