

LEVI FOLIATIONS IN PSEUDOCONVEX BOUNDARIES AND VECTOR FIELDS THAT COMMUTE APPROXIMATELY WITH $\bar{\partial}$

EMIL J. STRAUBE AND MARCEL K. SUCHESTON

ABSTRACT. Boas and Straube proved a general sufficient condition for global regularity of the $\bar{\partial}$ -Neumann problem in terms of families of vector fields that commute approximately with $\bar{\partial}$. In this paper, we study the existence of these vector fields on a compact subset of the boundary whose interior is foliated by complex manifolds. This question turns out to be closely related to properties of interest from the point of view of foliation theory.

1. INTRODUCTION

In [7], Boas and Straube established a general sufficient condition for global regularity of the $\bar{\partial}$ -Neumann problem in terms of (families of) vector fields that commute approximately with $\bar{\partial}$. They then showed that when the infinite type points lie in certain submanifolds, in particular in complex submanifolds, of the boundary, then there is an element in the first de Rham cohomology of the submanifold that is the obstruction to the existence of the required vector fields: these vector fields exist if and only if this cohomology class is zero ([7], Main Theorem and Remark 5). In the case of a complex submanifold of the boundary, this cohomology class has a geometric interpretation as a measure of the winding of the boundary around the submanifold ([3]).

In this paper, we study the situation where there is a Levi foliation in the boundary. More precisely, we assume that the set K of infinite type points (which is compact) is the closure of its interior $\overset{\circ}{K}$ (in the boundary), and that all these points are Levi flat (the Levi form vanishes identically). By the Frobenius theorem, $\overset{\circ}{K}$ is foliated by complex manifolds of codimension 1; this foliation is usually referred to as the Levi foliation of $\overset{\circ}{K}$. We start in section 2 by briefly recalling from [7] what is needed concerning vector fields that commute approximately with $\bar{\partial}$. In section 3, we assume that the domain Ω is in \mathbb{C}^2 , and that the boundary of K (relative to $b\Omega$) is a smooth surface with only isolated generic complex tangencies. Under suitable hypotheses, we show that the (families of) vector fields from [7] exist (Theorem 1). The cohomology class mentioned above, on each leaf of the Levi foliation, turns out to coincide with the infinitesimal holonomy of the leaf. Thus the question of existence of vector fields commuting approximately with $\bar{\partial}$ is closely

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related to foliation theoretic properties of the Levi foliation. We elaborate on this theme in section 4; in particular, Sullivan's theory of foliation currents ([17], [18], [10]) yields a homological necessary and sufficient condition for the existence of the vector fields (Theorem 3).

2. VECTOR FIELDS THAT COMMUTE APPROXIMATELY WITH $\bar{\partial}$

On a bounded pseudoconvex domain, the $\bar{\partial}$ -Neumann operator N_q ($1 \leq q \leq n$) is the inverse of the complex Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ acting on $(0, q)$ -forms. For background on the $\bar{\partial}$ -Neumann problem, we refer the reader to [12] and to the recent survey [8]. We say that the $\bar{\partial}$ -Neumann problem is globally regular in Sobolev spaces on a domain Ω if the operators N_q map the spaces $W_{0,q}^s(\Omega)$ of $(0, q)$ -forms with coefficients in $W^s(\Omega)$ continuously to themselves for $s \geq 0$, $1 \leq q \leq n$. Here, $W^s(\Omega)$ denotes the usual \mathcal{L}^2 -Sobolev space of order s .

In [7], Boas and Straube showed that a sufficient condition for global regularity is the existence of families of vector fields that have certain approximate commutation properties with $\bar{\partial}$. Let Ω be a smooth (C^∞) bounded pseudoconvex domain in \mathbb{C}^n with defining function ρ . Suppose there is a constant $C > 0$ such that for every $\varepsilon > 0$ there exists a vector field X_ε of type $(1,0)$ whose coefficients are smooth in a neighborhood U_ε in \mathbb{C}^n of the set of boundary points of Ω of infinite type (which is compact, [11]) and such that

$$(1) \quad |\arg X_\varepsilon \rho| < \varepsilon \quad \text{and} \quad C^{-1} < |X_\varepsilon \rho| < C \quad \text{on } U_\varepsilon$$

and

$$(2) \quad |\partial \rho([X_\varepsilon, \partial/\partial \bar{z}_j])| < \varepsilon \quad \text{on } U_\varepsilon, \quad 1 \leq j \leq n.$$

Condition (2) says that the normal $(1,0)$ -component of the commutators $[X_\varepsilon, \partial/\partial \bar{z}_j]$ should have modulus less than ε , i.e., X_ε commutes approximately with $\partial/\partial \bar{z}_j$, $1 \leq j \leq n$, modulo terms of type $(0,1)$ and terms that are complex tangential. These terms are benign (they can be absorbed) as far as Sobolev estimates for the $\bar{\partial}$ -Neumann problem are concerned (see [6], p. 83, formulas (2) and (3) for a precise statement), and so we still say that the fields X_ε commute approximately with $\bar{\partial}$, although only the normal $(1,0)$ -component of the commutator is required to be small. We also note that in (2), it suffices to consider commutators with tangential fields of type $(0,1)$, as one can always suitably extend fields from the boundary of Ω to the inside (see [6], p. 86).

