Nondegenerate Multidimensional Matrices and Instanton Bundles

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Abstract. In this paper we prove that the moduli space of rank \( 2n \) symplectic instanton bundles on \( \mathbb{P}^{2n+1} \), defined from the well-known monad condition, is affine. This result was not known even in the case \( n = 1 \), where by Atiyah, Drinfeld, Hitchin, and Manin in 1978 the real instanton bundles correspond to self-dual Yang Mills \( Sp(1) \)-connections over the 4-dimensional sphere. The result is proved as a consequence of the existence of an invariant of the multidimensional matrices representing the instanton bundles.

1. Introduction

A symplectic instanton bundle on \( \mathbb{P}^{2n+1} \) is a bundle of rank \( 2n \) defined as the cohomology bundle of a well-known monad (see Definition 2.2).

In [ADHM78] it was shown that instanton bundles on \( \mathbb{P}^3 \) satisfying a reality condition correspond to self-dual Yang Mills \( Sp(1) \)-connections over the 4-dimensional sphere \( S^4 = \mathbb{P}^3_\mathbb{H} \). This correspondence was generalized by Salamon (Sal84) who showed that instanton bundles on \( \mathbb{P}^{2n+1} \) which are trivial on the fiber of the twistor map \( \mathbb{P}^{2n+1} \to \mathbb{P}^n_\mathbb{H} \) correspond to \( Sp(n) \)-connections which minimize a certain Yang Mills functional over \( \mathbb{P}^n_\mathbb{H} \). We denote by \( MI_{\mathbb{P}^{2n+1}}(k) \) the moduli space of symplectic instanton bundles on \( \mathbb{P}^{2n+1} \) with \( c_2 = k \) (see Definition 2.4) and we denote by \( I_{\mathbb{P}^{2n+1}}(k) \) the moduli space of \( k \)-instanton bundles on \( \mathbb{P}^{2n+1} \) (see Definition 4.1).

Up to now, very little is known concerning the geometry of the moduli spaces \( I_{\mathbb{P}^{2n+1}}(k) \) and a few results have been proved regarding \( MI_{\mathbb{P}^{2n+1}}(k) \). For instance, up to the authors’ knowledge, the only results concerning \( MI_{\mathbb{P}^{2n+1}}(k) \) deal with small values of \( n \) and \( k \). Indeed, it is known (ADHM78) that \( MI_{\mathbb{P}^{2n+1}}(k) \) has a component of dimension \( 8k - 3 \) for \( n = 1 \), that it is smooth for \( n = 1 \) and \( k \leq 5 \) (KO99) but, it is conjectured that it is singular and reducible for \( n \geq 2 \) and \( k \geq 4 \) (see AO00).

The goal of this paper is to show that all the moduli spaces \( MI_{\mathbb{P}^{2n+1}}(k) \), for any \( n \geq 1 \) and any \( k \geq 1 \), share the following surprising property:

Theorem 1.1. \( MI_{\mathbb{P}^{2n+1}}(k) \) is affine.

In addition, we will see that the same holds for all moduli spaces parametrizing \( k \)-instanton bundles on \( \mathbb{P}^{2n+1} \). Indeed, we will prove

Theorem 1.2. \( I_{\mathbb{P}^{2n+1}}(k) \) is affine.
As a by-product of Theorems 1.1 and 1.2, we will contribute to the study of a problem posed in the 80’s (see for instance [HH86]) that, in the context of instanton bundles on $\mathbb{P}^{2n+1}$, reads as follows:

**Problem.** Determine the maximal dimension of complete subvarieties lying on $MI_{P_{2n+1}}(k)$ (resp. $I_{P_{2n+1}}(k)$).

More precisely, in this case, we will completely solve the problem and in Corollaries 3.3 and 4.6 we will see that $MI_{P_{2n+1}}(k)$ (resp. $I_{P_{2n+1}}(k)$) does not contain any complete subvariety of positive dimension.

The technique we use to prove our main results is to exhibit $MI_{P_{2n+1}}(k)$ (resp. $I_{P_{2n+1}}(k)$) as the GIT-quotient of an affine variety $Q^0$ (resp. $P^0$) and then use standard results in invariant theory. The fact that $Q^0$ (resp. $P^0$) is affine is a consequence of the existence of an invariant of multidimensional matrices representing the instanton bundles, which generalizes the hyperdeterminant (see [GKZ94] and [AO99]).

