

THE L^p DIRICHLET PROBLEM AND NONDIVERGENCE HARMONIC MEASURE

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In memory of E. Fabes

ABSTRACT. We consider the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } D, \\ u = g & \text{on } \partial D \end{cases}$$

for two second-order elliptic operators $\mathcal{L}_k u = \sum_{i,j=1}^n a_k^{i,j}(x) \partial_{ij} u(x)$, $k = 0, 1$, in a bounded Lipschitz domain $D \subset \mathbb{R}^n$. The coefficients $a_k^{i,j}$ belong to the space of bounded mean oscillation BMO with a suitable small BMO modulus. We assume that \mathcal{L}_0 is regular in $L^p(\partial D, d\sigma)$ for some p , $1 < p < \infty$, that is, $\|Nu\|_{L^p} \leq C \|g\|_{L^p}$ for all continuous boundary data g . Here σ is the surface measure on ∂D and Nu is the nontangential maximal operator. The aim of this paper is to establish sufficient conditions on the difference of the coefficients $\varepsilon^{i,j}(x) = a_1^{i,j}(x) - a_0^{i,j}(x)$ that will assure the perturbed operator \mathcal{L}_1 to be regular in $L^q(\partial D, d\sigma)$ for some q , $1 < q < \infty$.

1. INTRODUCTION

In the present note we consider linear elliptic second-order differential operators in nondivergence form $\mathcal{L} = \sum_{i,j=1}^n a^{i,j}(x) \partial_{ij}$, where $A(x) = (a^{i,j}(x))_{i,j=1}^n$ is a symmetric matrix verifying the uniform ellipticity and boundedness condition

$$(1.1) \quad \lambda |\xi|^2 \leq \xi^t A(x) \xi \leq \Lambda |\xi|^2, \quad x, \xi \in \mathbb{R}^n$$

for some fixed $0 < \lambda \leq \Lambda < \infty$ and $n \geq 2$. We study the Dirichlet problem

$$(1.2) \quad \begin{cases} \mathcal{L}u = 0 & \text{in } D, \\ u = g & \text{on } \partial D \end{cases}$$

on a bounded Lipschitz domain $D \subset \mathbb{R}^n$. From [1] and a standard approximation argument, it follows that if the coefficients $a^{i,j}$ are in VMO (BMO_{ϱ_0}) and $g \in \mathcal{C}(\partial D)$, problem (1.2) has a unique solution $u = u_g \in \mathcal{C}(\overline{D}) \cap W_{\text{loc}}^{2,p}(D)$ for all p , $1 < p < \infty$ ($1 < p < p_0(\varrho_0)$). We denote by σ the surface measure on ∂D and we say that the operator \mathcal{L} is regular in $L^p(\partial D, d\sigma)$ or that \mathcal{D}_p holds for \mathcal{L} in D , $1 < p < \infty$, if there exists a constant C_p which depends on n , λ , Λ , D , p and the

Received by the editors April 5, 2002 and, in revised form, May 17, 2002.

2000 *Mathematics Subject Classification*. Primary 35J25; Secondary 35B20, 31B35.

Key words and phrases. Nondivergence elliptic equations, Dirichlet problem, harmonic measure.

BMO modulus of the coefficients such that for all continuous boundary data g , the solution u of (1.2) verifies

$$(1.3) \quad \|Nu\|_{L^p(\partial D, d\sigma)} \leq C_p \|g\|_{L^p(\partial D, d\sigma)},$$

where Nu is the nontangential maximal operator

$$Nu(Q) = \sup_{\Gamma_\alpha(Q)} |u(x)|.$$

Here and henceforth, $\Gamma_\alpha(Q)$ denotes the interior truncated cone (of opening α)

$$(1.4) \quad \Gamma_\alpha(Q) = \{x \in D : |x - Q| \leq (1 + \alpha) \delta(x)\} \cap B_{r^*}(Q),$$

$\delta(x) = \text{dist}(x, \partial D)$, $B_r(x)$ denotes the ball in \mathbb{R}^n centered at x of radius r and $\alpha \geq \alpha^* = \alpha^*(D) > 0$, $r^* = r^*(D, \lambda, \Lambda, \eta) > 0$ are fixed (here η is the BMO modulus of the coefficients of \mathcal{L} ; see Section 2 and (2.10)). When necessary, we will write $N_\alpha u$ for the nontangential maximal operator of opening α .

The purpose of this note is to give sufficient conditions for the preservation of the regularity of the L^p Dirichlet problem under small perturbations on the coefficients. Given two elliptic operators $\mathcal{L}_k = \sum_{i,j=1}^n a_k^{i,j}(x) \partial_{x_i} \partial_{x_j}$, where $A_k(x) = (a_k^{i,j}(x))_{i,j=1}^n$, $k = 0, 1$, are symmetric matrices verifying (1.1), let $\varepsilon(x) = (a_1^{i,j}(x) - a_0^{i,j}(x))_{i,j=1}^n$ be the difference between the coefficients and $B(x) = B_{\delta(x)/2}(x)$, $x \in D$, and consider the quantity

$$(1.5) \quad \mathbf{a}(x) = \max_{1 \leq i,j \leq n} \text{ess sup}_{y \in B(x)} |\varepsilon^{i,j}(y)|.$$

For $Q \in \partial D$ and $r > 0$ we denote the boundary ball of radius r at Q by $\Delta_r(Q) = B_r(Q) \cap \partial D$, and the Carleson region at Q of radius r by $T_r(Q) = B_r(Q) \cap D$. Our main result is the following:

Theorem 1.1. *Suppose that \mathcal{L}_0 verifies \mathcal{D}_p for some p , $1 < p < \infty$. Then there exists $\varrho_0 = \varrho_0(n, \lambda, \Lambda, D, C_p) > 0$ such that if $a_k^{i,j} \in \text{BMO}_{\varrho_0}(\mathbb{R}^n)$, $1 \leq i, j \leq n$, $k = 0, 1$, and*

$$(1.6) \quad \sup_{Q \in \partial D, r > 0} \frac{1}{\sigma(T_r(Q))} \int_{T_r(Q)} \frac{\mathbf{a}^2(x)}{\delta(x)} dx = M < \infty,$$

then \mathcal{L}_1 verifies \mathcal{D}_q for some q , $1 < q < \infty$.

A similar result was established in [2] for divergence form operators with coefficients in $L^\infty(\mathbb{R}^n)$. We are able to adapt the divergence case techniques and obtain the results in [3], [4] and [2] (under extra assumptions on the coefficients) for the nondivergence case (see [5]). This gives a partial answer to the problem posed by C. Kenig in [6] (Problem 3.3.9). Condition (1.6) says that the measure $\mathbf{a}^2/\delta dx$ is a Carleson measure with respect to σ with Carleson norm bounded by M .

By the maximum principle, the correspondence $g \mapsto u_g(x)$ is a positive linear functional on $\mathcal{C}(\partial D)$ for each fixed $x \in D$. The Riesz representation theorem implies that there exist a unique regular positive Borel measure $\omega^x = \omega_{\mathcal{L}, D}^x$ such that

$$u(x) = \int_{\partial D} g(Q) d\omega^x(Q).$$

The measure ω^x is called the *harmonic measure* for \mathcal{L} and D at x and constitutes one of our main tools in the proof of Theorem 1.1. Also crucial for this task is the concept of *normalized adjoint solution* (n.a.s.), first introduced in [7] (see

also [8], [9]). In [10] n.a.s. are used to define a proper area function for solutions of nondivergence form operators with bounded coefficients. We also use the theory of Muckenhoupt weights [11], [12] and, in particular, the result in [13] which establishes that nonnegative adjoint solutions are A_p weights for all p , $p_0 \leq p \leq \infty$, where p_0 depends on the BMO modulus of the coefficients. Other important elements in our proofs are the a priori estimates for solutions [14], [1], basic properties of the harmonic measure [8], [15], [9] and weighted Poincaré inequalities [16].

Remark 1.2. It is known [17] [18] that the Laplacian operator $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ is regular in L^p for $2 - \varepsilon < p < \infty$ where $\varepsilon = \varepsilon(n, D)$. On the other hand, examples in [19] show the existence of a nondivergence operator \mathcal{L}_1 with continuous coefficients A_1 in the closure of the unit ball B in \mathbb{R}^n , such that $\mathcal{L}_1 = \Delta$ on ∂B and the harmonic measure $\omega_1 = \omega_{\mathcal{L}_1, B}$ is singular with respect to the surface measure σ . In particular, \mathcal{L}_1 is not regular in $L^p(\partial D, d\sigma)$ for any p (see Theorem 2.2). Setting $\mathcal{L}_0 = \Delta$, the modulus $\mathbf{a}(x)$ corresponding to this example violates condition (1.6). This shows that the perturbation problem addressed in Theorem 1.1 is nontrivial, even for continuous coefficients.

2. PRELIMINARIES

In general, we write $X \lesssim Y$ when there exists a constant $C > 0$ which depends at most on $n, \lambda, \Lambda, \eta$ and D such that $X \leq CY$. Similarly, we define the expression $X \gtrsim Y$ and write $X \approx Y$ when $X \lesssim Y$ and $X \gtrsim Y$.

If $G \subset \mathbb{R}^n$ is a Borel set, we denote by $\mathcal{C}(G)$ the space of real-valued continuous functions on G . If μ is a σ -finite Borel measure on G , $L^p(G, d\mu)$, $1 \leq p < \infty$ denotes the Banach space of μ -measurable functions f on G such that $\|f\|_{L^p(G, d\mu)} = (\int_G |f|^p d\mu)^{\frac{1}{p}} < \infty$. We use dx to denote the Lebesgue measure in \mathbb{R}^n , $|E| = \int_E dx$ for any Borel set E in \mathbb{R}^n and we write $L^p(G) = L^p(G, dx)$. The spaces $L^\infty(G, d\mu)$, $L^p_{loc}(G, d\mu)$ are also defined in a standard way. If $G \subset \mathbb{R}^n$ is open, k is a nonnegative integer and $1 \leq p \leq \infty$, we set $W^{k,p}(G)$ to be the Sobolev space of functions f with k weak derivatives in $\mathcal{L}^p(G)$ (see [20], Chapter 7).

Given $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$, we set

$$\eta(r, x) = \eta_f(r, x) = \sup_{s \leq r} \frac{1}{|B_s(x)|} \int_{B_s(x)} |f(y) - f_{B_s(x)}| dy$$

where $f_E = \frac{1}{|E|} \int_E f(y) dy$. We say that f has *bounded mean oscillation* or that $f \in \text{BMO}(\mathbb{R}^n)$ if $\eta \in L^\infty(\mathbb{R}^+, \mathbb{R}^n)$ and set $\|f\|_{\text{BMO}(\mathbb{R}^n)} = \|\eta_f\|_{L^\infty(\mathbb{R}^+, \mathbb{R}^n)}$.

Definition 2.1. Given $\varrho > 0$ and $\zeta > 0$, we let $\Phi(\varrho, \zeta)$ be the set

$$\Phi(\varrho, \zeta) = \{\eta : \mathbb{R}^+ \mapsto \mathbb{R}^+, \eta \text{ nondecreasing, } \eta(r) \leq \varrho \text{ whenever } r < \zeta\}.$$

We also set $\Phi(\varrho) = \bigcup_{\zeta > 0} \Phi(\varrho, \zeta)$, and given $\eta \in \Phi(\varrho)$, we denote $\zeta(\eta, \varrho) = \zeta(\eta) = \sup\{\zeta > 0 : \eta \in \Phi(\varrho, \zeta)\}$.

