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IITAKA'S FIBRATIONS VIA MULTIPLIER IDEALS

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Dedicated to Professor Shigeru Iitaka on his sixtieth birthday

Abstract. We give a new characterization of Iitaka's fibration of algebraic varieties associated to line bundles. Introducing an "intersection number" of line bundles and curves by using the notion of multiplier ideal sheaves, Iitaka's fibration can be regarded as a "numerically trivial fibration" in terms of this intersection theory.

1. Introduction

In this paper, we would like to discuss reduction maps of varieties associated to line bundles, such as Iitaka's fibration [I], Tsuji's fibration [T], and that of Bauer et al. [B]. These fibrations are obtained by contracting subvarieties on which line bundles under consideration are "trivial" in an appropriate sense. Here we propose a new such kind of fibration by using the notion of multiplier ideal sheaves. The fibration of Tsuji and that of Bauer et al. are characterized numerically; however, neither of them reflects properties of linear systems of line bundles, in contrast with Iitaka's. Our fibration is also characterized numerically, similarly as in [T] and [B] (in particular, it is closely related with Tsuji's). However, it reflects properties of linear systems of line bundles. In fact, it coincides with Iitaka's. In this sense our fibration is a new construction and/or a new characterization of Iitaka's fibration by multiplier ideals.

In any case, we need to clarify in which sense a line bundle is "trivial". We consider a smooth complex projective variety X, and a line bundle L on X with non-negative Kodaira-Iitaka dimension: $\kappa(L) \geq 0$. We shall define, modeled after Tsuji [T, §2.4], an "intersection number" $||L;C||$ of L and a curve C by using the notion of asymptotic multiplier ideal sheaves, which was introduced by Ein and Kawamata [K1], [K2] (we will recall the definition in §2). We may call L "numerically trivial" if $||L;C|| = 0$ for (almost) all curves C. The asymptotic multiplier ideal $\mathcal{J}(||L||) \subset \mathcal{O}_X$ is a coherent ideal sheaf which reflects the asymptotic behavior of the base scheme of the linear system $|kL|$ for large k. Roughly speaking, we define $\mathcal{J}(||L||)$ as a limit, when k goes to infinity, of "the base ideal of $|kL|$ divided by k", and we regard a limit, when m goes to infinity, of "$mL \otimes \mathcal{J}(||mL||)$ divided by m" as the positive part of L. We will see that these rough ideas are justified, at least numerically. We define an "intersection number" $||L;C||$ as the intersection number ("the positive part of L")$\cdot C$ as follows (see Definition 2.7):

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Definition 1.1. Let C be an integral curve in X such that $C \not\subset \mathrm{SBs}\,|L| := \bigcap_{m\in\mathbb{N}} \mathrm{Bs}\,|mL|$, which stands for the stable base locus of L. The *intersection number* $||L;C||$ is defined by

$$||L;C|| := \lim_{m\to\infty} m^{-1} \deg_C mL \otimes \mathcal{J}(||mL||).$$

As for the right-hand side, we define

$$\deg_C mL \otimes \mathcal{J}(||mL||) := mL\cdot C + \deg_C \mathcal{J}(||mL||),$$

and "$\deg_C$" as follows. Let $\nu : C' \longrightarrow C \subset X$ be the normalization, and $\mathcal{J} \subset \mathcal{O}_X$ a coherent ideal sheaf such that $C \not\subset V\mathcal{J}$ (the subscheme defined by $\mathcal{J}$); then we set

$$\begin{aligned}\deg_C \mathcal{J} &:= \deg_{C'} \nu^{-1}\mathcal{J}\cdot\mathcal{O}_{C'}\\ &= \text{the degree of the invertible sheaf } \nu^{-1}\mathcal{J}\cdot\mathcal{O}_{C'} \text{ on } C'.\end{aligned}$$

For example, in case L is semi-ample, we will have $\mathcal{J}(||mL||) = \mathcal{O}_X$ for every $m \in \mathbb{N}$, and hence $||L;C|| = L\cdot C$. In case L is big and admits a decomposition $L = P + N$ with $\mathbb{Q}$-divisors P and N such that P is nef, N is effective, and the natural injection $H^0(X, \llcorner mP \lrcorner) \longrightarrow H^0(X, mL)$ is bijective for all $m \in \mathbb{N}$, then $||L;C|| = P\cdot C$ holds for $C \not\subset \mathrm{SBs}\,|L|$ (Example 2.11). Such a decomposition is called a Zariski decomposition of L. The $\mathbb{Q}$-divisor P (respectively N) is called the positive (respectively negative) part of L (cf. [KMM, §7-3]). The relation $||L;C|| = P\cdot C$ in this special but important case is supporting evidence for the principle $||L;C|| =$ ("the positive part of L")$\cdot C$ for not necessarily big L. In this principle, the following theorem, which is a crucial property of our intersection theory, says that $\kappa(L) = 0$ if and only if L has no positive part.

Theorem 1.2. *Let X be a smooth complex projective variety, and L a line bundle on X with $\kappa(L) \geq 0$. Then $\kappa(L) = 0$ if and only if L is numerically trivial in the sense that $||L;C|| = 0$ for every integral curve $C \subset X$ with $C \not\subset \mathrm{SBs}\,|L|$.*

By using these $||L;C||$'s as a measure to measure how positive or how trivial a line bundle L is, we have a fibration by contracting "numerically trivial" loci of L.

