TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 12, Pages 4893–4907
S 0002-9947(02)03097-0
Article electronically published on August 1, 2002

NONISOTROPIC STRONGLY SINGULAR INTEGRAL OPERATORS

BASSAM SHAYYA

ABSTRACT. We consider a class of strongly singular integral operators which include those studied by Wainger, and Fefferman and Stein, and extend the results concerning the L^p boundedness of these operators to the nonisotropic setting. We also describe a geometric property of the underlying space which helps us show that our results are sharp.

1. INTRODUCTION

Let $0 < a_1 \leq a_2$, $\nu = a_1 + a_2$, and consider the one-parameter group $\{\delta_t\}_{t>0}$ of nonisotropic dilations on $\mathbb{R}^2$ given by $\delta_t : (x_1, x_2) \longmapsto (t^{a_1}x_1, t^{a_2}x_2)$. Following Stein and Wainger [9], we define a function $\rho : \mathbb{R}^2 \to [0, \infty)$ as follows. If $x \neq 0$, $|\delta_{\frac{1}{t}}x|$ as a function of t is strictly decreasing and is therefore equal to 1 for a unique value of t. Define $\rho(x)$ to be this unique t. If $x = 0$, set $\rho(x) = 0$. Then ρ is continuous, $\rho(x + y) \leq C(\rho(x) + \rho(y))$ for some $C > 0$, and $\rho(\delta_t x) = t\rho(x)$ for every $t > 0$. This function ρ is often called a δ_t–homogeneous distance function. The purpose of this paper is to study the L^p boundedness of the singular integral operator defined on the space $C_0^\infty(\mathbb{R}^2)$ of infinitely differentiable functions of compact support by

$$T\varphi(x) = \lim_{\epsilon \to 0} \int_{1 \geq \rho(y) \geq \epsilon} \frac{e^{i/\rho(y)^\beta}}{\rho(y)^\alpha} \varphi(x - y)dy, \tag{1}$$

where $\alpha, \beta > 0$. Using the generalized system of polar coordinates that one has in this setting, it is easy to see that the function $1/\rho(y)^\alpha$ is integrable near the origin if $\alpha < \nu$. So we assume $\alpha \geq \nu$. Then a straightforward argument of integration by parts shows us that the limit in (1) exists if $\beta > \alpha - \nu$.

In the special case $\rho(y) = |y|$ ($a_1 = a_2 = 1$), and in the setting of $\mathbb{R}^n$, it was shown in Wainger [10] that T extends to a bounded operator on $L^p(\mathbb{R}^n)$ for $|1/p - 1/2| < ((n/2)\beta - \alpha + n)/n\beta$, and that T is not bounded on $L^p(\mathbb{R}^n)$ if $|1/p - 1/2| > ((n/2)\beta - \alpha + n)/n\beta$. This was obtained by fully describing the asymptotic behavior near ∞ of the Fourier transform of the kernel of T. The question of whether or not T remains bounded on $L^p(\mathbb{R}^n)$ when $|1/p - 1/2| = ((n/2)\beta - \alpha + n)/n\beta$ ($\alpha > n$) was answered positively in Fefferman and Stein [3] using complex interpolation on Hardy spaces after proving the following theorem:

Received by the editors May 6, 1997.
2000 *Mathematics Subject Classification.* Primary 42B20; Secondary 42B15.

Theorem A. *Let L be an integrable function on $\mathbb{R}^n$ with $L(x) = 0$ for $|x| > 1$. Assume there exists $\theta \in (0,1)$ such that*

$$\int_{|x|>2|y|^{1-\theta}} |L(x-y) - L(x)|\, dx \leq B,$$

for $|y| < 1$, and

$$\left|\widehat{L}(\xi)\right| \leq \frac{B}{(1+|\xi|)^{n\theta/2}}.$$

Then the transformation $S(f) = L * f$ is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with a bound that depends on θ and B but not on the L^1 norm of L.

The function defined by $L_\epsilon(x) = e^{i/|x|^\beta}/|x|^n$ for $\epsilon \leq |x| \leq 1$, and $L_\epsilon(x) = 0$ otherwise, satisfies the hypothesis of Theorem A with $\theta = \beta/(\beta+1)$ and B independent of ϵ (see [2], [3], and [10]). For further results in the radial case, we refer the reader to [4], [5], and [6].

We are going to extend the above results to the nonisotropic setting. To extend Theorem A, we introduce another distance function ρ_β which will better describe the smoothness of the kernel of a nonisotropic strongly singular integral operator and the decay of its Fourier transform. It will turn out that the balls associated to ρ, and those associated to ρ_β, are related by a geometric property which will play an important role in studying the operator T. Our main results on the L^p boundedness of T are stated in the following theorem.

Theorem 1. *Suppose $\beta > \alpha - \nu \geq 0$. For $\varphi \in C_0^\infty$, define*

$$T\varphi(x) = \lim_{\epsilon\to 0} \int_{1\geq\rho(y)\geq\epsilon} \frac{e^{i/\rho(y)^\beta}}{\rho(y)^\alpha} \varphi(x-y)dy.$$

Then:

(i) *If $\alpha > \nu$, then T extends to a bounded linear operator on $L^p(\mathbb{R}^2)$ for*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{\beta - \alpha + \nu}{2\beta}.$$

If $\alpha = \nu$, then T is bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$. On the other hand,

(ii) *if*

$$\left|\frac{1}{p} - \frac{1}{2}\right| > \frac{\beta - \alpha + \nu}{2\beta},$$

then T is not bounded on $L^p(\mathbb{R}^2)$.

If $x_0 \in \mathbb{R}^2$, and $r \geq 0$, we define a ρ–ball by $B(x_0, r) = \{x \in \mathbb{R}^2 : \rho(x - x_0) \leq r\}$. A 1–atom is a function $a \in L^\infty(\mathbb{R}^2)$ supported in a ρ–ball $B(x_0, r)$ such that

(i) $\|a\|_{L^\infty} \leq r^{-\nu}$, and

(ii) $\int a(x)dx = 0$.

Following Coifman and Weiss [1], we define $H^1_\rho(\mathbb{R}^2)$ as the set of all $f \in S'$ that can be represented in the form $f = \sum_{i=0}^\infty \mu_i a_i$, where each a_i is a 1–atom and $\sum_{i=0}^\infty |\mu_i| < \infty$. Also, for $f \in H^1_\rho(\mathbb{R}^2)$ we have $\|f\|_{H^1_\rho} = \inf\{\sum |\mu_i| : f = \sum \mu_i a_i\}$. Throughout this paper a constant is a positive real number that depends only on α, β, a_1, and a_2. c will always denote a constant which does not necessarily have the same value every time it appears.