When computing the normal $(1,0)$ -components of the commutators in (2), a certain 1-form appears naturally. Let $\eta := i(\partial\rho - \bar{\partial}\rho)$ and $T := -i(L_n - \bar{L}_n)$, where $L_n = (2/|\nabla\rho|^2) \sum_{j=1}^n \frac{\partial\rho}{\partial\bar{z}_j} \frac{\partial}{\partial z_j}$. L_n is the complex normal, normalized so that $\eta(T) \equiv 1$ on $b\Omega$. Let $\alpha := -\mathcal{L}_T\eta$, minus the Lie derivative of η in the direction of T ([11], [7]). (Note that η and T as defined here differ by factors of i and $-i$, respectively, from the notation in [11], [7], so that both η and T are real. This does not affect α .) If Y denotes a (local) section of $T^{0,1}(b\Omega)$, then

$$(3) \quad \alpha(Y) = 2\partial\rho([L_n, Y])$$

(see [11], p. 92, [7], section 2). Let $\bar{L}_1, \dots, \bar{L}_{n-1}$ be local sections of $T^{0,1}(b\Omega)$ which span the complex tangent space to $b\Omega$ near a boundary point P . Near P , a vector field X on $b\Omega$ of type $(1,0)$, with $X\rho \neq 0$ on $b\Omega$, can be written as

$X = e^h L_n + \sum_{j=1}^{n-1} a_j L_j$ (the $e^h L_n$ term is actually global). We have (note that $L_n \rho = 1/2$)

$$\begin{aligned}
 (4) \quad \partial\rho([X, \bar{L}_k]) &= e^h \left[\frac{1}{2} \bar{L}_k h + \partial\rho([L_n, \bar{L}_k]) \right] + \sum_{j=1}^{n-1} a_j \partial\rho([L_j, \bar{L}_k]) \\
 &= \frac{1}{2} e^h [\bar{L}_k h + \alpha(\bar{L}_k)] + \sum_{j=1}^{n-1} a_j \partial\rho([L_j, \bar{L}_k]).
 \end{aligned}$$

In general, one can use both h and the a_j 's to adjust the commutators ([6], [7], [8], Remark 1 below). Assume now there is a Levi foliation in the boundary, as discussed in section 1. We denote this foliation by \mathcal{F} . If \bar{L}_k is tangent to the leaves of \mathcal{F} , then $\partial\rho([L_j, \bar{L}_k]) = 0$, $1 \leq j \leq n - 1$ (because \bar{L}_k is a null direction of the Levi form, which is positive semi-definite on the complex tangent space). So we only have the function h to work with. If there exists a real function h , smooth in a neighborhood of the set K of points of infinite type, such that

$$(5) \quad dh|_{\text{leaf}} = -\alpha|_{\text{leaf}}$$

for all leaves of the Levi foliation, then the field $X := e^h L_n$ will have properties (1) and (2) above in a suitably small neighborhood U_ε of K (by continuity). We note that (5) requires of course that the restriction of α to any leaf of the Levi foliation is closed, but this was shown to be the case in [7] (see the lemma in section 2). Furthermore, the definition of α involves a choice of defining function. However, it is easy to see that α 's resulting from different defining functions differ by an exact form on the leaves of \mathcal{F} , so that whether or not (5) can be solved does not depend on the defining function. Indeed, if $\tilde{\rho} = e^g \rho$, then one computes that the restriction of α to leaves of \mathcal{F} changes by dg .

The following property of α will be useful. In terms of η , the Frobenius condition reads $d\eta \wedge \eta = 0$. Consequently, there exists a one-form β such that $d\eta = \beta \wedge \eta$. In fact, α is such a form, that is $d\eta = \alpha \wedge \eta$ on K . For this, see for example the discussion in chapter 2 of [19], in particular Proposition 2.2.

Remark 1. At this point, we can easily consider the case of a Levi foliation of higher codimension. That is, assume that the rank of the Levi form is constant on K , but possibly greater than zero (which is the flat case). If that rank is p , then $\overset{\circ}{K}$ is foliated by complex manifolds of dimension $n - 1 - p$. If there is a function h (smooth in a neighborhood of K) satisfying (5), then vector fields X_ε satisfying (1) and (2) may be obtained using the complex tangential term $\sum_{j=1}^{n-1} a_j L_j$ to adjust the commutators when L_k is tangential to $b\Omega$, but is not tangential to a leaf of the foliation (note that in this case, $\partial\rho([L_k, \bar{L}_k]) \neq 0$, so that we do get a contribution in (4) from these terms). Details on how to obtain these fields locally, and on how to patch the local fields to fields defined in neighborhoods U_ε of K , may be found in [6] (proof of the lemma) and [7], section 3.