Theorem 1.3. *Let X be a smooth complex projective variety, and L a line bundle on X with $\kappa(L) \geq 0$. Then there exist a proper birational morphism $\mu : X' \longrightarrow X$ from a smooth projective variety X', and a proper surjective morphism $f : X' \longrightarrow Y$ to a variety Y with connected fibers, with the following two properties.*

(1) *μ^*L is numerically trivial on every very general fibre of f, i.e., if $y \in Y$ is very general and $C' \subset X'_y := f^{-1}(y)$ is an integral curve with $C' \not\subset \mathrm{SBs}\,|\mu^*L|$, then $||\mu^*L;C'|| = 0$.*

(2) *If $x' \in X'$ is general and $C' \subset X'$ is any integral curve passing through x' with $\dim f(C') > 0$, then $||\mu^*L;C'|| > 0$.*

Moreover such $f : X' \longrightarrow Y$, or the corresponding rational map $X \dashrightarrow Y$, is unique up to birational equivalence. In fact, the map $X \dashrightarrow Y$ is (birationally equivalent to) the Iitaka fibration associated to L.

Conventions and Notation

(1) We work over a complex number field.

(2) For a real number a, we let $\llcorner a \lrcorner$ be the largest integer that is less than or equal to a, and let $\ulcorner a \urcorner$ be the smallest integer that is greater than or equal to a. We also use the notation $\llcorner D \lrcorner$ and $\ulcorner D \urcorner$ for $\mathbb{R}$-divisors D.

(3) We do not distinguish between line bundles, linear equivalence classes of divisors, and invertible sheaves.

(4) For a coherent ideal sheaf $\mathcal{J}$, we denote by $V\mathcal{J}$ the associated subscheme.

(5) For a divisor (or a line bundle) D on a proper variety X, we denote by $\mathrm{Bs}\,|D|$ the (set-theoretic) base locus of the linear system $|D|$, and by $\mathrm{SBs}\,|D| := \bigcap_{m\in\mathbb{N}} \mathrm{Bs}\,|mD|$ the stable base locus of D. In case $|D| \neq \emptyset$, we have a rational map, denoted by $\Phi_{|D|} : X \dashrightarrow \mathbb{P}^{\dim|D|}$. We denote by $\Phi_{|D|}(X)$ the closure of $\Phi_{|D|}(X - \mathrm{Bs}\,|D|)$ in $\mathbb{P}^{\dim|D|}$.

(6) Let $\mu : X' \longrightarrow X$ be a birational morphism of varieties. The exceptional locus $\mathrm{Exc}(\mu)$ of μ is the smallest closed subset of X' such that $\mu|_{X'-\mathrm{Exc}(\mu)}$ is an isomorphism.

(7) By a very general point, we mean a point that belongs to the complement of a countable union of proper Zariski closed subsets.

Throughout this paper, unless otherwise stated, we let X be a smooth complex projective variety and L a line bundle (or a divisor) on X with nonnegative Kodaira-Iitaka dimension: $\kappa(L) \geq 0$, i.e., $H^0(X, mL) \neq 0$ for some $m \in \mathbb{N}$, and then $\kappa(L) := \max\{\dim \Phi_{|mL|}(X);\ m \in \mathbb{N}\}$.

2. Multiplier Ideals and Intersection Numbers

2A. Multiplier ideals. We recall the definitions of multiplier ideal sheaves (see [DEL, §1] [L, §2, §7]). We let c be a positive rational number throughout this subsection.

Definition 2.1. Let D be an effective $\mathbb{Q}$-divisor on X. We take a log-resolution $\mu : X' \longrightarrow X$ of D, i.e., a proper birational morphism from a smooth variety such that $\mu^*D + \mathrm{Exc}(\mu)$ is a divisor with simple normal crossing support. Then the *multiplier ideal sheaf* $\mathcal{J}(D) \subset \mathcal{O}_X$ associated to D is

$$\mathcal{J}(D) = \mathcal{J}(X, D) := \mu_*\mathcal{O}_{X'}(K_{X'/X} - \llcorner \mu^*D \lrcorner).$$

Here $K_{X'/X} = K_{X'} - \mu^*K_X$ is the relative canonical divisor of X' over X.

Definition 2.2. Let $V \subset H^0(X, L)$ be a vector subspace, and $|V| \subset |L|$ the corresponding linear subsystem. Let $\mu : X' \longrightarrow X$ be a log-resolution of $|V|$, i.e., a proper birational morphism from a smooth variety such that $\mu^*|V| = |W| + F$, where $F + \mathrm{Exc}(\mu)$ is a divisor with simple normal crossing support, and $W \subset H^0(X', \mu^*L - F)$ defines a base-point-free linear system. The *multiplier ideal sheaf* $\mathcal{J}(c \cdot |V|) \subset \mathcal{O}_X$ corresponding to c and $|V|$ is

$$\mathcal{J}(c \cdot |V|) = \mathcal{J}(X, c \cdot |V|) := \mu_*\mathcal{O}_{X'}(K_{X'/X} - \llcorner cF \lrcorner).$$

The semi-group of L is $N(L) = N(X, L) := \{m \in \mathbb{N};\ H^0(X, mL) \neq 0\}$. All sufficiently large elements of $N(L)$ are multiples of a largest single $e(L) \in \mathbb{N}$, which we call the exponent of L, and all sufficiently large multiples of $e(L)$ appear in $N(L)$. The exponent $e(L)$ is the greatest common divisor of all elements of $N(L)$. We define the Iitaka threshold of L to be the least integer $m_0 = m_0(L)$ such that $H^0(X, mL) \neq 0$ for all $m \geq m_0(L)$ and $e(L)|m$.