2. Acknowledgements

This paper is based in part on my Ph.D. thesis written at the University of Wisconsin–Madison, and I would like to take this opportunity to thank my teacher and advisor, S. Wainger, for his support and valuable guidance. I would also like to thank A. Seeger for many helpful comments and suggestions concerning the content of this paper.

3. The L^p Inequality

We start this section by stating some further properties of the function ρ.

Proposition 1. (i) *$\rho(x)$ is infinitely differentiable in $\mathbb{R}^2 - 0$. Also, for $x \neq 0$,*

$$\left|\frac{\partial \rho}{\partial x_1}(x)\right| \leq C\rho(x)^{1-a_1} \text{ and } \left|\frac{\partial \rho}{\partial x_2}(x)\right| \leq C\rho(x)^{1-a_2}$$

for some $C > 0$.

(ii) *If $|x| \geq 1$, then $\rho(x)^{a_1} \leq |x| \leq \rho(x)^{a_2}$.*

(iii) *If $|x| \leq 1$, then $\rho(x)^{a_1} \geq |x| \geq \rho(x)^{a_2}$.*

(iv) *If $f \in L^1(\mathbb{R}^2)$ or $f \geq 0$, then*

$$\int_{\mathbb{R}^2} f(x)dx = \int_0^{2\pi} \Omega(\theta) \left[\int_0^\infty f(\delta_r(\cos\theta, \sin\theta))r^{\nu-1}dr\right] d\theta$$

where $\Omega(\theta) = a_1 + (a_2 - a_1)\sin^2\theta$.

Part (iv) describes the generalized polar coordinates mentioned above. For a proof of Proposition 1, see [9].

For $\beta > 0$ we associate to ρ a function ρ_β as follows. For $t > 0$ and $x \in \mathbb{R}^2$, define

$$\gamma_t(x) = t^\beta \delta_t(y) = (t^{a_1+\beta}x_1, t^{a_2+\beta}x_2),$$

and let ρ_β be the distance function corresponding to the group $\{\gamma_t\}_{t>0}$. The geometric property, mentioned before, that relates ρ_β–balls to ρ–balls will be described in detail in the next section. For now let us note that

$$\rho(x) \leq \rho_\beta(x), \text{ if } \rho(x) \leq 1. \tag{2}$$

We start by proving the following generalization of Theorem A.

Theorem 2. *Let $K_0 \in L^1(\mathbb{R}^2)$ with $K_0(x) = 0$ for $\rho(x) > 1$. Assume there exist $\beta > 0$ and a constant C such that*

$$\int_{\rho(x)>C\rho_\beta(y)} |K_0(x-y) - K_0(x)|\, dx \leq B_0$$

for $\rho(y) < 1$, and

$$\left|\widehat{K_0}(\xi)\right| \leq \frac{B_0}{(1+\rho_\beta(\xi))^\beta}.$$

Then the transformation $T_0(f) = K_0 * f$ is bounded from $H^1_{\rho_\beta}(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$ with a bound that depends on β, B_0 and C but not on the L^1 norm of K_0.

Proof. It suffices to show that $\|T_0(a)\|_{L^1} \leq c$ for each 1–atom a, with c independent of the L^1 norm of K_0 and the choice of a. Let a be a 1–atom supported in a ρ_β–ball $B = B(x_0, r)$. Since T_0 is translation invariant, we can take $x_0 = 0$. Then $T_0(a)$ is supported in a ρ_β–ball $B(0, c(1+r))$. By part (iv) of Proposition 1, the Lebesgue measure $|B(0, c(1+r))|$ of $B(0, c(1+r))$ is $\leq c(1+r)^{2\beta+\nu}$. So if $r \geq 1$,

$$\begin{aligned}
\|T_0(a)\|_{L^1} &\leq c(1+r)^{\beta+\nu/2}\|T_0(a)\|_{L^2} \\
&\leq c(1+r)^{\beta+\nu/2}\|a\|_{L^2} \\
&\leq c(1+r)^{\beta+\nu/2}\frac{1}{r^{\beta+\nu/2}} \\
&\leq c.
\end{aligned}$$

Suppose $r < 1$ and consider the ρ–ball $B^* = B(0, Cr)$. Then

$$\begin{aligned}
\|T_0(a)\|_{L^1(\mathbb{R}^2 - B^*)} &= \int_{\mathbb{R}^2 - B^*} \left| \int K_0(x-y)a(y)dy \right| dx \\
&= \int_{\mathbb{R}^2 - B^*} \left| \int \left(K_0(x-y) - K_0(x)\right) a(y)dy \right| dx \\
&\leq \int |a(y)| \int_{\mathbb{R}^2 - B^*} |K_0(x-y) - K_0(x)|\, dxdy \\
&\leq \int |a(y)| \int_{\rho(x) > C\rho_\beta(y)} |K_0(x-y) - K_0(x)|\, dxdy \\
&\leq B_0\|a\|_{L^1} \\
&\leq c\,,
\end{aligned}$$

and

$$\begin{aligned}
\|T_0(a)\|^2_{L^1(B^*)} &\leq |B^*|\, \|T_0(a)\|^2_{L^2} \\
&\leq c\, r^\nu \|\widehat{T_0(a)}\|^2_{L^2} \\
&= c\, r^\nu \int |\widehat{K_0}(\xi)|^2 |\widehat{a}(\xi)|^2 d\xi \\
&= c\, r^\nu \int_{\rho_\beta(\xi) \geq 1/r} |\widehat{K_0}(\xi)|^2 |\widehat{a}(\xi)|^2 d\xi \\
&\quad + c\, r^\nu \int_{\rho_\beta(\xi) \leq 1/r} |\widehat{K_0}(\xi)|^2 |\widehat{a}(\xi)|^2 d\xi \\
&\leq c\, r^\nu \int_{\rho_\beta(\xi) \geq 1/r} \rho_\beta(\xi)^{-2\beta} |\widehat{a}(\xi)|^2 d\xi \\
&\quad + c\, r^\nu \|\widehat{a}\|^2_{L^\infty} \int_{\rho_\beta(\xi) \leq 1/r} \rho_\beta(\xi)^{-2\beta} d\xi \\
&\leq c\, r^{\nu+2\beta} \int_{\rho_\beta(\xi) \geq 1/r} |\widehat{a}(\xi)|^2 d\xi \\
&\quad + c\, r^\nu \|a\|^2_{L^1} \int_0^{1/r} s^{-2\beta} s^{\nu+2\beta-1} ds \\
&\leq c\, r^{\nu+2\beta} \|a\|^2_{L^2} + c\, r^\nu \int_0^{1/r} s^{\nu-1} ds \\
&\leq c\,.
\end{aligned}$$

Hence $\|T_0(a)\|_{L^1} = \|T_0(a)\|_{L^1(B^*)} + \|T_0(a)\|_{L^1(\mathbb{R}^2-B^*)} \leq c$. This completes the proof. □

For $y \neq 0$ define $K(y) = e^{i/\rho(y)^\beta}/\rho(y)^\alpha$, and set

$$K_\epsilon(y) = \begin{cases} K(y) & \text{if } \epsilon \leq \rho(y) \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

$(0 < \epsilon \leq 1)$. Now for $f \in L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$, define $T_\epsilon f = K_\epsilon * f$. Then if $\beta > \alpha - \nu \geq 0$ and $\varphi \in C_0^\infty(\mathbb{R}^2)$, it follows that $T\varphi(x) = \lim_{\epsilon\to 0} T_\epsilon\varphi(x)$ for every $x \in \mathbb{R}^2$.

