TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 11, Pages 4493–4504
S 0002-9947(02)03102-1
Article electronically published on July 2, 2002

# CLASSIFICATION OF COMPACT COMPLEX HOMOGENEOUS SPACES WITH INVARIANT VOLUMES

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ABSTRACT. We solve the problem of the classification of compact complex homogeneous spaces with invariant volumes (see Matsushima, 1961).

## 1. INTRODUCTION

We call a $2n$-dimensional manifold $M$ a *complex homogeneous space with invariant volume* if there is a complex structure and a nonzero $2n$-form on $M$ such that a transitive Lie transformation group keeps both the complex structure and the $2n$-form invariant. There are many papers published in the direction of classification of such manifolds, e.g., [4], [6], [7], [8], [9], [11], [12], [16], [19], [20], [27], [29], [30], [35] and the references there (see also [2], [5], [10], [17], [11], [12], [14], [33], [34] for related topics involving compact complex homogeneous spaces). In this paper, we shall deal with the compact case and finish the classification up to certain better-understood building blocks.

In this paper, we shall always deal with compact manifolds except those manifolds in the Preliminaries and in Theorem 1.

After [30], not much has been done for the classification of compact complex homogeneous spaces with invariant volumes until very recently. Two breakthroughs in [6] are: first, the proof that the Hano-Kobayashi fiberation (we might also call it the Ricci form reduction) is holomorphic and is the same as the anticanonical fibering in the compact case; second, the classification of compact complex homogeneous spaces with invariant pseudo-Kähler structures (see [6], [19] and [11], [12], also [13]).

In [19] Huckleberry observed that one can handle the pseudo-Kähler case by using methods from symplectic geometry.

Huckleberry's method was used in [11], [12] to obtain following theorem:

**Proposition 1.** *Every compact homogeneous complex manifold with a* 2*-cohomology class* $\omega$ *such that* $\omega^n$ *is not zero in the top cohomology is a product of a rational homogeneous space and a complex parallelizable solv-manifold with a symplectic structure which is right-invariant in its universal covering.*

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Received by the editors September 28, 2001 and, in revised form, April 21, 2002.

2000 *Mathematics Subject Classification.* Primary 53C30, 32M10, 32M05 14M17; Secondary 14M20, 53C10, 53C56.

*Key words and phrases.* Invariant volume, homogeneous, product, fiber bundles, complex manifolds, parallelizible manifolds, discrete subgroups, classifications.

Supported by NSF Grants DMS-9401755 and DMS-9627434.

This generalized the result of [5] for the Kähler case (one does not assume that the Kähler form is invariant).

For a general compact complex homogeneous space with an invariant volume, the symplectic method does not apply. However, our original method (see [20], [30] and [6]) gives a classification (see Theorem 4).

**Main Theorem 1.** *Every compact complex homogeneous space with an invariant volume form is a principal homogeneous complex torus bundle over the product of a projective rational homogeneous space and a parallelizable manifold. Conversely, every compact complex homogeneous space that is a complex homogeneous torus bundle over a product of a projective rational homogeneous space and a complex parallelizable manifold admits a transitive real Lie group $G$, acting on $M$ by holomorphic transforms and preserving a volume form on $M$.*

It is this theorem that motivates the torus bundle type of structure theorems in [14].

For more details of the Main Theorem 1, one might look at sections 3, 4 and 5. We actually give a very explicit construction of this kind of manifold as a complex quotient manifold of a product of a manifold in [35] and a manifold in [34] by the anti-diagonal action of a complex torus which acts on both manifolds and is in the center of the latter manifold. See section 5 for the details.

The proof of Theorem 3 is critical. By applying a straightforward argument in the proof of Theorem 3, we can obtain the second part of the Main Theorem 3 in [15].

Theorem 1 may be applied to the noncompact case and Theorem 5 is stronger than our Main Theorem 1 in some sense. For some technical reasons, we do not state them here but refer the interested readers to the detailed sections.

We also note that every compact complex homogeneous space $M$ with a 2-cohomology class such that its top power is nonzero in the top cohomology group, admits a transitive real Lie transformation group $G$, which acts on $M$ by holomorphic transforms and preserves a volume form. It is also well-known that all simply connected compact complex homogeneous spaces studied in [35] have invariant volumes since the transitive group can be chosen to be a compact Lie group.

In [30] Matsushima considered the special case of a semisimple group action. He proved:

**Proposition 2.** *If $G/H$ is a compact complex homogeneous space with a $G$-invariant volume and $G$ semisimple, then $G/H$ is a holomorphic fiber bundle over a rational homogeneous space with a complex reductive parallelizable manifold as a fiber.*

Applying our Main Theorem 1 to this situation, we immediately obtain that the result of Matsushima can be generalized to the case when $G$ is reductive. Moreover, we have the following stronger result.

**Main Theorem 2.** *Assume that $G/H$ is a compact complex homogeneous space with a $G$-invariant volume and that $G$ is reductive. Then $G/H$ is a principal holomorphic torus bundle over the product of a projective rational homogeneous space and a complex parallelizable homogeneous space of a semisimple complex Lie group.*

Theoretically our arguments can be applied to solve a more general problem of classifying all compact complex homogeneous spaces up to some better-understood building blocks. We shall leave it to [14].