Lemma 2.3 ([L, 7.5])**.** *Assume $e(L) = 1$.*

(1) *Let $p \geq m_0(L)$ be a positive integer. Then $\mathcal{J}(\frac{c}{p} \cdot |pL|) \subset \mathcal{J}(\frac{c}{pk} \cdot |pkL|)$ for every positive integer k.*

(2) *The family of ideals $\{\mathcal{J}(\frac{c}{p} \cdot |pL|)\}_{p \geq m_0(L)}$ has a unique maximal element.*

Definition 2.4. The *asymptotic multiplier ideal sheaf*

$$\mathcal{J}(c\cdot||L||)=\mathcal{J}(X,c\cdot||L||)\subset\mathcal{O}_X$$

associated to c and $|L|$ is defined as follows.

(1) For $e(L)=1$, it is the unique maximal member among $\{\mathcal{J}(\frac{c}{p}\cdot|pL|)\}_{p\geq m_0(L)}$.

(2) For $e=e(L)$ general, $\mathcal{J}(c\cdot||L||):=\mathcal{J}(\frac{c}{e}\cdot||eL||)$, where the right-hand side is defined in (1).

Remark 2.5. By the definitions (and Lemma 2.3), we can see immediately that (1) $V\mathcal{J}(c\cdot|L|)\subset\mathrm{Bs}\,|L|$ for any $c\leq 1$, (2) $V\mathcal{J}(c\cdot||L||)\subset\mathrm{SBs}\,|L|$ for any c, and (3) $V\mathcal{J}(c\cdot||mL||)\subset\mathrm{SBs}\,|L|$ for any c and $m\in\mathbb{N}$.

2B. Intersection number. We will state several fundamental properties of our intersection number. The following lemma implies that the assumption $\kappa(L)\geq 0$ guarantees $||L;C||\geq 0$ for almost all curves C.

Lemma 2.6. *Let $C\subset X$ be an integral curve with $C\not\subset\mathrm{SBs}\,|L|$. Then*:

(1) *$C\not\subset V\mathcal{J}(||mL||)$ (by Remark 2.5) and $\deg_C\mathcal{J}(||mL||)\geq -mL\cdot C$ for every $m\in\mathbb{N}$.*

(2) *The limit $\lim_{m\to\infty}m^{-1}\deg_C mL\otimes\mathcal{J}(||mL||)$ exists.*

Proof. (1) We take a sufficiently large and divisible positive integer p such that $C\not\subset D$ for some $D\in|mpL|$ (by $C\not\subset\mathrm{SBs}\,|L|$), and such that $\mathcal{J}(||mL||)=\mathcal{J}(\frac{1}{p}\cdot|pmL|)$ (by [L, 7.9, 7.12]). Then we have $\mathcal{J}(\frac{1}{p}D)\subset\mathcal{J}(\frac{1}{p}\cdot|pmL|)$ by the definition, and $\mathcal{J}(D)=\mathcal{O}_X(-D)$ by definition and the projection formula ([L, 2.17]). By the subadditivity property ([DEL, §2], [L, 5.13]), we have $\mathcal{J}(D)\subset\mathcal{J}(\frac{1}{p}D)^p$. Putting everything together, we finally have $\mathcal{O}_X(-D)\subset\mathcal{J}(||mL||)^p$. Then

$$\deg_C\mathcal{J}(||mL||)=p^{-1}\deg_C\mathcal{J}(||mL||)^p\geq p^{-1}\deg_C\mathcal{O}(-D)=-mL\cdot C.$$

(2) By (1) we have $\deg_C mL\otimes\mathcal{J}(||mL||)\geq 0$. By the subadditivity property ([DEL, §2], [L, 8.7]), we have $\mathcal{J}(||(\ell+m)L||)\subset\mathcal{J}(||\ell L||)\cdot\mathcal{J}(||mL||)$ for every $\ell,m\in\mathbb{N}$; in particular, we have the inequality

$$\deg_C(\ell+m)L\otimes\mathcal{J}(||(\ell+m)L||)\leq\deg_C\ell L\otimes\mathcal{J}(||\ell L||)+\deg_C mL\otimes\mathcal{J}(||mL||).$$

Then the existence of our limit follows from the elementary general theory of functions. □

Thanks to this lemma, we have the following:

Definition 2.7. Let C be an integral curve in X such that $C\not\subset\mathrm{SBs}\,|L|$. The *intersection number* $0\leq||L;C||\leq L\cdot C$ is defined by

$$||L;C||:=\lim_{m\to\infty}m^{-1}\deg_C mL\otimes\mathcal{J}(||mL||).$$

In case $\mathcal{J}(||mL||)=\mathcal{O}_X$ for every sufficiently large m, for example in case L is semi-ample, or L is nef and big ([L, 8.11]), we have $||L;C||=L\cdot C$ for every integral curve $C\subset X$ with $C\not\subset\mathrm{SBs}\,|L|$. However, as in Example 2.8 below, for a general nef (but not big) line bundle L, the numbers $||L;C||$ and $L\cdot C$ can be totally different. That difference is exactly the difference between our fibration and those of Tsuji [T], or of Bauer et al. [B].

Example 2.8 ([DEL, §1])**.** Let E be a unitary flat vector bundle on a smooth variety Y such that no nontrivial symmetric power of E and E^* has sections, and set $U = \mathcal{O}_Y \oplus E$. We consider $X = \mathbb{P}(U)$ and $L = \mathcal{O}_{\mathbb{P}(U)}(1)$. Then for every $p \geq 1$, pL has a unique nontrivial section that vanishes to order p along the "divisor at infinity" $H = \mathbb{P}(\mathcal{O}_Y) \subset X$. Therefore $\mathcal{J}(||mL||) = \mathcal{O}_X(-mH)$, and $mL \otimes \mathcal{J}(||mL||) \cong \mathcal{O}_X$ for every positive integer m. We have $||L; C|| = 0$ for every integral curve $C \subset X$ that is not contained in H (this is the model case of Theorem 1.2).