If $\varrho > 0$, we say that $f \in \text{BMO}_\varrho(\mathbb{R}^n)$ if $\eta(r) = \|\eta(r, \cdot)\|_{L^\infty(\mathbb{R}^n)}$ lies in $\Phi(\varrho)$. If $\lim_{r \rightarrow 0^+} \eta(r) = 0$, we say that f has *vanishing mean oscillation* or that $f \in \text{VMO}(\mathbb{R}^n)$ (see [21]). We also define $\text{BMO}(G)$ and $\text{BMO}(G, d\mu)$ in a standard way through the modulus

$$\eta(r, x, G, \mu) = \sup_{s \leq r} \frac{1}{\mu(B_s(x) \cap G)} \int_{B_s(x) \cap G} |f(y) - f_{B_s(x) \cap G, d\mu}| d\mu,$$

where $f_{E,d\mu} = \frac{1}{\mu(E)} \int_E f(y) d\mu$, $G \subset \mathbb{R}^n$ is a Borel set and μ is a Borel measure.

Given a nondecreasing function $\eta : \overline{\mathbb{R}^+} \mapsto \overline{\mathbb{R}^+}$, we denote by $O(\lambda, \Lambda, \eta)$ the class of operators $\mathcal{L} = \sum_{i,j=1}^n a^{i,j}(x) \partial_{x_i x_j}$, with symmetric coefficients $A(x) = (a^{i,j}(x))_{i,j=1}^n$ verifying the ellipticity and boundedness conditions (1.1) and such that $a^{i,j} \in \text{BMO}(\mathbb{R}^n)$, $1 \leq i, j \leq n$, with BMO-modulus of continuity η in D . When there is no restriction on the regularity of the coefficients of \mathcal{L} , we say $\mathcal{L} \in O(\lambda, \Lambda)$.

We denote by D a bounded Lipschitz domain in \mathbb{R}^n , that is, a bounded, connected open set D such that its boundary ∂D can be covered by a finite number of open right circular cylinders whose bases have positive distance from ∂D and, corresponding to each cylinder C , there is a coordinate system (x', x_n) with $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$ with x_n axis parallel to the axis of C , and a function $\psi : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ satisfying a Lipschitz condition $(|\psi(x') - \psi(y')| \leq m_0 |x' - y'|)$ such that $C \cap D = \{(x', x_n) : x_n > \psi(x')\} \cap C$, and $C \cap \partial D = \{(x', x_n) : x_n = \psi(x')\} \cap C$. Whenever we say that a quantity depends on D , we mean it depends on the Lipschitz character of D . In what follows we assume that D is contained in the unit ball and contains the origin.

Let Δ denote a generic boundary ball in ∂D , i.e., $\Delta = \Delta_r(Q)$ for some $r > 0$, $Q \in \partial D$. Given two Borel measures μ and ν on ∂D , we say that μ is in A_∞ with respect to ν on ∂D and we write $\mu \in A_\infty(d\nu)$ if there exist $0 < \zeta < 1$ and $\kappa > 0$ such that

$$(2.1) \quad \frac{\nu(E)}{\nu(\Delta)} > \zeta \Rightarrow \frac{\mu(E)}{\mu(\Delta)} > \kappa,$$

whenever $E \subset \Delta$ and E is a Borel set. The theory of A_∞ weights originates in [11] and [22] where the results below can be found (see also [12] and [23]). We say that μ is in the reverse Hölder class $B_{p'}(d\nu)$, $1 < p' < \infty$, if μ is absolutely continuous with respect to ν and $k = \frac{d\mu}{d\nu}$ verifies

$$\left\{ \frac{1}{\nu(\Delta)} \int_\Delta k^{p'} d\nu \right\}^{\frac{1}{p'}} \leq C \frac{1}{\nu(\Delta)} \int_\Delta k d\nu,$$

for all boundary balls $\Delta \subset \partial D$. The weight k is in $A_p(d\nu)$, $1 < p < \infty$ if

$$(2.2) \quad \left\{ \frac{1}{\nu(\Delta)} \int_\Delta k d\nu \right\} \left\{ \frac{1}{\nu(\Delta)} \int_\Delta k^{-\frac{1}{p-1}} d\nu \right\}^{\frac{1}{p-1}} \leq C < \infty.$$

It is easy to see that A_∞ is an equivalence relation, and that $k \in A_p(d\nu)$ if and only if $k^{-1} \in B_{p'}(d\mu)$, $\frac{1}{p} + \frac{1}{p'} = 1$. The best constant C in (2.2) is called the $A_p(d\nu)$ “norm” of k and we denoted it by $\|k\|_{A_p(d\nu)}$ or $\|\mu\|_{A_p(d\nu)}$. We will also use the convention $k \in A_p$ (resp. $B_{p'}$, A_∞) whenever $k \in A_p(d\sigma)$ (resp. $B_{p'}(d\sigma)$, $A_\infty(d\sigma)$).

We say that a measure ν is a *doubling* measure, with doubling constant $c = c(\nu)$ if $\nu(\Delta_{2r}(Q)) \leq c \nu(\Delta_r(Q))$ for all $r > 0$ and $Q \in \partial D$. It is also well known that if $\mu \in A_p(d\nu)$, then μ is a doubling measure if and only if ν is a doubling measure and $c(\mu) = c(\nu)^p \|\mu\|_{A_p(d\nu)}$.

Given a Borel measure μ on ∂D , we denote by $M_\mu g(Q)$ the Hardy-Littlewood maximal operator at Q with respect to μ ; that is,

$$(2.3) \quad M_\mu g(Q) = \sup_{Q \in \Delta(Q)} \frac{1}{\mu(\Delta(Q))} \int_\Delta g(P) d\mu(P)$$

where $\Delta(Q)$ denotes a generic boundary ball in ∂D centered at Q . It is known that if μ is a doubling measure, then

$$\|M_\mu f\|_{L^p(\partial D, d\nu)} \leq C_p \|f\|_{L^p(\partial D, d\nu)}, \quad 1 < p \leq \infty,$$

with $C_p > 0$ independent of f , if and only if $\nu \in A_p(d\mu)$ [22] (see also [12]).

If u is the solution of (1.2) with boundary data $g \in \mathcal{C}(\partial D)$, then $Nu \approx M_\omega g$ ([7], Theorem 7.3). Here and henceforth, ω denotes the harmonic measure for \mathcal{L} and D at a fixed point $x_0 \in D$. Since the harmonic measure is a doubling measure [7] (see also [9]), we have that the maximal operator is bounded in $L^p(\partial D, d\omega)$, $1 < p \leq \infty$, and then $\|Nu\|_{L^p(\partial D, d\omega)} \lesssim \|g\|_{L^p(\partial D, d\omega)}$ for all p , $1 < p \leq \infty$. From the weighted maximal theorem ([12], IV.2.1) we then have that \mathcal{L} verifies \mathcal{D}_p for some $1 < p < \infty$ if and only if ω is a weight in the reverse Hölder class $B_{p'}(d\sigma)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Another basic fact of the theory of weights is that

$$A_\infty(d\nu) = \bigcup_{p>1} A_p(d\nu) = \bigcup_{p'>1} B_{q'}(d\nu);$$

hence, to prove Theorem 1.1 it is enough to show $\omega_1 \in A_\infty(d\sigma)$. The following theorem is a consequence of the weighted maximal theorem, the theory of weights, and the inequalities $Nu \approx M_\omega g$.

Theorem 2.2. *Let ω be the harmonic measure with respect to \mathcal{L} in D and let μ be a Borel measure on ∂D . The following are equivalent:*

- (i) $\omega \in A_\infty(d\mu)$.
- (ii) *There exist $1 < p < \infty$ such that $\mathcal{D}_p(d\mu)$ holds, that is,*

$$\|Nu\|_{L^p(\partial D, d\mu)} \leq C_p \|g\|_{L^p(\partial D, d\mu)}.$$

- (iii) *ω is absolutely continuous with respect to μ and $k = \frac{d\omega}{d\mu}$ belongs to $B_q(d\mu)$, ($\frac{1}{p} + \frac{1}{q} = 1$).*

2.1. A Priori Estimates and Properties of Solutions.

Theorem 2.3 (Maximum principle [20], 9.1). *Let $\mathcal{L} \in O(\lambda, \Lambda)$, D be a bounded domain and $u \in C(\overline{D}) \cap W_{\text{loc}}^{2,n}(D)$ verify $\mathcal{L}u \geq f$ with $f \in L^n(D)$. Then there exists $C > 0$ which depends only on n , $\text{diam}(D)$, λ and Λ such that*

$$\sup_D u \leq \sup_{\partial D} u^+ + C \|f\|_{L^n(D)}.$$

Theorem 2.4. *Let $w \in A_p$ and $p \in (1, \infty)$. There exist positive numbers $c = c(n, p, \lambda, \Lambda, \|[w]\|_{A_p})$ and $\tilde{\varrho}_p = \tilde{\varrho}_p(n, c)$ such that if $\eta \in \Phi(\tilde{\varrho}_p)$ and $\mathcal{L} \in O(\lambda, \Lambda, \eta)$, then for any open set $\Omega \subset \mathbb{R}^n$, $\text{diam}(\Omega) \leq \zeta(\eta)$, and any $u \in W_0^{2,p}(\Omega)$ we have*

$$\|\partial_{ij}u\|_{L^p(\Omega, w)} \leq c \|\mathcal{L}u\|_{L^p(\Omega, w)} \quad \forall i, j = 1, \dots, n.$$

Proof. This theorem is an immediate consequence of the techniques in [14] and weighted estimates for singular integral operators and commutators. We give a sketch of the proof. Given $u \in W_0^{2,p}(\Omega)$, we have the following representation formula [14]:

$$(2.4) \quad \begin{aligned} \partial_{ij}u(x) &= \mathcal{K}_{i,j} \left(\sum_{h,k=1}^n (a^{hk}(x) - a^{hk}(\cdot)) \partial_{hk}u(\cdot) + \mathcal{L}u(\cdot) \right) \\ &+ \mathcal{L}u(x) \int_{|t|=1} \Gamma_i(x, t) t_j d\sigma(t). \end{aligned}$$

where

$$\mathcal{K}_{i,j}f(x) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B_\varepsilon(x)} \Gamma_{i,j}(x, x-y)f(y) dy$$

is a principal value operator and $\Gamma_i(x, t) = \frac{\partial}{\partial t_i} \Gamma(x, t)$, $\Gamma_{i,j}(x, t) = \frac{\partial^2}{\partial t_i \partial t_j} \Gamma(x, t)$. Here $\Gamma(x, t)$, $x \in \Omega$, is a fundamental solution of $\mathcal{L}_0 u(t) = \sum_{i,j=1}^n a^{i,j}(x) \partial_{ij} u(t)$ (see [14] for details). For each pair (i, j) , $1 \leq i, j \leq n$, $\mathcal{K}_{i,j}$ is a singular integral operator with a regular kernel and then $\mathcal{K}_{i,j}$ is bounded in $L^p(dw)$ with an operator norm which depends on n, λ, Λ, p and $\|w\|_{A_p}$ (cf. [12], Theorem IV3.1; see also [14], Theorem 2.11). From the weighted estimates for commutators in [24], we have that the commutators $\mathcal{C}_{i,j,h,k}$, $1 \leq i, j, h, k \leq n$, given by $\mathcal{C}_{i,j,h,k}f(x) = a^{h,k}(x)\mathcal{K}_{i,j}f(x) - \mathcal{K}_{i,j}(a^{i,j}f)(x)$ are bounded in $L^p(\mathbb{R}^n, dw)$ with norm $\|\mathcal{C}_{i,j,h,k}\| \leq \tilde{c} \|a^{i,j}\|_{\text{BMO}(\mathbb{R}^n, dw)}$. Moreover, whenever f is supported in Ω , we have the localized estimate (see [14], Theorem 2.13)

$$\|\mathcal{C}_{i,j,h,k}f\|_{L^p(\Omega, dw)} \leq c \|a^{i,j}\|_{\text{BMO}(\Omega)} \|f\|_{L^p(\Omega, dw)},$$

where we used that since $w \in A_p$, we have $\tilde{c} \|a^{i,j}\|_{\text{BMO}(\Omega, dw)} \leq c \|a^{i,j}\|_{\text{BMO}(\Omega)}$. Finally, it is not difficult to check that the factor multiplying $\mathcal{L}u(x)$ in (2.4) is uniformly bounded, with bound depending only on n, λ and Λ . From (2.4) and the mentioned estimates we have

$$\|\partial_{ij}u\|_{L^p(\Omega, dw)} \leq c \|\mathcal{L}u\|_{L^p(\Omega, dw)} + c \sum_{h,k=1}^n \|a^{h,k}\|_{\text{BMO}(\Omega)} \|\partial_{hk}u\|_{L^p(\Omega, dw)}.$$

The theorem follows taking $\tilde{\varrho}_p \leq (2n^2c)^{-1}$. □

The following theorem follows from the results in [1], the techniques just exposed and standard arguments (see Theorem 8.1 in [7]).