3. DOMAINS IN \mathbb{C}^2

We assume now that Ω is in \mathbb{C}^2 . Suppose that K is connected and has a smooth connected boundary; more precisely, K is the closure, relative to $b\Omega$, of the smoothly bounded domain $\overset{\circ}{K}$ (as a subset of $b\Omega$). Denote by Γ the boundary (relative to $b\Omega$)

of K (or $\overset{\circ}{K}$). Then Γ is a smooth compact orientable surface embedded in \mathbb{C}^2 . Recall that a generic complex tangency of Γ is either elliptic or hyperbolic: nearby, Γ can be written, in suitable coordinates, as $w = |z|^2 + \lambda \operatorname{Re} z^2 + O(|z|^3)$, with $0 \leq \lambda < 1$ (the elliptic case) or $1 < \lambda$ (the hyperbolic case). This goes back to Bishop's paper [5], as does the index formula $\#e - \#h = 2 - 2g$, where $\#e$ and $\#h$ denote the number of elliptic and hyperbolic tangencies, respectively, and g is the genus of Γ . (Compare the introduction in [13] for information on various indices relevant for real surfaces inside complex surfaces.) In our situation, the following point of view is useful. The Levi foliation on $\overset{\circ}{K}$ (still denoted by \mathcal{F}) is induced by the 1-form η , and there is a corresponding 1-dimensional foliation, with singularities, on Γ , induced by restricting η to Γ . The singularities (i.e., the points of Γ where $\eta/\Gamma = 0$) are precisely the complex tangencies. Moreover, this foliation is orientable (since Γ is), and so yields a vector field with singularities at the complex tangencies. The index is 1 at an elliptic point and -1 at a hyperbolic point. Thus the above index formula results in this case from the Poincaré-Hopf index theorem (compare also [4], section 3 for a discussion along these lines).

At a hyperbolic point of Γ , there are, locally, two leaves of \mathcal{F} that meet (they can be constructed by flowing along suitable complex tangential curves through the point). We make the (global) assumption that these two local leaves are distinct globally.

For terminology from foliation theory, in particular for the notions of (germinal) holonomy and of infinitesimal holonomy, respectively, of a leaf, we refer the reader to [9], [10].

Theorem 1. *Let Ω be a smooth bounded pseudoconvex domain in \mathbb{C}^2 . Suppose that the set K of infinite type points of $b\Omega$ is smoothly bounded (in $b\Omega$) and that its boundary Γ is connected and has only isolated generic complex tangencies. Assume that the two leaves meeting at a hyperbolic point of Γ have no other hyperbolic points in their closure (in K). If each leaf of the Levi foliation is closed (in $\overset{\circ}{K}$) and has trivial infinitesimal holonomy, then the vector fields discussed in section 2 exist. Consequently, the $\bar{\partial}$ -Neumann problem on Ω is globally regular.*

We now comment further on the assumptions.

Remark 2. Note again that it is understood that the two local leaves meeting at a hyperbolic point are distinct globally. The assumption that leaves have at most one hyperbolic point in their closure (relative to K) is also a kind of a genericity assumption; it simplifies the non-Hausdorff structure of the leaf space $\overset{\circ}{K}/\mathcal{F}$. It will be clear from the proof of Theorem 1 how to handle more general situations, provided certain "compatibility" conditions (arising from the fact that the local determination of h near hyperbolic points propagates along leaves, by virtue of the relation (5)) are met.

Remark 3. In order to solve $dh|_{\text{leaf}} = -\alpha|_{\text{leaf}}$, α must of course represent the trivial de Rham cohomology class on each leaf. However, the vanishing of this cohomology class is equivalent to the triviality of the infinitesimal holonomy of the leaf. In fact, the infinitesimal holonomy of a leaf L can be viewed naturally as an element in the first de Rham cohomology of L , and then it is the class of α ([10], Example 2.3.15). In this context, compare also Corollary 2 in the appendix of [1]; the computations in section 2 above (see in particular (4)) show that if $e^h L_n$ is holomorphic on L ,

then $dh/L + \alpha/L = 0$ (taking into account that both h and α are real). It will be seen in section 4 that the existence of a function h as above is equivalent to the existence of a (smooth) *closed* one-form that defines the Levi foliation, and thus, under the additional assumptions in Theorem 1, the holonomy itself of each leaf is trivial (see, e.g., [19], Theorem 3.29). For completeness, we point out that the condition of vanishing holonomy (hence vanishing infinitesimal holonomy) is trivially satisfied if the leaves are simply connected (that is, they are analytic discs, by the classification of simply connected Riemann surfaces).