On the other hand, L is nef (since $\mathcal{O}_Y \oplus E$ is) and nonzero effective. Therefore, X is covered by a family of curves $\{C_\lambda\}_{\lambda \in \Lambda}$ with $L \cdot C_\lambda > 0$, which is the major difference from [B]. In addition, the fibrations of [T] and [B] must coincide (up to birational equivalence) in this example, as we see below. We note that L has a smooth semi-positive Hermitian metric induced by the flat metric on E. In particular, $\mathcal{J}(h_{\min}^m) = \mathcal{O}_X$ for any $m > 0$; here $h_{\min}$ is a singular Hermitian metric with minimal singularity (cf. [D2, §4.1]). That means $(L, h_{\min}) \cdot C = L \cdot C$ for any integral curve $C \subset X$, where $(L, h_{\min}) \cdot C$ is an "intersection number" of Tsuji [T, §2.4]. Although we do not give the definition [T, Definition 2.9], it is the usual intersection number in this special case. In such cases, the fibrations of [T] and [B] have no difference (up to birational equivalence).

The following lemma will turn out to be useful (we may say that our intersection numbers are weakly linear).

Lemma 2.9. (1) *For every positive integer k,* $\mathrm{SBs}\,|kL| = \mathrm{SBs}\,|L|$.

(2) *Let $C \subset X$ be an integral curve with $C \not\subset \mathrm{SBs}\,|L|$. Then $||kL; C|| = k||L; C||$ for every positive integer k.*

Proof. (1) is clear (since we just consider "SBs" as subsets).

(2) The intersection number $||kL; C||$ is well-defined by (1). Then $||kL; C|| = k||L; C||$ follows from the fact that the limit $\lim_{m\to\infty} m^{-1} \deg_C mL \otimes \mathcal{J}(||mL||)$ exists. □

Remark 2.10. Although $||L; C||$ is weakly linear, it is not linear, as we can see as follows. Let us take the line bundles in [L, 8.8]. Let X be the blowing up of $\mathbb{P}^2$ at a point, and denote by H and E respectively the pull-back of a line and the exceptional divisor. Consider $L_1 := 2H - E$ and $L_2 := 2H + E$. Then L_1 as well as $L_1 + L_2$ are semi-ample. On the other hand, $\mathcal{J}(||mL_2||) = \mathcal{O}_X(-mE)$. For any integral curve $C \subset X$, $C \neq E$, we have $||L_1 + L_2; C|| = 4H \cdot C$ and $||L_1; C|| = (2H - E) \cdot C, ||L_2; C|| = 2H \cdot C$. Therefore, it can be true that

$$||L_1 + L_2; C|| > ||L_1; C|| + ||L_2; C||.$$

Example 2.11. Assume that L is big and admits a *Zariski decomposition*, i.e., there exist a proper birational morphism $\mu : X' \longrightarrow X$ from a smooth variety X' and a decomposition $\mu^* L = P + N$ with $\mathbb{Q}$-divisors P and N on X' such that (i) P is nef, (ii) N is effective, (iii) the natural injection $H^0(X', \llcorner mP \lrcorner) \longrightarrow H^0(X, mL)$ is bijective for all $m \in \mathbb{N}$. Then $||L; C|| = P \cdot C'$ holds for every integral curve $C \subset X$ with $C \not\subset \mathrm{SBs}\,|L| \cup \mu(\mathrm{Exc}(\mu))$, where C' is the strict transform of C in X'.

Proof. We first suppose that $\mu : X' \longrightarrow X$ is the identity, namely, $L = P + N$ on X. We may assume that both P and N are integral divisors. In fact, for a large and divisible k, $kL = kP + kN$ is a Zariski decomposition of kL with kP and kN integral. Therefore $||kL; C|| = (kP) \cdot C$ would imply $||L; C|| = P \cdot C$ by Lemma 2.9.

Under this assumption it is enough to show that $\mathcal{J}(||mL||) = \mathcal{O}_X(-mN)$ for every $m \in \mathbb{N}$. We fix $x \in X$ for a while. We note a lemma of Wilson [W, Theorem 2.2]: a function

$$m_x(k) = \mathrm{mult}_x\,|kP| : \text{ the multiplicity at } x \text{ of a general divisor in } |kP|$$

remains bounded as k goes to infinity (since P is nef and big). Then we can take a sufficiently large k such that (iv) $\mathcal{J}(||mL||) = \mathcal{J}(\frac{m}{k}|kL|)$ (note that $e(L) = 1$, since L is big), and that (v) $\frac{m}{k}m_x(k) < 1$. By property (iii), a general divisor in $|kL|$ is a form of $D + kN$ with D a general divisor in $|kP|$. Then

$$\mathcal{J}(||mL||) = \mathcal{J}(\frac{m}{k} \cdot |kL|) = \mathcal{J}(\frac{m}{k}(D + kN)) = \mathcal{J}(\frac{m}{k}D) \otimes \mathcal{O}_X(-mN).$$

The second equality follows from [L, 2.18], the third from [L, 2.17]. By (v) we have $\mathrm{mult}_x(\frac{m}{k}D) < 1$. Then we see that $\mathcal{J}(\frac{m}{k}D)_x = \mathcal{O}_{X,x}$ by [L, 5.11], and therefore $\mathcal{J}(||mL||) = \mathcal{O}_X(-mN)$.