## 2. Preliminaries

2.1. **Special Compact Complex Homogeneous Spaces.** A *rational homogeneous manifold* $Q$ is a compact complex manifold that can be realized as a closed orbit of a linear algebraic group in some projective space. Equivalently, $Q = S/P$ where $S$ is a complex semisimple Lie group and $P$ is a parabolic subgroup, i.e., a subgroup of $S$ that contains a maximal connected solvable subgroup (Borel subgroup). Every homogeneous rational manifold is simply-connected and is therefore an orbit of a compact group. In general, a quotient $K/L$ with $K$ compact and semisimple carries a $K$-invariant complex structure that is projective algebraic if and only if $L$ is the centralizer $C(T)$ of a torus $T \subset K$.

A *parallelizable complex manifold* is a compact quotient of a complex Lie group by a discrete subgroup. It is a *solv-manifold* or *nil-manifold* according as the complex Lie group is solvable or nilpotent. In the same way, we can define *reductive parallelizable manifolds* and *semisimple parallelizable manifolds*.

2.2. **Generalized Tits Fibrations.** In this subsection we recall some basic results on a generalization of the *Tits fibration*, introduced by A. Huckleberry and E. Oeljeklaus [21]. It coincides with a fibration considered by Hano [16] in case the isotropy group is connected. We call it the *$\mathcal{G}$-anticanonical fibering* as in [21] or the *HOT-fibration* as in [6]. Let $M = G/H$, $H^0$ be the identity component of $H$ and $\mathrm{Norm}_G(H^0)$ the normalizer of $H^0$ in $G$. Then we have:

**Proposition 3.** *Let $G$ be a connected real Lie group acting almost effectively and transitively as a group of holomorphic transformations on the complex manifold $M = G/H$ and let $G/H \to G/J$ be the HOT-fibration.*

*Then*

1. $J = \{k \in \mathrm{Norm}_G(H^0); R(k):\ G/H^0 \to G/H^0,\ gH^0 \to gkH^0,\ \textit{holomorphic}\}$ *where $G/H^0$ carries the complex structure induced by $G/H^0 \to G/H$. In particular, we have $J \subset \mathrm{Norm}_G(H^0)$.*
2. *$J/H^0$ is a complex Lie group and $G/H^0 \to G/J$ is a holomorphic $J/H^0$-principal fiber bundle. In particular, the fibering $G/H \to G/J$ is locally holomorphically trivial.*
3. *If $G$ is a connected complex Lie group and $H$ a closed complex subgroup, then $J = \mathrm{Norm}_G(H^0)$. Thus for a complex Lie group $G$, the HOT-fibration coincides with the Tits fibration.*

If $G$ is a complex Lie group and $H \subset G$ is a closed complex subgroup, then we have the *normalizer fibration* $G/H \to G/N$, where $N = N_G(H^0)$. Let $\mathcal{G}$ and $\mathcal{H}$ denote the Lie algebras of $G$ and $H$, respectively. The base space $G/N$ is realized as the $\mathrm{Ad}(G)$–orbit of the subspace $\mathcal{H}$ in the Grassmann manifold of subspaces of $\mathcal{G}$ that have the same dimension as that of $\mathcal{H}$. If $G/H$ is compact, then $G/N$ is a rational homogeneous manifold and $N/H$ is a compact parallelizable homogeneous manifold.

2.3. **More on HOT Fibrations.** We also recall Tits' result on the fibration of compact homogeneous spaces:

**Proposition 4.** *Let $G$ be a connected complex Lie group and $H$ a closed complex subgroup such that $G/H$ is compact. Then $G/\mathrm{Norm}_G(H^0)$ is a rational homogeneous space and $\mathrm{Norm}_G(H^0)/H$ is connected and parallelizable. Moreover, if $G/H \to G/R$ is a holomorphic fibration with parallelizable fiber $R/H$, then*

*$R \subset \mathrm{Norm}_G(H^0)$; if, in addition, the base $G/R$ is rational homogeneous, then $R = \mathrm{Norm}_G(H^0)$.*

Moreover, if $G$ is a real Lie group such that $G/H$ is a compact complex manifold with $G$ acting holomorphically and almost effectively, then by complexifying the vector fields corresponding to the Lie algebra, we can see that there exists a connected complex Lie group $G^{\mathbf{C}}$ such that $G \subset G^{\mathbf{C}}$ and $G/H = G^{\mathbf{C}}/H^{\mathbf{C}}$.

In general, $G^{\mathbf{C}}$ might not be a complexification of $G$, but we can always choose $G^{\mathbf{C}}$ such that $\mathcal{G}^{\mathbf{C}} = \mathcal{G} + i\mathcal{G}$.

