

NOETHERIAN PI HOPF ALGEBRAS ARE GORENSTEIN

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ABSTRACT. We prove that every noetherian affine PI Hopf algebra has finite injective dimension, which answers a question of Brown (1998).

0. INTRODUCTION

A classical result of Cartier states that every affine (more generally, locally algebraic) group scheme over a field of characteristic 0 is smooth [DG, II.§6.1.1]. Since there is a canonical anti-equivalence between the category of affine group schemes and the category of affine commutative Hopf algebras [DG, II.§1.1.6], every affine commutative Hopf algebra over a field of characteristic 0 is regular. An algebra is said to be *regular* (respectively, *Gorenstein*) if it has finite global dimension (respectively, finite injective dimension). In the case of positive characteristic, there are non-regular affine commutative Hopf algebras, but every affine commutative Hopf algebra is Gorenstein [Br, 2.3].

In the noncommutative case the geometric machinery becomes invalid and the structure of Hopf algebras seems more difficult to reveal. Some naive extensions of the commutative results do not hold in the noncommutative case. For example, over any field, there are finite dimensional Hopf algebras that are not regular. However, there are several surprising results in the finite dimensional case. Larson and Sweedler proved that every finite dimensional Hopf algebra is Frobenius, namely, Gorenstein of injective dimension 0 [LS]. Larson and Radford proved that if H is a finite dimensional Hopf algebra over a field of characteristic 0, then the following are equivalent: (a) H is semisimple, namely, H is regular of global dimension 0, (b) H is cosemisimple, and (c) its antipode is involutory, namely, $S^2 = \text{id}_H$ [LR1], [LR2].

During the last fifteen years, as quantum groups or “the function rings of quantum groups” have been studied extensively, more and more people have become interested in infinite dimensional Hopf algebras. A few years ago Brown and Goodearl started to study ring-theoretic properties of a class of infinite dimensional Hopf algebras [Br], [BG]. They verified that many examples of noetherian affine PI Hopf algebras, some of which are quantum groups at roots of unity, are Gorenstein, and

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showed that noetherian affine PI, Gorenstein, Hopf algebras have various good homological properties. In [Br], Brown posted a list of questions concerning general noetherian PI Hopf algebras, and one of them is the following [Br, Question A]:

Does every noetherian affine PI Hopf algebra have finite left and right injective dimension?

Under the additional hypothesis of “filtered noetherian” we showed that Brown’s question has an affirmative answer [WZ1, 0.3]. The proof of [WZ1, 0.3] is based on the dualizing complexes introduced by Yekutieli [Ye].

The first aim of the paper is to answer Brown’s question.

Theorem 0.1. *If H is a noetherian affine PI Hopf algebra over a field, then it is Gorenstein.*

A slightly more general version is given in Theorem 3.5. Note that this generalizes the result of Larson and Sweedler: finite dimensional Hopf algebras are Frobenius [LS].

The second aim is to show that the Gorenstein property is fundamental for other properties. We will use results of Ajitabh-Smith-Zhang [AjSZ], Brown [Br], Brown-Goodearl [BG] and Yekutieli-Zhang [YZ1] to prove that noetherian PI Hopf algebras have various properties that are useful for understanding the structure of Hopf algebras. Let GKdim denote the Gelfand-Kirillov dimension. Let id denote the injective dimension and let pd denote the projective dimension.

Theorem 0.2. *Let H be a noetherian affine PI Hopf algebra of injective dimension d .*

- (1) H is AS-Gorenstein, Auslander-Gorenstein, and Cohen-Macaulay.
- (2) H has a quasi-Frobenius artinian ring of fractions.
- (3) For every minimal prime \mathfrak{p} of H , $\text{GKdim } H/\mathfrak{p} = d$. As a consequence, $\text{GKdim } H = d$.
- (4) H has a resolution of H -bimodules

$$(0.2.1) \quad 0 \rightarrow H \rightarrow I^{-d} \rightarrow I^{-d+1} \rightarrow \dots \rightarrow I^0 \rightarrow 0$$

such that when restricted to the left-hand side (respectively, the right-hand side), (0.2.1) is a minimal injective resolution of ${}_H H$ (respectively, H_H), and each I^{-i} is pure of $\text{GKdim } i$ on the left and the right respectively.

- (5) (Auslander-Buchsbaum formula) If M is a noetherian left (or right) H -module of finite projective dimension, then $\text{pd } M + \text{depth } M = d$.
- (6) (Bass’s theorem) If M is a noetherian left (or right) H -module of finite injective dimension, then $\text{id } M = d$.

Some definitions are given in Section 3. Note that a version of Theorem 0.2 holds for commutative Gorenstein local rings. By the localization method, it can be extended to a class of commutative Gorenstein non-local rings. In contrast with the commutative case, we cannot use the localization method to prove any of the assertions in Theorem 0.2. It is fortunate that some “global techniques” work for PI Hopf algebras.

One consequence of the Gorenstein property, which we will state separately from Theorem 0.2, is the projectivity of Hopf algebras over Hopf subalgebras. Nichols and Zoeller proved the following Hopf freeness theorem: a finite dimensional Hopf algebra is a free module over any sub-Hopf algebra [NZ]. Hopf freeness was studied by other authors for infinite dimensional Hopf algebras (see [NR], [Ra], [Ta1], [Ta2]).

