

NONDEGENERATE MULTIDIMENSIONAL MATRICES AND INSTANTON BUNDLES

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ABSTRACT. In this paper we prove that the moduli space of rank $2n$ symplectic instanton bundles on \mathbb{P}^{2n+1} , defined from the well-known monad condition, is affine. This result was not known even in the case $n = 1$, where by Atiyah, Drinfeld, Hitchin, and Manin in 1978 the real instanton bundles correspond to self-dual Yang Mills $Sp(1)$ -connections over the 4-dimensional sphere. The result is proved as a consequence of the existence of an invariant of the multi-dimensional matrices representing the instanton bundles.

1. INTRODUCTION

A symplectic instanton bundle on $\mathbb{P}_{\mathbb{C}}^{2n+1}$ is a bundle of rank $2n$ defined as the cohomology bundle of a well-known monad (see Definition 2.2).

In [ADHM78] it was shown that instanton bundles on \mathbb{P}^3 satisfying a reality condition correspond to self-dual Yang Mills $Sp(1)$ -connections over the 4-dimensional sphere $S^4 = \mathbb{P}_{\mathbb{H}}^1$. This correspondence was generalized by Salamon ([Sal84]) who showed that instanton bundles on \mathbb{P}^{2n+1} which are trivial on the fiber of the twistor map $\mathbb{P}^{2n+1} \rightarrow \mathbb{P}_{\mathbb{H}}^n$ correspond to $Sp(n)$ -connections which minimize a certain Yang Mills functional over $\mathbb{P}_{\mathbb{H}}^n$. We denote by $MI_{\mathbb{P}^{2n+1}}(k)$ the moduli space of symplectic instanton bundles on \mathbb{P}^{2n+1} with $c_2 = k$ (see Definition 2.4) and we denote by $I_{\mathbb{P}^{2n+1}}(k)$ the moduli space of k -instanton bundles on \mathbb{P}^{2n+1} (see Definition 4.1).

Up to now, very little is known concerning the geometry of the moduli spaces $I_{\mathbb{P}^{2n+1}}(k)$ and a few results have been proved regarding $MI_{\mathbb{P}^{2n+1}}(k)$. For instance, up to the authors' knowledge, the only results concerning $MI_{\mathbb{P}^{2n+1}}(k)$ deal with small values of n and k . Indeed, it is known ([ADHM78]) that $MI_{\mathbb{P}^{2n+1}}(k)$ has a component of dimension $8k - 3$ for $n = 1$, that it is smooth for $n = 1$ and $k \leq 5$ ([KO99]) but, it is conjectured that it is singular and reducible for $n \geq 2$ and $k \geq 4$ (see [AO00]).

The goal of this paper is to show that all the moduli spaces $MI_{\mathbb{P}^{2n+1}}(k)$, for any $n \geq 1$ and any $k \geq 1$, share the following surprising property:

Theorem 1.1. *$MI_{\mathbb{P}^{2n+1}}(k)$ is affine.*

In addition, we will see that the same holds for all moduli spaces parametrizing k -instanton bundles on \mathbb{P}^{2n+1} . Indeed, we will prove

Theorem 1.2. *$I_{\mathbb{P}^{2n+1}}(k)$ is affine.*

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As a by-product of Theorems 1.1 and 1.2, we will contribute to the study of a problem posed in the 80's (see for instance [HH86]) that, in the context of instanton bundles on \mathbb{P}^{2n+1} , reads as follows:

Problem. Determine the maximal dimension of complete subvarieties lying on $MI_{\mathbb{P}^{2n+1}}(k)$ (resp. $I_{\mathbb{P}^{2n+1}}(k)$).

More precisely, in this case, we will completely solve the problem and in Corollaries 3.5 and 4.6 we will see that $MI_{\mathbb{P}^{2n+1}}(k)$ (resp. $I_{\mathbb{P}^{2n+1}}(k)$) does not contain any complete subvariety of positive dimension.