The general case is reduced to the above special case (for $\mu^*L = P + N$), by virture of Lemma 2.12 (2) below. □

Lemma 2.12. *Let $\mu : X' \longrightarrow X$ be a proper birational morphism from a smooth variety X'. Let $C' \subset X'$ be an integral curve with $C' \not\subset \mathrm{Exc}(\mu)$, and let $C := \mu(C')_{red} \not\subset \mathrm{SBs}\,|L|$. Then*

(1) *for every positive integer m, one has*

$$\deg_{C'} \mathcal{J}(||m\mu^*L||) \le \deg_C \mathcal{J}(||mL||) \le \deg_{C'} \mathcal{J}(||m\mu^*L||) + K_{X'/X} \cdot C'.$$

(2) $||\mu^*L; C'|| = ||L; C||$.

Proof. (1) We note that $N(X', \mu^*L) = N(X, L)$ and $e(X', \mu^*L) = e(X, L) =: e$. We take a sufficiently large p such that $\mathcal{J}(||mL||) = \mathcal{J}(\frac{m}{ep} \cdot |epL|)$ and $\mathcal{J}(||m\mu^*L||) = \mathcal{J}(\frac{m}{ep} \cdot |ep\mu^*L|)$. We note two basic relations [L, 5.7] and [L, 2.25]:

$$\mathcal{J}(\frac{m}{ep} \cdot |ep\mu^*L|) \subset \mu^{-1}\mathcal{J}(\frac{m}{ep} \cdot |epL|) \cdot \mathcal{O}_{X'};$$
$$\mathcal{J}(\frac{m}{ep} \cdot |epL|) = \mu_*(\mathcal{J}(\frac{m}{ep} \cdot |ep\mu^*L|) \otimes K_{X'/X}).$$

We also have a natural map

$$\mu^{-1}(\mu_*(\mathcal{J}(\frac{m}{ep} \cdot |ep\mu^*L|) \otimes K_{X'/X})) \overset{\text{gen. iso.}}{\longrightarrow} \mathcal{J}(\frac{m}{ep} \cdot |ep\mu^*L|) \otimes K_{X'/X},$$

which is generically isomorphic, since μ is birational. Putting everything together, we have

$$\mathcal{J}(||m\mu^*L||) \subset \mu^{-1}\mathcal{J}(||mL||) \cdot \mathcal{O}_{X'} \overset{\text{gen. iso.}}{\longrightarrow} \mathcal{J}(||m\mu^*L||) \otimes K_{X'/X}.$$

We let $\nu : C'' \longrightarrow C' \subset X'$ be the normalization of C' and C. Since $\nu(C'') = C' \not\subset \mathrm{Exc}(\mu)$, we have maps

$$\nu^{-1}\mathcal{J}(||m\mu^*L||) \cdot \mathcal{O}_{C''} \subset (\mu \circ \nu)^{-1}\mathcal{J}(||mL||) \cdot \mathcal{O}_{C''}$$
$$\overset{\text{gen. iso.}}{\longrightarrow} \nu^{-1}\mathcal{J}(||m\mu^*L||) \cdot \mathcal{O}_{C''} \otimes \nu^*K_{X'/X}.$$

Then by taking their degrees, we have (1).

(2) follows from (1) and the fact that $K_{X'/X} \cdot C'$ is a constant. □

3. Numerically Trivial Loci

In this section we shall discuss how to characterize loci that are covered by curves with trivial "intersection number". Theorem 1.2 is a direct consequence of Lemma 3.1 and Corollary 3.5.

Lemma 3.1. *Assume that $\kappa(L) = 0$. Then $||L; C|| = 0$ for every integral curve $C \subset X$ with $C \not\subset \mathrm{SBs}\,|L|$.*

Proof. We have $H^0(X, kL) \neq 0$ for some $k \in \mathbb{N}$. Since $||kL; C|| = k||L; C||$ by Lemma 2.9, we may assume that $H^0(X, L) \neq 0$ by considering kL instead of L. Moreover since $\kappa(L) = 0$, there exists a unique $D \in |L|$ such that $|pL| = pD$ for any $p \in \mathbb{N}$. In particular, $\mathrm{SBs}\,|L| = \mathrm{supp}\, D$ as sets. We take a sufficiently large p so that $\mathcal{J}(||mL||) = \mathcal{J}(\frac{m}{p} \cdot |pL|)$. Let $\mu : X' \longrightarrow X$ be a log-resolution of D. This also gives a log-resolution of $|pL|$ for any $p \in \mathbb{N}$. The fixed component of $\mu^*|pL|$ is nothing but $p\mu^*D$. Then

$$\mathcal{J}(\frac{m}{p} \cdot |pL|) = \mu_*\mathcal{O}_{X'}(K_{X'/X} - \llcorner \frac{m}{p} p\mu^* D \lrcorner) = \mathcal{O}_X(-mD).$$

We have $mL \otimes \mathcal{J}(||mL||) = mL \otimes \mathcal{O}_X(-mD) \cong \mathcal{O}_X$, and $||L; C|| = 0$. (Note that we are not saying that $mL \otimes \mathcal{J}(||mL||) \cong \mathcal{O}_X$ for every $m \in \mathbb{N}$ under the assumption $\kappa(L) = 0$.) □

The following lemma is a key for the proof of Theorem 1.3. Although it looks like just a relative version of Lemma 3.1, its proof requires a deep result concerning multiplier ideal sheaves: the generic restriction theorem [L, 5.19].