Theorem 2.5. *Let $w \in A_p$, $p \in [n, \infty)$ and $D \subset \mathbb{R}^n$ be a Lipschitz domain. There exist a positive $\varrho_p = \varrho_p(n, p, \lambda, \Lambda, \|w\|_{A_p})$ such that, if $\eta \in \Phi(\varrho_p)$ and $\mathcal{L} \in O(\lambda, \Lambda, \eta)$, then for any $f \in L^p(D, w)$, there exists a unique $u \in C(\overline{D}) \cap W_{\text{loc}}^{2,p}(D, w)$ such that $\mathcal{L}u = f$ in D and $u = 0$ on ∂D .*

Moreover, if ∂D is of class C^2 , then $u \in W_0^{1,p}(D, w) \cap W^{2,p}(D, w)$ and there exists a positive $c = c(n, p, \lambda, \Lambda, \|w\|_{A_p}, \chi, \partial D)$, with $\chi = \text{diam}(D)/\zeta(\eta, \varrho_p)$, such that

$$\|u\|_{W^{2,p}(D, w)} \leq c \|f\|_{L^p(D, w)}.$$

For each $x \in D$ and f, u as in Theorem 2.5, the maximum principle (Theorem 2.3) implies that the positive linear functional $f \mapsto -u(x)$ is bounded on $L^p(D)$. From Riesz' representation theorem, we have that there exists a unique nonnegative function $G_{\mathcal{L}, D}(x, \cdot) \in L^{p'}(D)$ with $p' = \frac{p}{p-1}$ such that

$$(2.5) \quad u(x) = - \int_D G_{\mathcal{L}, D}(x, y) f(y) dy.$$

Definition 2.6 (Green's function). The function $G_{\mathcal{L}, D}(x, y)$ is called the *Green's function* for \mathcal{L} in D . For simplicity we will often write $G(x, y) = G_{\mathcal{L}, D}(x, y)$.

Corollary 2.7. *For all $\varphi \in C_c^\infty(D)$ and $x \in D$ we have*

$$\varphi(x) = - \int_D \mathcal{L}\varphi(y) G(x, y) dy.$$

2.2. Properties of the Harmonic Measure.

Lemma 2.8 ([7], [15]). *Let $\Delta = \Delta_r(Q)$, $Q \in \partial D$:*

- (1) $\Delta' = \Delta_s(Q_0) \subset \Delta$, $x \in D \setminus T_{2r}(Q)$. Then $\omega^{x_r(Q)}(\Delta') \approx \frac{\omega^x(\Delta')}{\omega^x(\Delta)}$.
- (2) $\omega^{x_r(Q)}(\Delta) \approx 1$, $x_r(Q)$ verifies $\delta(x_r(Q)) \approx |x_r(Q) - Q|$.
- (3) (Doubling property) $\omega^x(\Delta) \approx \omega^x(\Delta_{2r}(Q))$, $x \in D \setminus T_{2r}(Q)$. In particular, the harmonic measure ω cannot have atoms.

2.3. Adjoint Solutions and Area Functions.

Definition 2.9 (Adjoint solution). Given $\mathcal{L} \in O(\lambda, \Lambda)$, a locally integrable function v is an adjoint solution of \mathcal{L} in a domain D and we write $\mathcal{L}^*v = 0$ in D if

$$\int_D v \mathcal{L}\varphi \, dx = 0$$

for all $\varphi \in C_c^\infty(D)$. More generally, if $f \in L^1_{loc}(D)$, we say that v is a solution to $\mathcal{L}^*v = f$ if

$$\int_D v \mathcal{L}\varphi \, dx = \int_D f \varphi \, dx$$

for all $\varphi \in C_c^\infty(D)$.

So v is an adjoint solution for \mathcal{L} in D if $\partial_{ij}(a^{ij}v) = 0$ in the sense of distributions. Suppose now that $\eta \in \Phi(\varrho_n)$, where ϱ_n is given by Theorem 2.5 for $p = n$ and $\Phi(\varrho_n)$ is as in Definition 2.1. If $\mathcal{L} \in O(\lambda, \Lambda, \eta)$ and G is the Green's function for \mathcal{L} in D (see Definition 2.6), then for each $x \in D$, $G(x, \cdot)$ is an adjoint solution of \mathcal{L} in $D \setminus \{x\}$. Indeed, from Corollary 2.7, we have

$$\int_D \mathcal{L}\varphi(y) G(x, y) \, dy = -\varphi(x) = 0 \quad \forall \varphi \in C_c^\infty(D \setminus \{x\}).$$

The following existence theorem for adjoint solutions is a consequence of the classical theory for smooth operators [20], Theorem 2.5 and the maximum principle. We omit the standard proof.

Theorem 2.10. *Let $\eta \in \Phi(\varrho_n)$, where ϱ_n is given by Theorem 2.5 for $p = n$ and $\Phi(\varrho_n)$ is as in Definition 2.1. If $\mathcal{L} \in O(\lambda, \Lambda, \eta)$ and G is the Green's function for \mathcal{L} in D , then for any $f \in L^{\frac{n}{n-1}}(D)$, there exists a solution $v \in L^{\frac{n}{n-1}}(D)$ to the problem*

$$(2.6) \quad \mathcal{L}^*v = f \quad \text{in } D.$$

Examples in [25] show that even if the coefficients of \mathcal{L} are continuous, adjoint solutions could not be in $L^\infty(D)$. Further, “weight type” regularity exists in the case of positive nonnegative solutions. In [9] it was shown that if w is a nonnegative adjoint solution for $\mathcal{L} \in O(\lambda, \Lambda, \eta)$, then $\log w$ lies in BMO. Next, Theorem 1.2 [13] is a more precise version of this result, more suitable to our applications.

Theorem 2.11. *Let $0 < \varrho \leq \varrho_n$ where ϱ_n is as in Theorem 2.5, $\eta \in \Phi(\varrho)$ (see Definition 2.1), $\mathcal{L} \in O(\lambda, \Lambda, \eta)$, and w a nonnegative adjoint solution to $\mathcal{L}^*w = \partial_{ij}(a^{i,j}w) = 0$ on B_{10} . Then there exists $\varrho_0 = \varrho_0(n, \lambda, \Lambda, \varrho) > 0$ such that $\log w$ is a function lying in BMO_{ϱ_0} . Moreover, $\varrho_0 \lesssim \varrho^\gamma$ for some $\gamma = \gamma(n, \lambda, \Lambda) > 0$.*

Recall that (the Lipschitz domain) $D \subset B_1$, where $B_r = B_r(0)$. We pick a point $\bar{x} \in \partial B_9$ and we let $\wp = \wp(\mathcal{L})$ be given by

$$(2.7) \quad \wp(y) = G_{\mathcal{L}, B_{10}}(\bar{x}, y) \quad \text{in } B_{10}$$

where $G_{\mathcal{L}, B_{10}}$ is the Green's function for \mathcal{L} in B_{10} (see Definition 2.6). From the previous theorem we have that there exists $\varrho^{**} > 0$ such that if $\mathcal{L} \in O(\lambda, \Lambda, \eta)$ with $\eta \in \Phi(\varrho^{**})$, then \wp is a weight in $A_{\frac{4}{3}}(B_8)$, and $\|[\wp]\|_{A_{\frac{4}{3}}}$ (see (2.2)) depends only on n, λ , and Λ . We set

$$(2.8) \quad \varrho^* = \frac{1}{2} \min\{\varrho^{**}, \varrho_n\},$$

where ϱ_n is given by Theorem 2.5 for $w = \wp$ and $p = n$. Since $\|[\wp]\|_{A_n} \leq \|[\wp]\|_{A_2} \leq \|[\wp]\|_{A_{\frac{4}{3}}}$, the constant ϱ^* depends only on n, λ and Λ . Note also that $\wp dx$ is a doubling measure in B_8 with doubling constant which depends only on n, λ and Λ .

Definition 2.12 ([7], n.a.s.). Let $\mathcal{L} \in O(\lambda, \Lambda)$. A *normalized adjoint solution* for \mathcal{L}^* in D is any function w of the form

$$w(x) = \frac{v(x)}{\wp(x)}$$

where v is a solution of the adjoint equation $\mathcal{L}^*v = 0$ in D and \wp is given by (2.7).

Normalized adjoint solutions, first introduced in [7], enjoy many desirable properties that adjoint solutions fail to verify. Following the techniques in [7], the Dirichlet problem for n.a.s. is uniquely solvable for continuous boundary data and coefficients in $O(\lambda, \Lambda, \eta)$, with $\eta \in \Phi(\varrho_n)$. A Harnack principle holds for nonnegative n.a.s., as well as a boundary Harnack inequality and a comparison principle (see [7], [8] and [15]). Although the definition of n.a.s. depends on the particular choice of the *normalizing function* \wp , this choice has no qualitative impact in our applications.