The technique we use to prove our main results is to exhibit $MI_{\mathbb{P}^{2n+1}}(k)$ (resp. $I_{\mathbb{P}^{2n+1}}(k)$) as the GIT-quotient of an affine variety \mathcal{Q}^0 (resp. \mathcal{P}^0) and then use standard results in invariant theory. The fact that \mathcal{Q}^0 (resp. \mathcal{P}^0) is affine is a consequence of the existence of an invariant of multidimensional matrices representing the instanton bundles, which generalizes the hyperdeterminant (see [GKZ94] and [AO99]).

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2. NOTATION AND PRELIMINARIES

We will start by fixing some notation and recalling some facts about k -instanton bundles on $\mathbb{P}^{2n+1} = \mathbb{P}(V)$, where V is a complex vector space of dimension $2n + 2$. (See, for instance, [OS86] and [AO94].)

Notation 2.1. $\mathcal{O}(d) = \mathcal{O}_{\mathbb{P}^{2n+1}}(d)$ denotes the invertible sheaf of degree d on \mathbb{P}^{2n+1} and for any coherent sheaf E on \mathbb{P}^{2n+1} we denote $E(d) = E \otimes \mathcal{O}_{\mathbb{P}^{2n+1}}(d)$.

Definition 2.2. A symplectic instanton bundle E over $\mathbb{P}^{2n+1} = \mathbb{P}(V)$ is a bundle of rank $2n$ which appears as a cohomology bundle of a monad,

$$(1) \quad I^* \otimes \mathcal{O}(-1) \xrightarrow{A} W \otimes \mathcal{O} \xrightarrow{A^t} I \otimes \mathcal{O}(1),$$

where (W, J) is a symplectic complex vector space of dimension $2n + 2k$ and I is a complex vector space of dimension k .

We do not assume in the definition that E is stable, so we have to recall some results.

The monad condition means that A is injective (as a bundle morphism), A^t is surjective and $\text{im}A \subset \ker A^t$ so that $E \simeq \ker A^t / \text{im}A$. The fact that the map $W \otimes \mathcal{O} \xrightarrow{A^t} I \otimes \mathcal{O}(1)$ is surjective, is equivalent to the fact that the matrix $A \in \text{Hom}(V^* \otimes I^*, W)$ representing E is nondegenerate according to [GKZ94] (see Definition 2.3 for the precise definition).

$\text{Hom}(V^* \otimes I^*, W)$ contains the subvariety \mathcal{Q} given by matrices A for which the sequence (1) is a complex, that is, such that $A^t J A = 0$. $GL(I) \times Sp(W)$ acts on \mathcal{Q} by $(g, s) \cdot A = sAg$.

Definition 2.3. A matrix $A \in \text{Hom}(V^* \otimes I^*, W)$ is called degenerate if the multilinear system $A(v \otimes i) = 0$ has a solution such that $0 \neq v \in V^*$ and $0 \neq i \in I^*$.

By [GKZ94], Theorem 14.3.1, this is equivalent to the standard definition of degeneracy given in chapter 14.1 of [GKZ94]. It is easy to check that degenerate matrices fill an irreducible subvariety N of $\text{Hom}(V^* \otimes I^*, W)$ of codimension k (see [WZ96]). Hence, only in the case $k = 1$ is it well-defined as a hyperdeterminant

according to [GKZ94]. In the next section we will define an $SL(I) \times Sp(W)$ -invariant on $Hom(V^* \otimes I^*, W)$, called D , which generalizes the hyperdeterminant and is suitable for our purposes.

It was shown in [AO94] that all instanton bundles are simple, so that they carry a unique symplectic form. Moreover, for $n = 1, 2$ it was proved in [AO94] that all instanton bundles are stable, and it is expected that the same result is true for $n \geq 3$.