Lemma 3.2. *Let $f : X \longrightarrow Y$ be a surjective morphism to a variety Y. Assume that $\dim \Phi_{|mL|}(X_y) = 0$ for general $y \in Y$ and every large and divisible m, where $X_y := f^{-1}(y)$. Then $||L; C|| = 0$ for every integral curve $C \subset X$ with $C \not\subset \mathrm{SBs}\,|L|$ such that $C \subset X_y$ for a very general $y \in Y$.*

Proof. By considering the Stein factorization, we may assume that f has connected fibers. As in the proof of Lemma 3.1, we may also assume that $H^0(X, L) \neq 0$ by Lemma 2.9. Then we have a Zariski closed subset $Y_1 \subset Y$ such that (i) $\emptyset \neq Y - Y_1 \subset \mathrm{Reg}\, Y$ (the nonsingular locus of Y), (ii) f is smooth over $Y - Y_1$, and (iii) $X_y \not\subset \mathrm{Bs}\,|L|$ for every $y \in Y - Y_1$. For every general $y \in Y - Y_1$, we can write $|L|_{X_y} = B_y$ for an effective divisor B_y on X_y, and $|pL|_{X_y} = pB_y$ for every $p \in \mathbb{N}$, by our assumption that $\dim \Phi_{|mL|}(X_y) = 0$. Here $|pL|_{X_y}$ is the restricted linear system from X to X_y. As in the proof of Lemma 3.1, we have

$$\mathcal{J}(X_y, \frac{m}{p} \cdot |pL|_{X_y}) = \mathcal{O}_{X_y}(-mB_y)$$

for every $m, p \in \mathbb{N}$.

For every $m \in \mathbb{N}$, we take p_m so large that $\mathcal{J}(||mL||) = \mathcal{J}(X, \frac{m}{p_m} \cdot |p_m L|)$. Then by applying the generic restriction theorem [L, 5.19] over $Y - Y_1$ countably many times, we obtain Y_2, a countable union of proper Zariski closed subsets of Y, such that, if $y \in Y - Y_1 - Y_2$, then

$$\mathcal{J}(X_y, \frac{m}{p_m} \cdot |p_m L|_{X_y}) = \mathcal{J}(X, \frac{m}{p_m} \cdot |p_m L|) \cdot \mathcal{O}_{X_y}$$

for any $m \in \mathbb{N}$.

Let us take a very general point $y \in Y$, such as $y \in Y - Y_1 - Y_2$, and an integral curve $C \subset X_y$ with $C \not\subset \mathrm{SBs}\,|L|$. Then

$$\begin{aligned}\deg_C \mathcal{J}(||mL||) &= \deg_C \mathcal{J}(X, \frac{m}{p_m} \cdot |p_m L|)\\ &= \deg_C \mathcal{J}(X, \frac{m}{p_m} \cdot |p_m L|) \cdot \mathcal{O}_{X_y}\\ &= \deg_C \mathcal{J}(X_y, \frac{m}{p_m} \cdot |p_m L|_{X_y})\\ &= \deg_C \mathcal{O}_{X_y}(-mB_y)\\ &= -mL \cdot C.\end{aligned}$$

Thus we have $||L; C|| = 0$. □

Some well-known arguments of the usual intersection theory can be generalized in our case.

Lemma 3.3. *Let $x \in X$ be a general point, such as $x \notin \mathrm{SBs}\,|L|$. Let C be an integral curve with $x \in C$, and let $0 \neq D \in |mL|$ with $x \in D$ for some $m \in \mathbb{N}$.*

(1) *Assume $C \not\subset D$. Then*

$$\deg_C mL \otimes \mathcal{J}(||mL||) \geq \mathrm{mult}_x D \,\mathrm{mult}_x C.$$

Here $\mathrm{mult}_x V$ is the multiplicity of a subscheme V at x.

(2) *Assume $m||L; C|| < \mathrm{mult}_x C$. Then $C \subset D$.*

Proof. (1) We note that $\mathcal{O}_X(-D) = \mathcal{J}(D) \subset \mathcal{J}(||mL||)$. Then

$$\begin{aligned}\deg_C mL \otimes \mathcal{J}(||mL||) &= D \cdot C + \deg_C \mathcal{J}(||mL||)\\ &\geq \sum_{P \in D \cap C, P \notin V\mathcal{J}(||mL||)} (D \cdot C)_P\\ &\geq \mathrm{mult}_x D \,\mathrm{mult}_x C.\end{aligned}$$

Here $(D \cdot C)_P$ is the local intersection multiplicity of D and C at P.

(2) Assume on the contrary that $C \not\subset D$. We take $\varepsilon_1 > 0$ such that

$$(m + \varepsilon_1)||L; C|| < \mathrm{mult}_x C,$$

and $\varepsilon_2 > 0$ such that

$$\varepsilon_2 m(m + \varepsilon_1) < \varepsilon_1 \,\mathrm{mult}_x C.$$

By definition of $||L; C||$, there exists $k_0 = k_0(\varepsilon_2) \in \mathbb{N}$ such that

$$\deg_C kL \otimes \mathcal{J}(||kL||) < k(||L; C|| + \varepsilon_2)$$

for any $k > k_0$. We take an integer $m_0 > k_0$. Applying (1) for $m_0 D \in |m_0 mL|$, we have

$$\mathrm{mult}_x(m_0 D) \,\mathrm{mult}_x C \leq \deg_C m_0 mL \otimes \mathcal{J}(||m_0 mL||) < m_0 m(||L; C|| + \varepsilon_2).$$