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2. **Notation and preliminaries**

We will start by fixing some notation and recalling some facts about $k$-instanton bundles on $\mathbb{P}^{2n+1} = \mathbb{P}(V)$, where $V$ is a complex vector space of dimension $2n + 2$. (See, for instance, [OS86] and [AO94].)

**Notation 2.1.** $O(d)$ denotes the invertible sheaf of degree $d$ on $\mathbb{P}^{2n+1}$ and for any coherent sheaf $E$ on $\mathbb{P}^{2n+1}$ we denote $E(d) = E \otimes O_{\mathbb{P}^{2n+1}}(d)$.

**Definition 2.2.** A symplectic instanton bundle $E$ over $\mathbb{P}^{2n+1} = \mathbb{P}(V)$ is a bundle of rank $2n$ which appears as a cohomology bundle of a monad,

$$ I^* \otimes O(-1) \xrightarrow{A} W \otimes O \xrightarrow{A^t} I \otimes O(1), $$

where $(W, J)$ is a symplectic complex vector space of dimension $2n + 2k$ and $I$ is a complex vector space of dimension $k$.

We do not assume in the definition that $E$ is stable, so we have to recall some results.

The monad condition means that $A$ is injective (as a bundle morphism), $A^t$ is surjective and $\text{im} A \subset \ker A^t$ so that $E \simeq \ker A^t / \text{im} A$. The fact that the map $W \otimes O \xrightarrow{A^t} I \otimes O(1)$ is surjective, is equivalent to the fact that the matrix $A \in \text{Hom}(V^* \otimes I^*, W)$ representing $E$ is nondegenerate according to [GKZ94] (see Definition 2.3 for the precise definition).

$\text{Hom}(V^* \otimes I^*, W)$ contains the subvariety $Q$ given by matrices $A$ for which the sequence (11) is a complex, that is, such that $A^t J A = 0$. $\text{GL}(I) \times \text{Sp}(W)$ acts on $Q$ by $(g, s) \cdot A = s A g$.

**Definition 2.3.** A matrix $A \in \text{Hom}(V^* \otimes I^*, W)$ is called degenerate if the multilinear system $A(v \otimes i) = 0$ has a solution such that $0 \neq v \in V^*$ and $0 \neq i \in I^*$.

By [GKZ94], Theorem 14.3.1, this is equivalent to the standard definition of degeneracy given in chapter 14.1 of [GKZ94]. It is easy to check that degenerate matrices fill an irreducible subvariety $N$ of $\text{Hom}(V^* \otimes I^*, W)$ of codimension $k$ (see [WZ96]). Hence, only in the case $k = 1$ is it well-defined as a hyperdeterminant.
according to \cite{GKZ94}. In the next section we will define an \(SL(I) \times Sp(W)\)-invariant on \(Hom(V^* \otimes I^*, W)\), called \(D\), which generalizes the hyperdeterminant and is suitable for our purposes.

It was shown in \cite{AO94} that all instanton bundles are simple, so that they carry a unique symplectic form. Moreover, for \(n = 1, 2\) it was proved in \cite{AO94} that all instanton bundles are stable, and it is expected that the same result is true for \(n \geq 3\).

Recall that given \(X = Spec(A)\), an affine scheme, and a reductive group \(G\) acting on \(X\), then a theorem of Hilbert and Nagata shows that the ring of invariants \(A^G\) is finitely generated and \(X/G := Spec(A^G)\) is what is called the affine algebro-geometric quotient of \(X\) by \(G\). In addition, \(X/G\) is a good quotient and it is a geometric quotient if and only if all orbits are closed. In this setting, every orbit contains a unique closed orbit in its closure and a point in \(X\) is called stable if its orbit is closed and has the maximal dimension (see \cite{PV89}).

In \cite{BH78} it was essentially proved that there is a natural one-to-one correspondence between

i) isomorphism classes of symplectic instanton bundles, and

ii) orbits of \(GL(I) \times Sp(W)\) on the open subvariety \(Q^0\) of \(Q\) given by nondegenerate matrices.

In fact, using the quoted results of \cite{AO94}, one can see that \cite{BH78}, Section 4 and the Theorem on page 19, adapt literally to our situation.

Moreover, in Theorem 3.3 we will see that \(Q^0\) is affine. Hence, if we denote by \(G\) the quotient of \(GL(I) \times Sp(W)\) by \(\pm(id, id)\), Barth and Hulek proved in \cite{BH78} that \(G\) acts freely on \(Q^0\) and, in particular, all orbits are closed (in fact, any orbit contains in the closure orbits of smaller dimension). Therefore, all points of \(Q^0\) are stable for the action of \(GL(I) \times Sp(W)\) and \(Q^0 \rightarrow Q^0/G\) is a geometric quotient.