Lemma 2.13. *Let ϱ^* be given by (2.8). Then if $\eta \in \Phi(\varrho^*)$, $\mathcal{L} \in O(\lambda, \Lambda, \eta)$ and u satisfy $\mathcal{L}u = 0$ in D , and $v \in L^1_{\text{loc}}(D)$ is a nonnegative adjoint solution for \mathcal{L} in $B_{3r}(x_0) \subset D$, then for $0 < 2r \leq \zeta(\eta)$ and any constants β and γ , the following holds:*

$$\begin{aligned} \int_{B_r(x_0)} |\nabla^2 u(x)|^2 v(x) dx &\lesssim r^{-4} \int_{B_{2r}(x_0)} |u(x) - \beta - \gamma x|^2 v(x) dx \\ &\quad + r^{-2} \int_{B_{2r}(x_0)} |\nabla(u(x) - \gamma x)|^2 v(x) dx. \end{aligned}$$

Proof. We choose a nonnegative $\phi \in C_c^\infty(B_{2r}(x_0))$ such that $\phi \equiv 1$ in $B_r(x_0)$ and $|\partial_j \phi| \leq M r^{-j}$, $j = 0, 1, 2$, with $M > 0$ a universal constant. Applying Theorem 2.4 to the function $(u - \beta - \gamma x)\phi$, we have

$$(2.9) \quad \int_{B_r(x_0)} |\nabla^2 u(x)|^2 \wp(x) dx \lesssim \int |\mathcal{L}((u - \beta - \gamma x)\phi(x))|^2 \wp(x) dx.$$

Developing the derivatives, rearranging terms, applying Hölder inequality and since u is a solution for \mathcal{L} , we have that the right-hand side of (2.9) is bounded by

$$C \int (|u(x) - \beta - \gamma x|^2 |\nabla^2 \phi(x)|^2 + |\nabla(u(x) - \gamma x) \cdot \nabla \phi(x)|^2) \wp(x) dx,$$

which proves the lemma in the case $v = \varphi$. Now let $w = v/\varphi$, with v as in the statement of Lemma 2.13. Then w is a *normalized adjoint solution* of the operator \mathcal{L} in D . The lemma follows from the Harnack inequality for n.a.s. (cf. [7], [15]). \square

Lemma 2.14 ([10], Lemma 2). *Let $G(x, y)$ be the Green's function in D for $\mathcal{L} \in O(\lambda, \Lambda)$. Then there is a constant r_0 depending on the Lipschitz character of D , such that for all $Q \in \partial D$, $r \leq r_0$, $y \in \partial B_r(Q) \cap \Gamma_1(Q)$, and $x \notin T_{4r}(Q)$, the following holds:*

$$\frac{G(x, y) \varphi(B(y))}{\delta(y)^2 \varphi(y)} \sim \omega^x(\Delta_r(Q)).$$

The following lemma establishes that the regularity of the Dirichlet problem depends on the coefficients locally at the boundary.

Lemma 2.15. *Let $\mathcal{L}_0, \mathcal{L}_1 \in O(\lambda, \Lambda, \eta)$ be such that if A_0 and A_1 denote their respective matrices of coefficients, we have that for some $s_0 > 0$,*

$$A_1(x) = A_2(x) \quad \text{for all } x \in D \text{ such that } \delta(x) \leq s_0.$$

Then there exists $C > 0$, depending only on n , ellipticity, D and s_0 such that

$$C^{-1} \omega_0(\Delta_r(Q)) \leq \omega_1(\Delta_r(Q)) \leq C \omega_0(\Delta_r(Q)), \quad \forall Q \in \partial D, r > 0,$$

where ω_0 and ω_1 denote the harmonic measures for \mathcal{L}_0 and \mathcal{L}_1 , respectively. In particular, $\omega_1 \in \bigcap_{p>1} A_p(d\omega_0) \cap \bigcap_{q>1} B_q(d\omega_0)$.

Proof. Let $Q \in \partial D$, $r = \frac{s_0}{8}$, $\Omega_s = \{x \in D : \delta(x) < s\}$, and G_0, G_1 denote the Green's functions in D for \mathcal{L}_0 and \mathcal{L}_1 , respectively. Then for any $x \in D$ we have that the functions $G_0(x, \cdot)$ and $G_1(x, \cdot)$ are adjoint solutions for \mathcal{L}_0 in $\Omega_{s_0} \setminus \{x\}$. From the comparison principle for normalized adjoint solutions [15], we have that

$$\frac{G_0(x, y)}{G_1(x, y)} \approx \frac{G_0(x, y_r(Q))}{G_1(x, y_r(Q))}, \quad y \in T_r(Q), x \in D \setminus \Omega_{4r}.$$

From Lemma 2.14, [10] we have

$$\frac{\omega_0^x(\Delta_y)}{\omega_1^x(\Delta_y)} \approx \frac{\omega_0^x(\Delta_r(Q))}{\omega_1^x(\Delta_r(Q))},$$

where $\Delta_y = \Delta_s(P)$ with $s \approx \delta(y) \approx |y - P|$, $P \in \partial D$. From Lemma 2.8(2) and the interior Harnack inequality, we have

$$\omega_0^x(\Delta_y) \approx \omega_1^x(\Delta_y), \quad y \in T_r(Q), x \in T_{5r}(Q) \setminus \Omega_{4r}.$$

From the interior Harnack inequality for solutions, [26], the harmonic measures ω_i^x , $i = 0, 1$, where $x \in \Omega_{5r} \setminus \Omega_{4r}$, are comparable to the respective harmonic measures at the center of D , ω_0 and ω_1 (with constants depending on D and s_0); thus, we obtain for some $C > 0$ as wanted

$$C^{-1} \omega_0(\Delta_y) \leq \omega_1(\Delta_y) \leq C \omega_0(\Delta_y), \quad \forall y \in \Omega_{\frac{s_0}{8}}.$$

The general case follows from Lemma 2.8(3). \square

Definition 2.16 (Area functions). For a function u defined on D , the area function of aperture α , $S_\alpha u$ and the second area function of aperture α , $A_\alpha u$, are defined respectively as

$$S_\alpha u(Q)^2 = \int_{\Gamma_\alpha(Q)} \frac{\delta(x)^2}{\varphi(B(x))} |\nabla u(x)|^2 \varphi(x) dx$$

and

$$A_\alpha u(Q)^2 = \int_{\Gamma_\alpha(Q)} \frac{\delta(x)^4}{\wp(B(x))} |\nabla^2 u(x)|^2 \wp(x) dx,$$

where \wp is as in (2.7), $B(x) = B_{\delta(x)/2}(x)$, and $Q \in \partial D$.

Theorem 2.17 ([10]). *Let $\mathcal{L} \in O(\lambda, \Lambda)$, $u \in C(\overline{D})$ be a solution to $\mathcal{L}u = 0$ in D , $u(Q) = g(Q)$ on ∂D , where $g \in C(\partial D)$. If ν is a positive Borel measure on ∂D , which is in $A_\infty(\omega)$, where ω is the harmonic measure for \mathcal{L} in D evaluated at 0, then given $0 < p < \infty$, $\alpha > 0$, $\beta > 0$, there exists a constant C which depends on n , ellipticity, p , α , β , the A_∞ constant of ν and the Lipschitz character of D such that*

$$\|S_\alpha u\|_{L^p(\partial D, d\nu)} \leq C \|N_\beta u\|_{L^p(\partial D, d\nu)}.$$

Moreover, if $u(0) = 0$, then

$$\|N_\alpha u\|_{L^p(\partial D, d\nu)} \leq C \|S_\beta u\|_{L^p(\partial D, d\nu)}.$$

Lemma 2.18. *Let $\mathcal{L} \in O(\lambda, \Lambda, \eta)$, u a solution to $\mathcal{L}u = 0$ in D and w an A_2 weight. Then if $B_{2r}(x) \subset D$ and $2r \leq \zeta(\eta)$, we have*

$$\int_{B_r(x)} |\nabla^2 u(y)|^2 w(y) dy \leq \frac{C}{r^2} \int_{B_{2r}(x)} |\nabla u(y)|^2 w(y) dy,$$

where C depends only on n , D , ellipticity and $\|w\|_{A_2}$.

Proof. Choose a cut-off function $\phi \in C_c^\infty(B_{2r}(x))$, with $\phi = 1$ on $B_r(x)$,

$$\|\partial_i \phi\|_{L^\infty(B_{2r}(x))} \leq M r^{-i}, \quad i = 0, 1, 2,$$

and M a universal constant. Applying Lemma 2.13 to $u\phi$, we obtain

$$\int_{B_r(x)} |\nabla^2 u|^2 dw \lesssim \frac{1}{r^2} \int_{B_{2r}(x)} |\nabla u|^2 dw + \frac{1}{r^4} \int_{B_{2r}(x)} |u - u_{B_{2r}(x),w}|^2 dw,$$

where $u_{B,w} = \frac{1}{w(B)} \int_B u dw$. By the weighted Poincaré type inequality (Theorem 1.5 in [16]) we have

$$\int_{B_{2r}(x)} |u - u_{B_{2r}(x),w}|^2 w(y) dy \leq C r^2 \int_{B_{2r}(x)} |\nabla u|^2 w(y) dy,$$

where C has the required dependence. This finishes the proof. □

We will fix from now on the length of the truncation r^* for the cones $\Gamma_\alpha(Q) = \{x \in D : |x - Q| \leq (1 + \alpha)\delta(x)\} \cap B_{r^*}(Q)$ defined in (1.4). We set

$$(2.10) \quad r^* = \min\{r_0, (2\sqrt{n})^{-1} \zeta(\eta)\},$$

where $r_0 = r_0(D, \lambda, \Lambda)$ is as in Lemma 2.14, η is the common modulus of continuity for \mathcal{L}_0 and \mathcal{L}_1 in Theorem 1.1 and $\zeta(\eta)$ is given by Definition 2.1.

Theorem 2.19. *Let ϱ^* be given by (2.8), $\eta \in \Phi(\varrho^*)$, $\mathcal{L} \in O(\lambda, \Lambda, \eta)$ and u a solution to $\mathcal{L}u = 0$ in D . For any $\alpha \geq \alpha^*(D) > 0$ we have the point-wise inequality on ∂D ,*

$$A_\alpha u(Q) \leq C S_{c\alpha} u(Q),$$

where $c > 1$ depends only on dimension n and C depends on n and the ellipticity constants.

Proof. Let $\{Q_j\}_{j=1}^\infty$ be a Whitney decomposition of D into cubes, that is, $D = \bigcup_{j=1}^\infty Q_j$, Q_j and Q_k have disjoint interiors for $j \neq k$, and $r_j = \text{side-length}(Q_j) \approx \text{dist}(Q_j, \partial D)$. Denote by x_j the center of Q_j . We assume that $B_{2\sqrt{n}r_j}(x_j) \subset D$, and denote \tilde{Q}_j the cube with center x_j and side-length $\sqrt{n}r_j$. Note that there exist constants N and $c > 1$ depending only on the dimension n such that

$$(2.11) \quad \sum_{\substack{j=1 \\ Q_j \cap \Gamma_\alpha(Q) \neq \emptyset}}^\infty \chi_{\tilde{Q}_j} \leq N \chi_{\Gamma_{c\alpha}(Q)},$$

where $\chi_E(x)$ denotes the characteristic function of the set E . We have

$$\begin{aligned} A_\alpha^2 u(Q) &= \sum_{Q_j \cap \Gamma_\alpha(Q) \neq \emptyset} \int_{Q_j \cap \Gamma_\alpha(Q)} \frac{\delta(x)^4}{\varphi(B(x))} |\nabla^2 u(x)|^2 \varphi(x) dx \\ &\leq C \sum_{Q_j \cap \Gamma_\alpha(Q) \neq \emptyset} \frac{r_j^4}{\varphi(B(x_j))} \int_{Q_j} |\nabla^2 u(x)|^2 \varphi(x) dx. \end{aligned}$$

Since $\eta \in \Phi(\varrho^*)$, φ is an A_2 weight with A_2 constant depending only on n , λ and Λ . Note that by assumption (2.10) we have $2\sqrt{n}r_j \leq \zeta(\eta)$. We apply Lemma 2.18 and (2.11) to obtain

$$\begin{aligned} A_\alpha^2 u(Q) &\leq C \sum_{Q_j \cap \Gamma_\alpha(Q) \neq \emptyset} \frac{r_j^2}{\varphi(B(x_j))} \int_{\tilde{Q}_j} |\nabla u(x)|^2 \varphi(x) dx \\ &\leq C \sum_{Q_j \cap \Gamma_\alpha(Q) \neq \emptyset} \int_{\tilde{Q}_j} \frac{\delta(x)^2}{\varphi(B(x))} |\nabla u(x)|^2 \varphi(x) dx \\ &\leq NC \int_{\Gamma_{c\alpha}(Q)} \frac{\delta(x)^2}{\varphi(B(x))} |\nabla u(x)|^2 \varphi(x) dx \\ &= NC S_{c\alpha}^2 u(Q). \end{aligned}$$

□

We will also find useful an averaged version of the nontangential maximal function.