Remark 4. An example where there is a leaf with nontrivial holonomy may be found in [3], p. 21. Consider

$$S_R := \{(z, w) \in \mathbb{C}^2 \mid |z| \leq 1, 1 \leq |w| \leq R, \operatorname{Re}(ze^{i \log |w|^2}) = 0\}.$$

This is a Levi flat hypersurface in \mathbb{C}^2 . It is shown in [3] how to “cap off” S_R to obtain a pseudoconvex domain $\Omega \subseteq \mathbb{C}^2$ such that $S_R \subseteq b\Omega$ and $b\Omega \setminus S_R$ is strongly pseudoconvex. Ω is a worm domain, but with the critical annulus fattened. The leaves of the Levi foliation on S_R are given by

$$L(c) := \{z = icw^{-2i} \mid 1 < |w| < R, |z| < 1\},$$

where $c \in \mathbb{R}$. On $L(0)$, α does not represent the trivial cohomology class; its integral around a concentric circle inside $L(0)$ (which is an annulus) is nonzero. Also note that the leaves $L(c)$ for $c \neq 0$ are not closed (they spiral towards $L(0)$). Barrett’s arguments carry over from the case of the “standard” worm domains ([2]) to show that the conclusion of Theorem 1 fails for such Ω .

Remark 5. Given the results of [6], it is natural to ask whether in the situation of Theorem 1, there is a defining function that is plurisubharmonic on the boundary. It is not hard to see that $e^h \rho$ (h as in section 2) gives a defining function that is plurisubharmonic at points of K if h is extended from the boundary with suitable normal derivative. While this implication does not hold in general ([7], Remark 3), in the case at hand it is a special case of results in [16].

To prove Theorem 1, it suffices, by the discussion in section 2, to find a function $h \in C^\infty(K)$ such that

$$(6) \quad dh|_{\text{leaf}} = -\alpha|_{\text{leaf}}.$$

We will construct h on $\overset{\circ}{K}$ satisfying (6) and, in addition, having all its derivatives bounded on $\overset{\circ}{K}$. Such a function will then extend to a function in $C^\infty(K)$ (since $\overset{\circ}{K}$ has smooth boundary). The construction proceeds in three stages: in saturated neighborhoods of the complex tangencies of Γ , in a saturated neighborhood of the closure of the remaining points, and, finally, patching to obtain a globally defined h . In the last step, the key is that we can use cutoff functions which are constant on the leaves (i.e., we only need to cut off in the direction transverse to the Levi foliation), so that (6) is not affected (this technique occurs in a related context already in [3]).

We start with a hyperbolic point P . There is a diffeomorphism of a neighborhood of P (in $b\Omega$) onto a neighborhood of $O \in \mathbb{R}^3$ such that in these coordinates, K is given locally by $x_3 \leq x_1^2 - x_2^2$ (by the Morse theorem). The complex tangent plane to $b\Omega$ at P corresponds to the (x_1, x_2) -plane. Denote by L_1 the leaf of \mathcal{F} that meets O (i.e. P) from the side where $x_1 < 0$. (Near the origin, this leaf is obtained by

flowing along complex tangential curves emanating from 0; alternatively, one starts at points on the (unique) complex tangential curve in the (x_1, x_3) -plane through 0, and then flows along complex tangential directions transverse to it.) Then if $a > 0$ is small enough, there is a unique t_0 such that $(-a, 0, t_0) \in L_1$. Then, for t close enough to t_0 , the curve $t \mapsto (-a, 0, t)$ is transversal to the leaves of \mathcal{F} , and it meets each leaf at most once. This last property is clear locally, but it also holds globally by [14], Proposition 3.1 (note that the assumption that the first Betti number of the underlying manifold, in our case \mathring{K} , is finite, is satisfied). On these leaves, we define h (globally) by the requirements that