Theorem 3. *Suppose $\beta > 0$ and $\beta \geq \alpha - \nu \geq 0$. If $|1/p - 1/2| \leq (\beta - \alpha + \nu)/(2\beta)$ $(\alpha > \nu)$ or $1 < p < \infty$ $(\alpha = \nu)$, we have*

$$\|T_\epsilon f\|_{L^p} \leq A_p \|f\|_{L^p}$$

for every $f \in L^p$. The constant A_p is independent of ϵ.

A standard limiting argument shows that part (i) of Theorem 1 is an immediate consequence of Theorem 3. Part (i) of Proposition 1 tells us that

$$\left|\frac{\partial K_\epsilon}{\partial x_1}(x)\right| \leq c\rho(x)^{-\alpha-\beta-a_1} \text{ and } \left|\frac{\partial K_\epsilon}{\partial x_2}(x)\right| \leq c\rho(x)^{-\alpha-\beta-a_2}.$$

So, if $\alpha = \nu$, it can be easily checked that

$$\int_{\rho(x)>C\rho_\beta(y)} |K_\epsilon(x-y) - K_\epsilon(x)|\, dx \leq B_0,$$

uniformly in ϵ. In the next theorem, we estimate the Fourier transform of K_ϵ, and it will turn out that if $\alpha = \nu$, then $\left|\widehat{K_\epsilon}(\xi)\right| \leq B_0\,(1 + \rho_\beta(\xi))^{-\beta}$. Theorem 2 then tells us that T_ϵ is bounded from $H^1_{\rho_\beta}(\mathbb{R}^2)$ to $L^1(\mathbb{R}^2)$ with a bound that is independent of ϵ. So our next task is to estimate $\widehat{K_\epsilon}$, and for this we need the following lemma of van der Corput, which can be found in [8, pages 332–334].

Proposition 2. *Suppose ϕ is real-valued and smooth in (a, b), and that $|\phi^{(k)}(x)| \geq \lambda > 0$ for all $x \in (a, b)$. Then*

$$\left|\int_a^b e^{i\phi(x)} dx\right| \leq c_k \lambda^{-1/k} \tag{3}$$

holds when:

(i) *$k \geq 2$, or*

(ii) *$k = 1$ and $\phi''(x)$ has at most one zero.*

Also, $c_k = 5(2^k) - 4$.

Now if $0 < a < b$, ϕ and ψ are real-valued and smooth in (a, b), and $|\phi^{(k)}(x)| \geq \lambda/x^s$ $(s \geq 0)$ (when $k = 1$ we also assume that $\phi''(x)$ has at most one zero), then

$$\int_a^b e^{i\phi(x)}\psi(x)dx = \int_a^b \psi(x)F'(x)dx,$$

where $F(x) = \int_a^x e^{i\phi(t)}dt$. By Proposition 2, $|F(x)| \leq c_k\lambda^{-1/k}x^{s/k}$ for $x \in [a, b]$, and on integrating the above integral by parts it follows that

$$\left|\int_a^b e^{i\phi(x)}\psi(x)dx\right| \leq c_k\lambda^{-1/k}\left[b^{s/k}|\psi(b)| + \int_a^b x^{s/k}|\psi'(x)|dx\right]. \tag{4}$$

In particular, if $s = 0$, then

$$\left| \int_a^b e^{i\phi(x)} \psi(x) dx \right| \leq c_k \lambda^{-1/k} \left[|\psi(b)| + \int_a^b |\psi'(x)| dx \right]. \tag{5}$$

Theorem 4. *Suppose $\beta > 0$ and $\beta \geq \alpha - \nu \geq 0$. Then*

$$\left| \widehat{\left(\frac{K_\epsilon}{\rho(.)^{iv}} \right)}(\xi) \right| \leq B \frac{1 + |v|}{(1 + \rho_\beta(\xi))^{\beta - \alpha + \nu}},$$

$-\infty < v < +\infty$. The constant B is independent of ϵ.

Proof. If ρ' is the distance function corresponding to the group $\{\delta'_t\}_{t>0}$, where $\delta'_t x = (tx_1, t^{a_2/a_1} x_2)$, then it is not hard to see that $\rho(y) = \rho'(y)^{1/a_1}$ and $\rho_\beta(y) = \rho'_{\beta/a_1}(y)^{1/a_1}$ for every $y \in \mathbb{R}^2$. Therefore, we can assume $a_1 = 1$ (then $\nu = 1 + a_2 \geq 2$). If $\rho_\beta(\xi)$ is small, an easy argument of integration by parts shows that the Fourier transform of $K_\epsilon / \rho(.)^{iv}$ is bounded. So it suffices to prove the theorem for large values of $\rho_\beta(\xi)$. Furthermore, since $\rho(x_1, x_2) = \rho(-x_1, x_2) = \rho(-x_1, -x_2)$, it is enough to look at $\xi = (\xi_1, \xi_2)$ with $\xi_1, \xi_2 \geq 0$. Write

$$\widehat{\left(\frac{K_\epsilon}{\rho(.)^{iv}} \right)}(\xi) = I_1 + I_2,$$

where

$$I_1 = \int_{\rho_\beta(x) \leq C_0 \lambda(\xi)} \frac{K_\epsilon(x)}{\rho(x)^{iv}} e^{i\xi \cdot x} dx$$

and

$$I_2 = \int_{\rho_\beta(x) \geq C_0 \lambda(\xi)} \frac{K_\epsilon(x)}{\rho(x)^{iv}} e^{i\xi \cdot x} dx.$$