Definition 2.20. For a function u in D and $\alpha > 0$, define the *modified nontangential maximal function* $N_\alpha^0(u)$ by

$$N_\alpha^0 u(Q) = \sup_{\Gamma_\alpha(Q)} \left\{ \int_{B_0(x)} u^2(y) \frac{\varphi(y)}{\varphi(B(y))} dy \right\}^{\frac{1}{2}},$$

where $B_0(x) = B_{\frac{\delta(x)}{6}}(x)$.

From the doubling property of the weight φ , we have $N_\alpha^0 u \lesssim N_\alpha u$. When u is a solution of any elliptic operator in $O(\lambda, \Lambda, \eta)$, we also have:

Lemma 2.21. *Let ϱ^* be given by (2.8), $\eta \in \Phi(\varrho^*)$, $\mathcal{L}_1 \in O(\lambda, \Lambda, \eta)$, and u a solution of $\mathcal{L}_1 u = 0$ in D . Then*

$$N_\alpha u(Q) \lesssim N_\alpha^0 u(Q) \quad \forall Q \in \partial D.$$

Proof. Let $Q \in \partial D$ and $x \in \Gamma_\alpha(Q)$. From a known reverse Hölder inequality for solutions ([27], see also [20] Theorem 8.17) we have $|u(x)| \lesssim \|u\|_{L^{3/2}(B_0(x))} |B_0(x)|^{-3/2}$. Then from the Hölder inequality we get

$$|u(x)| \lesssim \frac{1}{|B_0(x)|^{\frac{2}{3}}} \left(\int_{B_0(x)} |u|^2 \frac{\varphi(y)}{\varphi(B(y))} dy \right)^{\frac{1}{2}} \left(\int_{B_0(x)} \frac{\varphi(B(y))^3}{\varphi(y)^3} dy \right)^{\frac{1}{6}}.$$

Since φ is an $A_{\frac{4}{3}}$ -weight, we have $\|1/\varphi\|_{L^3(B_0(x))} \lesssim |B_0(x)|^{\frac{4}{3}} \varphi(B_0(x))^{-1}$, and from the doubling property of φ , we have $\varphi(B(y)) \lesssim \varphi(B(x))$ for all $y \in B_0(x)$; thus,

$$|u(x)| \lesssim N_\alpha^0 u(Q) \frac{\varphi(B(x))^{\frac{1}{2}}}{|B_0(x)|^{\frac{1}{2}}} \frac{|B_0(x)|^{\frac{1}{2}}}{\varphi(B_0(x))^{\frac{1}{2}}} \lesssim N_\alpha^0 u(Q).$$

The lemma follows by taking the supremum for $x \in \Gamma_\alpha(Q)$ on the above inequality. □

3. THE MAIN LOCAL ESTIMATE

We present here a version of Theorem 1.1 in which surface measure $d\sigma$ is replaced by the harmonic measure $d\omega_0$ and (1.6) is modified accordingly. A similar result was proved in [2] for divergence form operators.

Theorem 3.1. *Let $G_0(x, y)$ be the Green’s function for \mathcal{L}_0 in D and set $G_0(y) = G_0(0, y)$. There exists $\varepsilon_0 > 0$ which depends only on n, λ, Λ and D such that if $a_k^{i,j} \in \text{BMO}_{\varrho^*}$, $1 \leq i, j \leq n, k = 0, 1$, and*

$$(3.1) \quad \sup_{r>0, Q \in \partial D} \left\{ \frac{1}{\omega_0(\Delta_r(Q))} \int_{T_r(Q)} G_0(y) \frac{\alpha^2(y)}{\delta^2(y)} dy \right\}^{\frac{1}{2}} \leq \varepsilon_0,$$

then $\omega_1 \in B_2(d\omega_0)$.

We follow the ideas in [2] (see also [6], Theorem 2.7.1). Our proof is specialized to the case in which the domain D is the unit ball $B = B_1(0)$. Let φ be as in (2.7), $g \in C(\partial B)$, and let u_1 be the solution of

$$\begin{cases} \mathcal{L}_1 u_1 = 0 & \text{in } B, \\ u_1 = g & \text{on } \partial B. \end{cases}$$

To prove Theorem 3.1 it is enough to show that for ε_0 small enough, there exists $C > 0$, which depends on $\alpha > 0$ and the same parameters as ε_0 , such that

$$(3.2) \quad \|N_\alpha u_1\|_{L^2(\partial B, d\omega_0)} \leq C \|g\|_{L^2(\partial B, d\omega_0)}.$$

Indeed, (3.2) is condition (ii) on Theorem 2.2 when we take $\mu = \omega_0$ and $\omega = \omega_1$. Theorem 3.1 then follows from Theorem 2.2.

Now let u_0 solve

$$\begin{cases} \mathcal{L}_0 u_0 = 0 & \text{in } B, \\ u_0 = g & \text{on } \partial B. \end{cases}$$

Then from Corollary 2.7 we have

$$u_1(x) = u_0(x) - \int_B G_0(x, y) \mathcal{L}_0 u_1(y) dy = u_0(x) - F(x).$$

Lemma 3.2. *Under the hypothesis of Theorem 3.1 we have*

$$N_\alpha^0 F(Q) \lesssim \varepsilon M_{\omega_0}(A_\alpha u_1)(Q),$$

where N_α^0 denotes the modified nontangential maximal operator (Definition 2.20).

Lemma 3.3. *Under the hypothesis of Theorem 3.1 we have*

$$\int_{\partial B} S_{c\alpha}^2 u_1 d\omega_0 \lesssim \int_{\partial B} N_\alpha u_1^2 d\omega_0.$$

Let us take Lemmas 3.2 and 3.3 for granted, and let us show how we can then conclude the proof of Theorem 3.1. From Theorem 2.2 we have

$$(3.3) \quad \|N_\alpha u_0\|_{L^p(\partial D, d\omega_0)} \leq C \|g\|_{L^p(\partial D, d\omega_0)}.$$

Now, from (3.3), Lemmas 2.21, 3.2 and 3.3:

$$\begin{aligned} \int_{\partial B} (N_\alpha u_1)^2 d\omega_0 &\lesssim \int_{\partial B} (N_\alpha^0 u_1)^2 d\omega_0 \\ &\lesssim \int_{\partial B} [(N_\alpha^0 u_0)^2 + (N_\alpha^0 F)^2] d\omega_0 \\ &\lesssim \int_{\partial B} N_\alpha u_0^2 d\omega_0 + \varepsilon^2 \int_{\partial B} M_{\omega_0}(A_\alpha u_1)^2 d\omega_0 \\ &\lesssim \int_{\partial B} g^2 d\omega_0 + \varepsilon^2 \int_{\partial B} (A_\alpha u_1)^2 d\omega_0 \\ &\lesssim \int_{\partial B} g^2 d\omega_0 + \varepsilon^2 \int_{\partial B} (S_{c\alpha} u_1)^2 d\omega_0 \\ &\lesssim \int_{\partial B} g^2 d\omega_0 + \varepsilon^2 \int_{\partial B} (N_\alpha u_1)^2 d\omega_0, \end{aligned}$$

which proves Theorem 3.1 given that ε is small enough.

3.1. Proof of Lemma 3.2. We assume, without loss of generality, that

$$(3.4) \quad \varepsilon(x) = 0 \quad \text{for } \delta(x) \geq \min\{1/2, r_0\}.$$

We note that from Lemma 2.8(1) and Lemma 2.14 we have that if $\text{dist}(y, \partial B) = \delta(y) \leq 1/4$, then

$$\frac{1}{\omega_0(\Delta_y)} \frac{G_0(y)}{\delta(y)^2} \approx \frac{\varphi(y)}{\varphi(B(y))}.$$

Then, (3.1) gives

$$(3.5) \quad \left\{ \int_{B(x)} a^2(y) \frac{\varphi(y)}{\varphi(B(y))} dy \right\}^{1/2} \lesssim \varepsilon_0.$$

Let $Q_0 \in \partial B$ and $x_0 \in \Gamma_\alpha(Q_0)$. Denote by $\delta_0 = \delta(x_0)$, $B_0 = B_{\frac{\delta_0}{6}}(x_0)$, $2B_0 = B_{\frac{\delta_0}{3}}(x_0)$ and $B(x) = B_{\frac{\delta(x)}{2}}(x)$. Also, let $\tilde{G}(x, y)$ be the Green's function for \mathcal{L}_0 on

$B(x_0) = B_{\frac{\delta_0}{2}}$ and set

$$\begin{aligned} F_1(x) &= \int_{2B_0} \tilde{G}(x, y) \mathcal{L}_0 u_1(y) dy, \\ F_2(x) &= \int_{2B_0} [G_0(x, y) - \tilde{G}(x, y)] \mathcal{L}_0 u_1(y) dy, \\ F_3(x) &= \int_{B \setminus 2B_0} G_0(x, y) \mathcal{L}_0 u_1(y) dy, \end{aligned}$$

so that $F(x) = F_1(x) + F_2(x) + F_3(x)$, $x \in B(x_0)$. Remember $\varepsilon(x) = A_1(x) - A_0(x)$ and $\mathbf{a}(x) = \sup_{B(x)} |\varepsilon(x)|$. If $x \in B_0$, we have

$$\begin{aligned} |\varepsilon(x)| &\leq \frac{C(n)}{|B(x)|} \int_{B(x)} \mathbf{a}(y) dy \\ (3.6) \quad &\leq \frac{C(n)}{|B(x)|} \left\{ \int_{B(x)} \mathbf{a}^2(y) \frac{\varphi(y)}{\varphi(B(y))} dy \right\}^{1/2} \left\{ \int_{B(x)} \frac{\varphi(B(y))}{\varphi(y)} dy \right\}^{1/2} \\ &\leq C\varepsilon_0 \left\{ \frac{\varphi(B(x))}{|B(x)|} \frac{1}{|B(x)|} \int_{B(x)} \varphi^{-1}(y) dy \right\}^{1/2} \leq C\varepsilon_0, \end{aligned}$$

where we used (3.5), the doubling property of φ and the fact that $\varphi \in A_2$. Note that F_1 verifies $\mathcal{L}_0 F_1 = \chi(2B_0) \mathcal{L}_0 u_1$ in $B(x_0)$, $F_1 = 0$ on $\partial(B(x_0))$, with $\chi(2B_0)$ the characteristic function of $2B_0$. From the weighted Sobolev inequality (Theorem 1.2 in [16]), Theorem 2.5 and (3.6) we have

$$\begin{aligned} (3.7) \quad \left\{ \int_{2B_0} F_1^2(x) \frac{\varphi(x)}{\varphi(B(x))} dx \right\}^{\frac{1}{2}} &\lesssim \delta_0 \left\{ \int_{B(x_0)} |\nabla F_1(x)|^2 \frac{\varphi(x)}{\varphi(B(x))} dx \right\}^{\frac{1}{2}} \\ &\lesssim \left\{ \int_{B(x_0)} |\mathcal{L}_0 F_1(x)|^2 \delta^4(x) \frac{\varphi(x)}{\varphi(B(x))} dx \right\}^{\frac{1}{2}} \\ &\lesssim \left\{ \int_{2B_0} \varepsilon(x)^2 |\nabla^2 u_1(x)|^2 \delta^4(x) \frac{\varphi(x)}{\varphi(B(x))} dx \right\}^{\frac{1}{2}} \\ &\lesssim \varepsilon_0 A_\alpha u_1(Q_0). \end{aligned}$$