$$(7) \quad h(-a, 0, t) = 0$$

and

$$(8) \quad dh|_{\text{leaf}} = -\alpha|_{\text{leaf}}.$$

We use here that α represents the trivial de Rham cohomology class on each leaf, or, equivalently, that each leaf has trivial infinitesimal holonomy (see Remark 3 above). Now consider the leaf L_2 which meets 0 from the side where $x_1 > 0$. There is a unique t_1 such that $(a, 0, t_1) \in L_2$ (again, we take $a > 0$ small enough), and if t is close enough to t_1 , the curve $t \mapsto (a, 0, t)$ is transversal to \mathcal{F} and meets each leaf at most once. If $t < t_1$, $(a, 0, t)$ belongs to a leaf that passes through a point $(-a, 0, t')$ for some $t' < t_0$. On these leaves, h was defined above. Note that the function $h(a, 0, t)$ is *smooth up to* $t = t_1$ (from the side $t < t_1$). To see this, note that $h(a, 0, t)$ may be obtained by line integrals of α along integral curves of the vector field induced in the (x_1, x_3) -plane (near the origin) by intersecting with the complex tangent space. Both this vector field and α are smooth in a *full* neighborhood of the origin (in the (x_1, x_3) -plane). Denote by $g(t)$ a smooth continuation of $h(a, 0, t)$ across t_1 . On the leaves where $t > t_1$, we now define h (globally) by again requiring (8), but replacing (7) by $h(a, 0, t) = g(t)$. (As above, we use that, by assumption, α is exact on the leaves of \mathcal{F} .) In order to do this, we need to know that for t close enough to t_1 (and $t > t_1$), these leaves are distinct from the ones through points $(-a, 0, t')$, with $t' > t_0$ and close to t_0 . Since L_1 and L_2 have only one hyperbolic point, namely the origin, in their closure, and no elliptic point (no elliptic point is contained in the closure of a leaf by the regularity and uniqueness results in [15]), it follows that their closures intersect only at the origin, and they are smoothly bounded away from the origin (by an integral curve of the vector field induced on Γ by η/Γ). Consequently, away from the origin, there is transverse uniformity “up to the boundary” (see [9], Theorem 3, chapter III) for L_1 and L_2 . This implies that if t' and t are sufficiently close to t_0 and t_1 , respectively, the above leaves are indeed distinct. Note that we have used here the hypothesis that L_1 and L_2 are distinct. Finally, observe that an argument similar to the one used above to show that $h(a, 0, t)$ is smooth up to $t = t_1$ shows that the x_3 -derivatives of h are bounded near the origin. Since complex tangential derivatives of h are equal to components of α , we get inductively that near the origin, all derivatives of h are bounded, i.e., h is smooth up to the boundary.

We have now extended h into a saturated neighborhood of P (this is a slight abuse of language, since P is not in \mathring{K}). The extended function is C^∞ on this saturated subset of \mathring{K} (observe that the extension of h along the leaves can be obtained via line integrals of α , so that the smoothness of α , together with the

smoothness of h near P , gives differentiability of h also in directions transverse to the leaves; compare in this context the discussion of the holonomy maps in [9], chapter IV).

The function h is constructed in a saturated neighborhood of the union of the hyperbolic leaves (leaves having a hyperbolic point in their closure) by repeating the above construction for every hyperbolic point. We use here the assumption that two leaves meeting at a hyperbolic point have positive distance from the remaining hyperbolic points. This means that if L_1 and L_2 meet at P_1 , and L_3 and L_4 meet at P_2 , then there are disjoint saturated neighborhoods of $L_1 \cup L_2$ and $L_3 \cup L_4$ (this argument is similar to the one used in the paragraph before the last, or see below, where we show that two leaves whose closures contain no hyperbolic points can be separated by saturated neighborhoods). Consequently, this construction gives a well-defined function h that is C^∞ in a saturated neighborhood of the union of the hyperbolic leaves.

A similar construction works near the elliptic complex tangencies. Near such a point, $\overset{\circ}{K}$ is foliated by a smooth one-parameter family of disks, and there is a (uniformly) transverse curve emanating from the elliptic point (see [15]). Prescribing h to be zero on this curve and then solving $dh = \alpha$ on each leaf gives the required function h in a saturated subset of $\overset{\circ}{K}$ near the elliptic points. Note that the disks close to an elliptic point do not contain other complex tangencies in their closure (by the regularity up-to-the-boundary results in [15]), so that we choose the saturated neighborhoods small enough to be pairwise disjoint and disjoint from a saturated neighborhood of the union of the hyperbolic leaves.

For the second step of the construction, we consider the leaf space $\overset{\circ}{K}/\mathcal{F}$: points in the same leaf are identified, and the topology is the quotient topology. $\overset{\circ}{K}/\mathcal{F}$ is in general not Hausdorff (two hyperbolic leaves whose closure intersects in a hyperbolic point cannot be separated by saturated neighborhoods, and thus give rise to a pair of points in $\overset{\circ}{K}/\mathcal{F}$ which cannot be separated). In our situation, $\overset{\circ}{K}/\mathcal{F}$ carries a natural structure of a C^∞ one-dimensional real (in general non-Hausdorff) oriented manifold ([14], section 3). We use here that the leaves of \mathcal{F} are closed. Also, note that the first Betti number of $\overset{\circ}{K}$ is finite (this is assumed in all of section 3 in [14]). The local charts are obtained as follows. Pick a point P on a leaf L and choose a (short) transversal C^∞ curve through P that meets each leaf of \mathcal{F} at most once ([14], Proposition 3.1). The projection of this curve gives a neighborhood U of L in $\overset{\circ}{K}/\mathcal{F}$, and we take a smooth parameter on the curve (for instance, arclength) to be a local coordinate in U . Then the coordinate transformations are given by holonomy maps and are thus C^∞ ([9], chapter IV). Because \mathcal{F} is transversely oriented (by the vector field T), $\overset{\circ}{K}/\mathcal{F}$ is oriented. Note that the projection $\pi: \overset{\circ}{K} \rightarrow \overset{\circ}{K}/\mathcal{F}$ is a C^∞ submersion.