**Definition 2.4.** The GIT-quotient \(Q^0/GL(I) \times Sp(W)\) is denoted by \(M_{IP_{2n+1}}(k)\) and is called the moduli space of symplectic \(k\)-instanton bundles on \(\mathbb{P}^{2n+1}\). It is a geometric quotient.

The above discussion shows that \(M_{IP_{2n+1}}(k)\) coincides for \(n = 1, 2\) with the open subset \(\mathcal{M}_{IP_{2n+1}}(k)\) of the Maruyama scheme of symplectic stable bundles on \(\mathbb{P}^{2n+1}\) of rank \(2n\) and Chern polynomial \(\frac{1}{(1-t)^2}\) which are instanton bundles (this is an open condition because by Beilinson’s theorem, it is equivalent to certain vanishing in cohomology; see \cite{OS86}). In particular, our notation for \(M_{IP_3}(k)\) is consistent with the usual one. For \(n \geq 3\) it is expected that the same result is true, but at present we can only say that \(\mathcal{M}_{IP_{2n+1}}(k)\) is an open subset of \(M_{IP_{2n+1}}(k)\).

3. The Invariant \(D\) and the Proof of the Main Result

First, we remark that the vector spaces \(W \otimes S^n I\) and \(V \otimes S^{n+1} I\) have the same dimension \((2n + 2k)(k+n-1) = (2n + 2)(n+1)\). We can construct from

\[ W \xrightarrow{A^t} V \otimes I, \]

the morphisms

\[ A^t \otimes id_{S^nI} : W \otimes S^n I \to V \otimes I \otimes S^n I, \]
\[ id_V \otimes \pi : V \otimes I \otimes S^n I \to V \otimes S^{n+1} I, \]

where \(\pi\) is the natural projection, and we consider the composition

\[ \Delta_A = (id_V \otimes \pi) \cdot (A^t \otimes id_{S^n I}) : W \otimes S^n I \to V \otimes S^{n+1} I. \]
Definition 3.1. Let $A \in \text{Hom}(V^* \otimes I^*, W)$. We define $D(A)$ to be the usual determinant of the morphism $\Delta_A$ in (2) induced by $A$.

Notice that

$$D: \text{Hom}(V^* \otimes I^*, W) \to (\det W)^{\alpha} \otimes (\det V)^{\beta}$$

where $\alpha = -(\binom{k+n-1}{n})$ and $\beta = (\binom{k+n}{n+1})$ is a $GL(V) \times GL(I) \times Sp(W)$-equivariant map and $D(A) = 0$ defines a homogeneous hypersurface of degree $(2n+2k)(\binom{k+n-1}{n}) = (2n+2)(\binom{k+n}{n+1})$. After a basis has been fixed in each of the vector spaces $V$, $I$ and $W$, the map $D$ can be seen as an $SL(V) \times SL(I) \times Sp(W)$-invariant.

In fact, this definition generalizes the hyperdeterminant of boundary format as introduced in Theorem 14.3.3 of [GKZ94].

Lemma 3.2. If $A$ is degenerate, then $D(A) = 0$.

Proof. There are $0 \neq v \in V^*$ and $0 \neq i \in I^*$ such that $A(v \otimes i) = 0$. Hence, $v \otimes S^{n+1}i \in V^* \otimes S^{n+1}I^*$ goes to zero under the dual of (2).

If $A$ is nondegenerate, we get $D(A) \neq 0$ only in the case $k = 1$ and, in general, it can happen that $D(A) = 0$, because the codimension of $N$ is $k$. Our main technical result is the following.

Theorem 3.3. If $A$ defines an instanton (that is, $A$ belongs to $Q^0$), then $D(A) \neq 0$.

Proof. From (1) we get the exact sequence

$$0 \to K \to W \otimes O \to I \otimes O(1) \to 0.$$  

The $(n+1)$-th wedge power twisted by $O(-n)$ gives the exact sequence

$$0 \to \wedge^{n+1}K(-n) \to \wedge^{n+1}W(-n) \to \ldots$$

$$\ldots \to \wedge^2W \otimes S^{n-1}I(-1) \to W \otimes S^nI \to S^{n+1}I(1) \to 0$$

where the $H^0$ of the last morphism corresponds to $\Delta_A$ in (2). Taking cohomology, it is enough to prove

$$H^n(\wedge^{n+1}K(-n)) = 0.$$  

The $(n+1)$-th wedge power twisted by $O(-n)$ of the sequence

$$0 \to I^* \otimes O(-1) \to K \to E \to 0$$

gives the sequence

$$0 \to S^{n+1}I^* \otimes K(-2n-1) \to \ldots \to \wedge^{n-1}K \otimes S^2I^*(-n-2) \to \ldots$$

$$\ldots \to \wedge^nK \otimes I^*(-n-1) \to \wedge^{n+1}K(-n) \to \wedge^{n+1}E(-n) \to 0.$$  