C_0 and $\lambda(\xi)$ are going to be chosen. For $r > 0$, set $f(r) = \frac{d}{dr} |\delta_r \xi|$. Then $f'(r) > 0$, and it follows that the equation $\beta r^{-\beta - 1} = f(r)$ has a unique solution in $(0, \infty)$. Define $\lambda(\xi)$ to be this unique solution. An easy computation then shows that $\lambda(\gamma_t \xi) = (1/t)\lambda(\xi)$ for $t > 0$, and that there exist constants C_1 and C_2 such that $0 < C_1 \leq \lambda(\xi) \leq C_2$ whenever $|\xi| = 1$. So, writing $\xi = \gamma_{\rho_\beta(\xi)} \xi'$ with $|\xi'| = 1$, we conclude that

$$\frac{C_1}{\rho_\beta(\xi)} \leq \lambda(\xi) \leq \frac{C_2}{\rho_\beta(\xi)}. \tag{6}$$

In generalized polar coordinates,

$$I_1 = \int_0^{2\pi} \Omega(\theta) \left[\int_\epsilon^{C_0 \lambda(\xi)} \frac{e^{-iv \ln r}}{r^{\alpha - \nu + 1}} e^{i/r^\beta} e^{i\xi \cdot \delta_r (\cos\theta, \sin\theta)} dr \right] d\theta.$$

Writing $e^{i/r^\beta} = \frac{i}{\beta}(e^{i/r^\beta})' r^{\beta+1}$ and integrating the inner integral by parts, it follows that

$$\begin{aligned} |I_1| \quad \leq \quad & c\,\lambda(\xi)^{\beta-\alpha+\nu} + c\,(1+|v|)\int_0^{2\pi}\left|\int_\epsilon^{C_0\lambda(\xi)} e^{-iv\ln r} r^{\beta-\alpha+\nu-1} r^{i\Phi_\theta(r)} dr\right| d\theta \\ & + c\,|\xi_1|\int_0^{2\pi}\left|\int_\epsilon^{C_0\lambda(\xi)} e^{-iv\ln r} r^{\beta-\alpha+\nu} r^{i\Phi_\theta(r)} dr\right| d\theta \\ & + c\,|\xi_2|\int_0^{2\pi}\left|\int_\epsilon^{C_0\lambda(\xi)} e^{-iv\ln r} r^{\beta-\alpha+2\nu-2} r^{i\Phi_\theta(r)} dr\right| d\theta \end{aligned}$$

where $\Phi_\theta(r) = r^{-\beta} + r\xi_1\cos\theta + r^{\nu-1}\xi_2\sin\theta$. Since $|\xi_1| \leq \rho_\beta(\xi)^{\beta+1}$ and $|\xi_2| \leq \rho_\beta(\xi)^{\beta+\nu-1}$, it follows by (6) that we can find a constant C_0 small enough that $|\Phi'_\theta(r)| \geq \beta/2r^{\beta+1}$ for $r \in (0, C_0\lambda(\xi)]$ (uniformly in θ). Applying (4) to each of the integrals on the right-hand side of the above inequality, we get

$$|I_1| \leq c\,(1+|v|)\,\lambda(\xi)^{\beta-\alpha+\nu}. \tag{7}$$

Estimating I_2 takes more work. As we did for I_1, we start by expressing the integral in polar coordinates:

$$I_2 = \int_{C_0\lambda(\xi)}^1 \frac{e^{-iv\ln r}}{r^{\alpha-\nu+1}} e^{i/r^\beta}\left[\int_0^{2\pi}\Omega(\theta)e^{i\xi\cdot\delta_r(\cos\theta,\sin\theta)}d\theta\right] dr.$$

Now using the observation that $\xi\cdot\delta_r(\cos\theta,\sin\theta) = |\delta_r\xi|\cos(\theta - h(r))$, where $h(r) = \arctan(r^{\nu-2}\xi_2/\xi_1)$, we get

$$I_2 = \int_{C_0\lambda(\xi)}^1 \frac{e^{-iv\ln r}}{r^{\alpha-\nu+1}} e^{i/r^\beta}\left[\int_0^{2\pi}\Omega(\theta+h(r))e^{i|\delta_r\xi|\cos\theta}d\theta\right] dr.$$

Note that $h'(r) \leq c/r$. By the method of stationary phase (as stated in [8, page 334]),

$$\begin{aligned} &\int_0^{2\pi}\Omega(\theta+h(r))e^{i|\delta_r\xi|\cos\theta}d\theta \\ &\qquad = \omega_1\frac{\Omega(h(r))}{|\delta_r\xi|^{1/2}}e^{i|\delta_r\xi|} + \omega_2\frac{\Omega(h(r))}{|\delta_r\xi|^{1/2}}e^{-i|\delta_r\xi|} + O(|\delta_r\xi|^{-3/2}) \end{aligned}$$

($\omega_1 = \sqrt{2\pi}\,e^{-i\pi/4}$ and $\omega_2 = \sqrt{2\pi}\,e^{i\pi/4}$). The bounds occurring in the error term in the above equation are independent of r because all derivatives of $\Omega(\theta+h(r))$ with respect to θ are bounded uniformly in r. Let $\psi(r) = e^{-iv\ln r}\Omega(h(r))/|\delta_r\xi|^{1/2}r^{\alpha-\nu+1}$ and $\phi_\theta(r) = r^{-\beta} + |\delta_r\xi|\cos\theta$. Then

$$I_2 = \omega_1\int_{C_0\lambda(\xi)}^1 \psi(r)e^{i\phi_0(r)}dr + \omega_2\int_{C_0\lambda(\xi)}^1 \psi(r)e^{i\phi_\pi(r)}dr + E,$$

with $|E| \leq c\int_{C_0\lambda(\xi)}^1 |\delta_r\xi|^{-3/2}r^{-\alpha+\nu-1}dr$. Now, using the definition of $\lambda(\xi)$, one can easily see that

$$\frac{1}{|\delta_r\xi|} \leq c\,\frac{\lambda(\xi)^{\beta+1}}{r} \tag{8}$$

for $C_0\lambda(\xi) \leq r \leq 1$. Therefore,

$$|E| \leq c\,\lambda(\xi)^{\frac{3}{2}\beta-\alpha+\nu}. \tag{9}$$

It remains to estimate

$$I_3 = \int_{C_0\lambda(\xi)}^{1} \psi(r) e^{i\phi_0(r)} dr$$

and

$$I_4 = \int_{C_0\lambda(\xi)}^{1} \psi(r) e^{i\phi_\pi(r)} dr.$$

But first let us notice that (8) tells us that if $C_0\lambda(\xi) \le r \le 1$, then

$$|\psi(r)| \le c\, \frac{\lambda(\xi)^{\frac{\beta}{2}+\frac{1}{2}}}{r^{\alpha-\nu+3/2}}$$

and

$$|\psi'(r)| \le c\,(1+|v|)\, \frac{\lambda(\xi)^{\frac{\beta}{2}+\frac{1}{2}}}{r^{\alpha-\nu+5/2}}.$$