Denote by L_1, L_2, \dots, L_m the hyperbolic leaves, and set $H := L_1 \cup L_2 \cup \dots \cup L_m$. Since the leaves are closed in $\overset{\circ}{K}$, $\overset{\circ}{K} \setminus H$ is open. Consider now the quotient $(\overset{\circ}{K} \setminus H)/\mathcal{F}$ (which is diffeomorphic to $(\overset{\circ}{K}/\mathcal{F}) \setminus \{L_1, \dots, L_m\}$). We claim that it is Hausdorff. The argument is similar to one used above. First note that any leaf L in $\overset{\circ}{K} \setminus H$ is smooth up to Γ and intersects Γ in an integral curve of the vector field induced on Γ by η/Γ . The reason is that the closure \bar{L} of L in K contains neither hyperbolic

points (by the definition of H) nor elliptic points (no elliptic point is contained in the closure of a leaf of $\overset{\circ}{K}$, by the results of [15]). But near every other point P , \mathcal{F} is smooth up to Γ and there is a unique leaf that intersects Γ (from one side) transversally in an integral curve through P of the field induced by η . This implies that if L_1 and L_2 are two distinct leaves in $\overset{\circ}{K} \setminus H$, then they (equivalently, their closures) are a positive distance apart. For if not, their closures (in K) would have to intersect (in points of Γ). This is impossible by the preceding discussion. Moreover, for these leaves, there is again transverse uniformity (see [9], Theorem 3, chapter III) “up to the boundary”. Consequently, they can be separated by *saturated* neighborhoods. This proves that $(\overset{\circ}{K} \setminus H)/\mathcal{F}$ is Hausdorff. With the differential structure induced as in the case of the full leaf space (equivalently, the restriction of the differential structure on $\overset{\circ}{K}/\mathcal{F}$ to $\overset{\circ}{K}/\mathcal{F} \setminus \{L_1, \dots, L_m\}$), $(\overset{\circ}{K} \setminus H)/\mathcal{F}$ is thus a bona fide real 1-dimensional manifold. By the classification of such manifolds, its connected components are diffeomorphic either to \mathbb{R} or to S^1 .

We now construct h on the components of $\overset{\circ}{K} \setminus H$ (which correspond to components of $(\overset{\circ}{K} \setminus H)/\mathcal{F}$) as follows. Let U be such a component. Then by the above, $\pi(U)$ is diffeomorphic to \mathbb{R} or to S^1 . Denote by γ the one-form on $\pi(U)$ corresponding to dx in the case of \mathbb{R} , and $d\theta$ in the case of S^1 . Note that γ is closed and nonvanishing. The form $\omega := \pi^*\gamma$ is then a nonsingular (π is a submersion) one-form on U that satisfies

$$(9) \quad d\omega = d\pi^*\gamma = \pi^*d\gamma = 0,$$

and

$$(10) \quad \omega/L = 0,$$

where L is any leaf of \mathcal{F}/U . Thus ω is of the form $e^g\eta$ for some C^∞ -function g (where $e^g = \omega(T)$). Therefore,

$$(11) \quad 0 = d\omega = e^g(dg \wedge \eta + d\eta) = e^g(dg \wedge \eta + \alpha \wedge \eta).$$

Here we have used that on K , $d\eta = \alpha \wedge \eta$ (see the discussion at the end of section 2). Consequently,

$$(12) \quad (dg + \alpha) \wedge \eta = 0,$$

which is equivalent to

$$(13) \quad dg/L = -\alpha/L$$

on each leaf of \mathcal{F}/U . (This computation will also show the equivalence of (i) and (ii) in Proposition 2 below.)

In the last step, we patch together the functions h and g constructed so far. By working in the leaf space (equivalently, on (short) integral curves of T), one constructs a smooth function ψ which is constant on the leaves of \mathcal{F} , is identically one in a (saturated) neighborhood of $L_1 \cup \dots \cup L_m$, is supported in a small (saturated) neighborhood of this set, is identically one near elliptic points and is supported close to these points. Then the function $h_1 := \psi h + (1 - \psi)g$ satisfies, for a leaf L

of \mathcal{F} ,

$$\begin{aligned}
 dh_1/L &= d(\psi h + (1 - \psi)g)/L = (\psi dh + (1 - \psi)dg)/L \\
 (14) \qquad &= \psi dh/L + (1 - \psi)dg/L \\
 &= \psi(-\alpha)/L + (1 - \psi)(-\alpha)/L = (-\alpha)/L.
 \end{aligned}$$

That is, h_1 satisfies (6).