Putting everything together, we have

$$\mathrm{mult}_x D \,\mathrm{mult}_x C < m(||L; C|| + \varepsilon_2) < m\left(\frac{\mathrm{mult}_x C}{m + \varepsilon_1} + \frac{\varepsilon_1 \,\mathrm{mult}_x C}{m(m + \varepsilon_1)}\right) = \mathrm{mult}_x C.$$

This is a contradiction. □

Lemma 3.4. *Let δ be a positive number. Assume that for every pair (x, x') of two very general points on X, there exists a chain of integral curves $C_1, \dots, C_n \not\subset \mathrm{SBs}\,|L|$ (n depends on (x, x')) such that (i) $x_1 := x \in C_1$, $x' \in C_n$, (ii) for every $i = 1, \dots, n-1$, some point $x_{i+1} \in C_i \cap C_{i+1}$ is a very general point of X, and (iii) $||L; C_i|| < \delta$ (more generally, $||L; C_i|| < \delta \,\mathrm{mult}_{x_i} C_i$) for every $i = 1, \dots, n$. Then $\dim H^0(X, mL) \le 1$ for any positive integer $m < 1/\delta$.*

Proof. Assume on the contrary that $\dim H^0(X, mL) > 1$ for some $0 < m < 1/\delta$. We take a very general point $x \in X$. Then there exists $0 \neq D \in |mL|$ such that $x \in D$. (We fix these x and D for a while.) Let us take another arbitrary very general point $x' \in X$. We can join x and x' by a chain of integral curves $C_1, \dots, C_n \not\subset \mathrm{SBs}\,|L|$ as in the statement. We apply Lemma 3.3 for C_1 and D at $x_1 = x \in C_1 \cap D$. Since $m||L; C_1|| < \mathrm{mult}_{x_1} C_1$, we have $C_1 \subset D$, and in particular $x_2 \in D$. Applying Lemma 3.3 again for C_2 and D at $x_2 \in C_2 \cap D$, we also have $C_2 \subset D$. Continuing these processes, we finally obtain $x' \in D$. Since $x' \in X$ is an arbitrary very general point, we have $D = X$. This is a contradiction. □

If we can take $\delta > 0$ arbitrary small, we have $\dim H^0(X, mL) \le 1$ for arbitrary large m. Therefore

Corollary 3.5. *Assume that for every pair (x, x') of two very general points on X, there exists a chain of integral curves $C_1, \dots, C_n \not\subset \mathrm{SBs}\,|L|$ (n depends on (x, x')) such that (i) $x \in C_1$, $x' \in C_n$, (ii) for every $i = 1, \dots, n-1$, some point in $C_i \cap C_{i+1}$ is a very general point of X, and (iii) $||L; C_i|| = 0$ for all i. Then $\kappa(L) = 0$.*

4. Numerically Trivial Fibrations and Iitaka's Fibrations

Proof of Theorem 1.3. In case $\kappa(L) = 0$, it follows from Theorem 1.2. In case $\kappa(L) = n := \dim X$, we can see that $\limsup_{m\to\infty} m^{-n} \dim H^0(X, mL) > 0$. By Kodaira's lemma, for a given very ample divisor A on X, there exists a positive integer k such that $kL = A + E$ for an effective divisor E. Then we have an inclusion $\mathcal{O}_X(-mE) \subset \mathcal{J}(||mkL||)$, and an injective homomorphism $\mathcal{O}_X(mA) \longrightarrow \mathcal{O}_X(mkL) \otimes \mathcal{J}(||mkL||)$ for every positive integer m. Thus, for every integral curve $C \not\subset E$, we have $||L; C|| \ge k^{-1} A \cdot C > 0$. Therefore, the identity map $X \longrightarrow X$ is (birationally equivalent to) the fibration.

We consider the case $0 < \kappa(L) < n$. We take a positive integer m such that $\kappa(L) = \dim \Phi_{|mL|}(X)$. Let $\mu : X' \longrightarrow X$ be a log-resolution of $|mL|$ such that $\mu^*|mL| = |W| + F$, with $|W|$ free and F the fixed component. Let $f' := \Phi_{|W|} : X' \longrightarrow Y' := \Phi_{|W|}(X') \subset \mathbb{P}^N$, $N = \dim |W|$, be the induced morphism, and let $f' = \phi \circ f : X' \longrightarrow Y \longrightarrow Y'$ be the Stein factorization. We shall show that this $f : X' \longrightarrow Y$ is the fibration. By the construction we have $\dim Y = \kappa(L)$.

We will denote $X'_y := f^{-1}(y)$ for $y \in Y$, and $X'_{y'} := f'^{-1}(y')$ for $y' \in Y'$. By Lemma 4.1 below, $\dim \Phi_{|k\mu^* L|}(X'_y) = 0$ for general $y \in Y$ and every large and divisible k. Then by Lemma 3.2, we have $||\mu^* L; C'|| = 0$ for every integral curve $C' \subset X'$ such that $C' \subset X'_y$ for a very general $y \in Y$ and $C' \not\subset \mathrm{SBs}\,|\mu^* L|$. This is property (1) in the statement.

Let us check property (2). We take a general $y \in Y$ so that $y' = \phi(y) \in Y'$ is also general. Let $H_1, \dots, H_{\kappa(L)}$ be general hyperplane sections of Y' in $\mathbb{P}^N$ passing through y'. We have effective divisors $D'_i := f'^* H_i + F \in |\mu^* mL|$ $(i = 1, \dots, \kappa(L))$ on X'. We see that $\bigcap_{i=1}^{\kappa(L)} D'_i = F \cup X'_{y'} \cup$ (other fibers of f'), since $H_1, \dots, H_{\kappa(L)}$

are general. We take any $x' \in X'_y - F$ and any integral curve $C' \subset X'$ passing through x' with $\dim f(C') > 0$ (in particular, $C' \not\subset X'_y \cup F$). If $||\mu^*L; C'|| = 0$, by Lemma 3.3 (2), we have $C' \subset \bigcap_{i=1}^{\kappa(L)} D'_i$. This is a contradiction. Therefore we have $||\mu^*L; C'|| > 0$, and property (2).