From the definition of the HOT-fibration [21, 1.7], it is easy to observe that $G/H$ and $G^{\mathbf{C}}/H^{\mathbf{C}}$ have the same HOT-fibration:

**Proposition 5.** *Let $G$ be a connected real Lie group acting almost effectively and transitively as a group of holomorphic transformations on the compact, complex manifold $G/H = G^{\mathbf{C}}/H^{\mathbf{C}}$.*

*Let $G/H \to G/J$ denote the HOT-fibration of $G/H$. Then the action of $G^{\mathbf{C}}$ on $G/H$ preserves this fibration. Moreover, let $G^{\mathbf{C}}/H^{\mathbf{C}} \to G^{\mathbf{C}}/J^{\mathbf{C}}$ denote the Tits fibration. Then $J = J^{\mathbf{C}} \cap G$, i.e., $G/J = G^{\mathbf{C}}/J^{\mathbf{C}}$. Thus, for compact $G/H$, the HOT-fibration and the Tits fibration are the same, and the HOT-fibration does not depend on the choice of $G$. In particular, $J$ is connected and $G/J$ is rational homogeneous.*

2.4. **Hano-Kobayashi Fibration.** Next we want to discuss the Hano-Kobayashi fibration. We shall call it the *HK-fibration* (or *Ricci form reduction*, or the *canonical fibration*). Let $M$ be a complex manifold and $\omega = K(z,\ \bar{z})dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$ be an invariant volume form. We also set

$$R_{i\bar{j}} = \frac{\partial^2 \log K}{\partial z^i \partial \bar{z}^j}$$

and

$$\chi = i \sum R_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Then $\chi$ is called the *Ricci form* of $M$. We recall the main result on the HK-fibration for homogeneous complex manifolds (see [20]):

**Proposition 6.** *Let $M$ be a connected complex manifold and $G$ a connected real Lie group acting holomorphically on $M$. Assume, moreover, that $M = G/H$ admits a $G$-invariant volume element $\omega$ and denote by $\chi$ the associated Ricci form of $M$.*

*Then there exists a unique closed subgroup $I$ of $G$ containing $H$ and a nondegenerate closed two-form $\hat{\chi}$ on $G/I$ such that*

1. *$G/I$ is a homogeneous symplectic manifold with respect to $\hat{\chi}$.*
2. *The fiber $I/H$ of the projection $G/H \to G/I$ is a complex connected submanifold of $G/H$ and $\chi|_{I/H} = 0$.*
3. *The pull-back of $\hat{\chi}$ to $M$ is equal to $\chi$.*
4. *If $I/H$ is compact, then it is (complex) parallelizable.*

In [11], [12] we notice that the definition of the HOT-fibration is *global* ($J/H$ might not be connected, for example) and the definition of the HK-fibration is *local* (see (2) in Proposition 6). Therefore, we might call the fibration in Proposition 5 the *global HOT-fibration* (or *global anticanonical fibration*) and define the *local HOT-fibration* by the fibering $G/H \to G/J_1$ where $J_1$ is the minimal closed and open subgroup of $J$ that contains $H$ (hence $J_1/H$ is connected). We might also call the fibration in Proposition 6 the *local HK-fibration* (or *local canonical fibration*) and define the *global HK-fibration* by the fibering $G/H \to G/I_1$ where $I_1 = \{g \in G \mid_{g^* \operatorname{div}(jX) = \operatorname{div}(jX) \text{ for all } X \in \mathcal{G}}\}$, where $j$ is the complex structure and div is the divergence (see [20, p. 236] for the proof of Theorem A).

2.5. **Koszul Algebra.** In the rest of this paper, we shall frequently use arguments at the Lie algebra level.

First we recall the following result due to Koszul [27]:

**Proposition 7.** *Let $G$ be a real Lie group and $H$ a closed subgroup. Then $G/H$ admits a $G$-invariant complex structure if and only if there exists an endomorphism $j$ of $\mathcal{G}$ such that for all $x, y \in \mathcal{G}$, $r \in H$ we have:*

$$\begin{aligned}
&j\mathcal{H} \subset \mathcal{H},\\
&j^2 x = -x (\operatorname{mod} \ \mathcal{H}),\\
&\operatorname{Ad} r(jx) = j \operatorname{Ad} r(x) (\operatorname{mod} \ \mathcal{H}),\\
&[jx, jy] = j[jx, y] + j[x, jy] + [x, y] (\operatorname{mod} \ \mathcal{H}).
\end{aligned}$$

Notice that $j$ is only determined modulo $\mathcal{H}$; so we might assume that $j\mathcal{H} = 0$. If we assume that $M = G/H$ has a $G$-invariant volume form $\omega$, we let

$$\phi(x) = \operatorname{tr}_{\mathcal{G}/\mathcal{H}}(\operatorname{ad} jx - j \circ \operatorname{ad} x), \quad x \in \mathcal{G}.$$

Then $\phi(x) = \operatorname{div}(jX)$ where $X$ is the holomorphic vector field corresponding to $x$ (see [20, p. 236]). From [27] (see also [20, Lemma 3.2]):

**Proposition 8.** *The Ricci form associated with $\omega$ is given by the formula*

$$\chi(x,\ y) = \phi([x,\ y]), \quad x,\ y \in \mathcal{G}.$$

*Moreover, the Ricci form satisfies for $x,\ y,\ z \in \mathcal{G}$,*

$$\begin{aligned}
&\chi(jx,\ jy) = \chi(x,\ y),\\
&\chi([x,\ y],\ z) + \chi([y,\ z],\ x) + \chi([z,\ x],\ y) = 0,\\
&\chi(\mathcal{G},\ \mathcal{H}) = 0.
\end{aligned}$$

With the notation of this subsection, we have an expression of global HOT-fibration $G/H \to G/J$ with

$$J = \{g \in G \mid gH^0g^{-1} = H^0,\ j \circ \operatorname{Ad}(g) = \operatorname{Ad}(g) j \circ \ (\operatorname{mod} \ \mathcal{H})\}.$$