Let $v_x(y) = G_0(x, y) - \tilde{G}(x, y)$, $x, y \in 2B(x_0)$. Then if \mathcal{L}_0^* denotes the adjoint operator to \mathcal{L}_0 , we have $\mathcal{L}_0^* v_x = 0$ in $B(x_0)$, and $v_x \geq 0$ in $B(x_0)$. In particular, $\tilde{v}(y) = v_x(y)/\varphi(y)$ is a nonnegative *normalized adjoint solution* (n.a.s.) of \mathcal{L}_0 in $B(x_0)$ (see Definition 2.12). From the maximum principle for n.a.s., the Harnack inequality for n.a.s., and Lemma 2.14 we have

$$(3.8) \quad \tilde{v} \leq \max_{y \in 2B_0} \tilde{v} \leq C \frac{G_0(x, \bar{y})}{\varphi(\bar{y})} \approx \frac{\omega^x(\Delta_{\bar{y}}) \delta^2(\bar{y})}{\varphi(B(\bar{y}))},$$

where $\bar{y} \in \partial(2B_0)$ verifies $\delta(\bar{y}) = \text{dist}(\partial(2B_0), \partial B)$. From Lemma 2.8(2) and the Harnack inequality it follows that $\omega^x(\Delta_{\bar{y}}) \approx 1$. Then, from Lemma 2.14, (3.8) and

(3.6) we get for $x \in B_0$,

$$\begin{aligned}
 (3.9) \quad |F_2(x)| &\lesssim \varepsilon_0 \int_{2B_0} \delta(\bar{y})^2 \omega^x(\Delta_{\bar{y}}) |\nabla^2 u_1(y)| \frac{\wp(y)}{\wp(B(\bar{y}))} dy \\
 &\lesssim \varepsilon_0 \left\{ \int_{2B_0} \delta(y)^4 |\nabla^2 u_1(y)|^2 \frac{\wp(y)}{\wp(B(y))} dy \right\}^{\frac{1}{2}} \left\{ \int_{B_0} \frac{\wp(y)}{\wp(B(y))} dy \right\}^{\frac{1}{2}} \\
 &\lesssim \varepsilon_0 A_\alpha u_1(Q_0).
 \end{aligned}$$

Now define

$$\Omega_0 = \left(B \cap B_{\frac{\delta_0}{2}}(Q_0) \right), \quad \Omega_j = \left(B \cap B_{2^{j-1}\delta_0}(Q_0) \right) \setminus \left(2B_0 \cup B_{2^{j-2}\delta_0}(Q_0) \right),$$

$j = 1, 2, \dots, j_0 - 1$, and

$$\Omega_{j_0} = B \setminus B_{2^{j_0-2}\delta_0}(Q_0),$$

where $\frac{1}{2} < \delta_0 2^{j_0-1} \leq 1$. We set

$$F_3^j(x) = \int_{\Omega_j} G_0(x, y) \mathcal{L}_0 u_1(y) dy, \quad j = 0, 1, \dots, j_0.$$

Thus,

$$(3.10) \quad F_3(x) = \sum_{j=0}^{j_0} F_3^j(x).$$

We now need to define the notion of a dyadic grid:

Definition 3.4. A *dyadic grid* on ∂B is a collection of Borel sets $\mathcal{I} = \bigcup_{k=1}^\infty \mathcal{I}_k$, such that

- (i) $\partial B = \bigcup_{I \in \mathcal{I}_k} I, \quad k = 1, 2, \dots,$
- (ii) $I, J \in \mathcal{I}_k, I \neq J \Rightarrow I$ and J have disjoint interiors,
- (iii) $I \in \mathcal{I}_k \Rightarrow \text{diam}(I) \approx 2^{-k},$
- (iv) for every $I \in \mathcal{I}_k, \exists! I' \in \mathcal{I}_{k-1} : I \subset I', k = 2, 3, \dots,$
- (v) for every $I \in \mathcal{I}$ there exists a boundary ball Δ_I such that

$$\Delta_I \subset I \subset c \Delta_I,$$

where $c > 1$ is a universal constant.

If $I \in \mathcal{I}$, we say that I is dyadic.

We consider now a Whitney decomposition of B into cubes, $B = \bigcup_{Q \in \mathcal{Q}} Q$. As in the proof of Theorem 2.19, it is easy to see that there exists a dyadic grid on ∂B such that for every $I \in \mathcal{I}$, we can assign a cube $I^+ \in \mathcal{Q}$ so that

$$(3.11) \quad \bigcup_{k \geq 1} \bigcup_{I \in \mathcal{I}_k} I^+ = B \setminus B_{\frac{1}{2}}(0).$$

Moreover, the correspondence $I \mapsto I^+$ can be defined so that for any boundary ball $\Delta_r(Q)$ we have $T_r(Q) \subset \bigcup_{I \in \mathcal{I}, I \subset \Delta_r(Q)} I^+$. Remember that by (3.4), $\varepsilon \equiv 0$ in $B_{\frac{1}{2}}(0)$. Let $\Delta_0 = \Delta_{\frac{\delta_0}{2}}(Q_0) = B_{\frac{\delta_0}{2}} \cap \partial B$, and suppose $I \subset \Delta_0$ is dyadic. For $x \in B_0$ and $y \in I^+$ we have from Lemma 2.8(1) and Lemma 2.14,

$$(3.12) \quad G_0(x, y) \approx \delta^2(y) \frac{\omega_0(\Delta_y)}{\omega_0(\Delta_0)} \frac{\wp(y)}{\wp(B(y))} \approx \frac{G_0(y)}{\omega_0(\Delta_0)}.$$

To estimate $F_3^0(x)$ we will use a “stopping time” argument. For $j = 0, \pm 1, \pm 2, \dots$, let

$$\begin{aligned} \mathcal{O}_j &= \{Q \in \Delta_0 : A_\alpha u_1(Q) > 2^j\}, \\ \tilde{\mathcal{O}}_j &= \{Q \in \Delta_0 : M_{\omega_0}(\chi(\mathcal{O}_j))(Q) > c^*\}, \\ \mathcal{J}_j &= \{I \text{ dyadic}, I \subset \Delta_0 : \omega_0(I \cap \mathcal{O}_j) \geq \frac{1}{2} \omega_0(I) \text{ but } \omega_0(I \cap \mathcal{O}_{j+1}) < \frac{1}{2} \omega_0(I)\}. \end{aligned}$$

If $I \in \mathcal{J}_j$, by Definition (3.4)(v) and the doubling property of ω_0 , it easily follows that for any $Q \in I$,

$$M_{\omega_0}(\chi(\mathcal{O}_j))(Q) > c^*$$

for some $c^* > 0$ depending only on dimension and ellipticity. Therefore, taking this choice of c^* in the definition of $\tilde{\mathcal{O}}_j$, we have $I \subset \tilde{\mathcal{O}}_j$, whenever $I \in \mathcal{J}_j$. Now, from (3.12) we get

$$\begin{aligned} |F_3^0(x)| &= \left| \sum_{\substack{I \subset \Delta_0 \\ I \text{ dyadic}}} \int_{I^+ \cap \Omega_0} G_0(x, y) \mathcal{L}_0 u_1(y) dy \right| \\ &\lesssim \frac{1}{\omega_0(\Delta_0)} \sum_j \sum_{I \in \mathcal{J}_j} \omega_0(I) \int_{I^+ \cap \Omega_0} \delta^2(y) \varepsilon(y) |\nabla^2 u_1(y)| \frac{\wp(y)}{\wp(B(y))} dy. \end{aligned}$$

On the other hand, if $I \subset \Delta_0$ is dyadic, then $I \in \mathcal{J}_j$ for some j . Now we set

$$\overline{\mathcal{J}}_j = \{J \text{ dyadic: } J \in \mathcal{J}_j, \text{ and } J \subset J', J' \text{ dyadic, } J' \in \mathcal{J}_j, \Rightarrow J' = J\},$$

that is, $\overline{\mathcal{J}}_j$ is the collection of “maximal” dyadic sets in \mathcal{J}_j . It is clear that $\mathcal{J}_j = \bigcup_{J \in \overline{\mathcal{J}}_j} J$, with a disjoint union. Let $T(J) = \bigcup_{\substack{I \subset J \\ I \text{ dyadic}}} I^+$, where J is dyadic. Then

if $J \in \overline{\mathcal{J}}_j$, from (3.1) and (3.12), we have

$$\begin{aligned} (3.13) \quad &\left\{ \sum_{\substack{I \text{ dyadic} \\ I \subset J}} \omega_0(I) \int_{I^+ \cap \Omega_0} a^2(y) \frac{\wp(y)}{\wp(B(y))} dy \right\}^{\frac{1}{2}} \\ &\lesssim \left\{ \int_{T(J) \cap \Omega_0} G_0 \frac{a^2}{\delta^2} dy \right\}^{\frac{1}{2}} \lesssim \varepsilon_0 (\omega_0(J))^{1/2}. \end{aligned}$$

By the Hölder inequality and (3.13),

$$\begin{aligned} |F_3^0(x)| &\lesssim \frac{\varepsilon_0}{\omega_0(\Delta_0)} \sum_j \sum_{J \in \overline{\mathcal{J}}_j} (\omega_0(J))^{1/2} \\ &\quad \left\{ \sum_{J \supset I \in \mathcal{J}_j} \omega_0(I) \int_{I^+ \cap \Omega_0} \delta^4(y) |\nabla^2 u_1(y)|^2 \frac{\wp(y)}{\wp(B(y))} dy \right\}^{1/2}. \end{aligned}$$

Now we make the following observation: There exists $\alpha = \alpha(n)$ such that if $I \subset \partial B$ is dyadic, and $E \subset I$ with $2\omega_0(E) \geq \omega_0(I)$, then

$$\int_{I^+} f(y) \omega_0(\Delta_y) dy \lesssim \int_E \int_{\Gamma_\alpha(Q)} f(y) dy d\omega_0(Q).$$

This is a consequence of Fubini's theorem and the fact that for appropriate $\alpha = \alpha(n)$ we have $I^+ \subset \Gamma_\alpha(Q)$ for all $Q \in I$. From the weak type inequality of the maximal operator M_{ω_0} we have $\omega_0(\tilde{\mathcal{O}}_j \setminus \mathcal{O}_{j+1}) \lesssim \omega_0(\mathcal{O}_j)$. Then, since $\omega_0(I \setminus \mathcal{O}_{j+1}) \geq \frac{1}{2} \omega_0(I)$ and $I \subset \tilde{\mathcal{O}}_j$ for all $I \in \mathcal{J}_j$, we have

$$\begin{aligned}
 |F_3^0(x)| &\lesssim \sum_j \frac{\varepsilon_0 \omega_0(\mathcal{O}_j)^{\frac{1}{2}}}{\omega_0(\Delta_0)} \left\{ \int_{\tilde{\mathcal{O}}_j \setminus \mathcal{O}_{j+1}} \int_{\Gamma_\alpha(Q)} |\nabla^2 u_1(y)|^2 \frac{\delta^4(y) \wp(y)}{\wp(B(y))} dy d\omega_0 \right\}^{\frac{1}{2}} \\
 &\lesssim \frac{\varepsilon_0}{\omega_0(\Delta_0)} \sum_j \omega_0(\mathcal{O}_j)^{\frac{1}{2}} \left\{ \int_{\tilde{\mathcal{O}}_j \setminus \mathcal{O}_{j+1}} A_\alpha^2 u_1(Q) d\omega_0(Q) \right\}^{\frac{1}{2}} \\
 &\lesssim \frac{\varepsilon_0}{\omega_0(\Delta_0)} \sum_j \omega_0(\mathcal{O}_j) 2^j \\
 &\lesssim \frac{\varepsilon_0}{\omega_0(\Delta_0)} \int_{\Delta_0} A_\alpha u_1(Q) d\omega_0(Q) \lesssim \varepsilon_0 M_{\omega_0}(A_\alpha u_1)(Q_0).
 \end{aligned}
 \tag{3.14}$$

Now we claim that for some $\theta > 0$ we have

$$|F_3^j(x)| \lesssim 2^{-j\theta} \varepsilon_0 M_{\omega_0}(A_\alpha u_1)(Q_0), \quad j = 1, 2, \dots.
 \tag{3.15}$$

If we take (3.15) for granted, adding in j in (3.15) and from (3.14) and (3.10), we obtain

$$|F_3(x)| \lesssim \varepsilon_0 M_{\omega_0}(A_\alpha u_1)(Q_0), \quad x \in B_0.
 \tag{3.16}$$

Then, from $F(x) = F_1(x) + F_2(x) + F_3(x)$, $x \in B_0$, (3.7), (3.9), (3.16) and the doubling property of \wp , it follows that

$$N_\alpha^0 F(Q_0) \lesssim \varepsilon_0 M_{\omega_0}(A_\alpha u_1)(Q_0), \quad x \in B_0,$$

which proves Lemma 3.2.