Now $\phi_\pi'(r) = -\beta r^{-\beta-1} - f(r)$, and since $f(r) > 0$, it follows that $|\phi_\pi'(r)| \ge c/\lambda(\xi)^{\beta+1}$ for $r \in [C_0\lambda(\xi), 3\lambda(\xi)/2]$. Also, for $3\lambda(\xi)/2 \le r \le 1$,

$$|\phi_\pi'(r)| = \beta r^{-\beta-1} + f(r) \ge f(r) \ge f(\lambda(\xi)) = \beta\lambda(\xi)^{-\beta-1}.$$

Thus $|\phi_\pi'(r)| \ge c/\lambda(\xi)^{\beta+1}$ on $[C_0\lambda(\xi), 1]$, and (5) then tells us that

$$\begin{aligned} |I_4| &\le c\,\lambda(\xi)^{\beta+1}\left[|\psi(1)| + \int_{C_0\lambda(\xi)}^{1} |\psi'(r)|\, dr\right] \\ &\le c\,(1+|v|)\,\lambda(\xi)^{\frac{3}{2}\beta-\alpha+\nu}. \end{aligned} \tag{10}$$

For I_3, we have

$$I_3 = \int_{C_0\lambda(\xi)}^{3\lambda(\xi)/2} \psi(r) e^{i\phi_0(r)} dr + \int_{3\lambda(\xi)/2}^{1} \psi(r) e^{i\phi_0(r)} dr = I_5 + I_6. \tag{11}$$

On $[3\lambda(\xi)/2, 1]$,

$$\begin{aligned} \phi_0'(r) &= -\beta r^{-\beta-1} + f(r) \\ &\ge -(2/3)^{\beta+1}\beta\lambda(\xi)^{-\beta-1} + f(\lambda(\xi)) \\ &= -(2/3)^{\beta+1}\beta\lambda(\xi)^{-\beta-1} + \beta\lambda(\xi)^{-\beta-1} \\ &\ge c\,\lambda(\xi)^{-\beta-1}, \end{aligned}$$

and, as before, (5) tells us that

$$|I_6| \le c\,(1+|v|)\,\lambda(\xi)^{\frac{3}{2}\beta-\alpha+\nu}. \tag{12}$$

For $C_0\lambda(\xi) \le r \le 3\lambda(\xi)/2$ we have

$$\phi_0''(r) = \beta(\beta+1)r^{-\beta-2} + f'(r) \ge \beta(\beta+1)r^{-\beta-2} \ge c/\lambda(\xi)^{\beta+2},$$

and applying (5) one more time, we get

$$\begin{aligned} |I_5| &\le c\,\lambda(\xi)^{\frac{\beta}{2}+1}\left[|\psi(\lambda(\xi)/2)| + \int_{C_0\lambda(\xi)}^{3\lambda(\xi)/2} |\psi'(r)|\, dr\right] \\ &\le c\,(1+|v|)\,\lambda(\xi)^{\beta-\alpha+\nu}. \end{aligned} \tag{13}$$

Combining (7), (9), (10), (12), and (13), we have

$$\left|\widehat{\left(\frac{K_\epsilon}{\rho(.)^{iv}}\right)}(\xi)\right| \leq c\ (1+|v|)\ \lambda(\xi)^{\beta-\alpha+\nu},$$

and by (6),

$$\left|\widehat{\left(\frac{K_\epsilon}{\rho(.)^{iv}}\right)}(\xi)\right| \leq c\ (1+|v|)\ \rho_\beta(\xi)^{-\beta+\alpha-\nu}.$$

This completes the proof. □

We are now ready to prove Theorem 3. We use interpolation of analytic families of operators on parabolic Hardy spaces (see [1]).

Proof of Theorem 3. As we mentioned before, if $\alpha = \nu$, then K_ϵ satisfies the hypothesis of Theorem 2 with bounds independent of ϵ, and it follows that T extends to a bounded linear operator on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$. Assume $\alpha > \nu$. For $z = u + iv \in \mathbb{C}$, set

$$M_z(y) = \begin{cases} \rho(y)^{\beta z-\beta-\nu} e^{i/\rho(y)^\beta} & \text{if } \epsilon \leq \rho(y) \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the family $\{R_z\}_{0\leq u\leq 1}$ of analytic operators defined on the domain of simple functions by

$$R_z f = M_z * f.$$

Clearly, $R_{\frac{\beta-\alpha+\nu}{\beta}} = T_\epsilon$.

If $u = 1$, then $\mathrm{Re}\,[-\beta z + \beta + \nu] = \nu$, and M_{1+iv} satisfies the hypothesis of Theorem 2 with $B_0 = (1+|v|)B_1$ and B_1 independent of ϵ. Thus

$$\|R_{1+iv} f\|_{L^1} \leq (1+|v|)A'\|f\|_{H^1_{\rho_\beta}}, \tag{14}$$

and the constant A' is independent of ϵ. On the other hand, Theorem 4 tells us that

$$\left|\widehat{M_{iv}}(\xi)\right| \leq B(1+|v|),$$

and it follows that

$$\|R_{iv} f\|_{L^2} \leq (1+|v|)A''\|f\|_{L^2}. \tag{15}$$

Now we interpolate between the inequalities in (14) and (15) to conclude that

$$\|R_u f\|_{L^p} \leq A(u,p)\|f\|_{L^p}$$

whenever $0 \leq u < 1$ and $\frac{1}{p} = \frac{1-u}{2} + u$. In particular,

$$\|T_\epsilon f\|_{L^p} = \|R_{\frac{\beta-\alpha+\nu}{\beta}} f\|_{L^p} \leq A_p\|f\|_{L^p}$$

for $\frac{1}{p} - \frac{1}{2} = \frac{\beta-\alpha+\nu}{2\beta}$. It follows that

$$\|T_\epsilon f\|_{L^p} \leq A_p\|f\|_{L^p}$$

for $0 \leq \frac{1}{p} - \frac{1}{2} \leq \frac{\beta-\alpha+\nu}{2\beta}$. Finally, a duality argument shows the corresponding result for $2 < p < \infty$.