Actually, this fibration is the same as the fibration given by the anticanonical line bundle in the compact case and an expression of the global HK-fibration $G/H \to G/I_1$ with

$$\begin{aligned}
I_1 = \{g \in G \mid \ &\operatorname{tr}_{\mathcal{G}/\mathcal{H}}(\operatorname{ad}(j(\operatorname{Ad} g(x) - x)) - j \circ \operatorname{ad}(\operatorname{Ad} g(x) - x)) = 0\\
&\text{for all } x \in \mathcal{G}\}.
\end{aligned}$$

This comes from the proof of Theorem A in [20, p. 236] (see also Lemmas 3.1 and 3.2 there). The condition in the bracket is the same as $\phi(x) = \phi(\operatorname{Ad} x)$. We observe that $x$ and $y$ in $G/H$ define the same point in $G/I_1$ if and only if $f_X(x) = f_X(y)$ (that is, if $y = gx$, then $\operatorname{div} jX = \operatorname{div} j \operatorname{Ad} g(X)$) for all $X \in \mathcal{G}$ regarding as right invariant vector fields on $G$, where $f_X$ is $\operatorname{div} jX$, the divergence of the vector field $jX$. In this situation, $\operatorname{div} j[X, Y] = \operatorname{div} j \operatorname{Ad} g([X, Y]) = \operatorname{div} j[\operatorname{Ad} g(X), \operatorname{Ad} g(Y)]$, i.e., the Ricci form comes from a 2-form on $G/I_1$.

Also, for any $x, y \in \mathcal{G}$, we have $\chi(x,\ y) = \mathrm{div}(j(\mathrm{L}_X(Y)))$ where $X,\ Y$ are the holomorphic vector fields corresponding to $x,\ y$.

From [16] we know that the Lie algebra $\mathcal{J}$ of $J$ can also be described as follows: Let $\mathcal{G}_- = \{x + ijx|_{x\in\mathcal{G}}\}$. Then $\mathcal{H} = \mathcal{G} \cap \mathcal{G}_-$ (this follows from the fact that $jx$ is the same as $jx$ in the holomorphic tangent space at the considered point and hence $x + ijx$ is $x + iix = x - x = 0$) and $\mathcal{J} = \mathcal{G} \cap \mathrm{norm}_{\mathcal{G}^{\mathbf{C}}}(\mathcal{G}_-)$.

2.6. **Representation Theory.** Here we collect some results that we need from the representation theory of the semisimple Lie algebras (Cf. [22, pp. 67–69, 113]). Let $s$ be a semisimple Lie algebra, $t$ a Cartan subalgebra, $\Delta$ an ordered root system, $\Delta^+$ the positive roots. We let $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ and $\{\alpha_1, \cdots, \alpha_l\}$ be the set of simple roots. We also let $\{H_1, \cdots, H_l\} \subset t$ be a set of elements dual to the simple roots such that $\frac{2(H_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$. We have:

**Proposition 9.** *Let $s$ be a semisimple Lie algebra. Then:*

1. *An element in $t$ is a highest weight for an irreducible representation if and only if it can be expressed as $\sum a_i H_i$ with $a_i$ nonnegative integers.*
2. $\delta = \sum H_i$.
3. $H_i = \sum_j a_{ij}\alpha_i$ *with positive* $a_{ij}$.
4. *Let $\pi_i$ be the representation corresponding to $H_i$. Then the unique irreducible representation with highest weight as in* (1) *is a submodule of $\bigotimes(\pi_i)^{a_i}$ generated by the highest weight vector which is the tensor product of the highest weight vectors of $\pi_i$.*

The statements (1), (2) and (4) come from the standard representation theory, while (3) can be found in [22, p. 69].

## 3. Four Fibrations

In this section, we shall prove the following generalization of a theorem in [6]:

**Theorem 1.** *For any complex homogeneous space $G/H$ with an invariant volume, compact or noncompact, $J \subset I_1$.*

*Proof.* Let $g \in J$ and $x \in \mathcal{G}$. Then $j\,\mathrm{Ad}\,g(x) = \mathrm{Ad}\,g(jx) + h$ for some $h \in \mathcal{H}$. Therefore,

$$\begin{aligned}
&\mathrm{ad}(j(\mathrm{Ad}\,g(x) - x)) - j \circ \mathrm{ad}(\mathrm{Ad}\,g(x) - x)\\
&\quad = \mathrm{ad}(\mathrm{Ad}\,g(jx) - jx) + \mathrm{ad}\,h - j \circ ((\mathrm{Ad}\,g - 1)(\mathrm{ad}\,x))\\
&\quad = ((\mathrm{Ad}\,g - 1)(\mathrm{ad}\,jx)) + \mathrm{ad}\,h - ((\mathrm{Ad}\,g - 1)(j\,\mathrm{ad}\,x)) + ([\mathrm{Ad}\,g,\ j](\mathrm{ad}\,x))\\
&\quad = ((\mathrm{Ad}\,g - 1)(\mathrm{ad}\,jx - j\,\mathrm{ad}\,x)) + \mathrm{ad}\,h + ([\mathrm{Ad}\,g,\ j](\mathrm{ad}\,x)).
\end{aligned}$$