It remains to check smoothness up to the boundary. As indicated at the beginning of the proof of Theorem 1, we will verify that derivatives of all orders are bounded. We have already noted that, by our construction, in local coordinates near a hyperbolic point of Γ , this is the case. The essence of the argument is the same near the other points of Γ . Let P be a point of Γ which is not a complex tangency. Near P , the Levi foliation is smooth up to Γ , the boundaries of the leaves being integral curves of the vector field on Γ induced by η/Γ . Denote by L_p the leaf “through” P . For $Q \in L_p$, sufficiently close to P , let $\sigma(t)$, $-\varepsilon < t < \varepsilon$, be a short integral curve of T through Q (say $\sigma(0) = Q$). For $z \in \overset{\circ}{K}$ near P , denote by L_z the leaf of \mathcal{F} through z , and by t_z the unique t such that $\sigma(t_z) \in L_z$. Writing $h_1(z)$ as $h_1(\sigma(t_z))$ plus a line integral of α in L_z shows, in view of the smoothness of α and of \mathcal{F} up to Γ , that derivatives of all orders of h_1 are bounded near P . Analogous arguments work near an elliptic point. This completes the proof of Theorem 1.

Remark 6. Near an elliptic point, one can take advantage of the fact that conditions (1) and (2) need only be satisfied to within ε . Specifically, if P is an elliptic point, define h_ε near P through $e^{h_\varepsilon}L_n = L_n(P)$. Then, close enough to P , $|\arg e^{h_\varepsilon}L_n(\rho)| < \varepsilon$, and $\partial\rho([e^{h_\varepsilon}L_n, \partial/\partial\bar{z}_j]) = 0$ (the latter holds since $e^{h_\varepsilon}L_n$ has holomorphic (namely constant) coefficients).

4. MORE FOLIATION THEORY

The existence of the function h constructed above also entails some properties of interest from the point of view of foliation theory, both of the Levi foliation and of the flow generated by the transverse field T . This leads to a homological condition that is necessary and sufficient for the construction from section 3, and thus for the vector fields from section 2, with $\varepsilon = 0$ on K , that uses Sullivan’s theory of foliation currents ([17], [18], [10]). Although this does not seem to simplify the proof of, for example, Theorem 1, it is of independent interest, and we expect this point of view to be useful in the more general situation where the set K of infinite type points is not the closure of a nicely foliated open subset of the boundary.

We keep the notation from section 2. For the moment, we let M be a C^∞ -smooth (open) Levi flat hypersurface in \mathbb{C}^n . \mathcal{F} again denotes the Levi foliation on M (in this case by complex manifolds of dimension $n - 1$).

A one-form ω is said to define the foliation \mathcal{F} if for each $P \in M$, the tangent space to the leaf through P is equal to the kernel of ω at P . So the form η defined in section 2 always defines \mathcal{F} .

A flow is said to be geodesible if there exists a Riemannian metric (on the underlying manifold) such that the orbits become geodesics (see, e.g., [19], chapter 6). We consider the flow generated by T .

Proposition 2. *The following are equivalent:*

(i) *there exists $h \in C^\infty(M)$ such that*

$$dh/L + \alpha/L = 0, \quad \forall L \in \mathcal{F};$$

- (ii) *there exists a smooth (C^∞) closed one-form ω on M that defines \mathcal{F} ;*
 (iii) *the flow generated by T is geodesible by a metric that is obtained from the Euclidean metric on M by scaling in the T -direction.*

Note that (ii) and (iii) are manifestly independent of a choice of defining function, while the definition of α involves such a choice. However, we have pointed out already in section 2 that one can check by direct computation that (i) is actually independent of this choice.

Proof. As noted in section 3, the equivalence of (i) and (ii) follows from the computation (11)–(13) there. Namely, since ω defines \mathcal{F} , it is nonsingular; so $\omega = e^h\eta$ for some real-valued function h (possibly after replacing ω by $-\omega$). Then

$$(15) \quad \begin{aligned} d\omega &= d(e^h\eta) = e^h(dh \wedge \eta + d\eta) \\ &= e^h(dh + \alpha) \wedge \eta. \end{aligned}$$

Therefore, $d\omega = 0 \Leftrightarrow (dh + \alpha) \wedge \eta = 0 \Leftrightarrow$ (i) holds. In the last equality in (15), we have again used that $d\eta = \alpha \wedge \eta$ (see section 2).