This establishes Theorem 3 and consequently part (i) of Theorem 1. □

4. The Sharp Result

In the last section we showed that, if $\alpha > \nu$, T extends to a bounded linear operator on L^p for $|1/p - 1/2| \leq (\beta - \alpha + \nu)/2\beta$. In this section we prove that this result is sharp. This was the assertion of part (ii) of Theorem 1, and for convenience, we restate it here as:

Theorem 5. *Suppose T extends to a bounded linear operator on L^p, $1 \leq p < \infty$. Then*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{\beta - \alpha + \nu}{2\beta}. \tag{16}$$

At this point, outlining the argument that is going to be used in the proof of Theorem 5 will help in understanding some of the details that will follow. We are going to consider an appropriate $\varphi \in C_0^\infty(\mathbb{R}^2)$ supported in a small neighborhood U of the origin. The goal is, of course, to find a lower bound for $\|T\varphi\|_{L^p}$. To achieve this, we examine $|T\varphi(x)|$ at those x's such that $e^{i/\rho(y)^\beta}$ does not oscillate rapidly for y near x. For example, suppose that $e^{i/\rho(y)^\beta}$ does not oscillate rapidly for $y \in B(0,b) - B(0,a)$, where $0 < a < b \leq 1$ ($B(0,a)$ and $B(0,b)$ are ρ–balls). For $x \in E \subset B(0,b) - B(0,a)$ let $U_x = \{y \in \mathbb{R}^2 : x - y \in U\} =$ support of φ translated by x. To gain the best possible lower bound for $|T\varphi(x)|$, U_x should lie entirely in $B(0,b) - B(0,a)$. Moreover, to gain a satisfactory lower bound for $\|T\varphi\|_{L^p}$, U_x should cover most of $B(0,b) - B(0,a)$ as x varies in E. For all of this to occur, $\rho_\beta(y-x)$, rather than $\rho(y-x)$, should be small for $y \in U_x$. This geometric property is the subject of the next lemma.

Lemma 1. *Let $0 < \epsilon \leq a < b$ and $2\epsilon^{a_1+\beta} < b^{a_1} - a^{a_1}$. Suppose*

$$\left(a^{a_1} + \epsilon^{a_1+\beta}\right)^{1/a_1} \leq \rho(x) \leq \left(b^{a_1} - \epsilon^{a_1+\beta}\right)^{1/a_1}$$

and $\rho_\beta(x-y) \leq \epsilon$. Then $a \leq \rho(y) \leq b$.

Proof. Since $\rho_\beta(x-y) \leq \epsilon$, we have $|\gamma_{\frac{1}{\epsilon}}(x-y)| \leq 1$. It follows that $|\delta_{\frac{1}{\epsilon}}(x-y)| \leq \epsilon^\beta$, and since $a/\epsilon \geq 1$, we get

$$\epsilon^\beta \geq \left|\delta_{\frac{1}{\epsilon}}(x-y)\right| = \left|\delta_{\frac{1}{a}\frac{a}{\epsilon}}(x-y)\right| \geq \left(\frac{a}{\epsilon}\right)^{a_1} \left|\delta_{\frac{1}{a}}(x-y)\right|,$$

or

$$\left|\delta_{\frac{1}{a}}(x-y)\right| \leq \frac{\epsilon^{a_1+\beta}}{a^{a_1}}. \tag{17}$$

Similarly,

$$\left|\delta_{\frac{1}{b}}(x-y)\right| \leq \frac{\epsilon^{a_1+\beta}}{b^{a_1}}. \tag{18}$$

Now, since $\left(a^{a_1} + \epsilon^{a_1+\beta}\right)^{1/a_1} \leq \rho(x) \leq \left(b^{a_1} - \epsilon^{a_1+\beta}\right)^{1/a_1}$, we have

$$\left|\delta_{\frac{1}{\left(b^{a_1} - \epsilon^{a_1+\beta}\right)^{1/a_1}}} x\right| \leq 1 \leq \left|\delta_{\frac{1}{\left(a^{a_1} + \epsilon^{a_1+\beta}\right)^{1/a_1}}} x\right|. \tag{19}$$

The second inequality in (19) tells us that

$$1 \leq \left|\delta_{\frac{1}{a\left(1+\frac{\epsilon^{a_1+\beta}}{a^{a_1}}\right)^{1/a_1}}} x\right| \leq \frac{1}{1 + \frac{\epsilon^{a_1+\beta}}{a^{a_1}}} \left|\delta_{\frac{1}{a}} x\right|.$$

Therefore,

$$\left|\delta_{\frac{1}{a}}\, x\right| \geq 1 + \frac{\epsilon^{a_1+\beta}}{a^{a_1}}. \tag{20}$$

Similarly,

$$1 \geq \left|\delta_{\frac{1}{\left(b^{a_1}-\epsilon^{a_1+\beta}\right)^{1/a_1}}}\, x\right| \geq \frac{1}{1-\frac{\epsilon^{a_1+\beta}}{b^{a_1}}}\, \left|\delta_{\frac{1}{b}}\, x\right|,$$

so that

$$\left|\delta_{\frac{1}{b}}\, x\right| \leq 1 - \frac{\epsilon^{a_1+\beta}}{b^{a_1}}. \tag{21}$$

Now (17) and (20) tell us that

$$\left|\delta_{\frac{1}{a}}\, y\right| = \left|\delta_{\frac{1}{a}}\, x - \delta_{\frac{1}{a}}(x-y)\right| \geq 1 + \frac{\epsilon^{a_1+\beta}}{a^{a_1}} - \frac{\epsilon^{a_1+\beta}}{a^{a_1}} = 1.$$

Also, by (18) and (21),

$$\left|\delta_{\frac{1}{b}}\, y\right| = \left|\delta_{\frac{1}{b}}(y-x) + \delta_{\frac{1}{b}}\, x\right| \leq \frac{\epsilon^{a_1+\beta}}{b^{a_1}} + 1 - \frac{\epsilon^{a_1+\beta}}{b^{a_1}} = 1.$$

Hence $a \leq \rho(y) \leq b$. □

Next we construct subintervals I_k of $(0,1]$ such that $e^{i/\rho(y)^\beta}$ does not oscillate rapidly when $\rho(y)^{a_1} \in I_k$.

Lemma 2. *There exist two positive numbers A_0 and B_0, with $B_0 < A_0^{1/\beta} < 1$, such that whenever $0 < \epsilon < B_0$ and $1 \leq k \leq A_0\epsilon^{-\beta}$ (k an integer), the following hold.*

(i) $4\epsilon^{a_1+\beta} < \frac{1}{(2\pi k-\pi/3)^{a_1/\beta}} - \frac{1}{(2\pi k+\pi/3)^{a_1/\beta}}$ *and* $\epsilon \leq \frac{1}{(2\pi k+\pi/3)^{1/\beta}}$.