To show (3.15), we proceed as in the estimate of F_3^0 . We set $\Delta_j = \Delta_{2^{j-1}\delta_0}(Q_0) = B_{2^{j-1}\delta_0}(Q_0) \cap \partial B$, and $\Delta_j^0 = \Delta_j \setminus \Delta_{j-1}$. From Definition 3.4 and simple geometrical considerations, it follows that there exists $\alpha > 0$ such that

$$\Omega_j \subset (\Gamma_\alpha(Q_0) \cap \Omega_j) \cup \left(\bigcup_{\substack{I \subset \Delta_j^0 \\ I \text{ dyadic}}} I^+ \right).$$

Let $x_j \in (\Gamma_\alpha(Q_0) \cap \Omega_j) = \Omega_j^0$, $j = 1, 2, \dots$. Since $G_0(\cdot, y)$ is a nonnegative solution of $\mathcal{L}_0 u = 0$ in $B \setminus \{y\}$ vanishing on ∂B , $G(\cdot, y)$ is Hölder continuous up to the boundary [7]; moreover, we have

$$G_0(x, y) \lesssim 2^{-j\theta} G_0(x_{j-1}, y), \quad y \in \Omega_j^0,$$

where C and θ only depend on ellipticity and dimension. From this inequality and the Harnack inequality for nonnegative solutions, we have $G_0(x, y) \lesssim 2^{-j\theta} G_0(x_{j+1}, y)$. From Lemma 2.14 we get

$$G_0(x_{j+1}, y) \approx \omega_0^{x_{j+1}}(\Delta_j) \wp(y) \wp(B(y))^{-1} \delta(y)^2 \approx \frac{\omega_0^{x_{j+1}}(\Delta_j)}{\omega_0(\Delta_j)} G_0(y).$$

From Lemma 2.8(2) we have $\omega_0^{x_{j+1}}(\Delta_j) \approx 1$; thus,

$$(3.17) \quad G_0(x, y) \lesssim \frac{2^{-j\theta}}{\omega_0(\Delta_j)} G_0(y), \quad x \in B_0, y \in \Omega_j^0.$$

We have for $x \in B_0$,

$$(3.18) \quad \begin{aligned} \left| \int_{\Omega_j^0} \mathcal{L}_0 u_1(x) G_0(x, y) dy \right| &\lesssim \frac{2^{-j\theta}}{\omega_0(\Delta_j)} \int_{\Omega_j^0} \varepsilon(x) |\nabla^2 u_1(x)| G_0(y) dy \\ &\lesssim \frac{2^{-j\theta}}{\omega_0(\Delta_j)} \left\{ \int_{\Omega_j^0} |\nabla^2 u_1(x)| \delta(y)^4 \frac{\wp(y)}{\wp(B(y))} dy \right\}^{\frac{1}{2}} \\ &\quad \left\{ \int_{\Omega_j^0} a^2(x) \delta(y)^{-4} G_0^2(y) \frac{\wp(B(y))}{\wp(y)} dy \right\}^{\frac{1}{2}} \\ &\lesssim 2^{-j\theta} \left\{ \int_{\Omega_j^0} |\nabla^2 u_1(x)| \delta(y)^4 \frac{\wp(y)}{\wp(B(y))} dy \right\}^{\frac{1}{2}} \\ &\quad \left\{ \frac{1}{\omega_0(\Delta_j)} \int_{\Omega_j^0} a^2(x) \delta(y)^{-2} G_0(y) dy \right\}^{\frac{1}{2}} \\ &\lesssim \varepsilon_0 2^{-j\theta} A_\alpha(u_1)(Q_0). \end{aligned}$$

On the other hand, an argument similar to the one applied to obtain the bound for F_0 and a consideration in the spirit of (3.17) yields

$$(3.19) \quad \left| \int_{\Omega_j \setminus \Gamma_\alpha(Q_0)} \mathcal{L}_0 u_1(x) G_0(x, y) dy \right| \lesssim \varepsilon_0 2^{-j\theta} M_{\omega_0}(A_\alpha(u_1))(Q_0).$$

(3.15) follows from (3.18) and (3.19). This concludes the proof of Lemma 3.2.

3.2. Proof of Lemma 3.3. From the identity $\mathcal{L}_0(u_1^2) = 2 A_0 \nabla u_1 \cdot \nabla u_1 + 2 u_1 \mathcal{L}_0 u_1$, we have

$$|\nabla u_1|^2 \lesssim 2 A_0 \nabla u_1 \cdot \nabla u_1 = \mathcal{L}_0(u_1^2) - 2 u_1 \mathcal{L}_0 u_1.$$

We let $B^* = B_{1-r_0}$. By Fubini’s theorem and the fact that $\varepsilon(x) = 0$ in $B_{\frac{1}{2}}$ we get

$$\begin{aligned} \int_{\partial B} S_{c\alpha}^2 u_1 d\omega_0 &\lesssim \int_{B \setminus B^*} |\nabla u_1|^2 \delta(x)^2 \frac{\wp(x)}{\wp(B(x))} \omega_0(\Delta_x) dx \\ &\lesssim \int_{B \setminus B^*} \{ \mathcal{L}_0(u_1^2) - 2 u_1 \mathcal{L}_0 u_1 \} \delta(x)^2 \frac{\wp(x)}{\wp(B(x))} \omega_0(\Delta_x) dx \\ &\lesssim \varepsilon_0 \int_{B \setminus B_{\frac{1}{2}}} |u_1| |\nabla^2 u_1| \delta(x)^2 \frac{\wp(x)}{\wp(B(x))} \omega_0(\Delta_x) dx, \end{aligned}$$

where we used Lemma 2.14 and $\int_B \mathcal{L}_0(u_1^2) G_0(x) dx \leq 0$. Now we apply a “stopping time” argument as in the proof of (3.14) to obtain

$$\int_{\partial B} S_{c\alpha}^2 u_1 d\omega_0 \lesssim \varepsilon_0 \int_{\partial B} N_\alpha u_1 \cdot A_\alpha u_1 d\omega_0.$$

From Theorem 2.19 and the inequality $|ab| \leq \mu^{-1} a^2 + \mu b^2$, $\mu > 0$ we get

$$\begin{aligned} \int_{\partial B} S_{c\alpha}^2 u_1 d\omega_0 &\leq C \varepsilon_0 \int_{\partial B} N_\alpha u_1 \cdot S_{c\alpha} u_1 d\omega_0 \\ &\leq C \frac{\varepsilon_0}{\mu} \int_{\partial B} (N_\alpha u_1)^2 d\omega_0 + C \varepsilon_0 \mu \int_{\partial B} S_{c\alpha}^2 u_1 d\omega_0. \end{aligned}$$

The lemma follows from choosing μ so that $C \varepsilon_0 \mu = \frac{1}{2}$.

4. PROOF OF THEOREM 1.1

We will obtain Theorem 1.1 as a consequence of the following special case in the spirit of [4].

Theorem 4.1. *Suppose that $\omega_0 \in A_\infty(d\sigma)$ and let $E(Q)$ be given by*

$$E_r(Q) = \left\{ \int_{\Gamma_\alpha(Q) \cap B_r(Q)} \frac{\mathbf{a}^2(x)}{\delta^n(x)} dx \right\}^{\frac{1}{2}} \quad r > 0, Q \in \partial D.$$

There exists $\varrho_0 = \varrho_0(n, \lambda, \Lambda, D, \zeta, \kappa) > 0$ such that if $\mathbf{a}_k^{i,j} \in \text{BMO}_{2\varrho^}(\mathbb{R}^n)$, $1 \leq i, j \leq n$, $k = 0, 1$, and*

$$(4.1) \quad \sup_{Q \in \partial D} E_{r_0}(Q) = M_1 < \infty,$$

then $\omega_1 \in A_\infty(d\sigma)$. Here, ζ and κ are the A_∞ constants of ω_0 with respect to σ as given in (2.1).

We postpone the proof of Theorem 4.1 to the next section, and we now show how Theorem 1.1 follows from this result. We assume that $\mathbf{a}(x) \equiv 0$ if $\delta(x) > r_0/2$. By Lemma 2.15, this assumption does not bring any loss of generality. Fix $Q \in \partial D$ and let $0 < r \leq r_0/2$. By Fubini’s theorem and (1.6) we have

$$\frac{1}{\sigma(\Delta_r(Q))} \int_{\Delta_r(Q)} E_r^2(P) d\sigma(P) \lesssim \frac{1}{\sigma(\Delta_r(Q))} \int_{T_{2r}(Q)} \frac{\mathbf{a}^2(x)}{\delta(x)} dx \lesssim M^2.$$

So there exists a closed set $F \subset \Delta_r(Q)$ such that $2\sigma(F) > \sigma(\Delta_r(Q))$ and $E(P) \lesssim M$ for all $P \in F$. Now, we need to introduce a “saw-tooth” region $\Omega = \Omega(F, r)$ over F , that is, for given $0 < \alpha < \beta$, Ω verifies (see [28], [10]):

- (i) for suitable $\alpha', \alpha'', c_1, c_2$ with $\alpha < \alpha' < \alpha'' < \beta$,

$$\bigcup_{P \in E} \{\bar{\Gamma}_{\alpha'}(P) \cap B_{c_1 r}(P)\} \subset \Omega \subset \bigcup_{P \in E} \{\bar{\Gamma}_{\alpha''}(P) \cap B_{c_2 r}(P)\};$$

- (ii) $\partial\Omega \cap \partial D = F$;
- (iii) there exists $x_0 \in \Omega$ with $\text{dist}(x_0, \partial\Omega) \approx r$; we call x_0 the center of Ω ;
- (vi) Ω is a Lipschitz domain with Lipschitz constant which depends only on B .