Recall that given $X = Spec(A)$, an affine scheme, and a reductive group G acting on X , then a theorem of Hilbert and Nagata shows that the ring of invariants A^G is finitely generated and $X/G := Spec(A^G)$ is what is called the affine algebro-geometric quotient of X by G . In addition, X/G is a good quotient and it is a geometric quotient if and only if all orbits are closed. In this setting, every orbit contains a unique closed orbit in its closure and a point in X is called stable if its orbit is closed and has the maximal dimension (see [PV89]).

In [BH78] it was essentially proved that there is a natural one-to-one correspondence between

- i) isomorphism classes of symplectic instanton bundles, and
- ii) orbits of $GL(I) \times Sp(W)$ on the open subvariety \mathcal{Q}^0 of \mathcal{Q} given by nondegenerate matrices.

In fact, using the quoted results of [AO94], one can see that [BH78], Section 4 and the Theorem on page 19, adapt literally to our situation.

Moreover, in Theorem 3.3 we will see that \mathcal{Q}^0 is affine. Hence, if we denote by G the quotient of $GL(I) \times Sp(W)$ by $\pm(id, id)$, Barth and Hulek proved in [BH78] that G acts freely on \mathcal{Q}^0 and, in particular, all orbits are closed (in fact, any orbit contains in the closure orbits of smaller dimension). Therefore, all points of \mathcal{Q}^0 are stable for the action of $GL(I) \times Sp(W)$ and $\mathcal{Q}^0 \rightarrow \mathcal{Q}^0/G$ is a geometric quotient.

Definition 2.4. The GIT-quotient $\mathcal{Q}^0/GL(I) \times Sp(W)$ is denoted by $MI_{\mathbb{P}^{2n+1}}(k)$ and is called the moduli space of symplectic k -instanton bundles on \mathbb{P}^{2n+1} . It is a geometric quotient.

The above discussion shows that $MI_{\mathbb{P}^{2n+1}}(k)$ coincides for $n = 1, 2$ with the open subset $\mathcal{MI}_{\mathbb{P}^{2n+1}}(k)$ of the Maruyama scheme of symplectic stable bundles on \mathbb{P}^{2n+1} of rank $2n$ and Chern polynomial $\frac{1}{(1-t^2)^k}$ which are instanton bundles (this is an open condition because by Beilinson's theorem, it is equivalent to certain vanishing in cohomology; see [OS86]). In particular, our notation for $MI_{\mathbb{P}^3}(k)$ is consistent with the usual one. For $n \geq 3$ it is expected that the same result is true, but at present we can only say that $\mathcal{MI}_{\mathbb{P}^{2n+1}}(k)$ is an open subset of $MI_{\mathbb{P}^{2n+1}}(k)$.

3. THE INVARIANT D AND THE PROOF OF THE MAIN RESULT

First, we remark that the vector spaces $W \otimes S^n I$ and $V \otimes S^{n+1} I$ have the same dimension $(2n + 2k) \binom{k+n-1}{n} = (2n + 2) \binom{k+n}{n+1}$. We can construct from

$$W \xrightarrow{A^t} V \otimes I$$

the morphisms

$$\begin{aligned} A^t \otimes id_{S^n I} &: W \otimes S^n I \rightarrow V \otimes I \otimes S^n I, \\ id_V \otimes \pi &: V \otimes I \otimes S^n I \rightarrow V \otimes S^{n+1} I, \end{aligned}$$

where π is the natural projection, and we consider the composition

$$(2) \quad \Delta_A = (id_V \otimes \pi) \cdot (A^t \otimes id_{S^n I}): W \otimes S^n I \rightarrow V \otimes S^{n+1} I.$$

Definition 3.1. Let $A \in \text{Hom}(V^* \otimes I^*, W)$. We define $D(A)$ to be the usual determinant of the morphism Δ_A in (2) induced by A .