(ii) *Let*

$$I_k = \left[\frac{1}{(2\pi k+\pi/3)^{a_1/\beta}} + \epsilon^{a_1+\beta}, \frac{1}{(2\pi k-\pi/3)^{a_1/\beta}} - \epsilon^{a_1+\beta}\right]$$

and

$$J_k = \left[\frac{1}{(2\pi(k+1)-\pi/3)^{a_1/\beta}} - \epsilon^{a_1+\beta}, \frac{1}{(2\pi k+\pi/3)^{a_1/\beta}} + \epsilon^{a_1+\beta}\right].$$

Also, let k' be "the k" such that $k' \leq A_0\epsilon^{-\beta} < k'+1$. Then $2A_0^{-a_1/\beta}\epsilon^{a_1} < 7^{-a_1/\beta}$ and

$$I_{k'} \cup \left[\bigcup_{k=1}^{k'-1}(I_k \cup J_k)\right] \supset [A_0^{-a_1/\beta}\epsilon^{a_1}, 7^{-a_1/\beta}].$$

(iii) $|J_k| \leq C|I_{k+1}|$ *for some constant C that only depends on a_1 and β.*

Proof. Set

$$A_0 = \text{ Min }\left[\frac{1}{4\pi}, \left(\frac{a_1\pi}{6\beta(3\pi)^{\frac{a_1+\beta}{\beta}}}\right)^{\frac{\beta}{a_1+\beta}}\right]$$

and

$$B_0 = \text{ Min }\left[(\tfrac{1}{2})^{1/a_1}(\tfrac{A_0}{7})^{1/\beta}, \left((\tfrac{3}{5\pi})^{a_1/\beta} - (\tfrac{1}{7})^{a_1/\beta}\right)^{\frac{1}{a_1+\beta}}\right].$$

(i) Let $f(x) = x^{-a_1/\beta}$ $(x > 0)$. Then $f'(x) = -(a_1/\beta)x^{-\frac{a_1+\beta}{\beta}}$. For $k \geq 1$,

$$\begin{aligned}\frac{1}{(2\pi k - \pi/3)^{a_1/\beta}} - \frac{1}{(2\pi k + \pi/3)^{a_1/\beta}} &= f(2\pi k - \pi/3) - f(2\pi k + \pi/3) \\ &= (-2\pi/3)f'(t) \\ &= \frac{2\pi a_1}{3\beta}\frac{1}{t^{\frac{a_1+\beta}{\beta}}},\end{aligned}$$

where $2\pi k - \pi/3 < t < 2\pi k + \pi/3 < 3\pi k$. Thus,

$$\begin{aligned}\frac{1}{(2\pi k - \pi/3)^{a_1/\beta}} - \frac{1}{(2\pi k + \pi/3)^{a_1/\beta}} &> \frac{2\pi a_1}{3\beta(3\pi)^{\frac{a_1+\beta}{\beta}}}\frac{1}{k^{\frac{a_1+\beta}{\beta}}} \\ &\geq 4A_0^{\frac{a_1+\beta}{\beta}}\frac{1}{k^{\frac{a_1+\beta}{\beta}}}.\end{aligned}$$

So for $1 \leq k \leq A_0\epsilon^{-\beta}$, we have

$$4\epsilon^{a_1+\beta} < \frac{1}{(2\pi k - \pi/3)^{a_1/\beta}} - \frac{1}{(2\pi k + \pi/3)^{a_1/\beta}}.$$

Also, since $A_0 \leq 1/(4\pi)$,

$$\epsilon \leq \frac{A_0^{1/\beta}}{k^{1/\beta}} \leq \frac{1}{(4\pi)^{1/\beta}}\frac{1}{k^{1/\beta}} \leq \frac{1}{(2\pi k + \pi/3)^{1/\beta}}.$$

(ii) By our choice of A_0 and B_0, we have

$$2^{1/a_1}A_0^{-1/\beta}\epsilon < 7^{-1/\beta} \text{ and } \tfrac{1}{(2\pi - \pi/3)^{a_1/\beta}} - \epsilon^{a_1+\beta} > 7^{-a_1/\beta}. \tag{22}$$

The second inequality in (22) tells us that $7^{-a_1/\beta} \in I_{k'} \cup [\bigcup_{k=1}^{k'-1}(I_k \cup J_k)]$. Now

$$2\pi k' - \frac{\pi}{3} > 4k' > k' + 1 > A_0\epsilon^{-\beta},$$

so that

$$A_0^{-a_1/\beta}\epsilon^{a_1} > \frac{1}{(2\pi k' - \pi/3)^{a_1/\beta}} > \frac{1}{(2\pi k' - \pi/3)^{a_1/\beta}} - \epsilon^{a_1+\beta}.$$

Thus,

$$I_{k'} \cup \left[\bigcup_{k=1}^{k'-1}(I_k \cup J_k)\right] \supset [A_0^{-a_1/\beta}\epsilon^{a_1}, 7^{-a_1/\beta}].$$

(iii) Let $a = 2\pi k + \pi/3$ and $d = 2\pi/3$. Then

$$|J_k| + |I_{k+1}| = f(a) - f(a + 3d) = (-3d)f'(s_1),$$

where $a < s_1 < a + 3d$. On the other hand,

$$|I_{k+1}| + 2\epsilon^{a_1+\beta} = f(a + 2d) - f(a + 3d) = (-d)f'(s_2)$$

with $a+2d<s_2<a+3d<2a<2s_1$. Then

$$\begin{aligned}
|I_{k+1}|+|J_k| &= \frac{3da_1}{\beta}(\frac{1}{s_1})^{\frac{a_1+\beta}{\beta}} \\
&\leq \frac{3da_1}{\beta}(\frac{2}{s_2})^{\frac{a_1+\beta}{\beta}} \\
&= 3(2^{\frac{a_1+\beta}{\beta}})\frac{da_1}{\beta}(\frac{1}{s_2})^{\frac{a_1+\beta}{\beta}} \\
&= 3(2^{\frac{a_1+\beta}{\beta}})(|I_{k+1}|+2\epsilon^{a_1+\beta}) \\
&\leq 6(2^{\frac{a_1+\beta}{\beta}})|I_{k+1}|.
\end{aligned}$$

Hence

$$|J_k|\leq C|I_{k+1}|.$$

This completes the proof. □

Proof of Theorem 5. If $\alpha=\nu$, the right-hand side of (16) is $1/2$ and there is nothing to prove. So we may assume $\alpha>\nu$. Moreover, since T is translation invariant, it is enough to prove the theorem for $1\leq p\leq 2$. Let A_0, B_0, I_k, J_k, k, k', and ϵ be as in Lemma 2. Fix $\varphi\in C_0^\infty$ such that $0\leq\varphi\leq 1$, $\varphi(x)=1$ for $\rho_\beta(x)\leq 1/2$, and $\varphi(x)=0$ for $\rho_\beta(x)\geq 1$. Define

$$\varphi_\epsilon(x)=\varphi(\gamma_{1/\epsilon}x).$$

Then

$$\int|\varphi_\epsilon(x)|^p dx = A_p\epsilon^{2\beta+\nu} \tag{23}$$

for some $A_p>0$.