Here we also regard $\mathrm{Ad}g$ as a map acting on the matrices $\mathrm{ad}y$ and $j\mathrm{ad}x$, i.e., $\mathrm{Ad}g(A) = m(g)A(m(g))^{-1}$ for any matrix $A$ with $m(g)$ being the transformation matrix of $\mathrm{Ad}g$ on $\mathcal{G}$ with respect to a given basis of $\mathcal{G}$. We notice that $\mathrm{ad}(jx) - j\mathrm{ad}x$ and $\mathrm{Ad}g$ leave $\mathcal{H}$ invariant. Therefore, by $\mathrm{tr}(\mathrm{Ad}g - 1)(A) = \mathrm{tr}(m(g)A(m(g))^{-1} - A) = 0$ for any matrix $A$, the trace of the first term vanishes. Since $M$ admits an invariant volume form, we have that the trace of the second term vanishes as well. Now we notice that by the definition of the HOT-fibration $[\mathrm{Ad}\,g,\ j]\mathcal{G} \subset \mathcal{H}$, we have

$$([\mathrm{Ad}\,g, j]A)y = [m(g), j]A(m(g))^{-1}y \in \mathcal{H},$$

i.e., the third term also vanishes. Altogether this shows that $g \in I_1$ for all $g \in J$, and hence $J \subset I_1$. $\square$

One might notice that Proposition 1 in [16] is a Corollary of this theorem.

Now we are ready to prove a generalization of a theorem in [6] for a compact complex homogeneous space with an invariant volume form (see also [11], [12]).

On a rational homogeneous manifold $Q$, there is a Kähler-Einstein metric, unique up to a scalar multiplier, which is invariant under a given maximal compact subgroup $K \subset \operatorname{Aut} Q$ (see [27]).

**Theorem 2.** *Let $M$ be a connected complex compact manifold and let $G$ be a connected real Lie group acting transitively and holomorphically on $M$. Assume that $M = G/H$ admits a $G$-invariant volume element.*

*Then the Lie groups $I$, $I_1$, $J$ and $J_1$ are all the same and the fibers of both the global HOT-fibration and HK-fibration are connected.*

*Moreover, the Ricci form comes down from $G/H$ to $G/I$ to be an invariant Kähler-Einstein metric which is in the Ricci class of $G/J$.*

*Proof* (Cf. [6]). From Proposition 5, we know that $J$ is connected. Hence, Theorem 1 implies $H \subset J_1 = J \subset I^0 \subset I \subset I_1$, where $I^0$ is the identity component of $I$. From Proposition 3, we know that $\mathcal{H}$ is an ideal of $\mathcal{J}$, and [34, Theorem 1] implies that $\mathcal{H}$ is an ideal of $\mathcal{I}$. Hence, $J/H^0$ is a complex Lie subgroup of $I^0/H^0$, where $H^0$ denotes the identity component of $H$. Thus, $I^0/J \subset G/J$ is a closed complex submanifold, and therefore a projective manifold. Since $G/J$ embeds equivariantly into $\mathbf{P}^n$, the maximal solvable subgroups of $(I^0)^{\mathbf{C}}$ have fixed points in $I^0/J$ by Borel's Fixed Point Theorem [21, Chapter I]. Therefore, the stablizer of $(I^0)^{\mathbf{C}}$ at $eJ$ is parabolic, and [21, Chapter I, Theorem 6] implies that $I^0/J$ is a rational homogeneous space. Now we consider the two complex fibrations $I^0/H \to I^0/J$ and $I^0/H \to I^0/I^0$. Both fibrations have rational homogeneous spaces as bases and parallelizable homogeneous spaces as fibers. Therefore, by the uniqueness of the Tits' fibration in Propositions 4 and 5, we have $J = I^0$. From Part (2) of Proposition 6, we see that $I/H$ is connected. Since $H \subset I^0$, this implies $J = I^0 = I$. Now by the definition of the global HK-fibration, the Ricci form comes down to be a closed two-form on $G/I_1$ and the image $I_2$ of $I_1$ in $\operatorname{Aut}(G/J)$ must be a closed and open subgroup of the centralizer of an element in the Lie algebra $\mathcal{S}$ of the image of $G$ in $\operatorname{Aut}(G/J)$ (see [4], [29]). Therefore, if we can prove that $\mathcal{S}$ is compact, then by [7, Proposition 4.2], $I_2$ must be connected. That is, $G/J = G/I_1$ and hence $I_1 = J$.

Now we consider the Ricci form on $G/J$. We have

$$\begin{aligned}
&\operatorname{tr}_{\mathcal{G}/\mathcal{H}}(\operatorname{ad} j[x,\ y] - j \circ \operatorname{ad}[x,\ y]) \\
&\qquad = \operatorname{tr}_{\mathcal{G}/\mathcal{J}}(\operatorname{ad} j[x,\ y] - j \circ \operatorname{ad}[x,\ y]) \\
&\qquad + \operatorname{tr}_{\mathcal{J}/\mathcal{H}}(\operatorname{ad} j[x,\ y] - j \circ \operatorname{ad}[x,\ y]).
\end{aligned}$$

The second term is always zero by the definition of $J$. Therefore, the Ricci form of $G/J$ is exactly the pushdown of the Ricci form of $G/H$. But the Ricci form of $G/J$ is exactly the Ricci form of a standard Kähler-Einstein metric. By a standard result in Riemannian geometry, we have that $\mathcal{S}$ is compact. The theorem is proved. □