Notice that

$$D: \text{Hom}(V^* \otimes I^*, W) \rightarrow (\det W)^\alpha \otimes (\det V)^\beta$$

where $\alpha = -\binom{k+n-1}{n}$ and $\beta = \binom{k+n}{n+1}$ is a $GL(V) \times GL(I) \times Sp(W)$ -equivariant map and $D(A) = 0$ defines a homogeneous hypersurface of degree $(2n + 2k)\binom{k+n-1}{n} = (2n + 2)\binom{k+n}{n+1}$. After a basis has been fixed in each of the vector spaces V , I and W , the map D can be seen as an $SL(V) \times SL(I) \times Sp(W)$ -invariant.

In fact, this definition generalizes the hyperdeterminant of boundary format as introduced in Theorem 14.3.3 of [GKZ94].

Lemma 3.2. *If A is degenerate, then $D(A) = 0$.*

Proof. There are $0 \neq v \in V^*$ and $0 \neq i \in I^*$ such that $A(v \otimes i) = 0$. Hence, $v \otimes S^{n+1}i \in V^* \otimes S^{n+1}I^*$ goes to zero under the dual of (2). \square

If A is nondegenerate, we get $D(A) \neq 0$ only in the case $k = 1$ and, in general, it can happen that $D(A) = 0$, because the codimension of N is k . Our main technical result is the following.

Theorem 3.3. *If A defines an instanton (that is, A belongs to \mathcal{Q}^0), then $D(A) \neq 0$.*

Proof. From (1) we get the exact sequence

$$(3) \quad 0 \rightarrow K \rightarrow W \otimes \mathcal{O} \rightarrow I \otimes \mathcal{O}(1) \rightarrow 0.$$

The $(n + 1)$ -th wedge power twisted by $\mathcal{O}(-n)$ gives the exact sequence

$$\begin{aligned} 0 \rightarrow \wedge^{n+1}K(-n) \rightarrow \wedge^{n+1}W(-n) \rightarrow \dots \\ \dots \rightarrow \wedge^2W \otimes S^{n-1}I(-1) \rightarrow W \otimes S^n I \rightarrow S^{n+1}I(1) \rightarrow 0 \end{aligned}$$

where the H^0 of the last morphism corresponds to Δ_A in (2). Taking cohomology, it is enough to prove

$$(4) \quad H^n(\wedge^{n+1}K(-n)) = 0.$$

The $(n + 1)$ -th wedge power twisted by $\mathcal{O}(-n)$ of the sequence

$$0 \rightarrow I^* \otimes \mathcal{O}(-1) \rightarrow K \rightarrow E \rightarrow 0$$

gives the sequence

$$\begin{aligned} 0 \rightarrow S^{n+1}I^* \otimes K(-2n-1) \rightarrow \dots \rightarrow \wedge^{n-1}K \otimes S^2I^*(-n-2) \rightarrow \dots \\ \dots \rightarrow \wedge^n K \otimes I^*(-n-1) \rightarrow \wedge^{n+1}K(-n) \rightarrow \wedge^{n+1}E(-n) \rightarrow 0. \end{aligned}$$

In order to prove (4), taking cohomology, we need $H^{n+i}(\wedge^{n-i}K(-n-i-1)) = 0$ for $i = 0, \dots, n$ and $H^n(\wedge^{n+1}E(-n)) = 0$. The first group of vanishing is easily obtained by taking suitable wedge powers of (3). The crucial point used to get the last vanishing is the isomorphism $\wedge^{n+1}E \simeq \wedge^{n-1}E$; it is true because E is a rank $2n$ vector bundle with $c_1 = 0$. From the sequence

$$\begin{aligned} 0 \rightarrow S^{n-1}I^*(-2n-1) \rightarrow S^{n-2}I^* \otimes K(-2n) \rightarrow \dots \\ \dots \rightarrow \wedge^{n-1}K(-n) \rightarrow \wedge^{n-1}E(-n) \rightarrow 0, \end{aligned}$$

in order to prove $H^n(\wedge^{n-1}E(-n)) = 0$, we only need to see that

$$H^{n+i}(\wedge^{n-1-i}K(-n-i)) = 0 \quad \text{for } i = 0, \dots, n,$$

which follows by using the exact sequence (3) exactly as above. □

Now, we can state and prove the main result of this section.