Suppose $\rho(x)^{a_1}\in I_k$ and $\rho_\beta(x-y)\leq\epsilon$. Then Lemma 1, together with part (i) of Lemma 2, tell us that

$$\frac{1}{(2\pi k+\pi/3)^{1/\beta}}\leq\rho(y)\leq\frac{1}{(2\pi k-\pi/3)^{1/\beta}},$$

or

$$2\pi k-\pi/3\leq\frac{1}{\rho(y)^\beta}\leq 2\pi k+\pi/3. \tag{24}$$

Now by (2), $\rho(x-y)\leq\epsilon$. Also by part (i) of Lemma 2, $\epsilon\leq\rho(x)$. Thus,

$$\rho(y)\leq C\left(\rho(x-y)+\rho(x)\right)\leq(\epsilon+\rho(x))\leq 2C\rho(x). \tag{25}$$

Choose ϵ' such that $0 < \epsilon' < \epsilon$. (24) and (25) tell us that if $\rho(x)^{a_1} \in I_k$, then

$$\begin{aligned}
\left|\int_{1\geq\rho(y)\geq\epsilon'} \frac{e^{i/\rho(y)^\beta}}{\rho(y)^\alpha}\varphi_\epsilon(x-y)dy\right| &\geq \left|\int_{1\geq\rho(y)\geq\epsilon'} \frac{\cos(1/\rho(y)^\beta)}{\rho(y)^\alpha}\varphi_\epsilon(x-y)dy\right| \\
&\geq \frac{1}{2}\int_{\rho_\beta(x-y)\leq\epsilon} \frac{1}{\rho(y)^\alpha}\varphi_\epsilon(x-y)dy \\
&\geq \frac{c}{\rho(x)^\alpha}\int_{\rho_\beta(x-y)\leq\epsilon} \varphi_\epsilon(x-y)dy \\
&= \frac{c}{\rho(x)^\alpha}\int \varphi_\epsilon(y)dy \\
&= \frac{c}{\rho(x)^\alpha}A_1\epsilon^{2\beta+\nu}.
\end{aligned}$$

Hence, if $\rho(x)^{a_1} \in I_k$,

$$|T\varphi_\epsilon(x)| = \lim_{\epsilon'\to 0}\left|\int_{1\geq\rho(y)\geq\epsilon'} \frac{e^{i/\rho(y)^\beta}}{\rho(y)^\alpha}\varphi_\epsilon(x-y)dy\right| \geq c\,\epsilon^{2\beta+\nu}\frac{1}{\rho(x)^\alpha}\,.$$

Then

$$\begin{aligned}
\int |T\varphi_\epsilon(x)|^p dx &\geq \sum_k \int_{\rho(x)^{a_1}\in I_k} |T\varphi_\epsilon(x)|^p dx \\
&\geq c\,\epsilon^{p(2\beta+\nu)}\sum_k \int_{\rho(x)^{a_1}\in I_k} \frac{dx}{\rho(x)^{\alpha p}}\,.
\end{aligned}$$

Changing $\int_{\rho(x)^{a_1}\in I_k}(1/\rho(x)^{\alpha p})\,dx$ into polar coordinates, and making a simple change of variables, we get

$$\int_{\rho(x)^{a_1}\in I_k} \frac{dx}{\rho(x)^{\alpha p}} \geq c\int_{I_k} \frac{dr}{r^{(\alpha p-a_2)/a_1}}.$$

Now, using the fact that $|J_k| \leq C|I_{k+1}|$ (part (iii) of Lemma 2), we have

$$\begin{aligned}
\int |T\varphi_\epsilon(x)|^p dx &\geq c\,\epsilon^{p(2\beta+\nu)}\sum_k \int_{I_k} \frac{dr}{r^{(\alpha p-a_2)/a_1}} \\
&\geq c\,\epsilon^{p(2\beta+\nu)}\left(\sum_{k=1}^{k'}\int_{I_k}\frac{dr}{r^{(\alpha p-a_2)/a_1}} + \sum_{k=1}^{k'-1}\int_{J_k}\frac{dr}{r^{(\alpha p-a_2)/a_1}}\right) \\
&\geq c\,\epsilon^{p(2\beta+\nu)}\int_{I_{k'}\cup[\bigcup_{k=1}^{k'-1}(I_k\cup J_k)]}\frac{dr}{r^{(\alpha p-a_2)/a_1}}\,.
\end{aligned}$$

Using part (ii) of Lemma 2, we get

$$\int |T\varphi_\epsilon(x)|^p dx \geq c\,\epsilon^{p(2\beta+\nu)}\int_{A_0^{-a_1/\beta}\epsilon^{a_1}}^{7^{-a_1/\beta}} \frac{dr}{r^{(\alpha p-a_2)/a_1}}\,.$$

By the assumptions made on α and p at the beginning of the proof, $\alpha p-\nu+1>1$. Hence

$$\int |T\varphi_\epsilon(x)|^p dx \geq c\,\epsilon^{p(2\beta+\nu)}\epsilon^{\nu-\alpha p}.$$

Now, since T is bounded on L^p,

$$A_p\epsilon^{2\beta+\nu} = \|\varphi_\epsilon\|_{L^p}^p \geq c\,\|T\varphi_\epsilon\|_{L^p}^p = c\,\epsilon^{p(2\beta+\nu)}\epsilon^{\nu-\alpha p}.$$

Letting $\epsilon \to 0$, it follows that

$$p(2\beta + \nu) + \nu - \alpha p \geq 2\beta + \nu,$$

or

$$p(2\beta - \alpha + \nu) \geq 2\beta.$$

Therefore,

$$\frac{1}{p} - \frac{1}{2} \leq \frac{\beta - \alpha + \nu}{2\beta}.$$

This completes the proof of the theorem. □

References

1. R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645. MR **56:**6264
2. C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36. MR **41:**2468
3. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193. MR **56:**6263
4. A. Miyachi, *On some Fourier multipliers for $H^p(\mathbb{R}^n)$*, J. Fac. Sci. Univ. Tokyo **27** (1980), 157–179. MR **81g:**42020
5. P. Sjölin, *L^p estimates for strongly singular convolution operators in $\mathbb{R}^n$*, Ark. Mat. **14** (1976), 59–64. MR **54:**844
6. ———, *An H^p inequality for strongly singular integrals*, Mat. Zeit. **165** (1979), 231–238. MR **81d:**42030
7. ———, *Convolution with oscillating kernels*, Indiana Univ. Math. J. **30** (1981), 47–56. MR **82d:**42018
8. E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, 1993. MR **95c:**42002
9. E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295. MR **80k:**42023
10. S. Wainger, *Special Trigonometric Series in k Dimensions*, Memoirs Amer. Math. Soc. # 59, American Mathematical Society, Providence, RI, 1965. MR **32:**320

Department of Mathematics, American University of Beirut, Beirut, Lebanon
E-mail address: bshayya@aub.edu.lb