## 4. The Splitting of the Lie Algebra

**Lemma 1.** *Let $M$ be a compact complex homogeneous space. Let $G$ be a connected complex group of holomorphic automorphisms acting on $M$ transitively and effectively, $H$ be the isotropy subgroup, and $J = N_G(H^0)$ be the normalizer of $H^0$ in*

*$G$. Let $G = SR$ be a Levi decomposition of $G$. Then, with respect to a Cartan subalgebra in $\mathcal{S} \cap \mathcal{J}$, $\mathcal{H}$ decomposes into eigenspaces.*

*If $h \in \mathcal{H}$ is an eigenvector with a nonzero eigenvalue, then $h = h_s + h_r$ such that $h_s \in \mathcal{S} \cap \mathcal{H}$ and $h_r \in \mathcal{R} \cap \mathcal{H}$.*

*Proof.* Since $J \cap S$ is parabolic, its Lie algebra contains a Cartan subalgebra in $\mathcal{S}$. Since $\mathcal{H}$ is an ideal of $\mathcal{J}$, it must be decomposed into its eigenvector spaces.

If $h$ is an eigenvector with nonzero eigenvalue such that $h$ is not in $\mathcal{R}$, then there is an $s = sl(2, \mathbf{C})$ generated by root vectors in $\mathcal{S}$ such that $h = h_s + h_r$ and $h_s \in s,\ h_r \in \mathcal{R}$ with weight $\alpha$.

If $h_r \neq 0$, then there is an $h_r^- \in \mathcal{R}$ which is an eigenvector with weight $-\alpha$ such that $[h_s,\ [h_s,\ h_r^-]] = -h_r$. We have $h_r,\ h_r^- \in \mathrm{nil}(\mathcal{G})$ and

$$\begin{aligned} h + [h,\ [h,\ h_r^-]] &= h_s + h_r + [h_s,\ [h_s,\ h_r^-]] \\ &\quad + [h_r,\ [h_s,\ h_r^-]] + [h_s,\ [h_r,\ h_r^-]] + [h_r,\ [h_r,\ h_r^-]] \\ &= h_s + [h_r,\ h_1] + [h_r^-,\ h_2] \\ &= h_s + h_r^2 \in \mathcal{H} \end{aligned}$$

where $h_1, h_2 \in \mathrm{nil}(\mathcal{G}) := n$. Hence $h_r^2 \in [n,\ n] := n_2$. In this way, we can find $h_r^k \in n_k := [n_{k-1},\ n_{k-1}]$ such that $h_s + h_r^k \in \mathcal{H}$. By $n$ being nilpotent, we obtain that $h_s \in \mathcal{H}$, and hence $h_r \in \mathcal{H}$ also. □

**Lemma 2.** *Let $M = G/H$ be a compact complex homogeneous space and $G^{\mathbf{C}}$ be the minimal conplex Lie group in $\mathrm{Aut}(M)$ that contains $G$, and $G^{\mathbf{C}} = S^{\mathbf{C}} R^{\mathbf{C}}$ be a complex Levi decomposition. If the image of $G$ in $\mathrm{Aut}(G/J)$ is compact, then all the root vectors in the nil radical of $p = \mathcal{S}^{\mathbf{C}} \cap \mathcal{J}^{\mathbf{C}}$ are in $\mathcal{H}^{\mathbf{C}}$.*

*Proof.* Let $s_1$ be the semisimple Lie algebra that contains all the simple factors of $s = \mathcal{G} \cap \mathcal{S}^{\mathbf{C}}$ acting nontrivially on $G/J$ and $s = s_1 + s_2$. We know that $\mathcal{S}^{\mathbf{C}} \cap \mathcal{J}$ is a centralizer of an element $w$ of $s_1$ in $s$ as we can see in the proof of Theorem 2. Choose a Cartan subalgebra in $\mathcal{S} \cap \mathcal{J}$ and an order in its complexification such that $\mathcal{S}^{\mathbf{C}} \cap \mathcal{J}^{\mathbf{C}}$ contains a Borel subalgebra. Let $e_\alpha$ be a positive root vector in $s_1$ such that $(w.\ \alpha) \neq 0$, $X = e_\alpha + e_{-\alpha}$. Then on $G/J$, $jX = i(e_\alpha - e_{-\alpha})$ and $X + ijX = 2e_{-\alpha} (\mathrm{mod}\ i(\mathcal{J} \cap \mathcal{S}))$ is an element in $p$. But we also have $x + ijx \in \mathcal{H}^{\mathbf{C}}$, where $x$ is the corresponding element of $X$ in $\mathcal{G}$ (see the end of 5 in section 2). Therefore, $2e_{-\alpha} + y = x + ijx \in \mathcal{H}^{\mathbf{C}}$ for some $y \in i(\mathcal{J} \cap \mathcal{S}) + \mathcal{R}^{\mathbf{C}}$ and by Lemma 1, we have $e_{-\alpha} \in \mathcal{H}^{\mathbf{C}}$ as desired since $e_{-\alpha}$ is not in $(\mathcal{J} \cap \mathcal{S})^{\mathbf{C}}$. □