Let $\tilde{\mathcal{L}} = \sum_{i,j=1}^n \tilde{a}^{i,j} \partial_{i,j}$ where $\tilde{A} = \{\tilde{a}^{i,j}(x)\}_{i,j=1}^n$ and

$$\tilde{A}(x) = \begin{cases} A_1(x) & x \in \Omega(F, r), \\ A_0(x) & x \in B \setminus \Omega(F, r). \end{cases}$$

From the definition of $\mathbf{a}(x)$ it is easy to see that

$$(4.2) \quad |\{y \in B(x) : \mathbf{a}(y) \geq \mathbf{a}(x)\}| \gtrsim |B(x)|.$$

From (4.2) and (1.6) we have that for all $x \in D$,

$$a^2(x) \lesssim \frac{1}{|B(x)|} \int_{B(x)} a^2(y) dy \lesssim M^2.$$

We set $\tilde{a}(x) = \sup_{y \in B(x)} |\tilde{A}(y) - A_0(y)|$. Then $\tilde{a}(x) \leq a(x) \lesssim M$ for all $x \in D$. Note that since for all $1 \leq i, j \leq n$, we have

$$\tilde{a}^{i,j} = a_0^{i,j} + (a_1^{i,j} - a_0^{i,j})\chi_{\Omega(F,r)},$$

where $\chi_{\Omega(F,r)}$ is the characteristic function of $\Omega(F,r)$, we have $\eta_{\tilde{a}} \leq \eta_{a_0} + CM$, where $\eta_{\tilde{a}}$ and η_{a_0} are the BMO moduli of continuity of \tilde{A} and A_0 , respectively. Therefore, if M is small enough, say $M \leq C^{-1}\varrho^*$, we have that the coefficients of $\tilde{\mathcal{L}}$ belong to the space $BMO_{2\varrho^*}$.

Our saw-tooth region $\Omega(F,r)$ can be constructed so that for any $Q \in \partial D$ such that $x \in \Gamma_\alpha(Q) \cap \Omega(F,r) \neq \emptyset$, there exists $P \in F$ such that $B(x) \subset \Gamma_\alpha(P)$. From this observation, we have that if $\tilde{E}_r(P)$ is as in the definition of $E_r(P)$ above but replacing a by \tilde{a} , then $\tilde{E}_r(P) \lesssim M$ for all $P \in \partial D$. Therefore, from Theorem 4.1 we have that the harmonic measure $\tilde{\omega}$ for $\tilde{\mathcal{L}}$ on ∂D is in $A_\infty(d\sigma)$ and by a known result in the theory of weights [11] there exists $\theta > 0$ and $c > 0$ (depending only on ζ and κ) such that

$$(4.3) \quad \left(\frac{\tilde{\omega}(Z)}{\tilde{\omega}(\Delta_r(Q))} \right)^\theta \geq c \frac{\sigma(Z)}{\sigma(\Delta_r(Q))}$$

for any set $Z \subset \Delta_r(Q)$. Since $\frac{\sigma(F)}{\sigma(\Delta_r(Q))} > \frac{1}{2}$ we have that if $\frac{\sigma(E)}{\sigma(\Delta_r(Q))} > \frac{3}{4}$, then $\frac{\sigma(E \cap F)\sigma(\Delta_r(Q))}{\sigma(\Delta_r(Q))} > \frac{1}{4}$. Therefore, from (4.3) we get

$$\frac{\tilde{\omega}(E \cap F)}{\tilde{\omega}(\Delta_r(Q))} \gtrsim 1.$$

Let x_0 be the ‘‘center’’ of the saw-tooth region $\Omega(R,r)$ and denote by $\tilde{\omega}_\Omega$ the harmonic measure for $\tilde{\mathcal{L}}$ on $\partial\Omega(F,r)$ evaluated at x_0 . From the ‘‘main lemma’’ in [10] (see also [28]) we have

$$\frac{\tilde{\omega}(E \cap F)}{\tilde{\omega}(\Delta_r(Q))} \leq C (\tilde{\omega}_\Omega(E \cap F))^\vartheta.$$

Thus, $\tilde{\omega}_\Omega(E \cap F) \geq C$. Since $\mathcal{L}_1 = \tilde{\mathcal{L}}$ in $\Omega(F,r)$, we obtain $\omega_{1\Omega}(E \cap F) \geq C$, where $\omega_{1\Omega}$ denotes the harmonic measure on $\partial\Omega(F,r)$ for the operator \mathcal{L}_1 . From the scaling of the harmonic measure (Lemma 2.8(1)) and the maximum principle we get

$$\frac{\omega_1(E \cap F)}{\omega_1(\Delta_r(Q))} \gtrsim \omega_1^{x_0}(E \cap F) \geq \omega_{1\Omega}(E \cap F) \gtrsim 1,$$

and then

$$\frac{\omega_1(E)}{\omega_1(\Delta_r(Q))} \geq \kappa_0,$$

for some positive κ_0 . This shows that condition (2.1) holds for the measures ω_1 and σ with $\zeta = \frac{3}{4}$ and $\kappa = \kappa_0$. Therefore, $\omega_1 \in A_\infty(d\sigma)$ as wanted, in the case $M \leq C^{-1}\varrho^*$.

For the general case, define $\mathcal{L}_t = (1 - t)\mathcal{L}_0 + t\mathcal{L}_1$ for $0 \leq t \leq 1$, let K be a positive integer such that $K^{-1}M \leq C^{-1}\varrho^*$, and for integers $0 \leq l < K$, let $\mathbf{a}_l(x) = \sup_{y \in B(x)} |A_{\frac{l+1}{K}}(y) - A_{\frac{l}{K}}(y)|$, where $A_t = (1 - t)A_0 + tA_1$. Note that since the set BMO_{ϱ^*} is convex, we have that $A_t \in \text{BMO}_{\varrho^*}$, $0 \leq t \leq 1$. By the same consideration, the matrices A_t are uniformly elliptic with ellipticity λ and their entries are bounded by Λ . Denote by ω_t the harmonic measure in D with respect to \mathcal{L}_t , $0 \leq t \leq 1$. Then since

$$\mathbf{a}_l(x) = \sup_{y \in B(x)} \left| \frac{1}{K} (A_1(y) - A_0(y)) \right| \leq \frac{1}{K} \mathbf{a}(x), \quad 0 \leq l < K,$$

we have that the pairs of operators $\mathcal{L}_{\frac{l}{K}}$ and $\mathcal{L}_{\frac{l+1}{K}}$, $0 \leq l < K$ verify

$$\sup_{\substack{0 < r \leq r_0 \\ Q \in \partial D}} h^l(r, Q) = M/K \leq C^{-1}\varrho^* < +\infty,$$

where

$$h^l(r, Q) = \left\{ \frac{1}{\sigma(\Delta_r(Q))} \int_{T_r(Q)} \frac{\mathbf{a}_l^2(x)}{\delta(x)} dx \right\}^{1/2}, \quad 0 \leq l < K.$$

From the previous special case we have

$$\omega_{\frac{l}{K}} \in A_\infty(d\sigma) \Rightarrow \omega_{\frac{l+1}{K}} \in A_\infty(d\sigma), \quad l = 0, 1, \dots, K - 1.$$

So $\omega_1 \in A_\infty(d\sigma)$, and this finishes the proof of Theorem 1.1.

5. PROOF OF THEOREM 4.1

Now we will show that Theorem 4.1 follows from Theorem 3.1. It is clear from the proof of Theorem 3.1 (see Section 3) that we might replace $\mathbf{a}(x)$ in (3.1) by $\mathbf{a}_0(x) = \sup_{y \in B_0(x)} |A_0(y) - A_1(y)|$, where $B_0(x)$ is the ball centered at x with radius $\delta(x)/c$ for any fixed constant $c \geq 2$. We claim that if we take $c = 8$ in the definition of \mathbf{a}_0 , M_1 is given by (4.1) and $0 < r \leq r_0$, we have

$$(5.1) \quad \frac{1}{\omega_0(\Delta_r(Q))} \int_{T_r(Q)} \mathbf{a}_0^2(x) \frac{G_0(x)}{\delta(x)^2} dx \lesssim M_1^2.$$

In fact, from Lemma 2.14 we have

$$\int_{T_r(Q)} \mathbf{a}_0^2(x) \frac{G_0(x)}{\delta(x)^2} dx \lesssim \int_{T_r(Q)} \mathbf{a}_0^2(x) \frac{\wp(x)}{\wp(B(x))} \omega_0(\Delta_x) dx.$$

From Besicovitch's covering lemma (I.8.17 in [23]), we can find a sequence $\{x_j\}_{j=1}^\infty \subset T_r(Q)$ such that $T_r(Q) \subset \bigcup_j B_j$, where $B_j = B_{\frac{\delta(x_j)}{8}}(x_j)$ and the balls B_j have finite

overlapping. Then

$$\begin{aligned}
 \int_{T_r(Q)} \mathbf{a}_0^2(x) \frac{G_0(x)}{\delta(x)^2} dx &\lesssim \sum_j \int_{B_j} \mathbf{a}_0^2(x) \frac{\wp(x)}{\wp(B(x))} \omega_0(\Delta_x) dx \\
 &\lesssim \sum_j \frac{\omega_0(\Delta_{x_j})}{\wp(B(x_j))} \int_{B_j} \mathbf{a}_0^2(x) \wp(x) dx \\
 &\lesssim \sum_j \omega_0(\Delta_{x_j}) \mathbf{a}^2(x_j) \\
 &\lesssim \sum_j \omega_0(\Delta_{x_j}) \int_{B(x_j)} \frac{\mathbf{a}^2(x)}{\delta(x)^n} dx \\
 &\lesssim \int_{\Delta_r(Q)} E_r^2(P) d\omega_0(P),
 \end{aligned}$$

where we have used Fubini's theorem; (5.1) now follows from (4.1). Then, if M_1 is small enough, say $M_1 \leq C^{-1}\varepsilon_0$, from Theorem 3.1 we have that $\omega_1 \in B_2(d\omega_0)$ and since $\omega_0 \in A_\infty(d\sigma)$, we have $\omega_1 \in A_\infty(d\sigma)$, proving Theorem 4.1 in the case $M_1 \leq C^{-1}\varepsilon_0$.

For the general case, we define as before $\mathcal{L}_t = (1-t)\mathcal{L}_0 + t\mathcal{L}_1$ for $0 \leq t \leq 1$. Let K be a positive integer such that $CK^{-1}M_1 \leq \varepsilon_0$ and for integers $0 \leq l < K$, let $\mathbf{a}_l(x) = \sup_{B(x)} |A_{\frac{l+1}{K}}(y) - A_{\frac{l}{K}}(y)|$, where $A_t = (1-t)A_0 + tA_1$. Denote by ω_t the harmonic measure in D with respect to \mathcal{L}_t , $0 \leq t \leq 1$. Then since

$$\mathbf{a}_l(x) = \sup_{y \in B(x)} \left| \frac{1}{K} (A_1(y) - A_0(y)) \right| \leq \frac{1}{K} \mathbf{a}(x),$$

from the previous result, we have

$$\omega_{\frac{l}{K}} \in A_\infty(d\sigma) \Rightarrow \omega_{\frac{l+1}{K}} \in A_\infty(d\sigma), \quad l = 0, 1, \dots, K-1.$$

So $\omega_1 \in A_\infty(d\sigma)$, and this completes the proof of Theorem 4.1.

ACKNOWLEDGMENT

The results of this paper were obtained during my Ph.D. studies at the University of Minnesota and are also contained in my thesis [5]. I would like to express immense gratitude to my supervisor Carlos Kenig, from the University of Chicago, whose guidance and support were essential for the successful completion of this project.

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