Theorem 3.4. *$MI_{\mathbb{P}^{2n+1}}(k)$ is affine.*

Proof. By Theorem 3.3, we get that $\mathcal{Q} \setminus N = \mathcal{Q}^0 = \mathcal{Q} \setminus \{D = 0\}$ is affine. It follows that $MI_{\mathbb{P}^{2n+1}}(k)$ is affine too, because it is the quotient of an affine variety by a reductive group; see, e.g., [PV89], section 4.4. □

As a consequence we deduce

Corollary 3.5. *$MI_{\mathbb{P}^{2n+1}}(k)$ does not contain any complete subvariety of positive dimension.*

Proof. This follows from the fact that a quasi-affine complete variety is a finite set. □

Remark 3.6. The invariant D is meaningful even in the case $n = 0$. In this case it corresponds to the usual determinant of the map $\mathbb{C}^{2k} \rightarrow \mathbb{C}^2 \otimes \mathbb{C}^k$. For example, for $n = 0$ and $k = 2$ the degenerate $2 \times 2 \times 4$ matrices fill a variety of codimension 2 and degree 12 ([BS]) in \mathbb{P}^{15} whose ideal is generated by one quartic (which is our invariant D), 10 sextics and one octic. We remark that the case $2 \times 2 \times 3$ is of boundary format. The case $2 \times 2 \times 5$ is interesting. Here degenerate matrices fill a variety of codimension 3 and degree 20, and its ideal is generated (at least) by 5 quartics, 50 sextics and 12 octics. The 5 quartics define a variety of codimension 2 and degree 10. Hence, in this case no analog of the invariant D can exist.

4. INSTANTON BUNDLES WITH STRUCTURE GROUP $GL(2n)$

Definition 4.1. A k -instanton bundle E on \mathbb{P}^{2n+1} is the cohomology bundle of a monad

$$(5) \quad K \otimes \mathcal{O}(-1) \xrightarrow{A} W \otimes \mathcal{O} \xrightarrow{B} I \otimes \mathcal{O}(1)$$

where W is a complex vector space of dimension $2n + 2k$ and I, K are complex vector spaces of dimension k .

Notice that E is not necessarily symplectic and that this notion is a true generalization of the one above only for $n \geq 2$, because all rank 2 bundles on \mathbb{P}^3 with $c_1 = 0$ are symplectic.

Let $(A, B) \in Hom(K \otimes V^*, W) \times Hom(W, I \otimes V)$ defining E . The monad condition is now equivalent to the fact that the matrices A and B are both nondegenerate and $B \cdot A = 0$.

$Hom(K \otimes V^*, W) \times Hom(W, I \otimes V)$ contains the subvariety \mathcal{P} given by pairs of matrices (A, B) for which the sequence (5) is a complex, that is, such that $B \cdot A = 0$. $GL(I) \times GL(K) \times GL(W)$ acts on \mathcal{P} by $(a, b, c) \cdot (A, B) = (cAb, aBc^{-1})$.

Arguing, as in the previous section, we can see that there is a natural one-to-one correspondence between

- i) isomorphism classes of instanton bundles, and
- ii) orbits of $GL(I) \times GL(K) \times GL(W)$ on the open subvariety \mathcal{P}^0 of \mathcal{P} given by pairs of nondegenerate matrices.

Moreover, as in the second section and using Theorem 4.4, if we denote by H the quotient of $GL(I) \times GL(K) \times GL(W)$ by $(\lambda \cdot id, \lambda^{-1} \cdot id, \lambda \cdot id)$, then H acts

freely on \mathcal{P}^0 . In particular, all points of \mathcal{P}^0 are stable for the action of $GL(I) \times GL(K) \times GL(W)$.

Definition 4.2. The GIT-quotient $\mathcal{P}^0/GL(I) \times GL(K) \times GL(W)$ is denoted by $I_{\mathbb{P}^{2n+1}}(k)$ and is called the moduli space of k -instanton bundles on \mathbb{P}^{2n+1} . It is a geometric quotient.