In order to prove (3), taking cohomology, we need $H^{n+i}(\wedge^nK(-n-i)) = 0$ for $i = 0, \ldots, n$ and $H^n(\wedge^{n+1}E(-n)) = 0$. The first group of vanishing is easily obtained by taking suitable wedge powers of (3). The crucial point used to get the last vanishing is the isomorphism $\wedge^{n+1}E \simeq \wedge^{n-1}E$; it is true because $E$ is a rank $2n$ vector bundle with $c_1 = 0$. From the sequence

$$0 \to S^{n-1}I^*(-2n-1) \to S^{n-2}I^* \otimes K(-2n) \to \ldots$$

$$\ldots \to \wedge^{n-1}K(-n) \to \wedge^{n-1}E(-n) \to 0$$

in order to prove $H^n(\wedge^{n-1}E(-n)) = 0$, we only need to see that

$$H^{n+i}(\wedge^{n-1-i}K(-n-i)) = 0 \quad \text{for } i = 0, \ldots, n,$$
which follows by using the exact sequence \[3\] exactly as above. \[\square\]

Now, we can state and prove the main result of this section.

**Theorem 3.4.** \(MI_{p2n+1}(k)\) is affine.

**Proof.** By Theorem \[3\] we get that \(Q \setminus N = Q^0 = Q \setminus \{D = 0\}\) is affine. It follows that \(MI_{p2n+1}(k)\) is affine too, because it is the quotient of an affine variety by a reductive group; see, e.g., [PYS9], section 4.4. \[\square\]

As a consequence we deduce

**Corollary 3.5.** \(MI_{p2n+1}(k)\) does not contain any complete subvariety of positive dimension.

**Proof.** This follows from the fact that a quasi-affine complete variety is a finite set. \[\square\]

**Remark 3.6.** The invariant \(D\) is meaningful even in the case \(n = 0\). In this case it corresponds to the usual determinant of the map \(\mathbb{C}^{2k} \to \mathbb{C}^2 \otimes \mathbb{C}^k\). For example, for \(n = 0\) and \(k = 2\) the degenerate \(2 \times 2 \times 4\) matrices fill a variety of codimension 2 and degree 12 (\[BS\]) in \(\mathbb{P}^{15}\) whose ideal is generated by one quartic (which is our invariant \(D\)), 10 sextics and one octic. We remark that the case \(2 \times 2 \times 3\) is of boundary format. The case \(2 \times 2 \times 5\) is interesting. Here degenerate matrices fill a variety of codimension 3 and degree 20, and its ideal is generated (at least) by 5 quartics, 50 sextics and 12 octics. The 5 quartics define a variety of codimension 2 and degree 10. Hence, in this case no analog of the invariant \(D\) can exist.

4. **Instanton bundles with structure group \(GL(2n)\)**

**Definition 4.1.** A \(k\)-instanton bundle \(E\) on \(\mathbb{P}^{2n+1}\) is the cohomology bundle of a monad

\[
K \otimes \mathcal{O}(-1) \xrightarrow{A} W \otimes \mathcal{O} \xrightarrow{B} I \otimes \mathcal{O}(1)
\]

where \(W\) is a complex vector space of dimension \(2n + 2k\) and \(I, K\) are complex vector spaces of dimension \(k\).

Notice that \(E\) is not necessarily symplectic and that this notion is a true generalization of the one above only for \(n \geq 2\), because all rank 2 bundles on \(\mathbb{P}^3\) with \(c_1 = 0\) are symplectic.

Let \((A, B) \in \text{Hom}(K \otimes V^*, W) \times \text{Hom}(W, I \otimes V)\) defining \(E\). The monad condition is now equivalent to the fact that the matrices \(A\) and \(B\) are both nondegenerate and \(B \cdot A = 0\).

\(\text{Hom}(K \otimes V^*, W) \times \text{Hom}(W, I \otimes V)\) contains the subvariety \(\mathcal{P}\) given by pairs of matrices \((A, B)\) for which the sequence \[13\] is a complex, that is, such that \(B \cdot A = 0\).