The equivalence of (ii) and (iii) is contained in Proposition 6.7 of [19], where various characterizations are given for a flow to be geodesible. But the arguments show that (ii) is equivalent to the flow of T being geodesible by a metric as in (iii). This completes the proof of Proposition 2. \square

Sullivan ([18]) characterizes geodesible flows by a homological condition. His arguments are easily modified to obtain a characterization of the conditions in Proposition 2 (actually, the versions on the compact set K , rather than on an open hypersurface, as in Proposition 2). We return to the analogue of the setup in section 3 for a domain in \mathbb{C}^n . Denote by \mathcal{T} the one-dimensional foliation whose leaves are the orbits of T . (This is actually a foliation of all of $b\Omega$.) We briefly recall the notions we need from the theory of foliation currents; for details, see [17] or [10], chapter 10. Denote by $\mathcal{D}_1(K)$ the space of one-forms that are C^∞ on K , with the usual Fréchet space topology. Its strong dual $\mathcal{D}_1(K)'$ comprises the one-currents on K . $\mathcal{D}_1(K)$ and $\mathcal{D}_1(K)'$ are mutually dual (i.e., $\mathcal{D}_1(K)$ is reflexive). The closed convex cone C_τ of foliation currents for \mathcal{T}/K is the closure, in $\mathcal{D}_1(K)'$, of the set of all finite linear combinations with positive coefficients of currents given by pairing at x with $T(x)$, for $x \in K$ (so-called Dirac currents). The crux of the matter is that C_τ has a compact base: $C_\tau \cap \eta^{-1}(0) = \{0\}$ and $C_\tau \cap \eta^{-1}(1)$ is compact (where η is viewed as an element of $\mathcal{D}_1(K)''$); this is analogous to Lemma 10.2.3 in [10]. We say that a p -chain is tangential to \mathcal{F} if it is a sum of p -chains, each of which is a p -chain in a leaf of \mathcal{F} .

Theorem 3. *There exists $h \in \mathcal{D}(K)$ satisfying $dh/L + \alpha/L = 0$ for each leaf L of \mathcal{F} if and only if no nontrivial foliation current for \mathcal{T}/K can be arbitrarily well approximated in $\mathcal{D}_1(K)'$ (i.e., uniformly on bounded subsets of $\mathcal{D}_1(K)$) by sums of the form (smooth 1-chain tangential to \mathcal{F}) plus (boundary of smooth 2-chain in $\overset{\circ}{K}$).*

The proof of Theorem 3 is completely analogous to the proof of (iii) in the theorem in [18]. Namely, the second statement in Theorem 3 means that the closed linear subspace of $\mathcal{D}_1(K)'$ generated by boundaries of smooth 2-chains in $\overset{\circ}{K}$ and smooth 1-chains tangential to \mathcal{F} intersects the cone C_τ only at $0 \in \mathcal{D}_1(K)'$. Since C_τ has a compact base (see above), a suitable version of the Hahn-Banach theorem (see, e.g., [10], Theorem 10.2.5 and the remark immediately following it) yields an element $\omega \in \mathcal{D}_1(K)'' = \mathcal{D}_1(K)$ that vanishes on the boundaries of smooth 2-chains and on smooth 1-chains tangential to \mathcal{F} , while it is strictly positive on $C_\tau \setminus \{0\}$. These properties imply, respectively, that ω is closed, vanishes at all points on vectors tangent to the leaf of \mathcal{F} through the point, and is strictly positive on T at all points of K . Consequently, $\omega = e^h \eta$, with $h \in \mathcal{D}(K)$, and h has the required property, see (the proof of) Proposition 2.

On the other hand, if there exists h as in the theorem, then $\omega = e^h \eta$ is closed (again by the proof of Proposition 2), and, if $\varphi \in C_\tau$, then $\varphi(\omega) > 0$, so that φ cannot be approximated as indicated in the theorem. This concludes the proof of Theorem 3.

Remark 7. As an illustration, consider the following. We know from section 3 that if there exists a function h as in Theorem 3, then the Levi foliation must be without holonomy. It is easily seen directly that the homological condition in Theorem 3 implies that the holonomy of \mathcal{F} is trivial. Consider a point x_0 in a leaf L and a simple loop σ in L , starting and ending at x_0 . Lift σ to a nearby leaf as in the construction of the holonomy maps. This yields a path $\tilde{\sigma}$. If $\tilde{\sigma}$ is not a loop, we have a two-dimensional surface whose boundary consists of σ plus $\tilde{\sigma}$ (both of which are tangential to the Levi foliation) plus a piece of an integral curve of T (i.e., a nontrivial foliation current for T). This is impossible by the homological condition. Since σ was arbitrary, L has trivial holonomy.

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843-3368

E-mail address: `straube@math.tamu.edu`