$I_{\mathbb{P}^{2n+1}}(k)$ coincides for $n = 1, 2$ with the open subset $\mathcal{I}_{\mathbb{P}^{2n+1}}(k)$ of the Maruyama scheme of stable bundles on \mathbb{P}^{2n+1} of rank $2n$ and Chern polynomial $\frac{1}{(1-t^2)^k}$, which are instanton bundles. For $n \geq 3$ we can say that $\mathcal{I}_{\mathbb{P}^{2n+1}}(k)$ is an open subset of $I_{\mathbb{P}^{2n+1}}(k)$. We remark that $MI_{\mathbb{P}^3}(k) = I_{\mathbb{P}^3}(k)$. $\mathcal{I}_{\mathbb{P}^{2n+1}}(k)$ is known to be singular for $n \geq 2$ and $k \geq 3$ (see [MO97]) and reducible for $n \geq 4$ (see [AO00]).

Definition 4.3. Let $(A, B) \in Hom(K \otimes V^*, W) \times Hom(W, I \otimes V)$. We define

$$\tilde{D}(A, B) := \det S(A) \cdot \det R(B)$$

where \det denotes the usual determinant and $S(A), R(B)$ are the morphisms

$$\begin{aligned} S(A) &: S^{n+1}K \otimes V^* \rightarrow S^n K \otimes W, \\ R(B) &: S^n I \otimes W \rightarrow S^{n+1}I \otimes V, \end{aligned}$$

induced by A and B respectively, as in Definition 3.1.

Theorem 4.4. *If (A, B) defines an instanton (that is, (A, B) belongs to \mathcal{P}^0), then $\tilde{D}(A, B) \neq 0$.*

Proof. First, we will see that $\det S(A) \neq 0$. From (5) we get the exact sequence

$$(6) \quad 0 \rightarrow K \otimes \mathcal{O}(-1) \rightarrow W \otimes \mathcal{O} \rightarrow Q \rightarrow 0.$$

The $(n+1)$ -th wedge power twisted by $\mathcal{O}(-n-2)$ gives the exact sequence

$$\begin{aligned} 0 \rightarrow S^{n+1}K \otimes \mathcal{O}(-2n-3) \rightarrow S^n K \otimes W \otimes \mathcal{O}(-2n-2) \rightarrow \dots \\ \dots \rightarrow \wedge^{n+1}W \otimes \mathcal{O}(-n-2) \rightarrow \wedge^{n+1}Q(-n-2) \rightarrow 0 \end{aligned}$$

where the H^{2n+1} of the first morphism corresponds to $S(A)$. Hence, taking cohomology, it is enough to prove

$$H^n(\wedge^{n+1}Q(-n-2)) = 0.$$

This is shown by considering the $(n+1)$ -wedge sequence of the exact sequence

$$0 \rightarrow E \rightarrow Q \rightarrow I \otimes \mathcal{O}(1) \rightarrow 0$$

and arguing as in the proof of Theorem 3.3.

In order to prove $\det R(B) \neq 0$, we proceed exactly as in Theorem 3.3 and we leave the details to the reader. \square

Theorem 4.5. *$I_{\mathbb{P}^{2n+1}}(k)$ is affine.*

Proof. First, notice that given $(A, B) \in \mathcal{P}$, if A or B is degenerate, then $\det S(A) \cdot \det R(B) = 0$. Hence, by Theorem 4.4 we get that $\mathcal{P}^0 = \mathcal{P} \setminus \{\tilde{D} = 0\}$ is affine. Therefore, by [PV89] section 4.4, $I_{\mathbb{P}^{2n+1}}(k)$ is affine also. \square

As a by-product of Theorem 4.5, we deduce

Corollary 4.6. *$I_{\mathbb{P}^{2n+1}}(k)$ does not contain any complete subvariety of positive dimension.*

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