The uniqueness (up to birational equivalence) follows from the properties (1) and (2). Let $\nu : X'' \longrightarrow X$ be another proper birational morphism from a smooth variety X'', and $g : X'' \longrightarrow Z$ another proper surjective morphism to a variety Z with connected fibers with the properties (1) and (2). We take a common modification $X' \longleftarrow X^+ \longrightarrow X''$, and denote by $f_+ : X^+ \longrightarrow Y$, $g_+ : X^+ \longrightarrow Z$ the compositions. By property (2) for f_+ (note Lemma 2.12(2)), every very general fibre of g_+ (which satisfies property (1) for g_+) must be contained in a fibre of f_+. By symmetry, every very general fibre of f_+ must be contained in a fibre of g_+. These imply that general fibres of f_+ and g_+ coincide. Therefore, the maps $X \dashrightarrow Y$ and $X \dashrightarrow Z$ are birationally equivalent.

We finally mention the relation with Iitaka's fibration. Iitaka's fibration is characterized by the five properties listed at the bottom of [I, p. 363]. By the construction above, $f : X' \longrightarrow Y$ satisfies the first four properties. We have $\kappa(X'_y, \mu^*L) = 0$ for general $y \in Y$ by Lemma 4.1 below, which is the final property of [I, p. 363]. □

We quote an argument from [I, pp. 362–363], which is quite standard. We have no specific reason to replace it by another argument. (Do not mix notation and situations with those in the proof of Theorem 1.3.)

Lemma 4.1. *Assume $|mL| \neq \emptyset$ for a positive integer m. Let $\mu : X' \longrightarrow X$ be a log-resolution of $|mL|$ such that $\mu^*|mL| = |W| + F$ with $|W|$ free and F the fixed component. Let $f := \Phi_{|W|} : X' \longrightarrow Y := \Phi_{|W|}(X') \subset \mathbb{P}^N$, $N = \dim |W|$, be the induced morphism, and let d be the number of irreducible components of a general fibre of it. Assume that, for some $k \in \mathbb{N}$, the rank of $f_*(\mu^*L^{\otimes k})$ at the generic point of Y is greater than d. Then $\kappa(L) > \dim Y$.*

Proof. For general $y \in Y$, we can decompose $X'_y := f^{-1}(y) = \coprod_{i=1}^d X'_{y,i}$ into its connected components with $X'_{y,i}$ smooth. Our assumption in fact implies that there exist $k \in \mathbb{N}$ and a nonempty Zariski open subset $Y_0 \subset Y$ such that (i) f is smooth over Y_0, and (ii) $f_*(\mu^*L^{\otimes k})|_{Y_0}$ and $f_*(\mu^*L^{\otimes mk})|_{Y_0}$ are vector bundles over Y_0 with rank $\geq 2d$. We take and fix such a k.

Let $H \subset Y$ be a hyperplane section in $\mathbb{P}^N$. We take a general $y \in Y_0$. Then, since $Y - H$ is affine, and by the base change theorem, we have a surjective morphism

$$H^0(Y - H, f_*(\mu^*L^{\otimes mk})) \longrightarrow f_*(\mu^*L^{\otimes mk})_y \cong \bigoplus_{i=1}^d H^0(X'_{y,i}, f_*(\mu^*L^{\otimes mk})).$$

Our assumption says that $\dim H^0(X'_{y,i}, f_*(\mu^*L^{\otimes mk})) > 1$ for every i. Since $f^*H \in |W|$, we have an effective divisor $D := f^*H + F \in |m\mu^*L|$. Then

$$\begin{aligned} H^0(Y - H, f_*(\mu^*L^{\otimes mk})) &= H^0(X' - f^{-1}(H), \mu^*L^{\otimes mk}) \\ &\subset H^0(X' - f^{-1}(H) - F, \mu^*L^{\otimes mk}) \\ &= \bigcup_{a=1}^{\infty} H^0(X', \mu^*L^{\otimes mk}(aD)). \end{aligned}$$

Here we denote by $H^0(X', \mu^*L^{\otimes mk}(aD))$ the space of rational sections φ of $\mu^*L^{\otimes mk}$ on X' such that the corresponding divisor $\mathrm{div}(\varphi) \geq -aD$. Fix an element $\eta \in$

$H^0(X', D)$ such that $\mathrm{div}(\eta) = D$. Then the map

$$H^0(X', \mu^* L^{\otimes mk}(aD)) \longrightarrow H^0(X', \mu^* L^{\otimes m(k+a)})$$

given by $\varphi \mapsto \varphi\eta^a$ is an isomorphism. Therefore, for example, there exist $s_1, s_2 \in H^0(X', \mu^* L^{\otimes m(k+a)})$ for some $a \in \mathbb{N}$ such that the elements

$$s_1|_{X'_{y,1}}, s_2|_{X'_{y,1}} \in H^0(X'_{y,1}, \mu^* L^{\otimes m(k+a)})$$

are linearly independent. Thus we can deduce that $\dim \Phi_{|m(k+a)\mu^* L|}(X') > \dim Y$ for some larger a, and hence $\kappa(L) > \dim Y$. □

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