**Lemma 3** (Cf. [2])**.** *Let $M$ be as in Lemma 1 and $\mathcal{S} = s_1 + s_2$ such that $s_1$ contains all the simple factors acting nontrivially on $G/J$. Then $\mathcal{G} = W_1 + \cdots + W_l + W_0$ where $W_i$ are nontrivial $s_1$ irreducible representations for $1 \leq i \leq l$ and $W_0$ is a vector space containing all the $s_1$ fixed vectors. If $w_1, \cdots, w_l$ are the highest weight vectors, then they are linearly independent modulo $\mathcal{H}$. Moreover, $\dim W_0 \leq \dim J/H$.*

*Proof.* The direct sum comes from the representation theory of semisimple Lie groups. If $w = \sum a_i w_i \in \mathcal{H}$ and $p = \mathcal{J} \cap s_1$, then $[p,\ w] \subset \mathcal{H}$ and $[s_2 + \mathcal{R},\ w] \subset \mathcal{H}$ since $\mathcal{H}$ is an ideal of $\mathcal{J}$. But $[B',\ w] = 0$, where $B$ is the Borel subalgebra that is the minimal parabolic subalgebra containing all the positive root vectors. We obtain that $[s_1,\ w] \subset \mathcal{H}$. Therefore, $m_1 = [\mathcal{G},\ w] \subset \mathcal{H}$, and $[B,\ m_1] = [[B,\ \mathcal{G}],\ w] \subset m_1$. If we let $m_k = [\mathcal{G},\ m_{k-1}]$ and assume that $m_k \subset \mathcal{H}$, $[B,\ m_k] \subset m_k$, then $m_{k+1} = [B + \mathcal{J},\ m_k] \subset [[B,\ \mathcal{G}],\ m_{k-1}] + [\mathcal{G},\ m_{k-1}] + \mathcal{H} \subset m_k + \mathcal{H} \subset \mathcal{H}$. Therefore, $w$

generates a $\mathcal{G}$-ideal in $\mathcal{H}$. This implies that $w = 0$. Hence, all the weight vectors $w_i$ are linearly independent modulo $\mathcal{H}$.

All the vectors in $W_0$ correspond to the actions, regarded as the action on the fiber of the bundle $G/H \to G/J$, being without any fixed point and invariant under $S_1$, which is the subgroup of $G$ corresponding to $s_1$. Since $S_1$ acts on $G/J$ transitively, each of these vector fields is determined by its value on any fixed fiber of $G/H \to G/J$. We obtain that $\dim W_0 \leq \dim J/H$. □

Now we are ready to prove the Splitting Theorem for the Lie algebra:

**Theorem 3.** *In the case of Lemma 2, we can apply Lemma 3 to $G^{\mathbf{C}}/H^{\mathbf{C}}$. All $W_i$ in Lemma 3 must be one of the simple factors in $s_1^{\mathbf{C}}$, i.e., $\mathcal{G} = s_1 \oplus (s_2 + \mathcal{R})$ where $s_2$ is a complex semisimple Lie algebra and $s_1$ is a compact semisimple Lie algebra. Moreover, if we let $c$ be the centralizer of $w$ in $s_1$, $c_1$ be its center and $c_2$ be the center of $s_2 + \mathcal{R}$, then $\mathcal{H}$ is a direct sum of the semisimple part of $s_1 \cap \mathcal{J}$ and a subalgebra of $c_1 + c_2$.*

*Proof.* By Proposition 9 and Lemmas 1,2,3, we see that $W_i$ can only be the simple factors of $s_1$; otherwise, $J/H^0$ cannot be unimodular by considering the effects of the actions of the fundamental weights $H_i$.

The semisimple part of $s_1 \cap \mathcal{J}$ is compact and hence must be in $\mathcal{H}$. Let $x \in \mathcal{H}$ be an eigenvector with weight zero. Then $x = h_s + h_r$ with $h_s \in c_1$ and $h_r \in (s_2 + \mathcal{R})$. $h_s$ cannot be zero; otherwise, $h_r$ generates an ideal of $\mathcal{G}$ in $\mathcal{H}$. Now $h_r$ must be in $c_2$; otherwise, $[\mathcal{J},\ h_r] = [\mathcal{J},\ x]$ will generate an ideal of $\mathcal{G}$ in $\mathcal{H}$. Therefore, we have the theorem. □

## 5. Global Structure Theorem

Now we are able to obtain a global structure for our manifolds. First, we mention two lemmas, which I believe have been known for a long time:

**Lemma 4** (Cf. [31, Theorem 6.15])**.** *If $G$ is a complex Lie group and $H$ is a co-compact discrete subgroup of $G$, then $H$ is finitely generated.*

This lemma also comes from the existence of finite triangularization for compact complex manifolds (see also [26]). We initially proved the following lemma by the methods in [32] and [12] in a situation in which we did not find a better reference, but then we found it in [36].