\(GL(I) \times GL(K) \times GL(W)\) acts on \(\mathcal{P}\) by \((a, b, c) \cdot (A, B) = (cAb, aBe^{-1})\).

Arguing, as in the previous section, we can see that there is a natural one-to-one correspondence between

i) isomorphism classes of instanton bundles, and

ii) orbits of \(GL(I) \times GL(K) \times GL(W)\) on the open subvariety \(\mathcal{P}^0\) of \(\mathcal{P}\) given by pairs of nondegenerate matrices.

Moreover, as in the second section and using Theorem \[13\] if we denote by \(H\) the quotient of \(GL(I) \times GL(K) \times GL(W)\) by \((\lambda \cdot \text{id}, \lambda^{-1} \cdot \text{id}, \lambda \cdot \text{id})\), then \(H\) acts
freely on $\mathcal{P}^0$. In particular, all points of $\mathcal{P}^0$ are stable for the action of $GL(I) \times GL(K) \times GL(W)$.

**Definition 4.2.** The GIT-quotient $\mathcal{P}^0/GL(I) \times GL(K) \times GL(W)$ is denoted by $I_{2n+1}(k)$ and is called the moduli space of $k$-instanton bundles on $\mathbb{P}^{2n+1}$. It is a geometric quotient.

$I_{2n+1}(k)$ coincides for $n = 1, 2$ with the open subset $I_{2n+1}(k)$ of the Maruyama scheme of stable bundles on $\mathbb{P}^{2n+1}$ of rank $2n$ and Chern polynomial $\frac{1}{1-t}$, which are instanton bundles. For $n \geq 3$ we can say that $I_{2n+1}(k)$ is an open subset of $I_{2n+1}(k)$. We remark that $MI_{3}(k) = I_{3}(k)$. $I_{2n+1}(k)$ is known to be singular for $n \geq 2$ and $k \geq 3$ (see [MO97]) and reducible for $n \geq 4$ (see [AO90]).

**Definition 4.3.** Let $(A, B) \in Hom(K \otimes V^*, W) \times Hom(W, I \otimes V)$. We define

$$\tilde{D}(A, B) := \det S(A) \cdot \det R(B)$$

where $\det$ denotes the usual determinant and $S(A), R(B)$ are the morphisms

$$S(A) : S^{n+1}K \otimes V^* \to S^nK \otimes W,$$
$$R(B) : S^nI \otimes W \to S^{n+1}I \otimes V,$$

induced by $A$ and $B$ respectively, as in Definition 4.1.

**Theorem 4.4.** If $(A, B)$ defines an instanton (that is, $(A, B)$ belongs to $\mathcal{P}^0$), then $\tilde{D}(A, B) \neq 0$.

**Proof.** First, we will see that $\det S(A) \not= 0$. From (5) we get the exact sequence

$$0 \to K \otimes O(-1) \to W \otimes O \to Q \to 0.$$ (6)

The $(n+1)$-th wedge power twisted by $O(-n - 2)$ gives the exact sequence

$$0 \to S^{n+1}K \otimes O(-2n - 3) \to S^nK \otimes W \otimes O(-2n - 2) \to \ldots$$
$$\ldots \to \wedge^{n+1}W \otimes O(-n - 2) \to \wedge^{n+1}Q(-n - 2) \to 0$$

where the $H^{2n+1}$ of the first morphism corresponds to $S(A)$. Hence, taking cohomology, it is enough to prove

$$H^n(\wedge^{n+1}Q(-n - 2)) = 0.$$ (7)

This is shown by considering the $(n + 1)$-wedge sequence of the exact sequence

$$0 \to E \to Q \to I \otimes O(1) \to 0$$

and arguing as in the proof of Theorem 3.3.

In order to prove $\det R(B) \neq 0$, we proceed exactly as in Theorem 3.3 and we leave the details to the reader. □

**Theorem 4.5.** $I_{2n+1}(k)$ is affine.

**Proof.** First, notice that given $(A, B) \in \mathcal{P}$, if $A$ or $B$ is degenerate, then $\det S(A) \cdot \det R(B) = 0$. Hence, by Theorem 4.4 we get that $\mathcal{P}^0 = \mathcal{P} \setminus \{D = 0\}$ is affine. Therefore, by [PV89] section 4.4, $I_{2n+1}(k)$ is affine also. □

As a by-product of Theorem 4.3 we deduce

**Corollary 4.6.** $I_{2n+1}(k)$ does not contain any complete subvariety of positive dimension.
References


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