**Lemma 5** (Cf. [32], see also [36, Cor. 3.4.14])**.** *Let $G$ be a complex Lie group and $G/H$ be a compact complex parallelizable manifold with $H$ being discrete. If $C$ is the center of $G$, then $C/(C \cap H)$ is compact.*

**Theorem 4.** *Suppose $M = G/H$ is a compact complex homogeneous space with an invariant volume. Then $G = S_1G_1$ is a local direct product with $S_1$ being compact semisimple, which acts on $G/J$ transitively, and $G_1$ having complex semisimple part. $H = H_sC_H$ is a local direct product with $H_s$ being the semisimple part of $S_1 \cap J$, which is compact, and the identity component of $C_H$ is in the center of $J$. $M$ is a holomorphic principal torus bundle over a product of a rational homogeneous space and a compact complex parallelizable manifold such that the torus action comes from the center of $J$. Conversely, any $G/H$ of this kind has an invariant volume.*

*Proof.* By Theorem 3, we have that $J = H_sC_1G_1$ where $C_1$ is a Lie group corresponding to $c_1$. Since $S_1 \cap J = H_sC_1$ is connected, we have that $C_1$ is connected. We obtain that $C_1$ is in the center of $J$. We also have that $H_s \subset H$. $J/H = C_1G_1/H \cap C_1G_1$ is a compact complex parallelizable manifold. The identity component of $H \cap C_1G_1$ is in $C_1C_2$, where $C_2$ is the center of $G_1$. $C_1C_2$ is the center of $C_1G_1$. Now by Lemma 5, we obtain that $J/H$ is a torus bundle over $(J/H^0)/(H/H^0)$. The torus action comes from the center $C^1$ of $J^1 = J/H^0$. Let $N^2$ be the intersection of the pullback of $C^1$ in $J$ with $G_1$ and $N^1 = C_1N^2$. Then $N^1$ is a nilpotent Lie group of at most two steps. Then $G/HN^1$ is a product of $S_1/H_sC_1$ and $G_1/N^2(HC_1 \cap G_1)$.

Now we want to prove that $\mathcal{N}^2 = c_2$. For any $n \in \mathcal{N}^2$, we have $[n,\ x] \in \mathcal{H}$ for all $x \in \mathcal{G}_1$ since $n$ represent a center element in $\mathcal{J}^1$. Therefore, $[n,\ x]$ generates an ideal of $\mathcal{G}$ in $\mathcal{H}$ and must be zero. We obtain $N^2 = C_2$ and $N^1 = C_1C_2$ is the center of $J$.

We can also get $M$ back, by forming the product of $M_1 = S_1C_2/H \cap S_1C_2$ and $M_2 = J/H$ and then taking the anti-diagonal equivalent relation $(x, y) \sim (xg, g^{-1}y)$ for all $g \in N^1 = C_1C_2$.

Conversely, if $G/H$ is a complex homogeneous torus bundle over a product of a compact rational homogeneous manifold $Q$ and a parallelizable manifold $P$ such that $G$ is a local direct product $S_1G_1$ and acts on $Q$ as the compact Lie group $S_1$ as in the first part of Theorem 4, then $G$ acts on $M_1$ and $M_2$ with invariant volumes. $M$, being the quotient space of $M_1 \times M_2$ by the anti-diagonal torus action, has invariant volumes. □

This is the first part of our Main Theorem 1. Applying this theorem to the case in which $G$ is reductive, we get Main Theorem 2.

We also notice that the main point from which this theorem is true comes from Lemma 2, i.e., if either $S_1$ is compact or the root vectors in the nil radical of $\mathcal{S}_1^{\mathbf{C}} \cap \mathcal{J}^{\mathbf{C}}$ are in $\mathcal{H}$, then all the arguments go through. We shall therefore have the same structure theorem. Hence, we have the following theorem and the second part of our Main Theorem 1:

**Theorem 5.** *Every compact complex homogeneous space $G/H$ with $G$ acting on $G/J$ as a compact Lie group admits an invariant volume.*

We can also notice that all the compact complex homogeneous spaces with a 2-cohomology class whose top power is nonzero in the top cohomology group admit invariant volumes.

In our Theorem 4, if $G_1$ is a compact torus, we have the manifolds in [35]; if $S_1$ is trivial, we have the manifolds in [34]. Therefore, we also notice that $M_1$ in the proof of Theorem 4 is exactly the manifold studied in [35] (by extending the center of the Lie group as big as possible, we can always assume $C_2$ to be the complex torus and hence $G$ can be compact), and $M_2$ is exactly the manifold studied in [34]. Our manifold is the quotient space of $M_1 \times M_2$ by the anti-diagonal torus action. Conversely, whenever we have an $M_1$ in [35] and an $M_2$ in [34] such that there is a complex torus action on both $M_1$ and $M_2$ that comes from a complex subgroup in the center of $M_2$, we can construct our manifold as the quotient space of $M_1 \times M_2$ by the anti-diagonal action of this given complex torus.

We shall try to classify those compact complex homogeneous spaces that do not admit any invariant volume in [14].

## 6. Acknowledgement

Here I take this opportunity to thank Professor S. Kobayashi for his interest in this work and encouragement, and his help in writing a latex file when I was modifying a certain part of [6]. I also thank Professor J. A. Wolf for his encouragement for this work and his help in papers [11], [12]. I thank Professor A. Borel for discussions and for showing me the paper [32]. I also thank Professor A. T. Huckleberry for his effort in [19] and Professor J. Dorfmeister for leading me to this direction and showing me [20] and [24]. I thank Professors F. A. Bogomolov, M. Gromov, Y. Siu, and K. Ding for their constant encouragement and support. Finally, I wish to express my thanks to the Department of Mathematics, Princeton University and to Professor W. C. Hsiang as well as NSF for their support, which made this work possible.

I also thank the referee of [15] in ERA-AMS for telling me about [17]. I also thank the referees for their helpful comments on this paper.

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