We consider a slightly more general situation, namely, the projectivity criterion of H' over H when there is an algebra homomorphism $H \rightarrow H'$ between two Hopf algebras.

Theorem 0.3. *Let H and H' be noetherian affine PI Hopf algebras of the same injective dimension. Suppose that there is an algebra homomorphism $H \rightarrow H'$ such that H' is a noetherian H -module on both sides. If either $\text{id}_H H' < \infty$ or $\text{pd}_H H' < \infty$ (e.g. H is regular), then both H'_H and ${}_H H'$ are projective.*

Theorem 0.3 plays a key role in the proof of the following result: Let H be an involutory Hopf algebra over a field of characteristic 0. If H is a finitely generated module over its affine center, then H is regular [WZ3, 0.1]. How to recognize when an infinite dimensional noetherian Hopf algebra is regular is another question asked by Brown [Br, Question C], which does not yet have a complete answer.

Some basic lemmas about PI algebras are collected in Section 1, and an inductive step of the proof of the main result is described in Section 2. The proofs of Theorems 0.1 and 0.2 are given in Section 3 and the proof of Theorem 0.3 is given in Section 4. Some slightly different versions of these results are discussed in Section 5.

1. PRELIMINARIES ON PI ALGEBRAS

Let k be a base field of arbitrary characteristic. A bimodule or an algebra is said to be *noetherian* if it is noetherian on both sides. An algebra is called *affine* if it is finitely generated over the base field. Let A° denote the opposite ring of A . We usually work with left modules and we adopt the convention of identifying A° -modules with right A -modules. A *polynomial identity algebra*, or *PI algebra* for short, is an algebra satisfying a polynomial identity. We refer to [MR, Ch. 13] for some basic materials about PI algebras.

We refer to [MR, 6.8.4] for the definition of dimension function. The basic dimension functions we will use are the Krull dimension, denoted by Kdim , and the Gelfand-Kirillov dimension, denoted by GKdim . Let δ be a dimension function. We say δ is *exact* if $\delta M = \max\{\delta L, \delta N\}$ for every short exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ of finitely generated A -modules.

All lemmas in this section are straightforward and well known. We choose to include some of the proofs because ideas in these proofs will be used in later sections. The details of this section could be skipped by readers who are familiar with the material. The following lemma is a collection of results that can be found in [KL].

Lemma 1.1. *Let A be a noetherian PI algebra and let M be a noetherian A -module.*

- (1) *If $\text{GKdim } M$ is finite, then it is an integer.*
- (2) *$\text{GKdim } A = \text{GKdim } A/\mathfrak{p}$ for some prime ideal \mathfrak{p} .*
- (3) *GKdim is exact.*
- (4) *GKdim is symmetric in the following sense: if A and B are two noetherian PI algebras and if M is a noetherian A - B -bimodule, then $\text{GKdim } {}_A M = \text{GKdim } M_B$.*
- (5) *$\text{GKdim } M \geq \text{Kdim } M$.*

Lemma 1.2. *Let A be a noetherian PI algebra and let M be a noetherian A -module.*

- (1) *Kdim is exact.*

- (2) Kdim is symmetric in the following sense: if A and B are two noetherian PI algebras and if M is a noetherian A - B -bimodule, then

$$\text{Kdim } {}_A M = \text{Kdim } M_B = \text{Kdim } ({}_A M_B).$$

- (3) If A is affine, then $\text{Kdim } M = \text{GKdim } M$.

Proof. (1) is [MR, 6.2.4]. (2) follows from [GW, 13.15] and [MR, 6.4.13], and (3) follows from [Row, 6.3.41, 6.3.44]. \square

When we have a noetherian bimodule M as in Lemma 1.2(2), we write $\text{Kdim } M$ for any of $\text{Kdim } {}_A M$, or $\text{Kdim } M_B$, or $\text{Kdim } ({}_A M_B)$.

Let A be a noetherian algebra. Let $\text{mod } A$ denote the category of noetherian A -modules. Recall that the Krull dimension is defined for all noetherian A -modules. For every integer s , let $\text{mod}_s A$ be the full subcategory of $\text{mod } A$ consisting of A -modules of $\text{Kdim} \leq s$. Hence $\text{mod}_s A$ is a dense subcategory of $\text{mod } A$ and we may form a quotient category $\text{mod } A / \text{mod}_s A$. The basic properties of a quotient category can be found in [Po, Ch.4].

Let $\text{Mod } A$ be the category of all A -modules. We can define the Krull dimension of a module to be the maximum of the Krull dimensions of its noetherian submodules. The subcategory $\text{Mod}_s A$ and the quotient category $\text{Mod } A / \text{Mod}_s A$ can be defined similarly.

For every A -module M , we denote by $\tau_s(M)$ the largest submodule of M of $\text{Kdim} \leq s$. So we have a filtration of submodules

$$\tau_0(M) \subset \tau_1(M) \subset \cdots \subset M.$$

If M is an A -module of $\text{Kdim } s$, we define the s -rank of M , denoted by $\text{rank}_s(M)$, to be the length of M as an object in the quotient category $\text{Mod } A / \text{Mod}_{s-1} A$. If M is a noetherian A -module of $\text{Kdim } s$, then the image of M in $\text{Mod } A / \text{Mod}_{s-1} A$ (also in $\text{mod } A / \text{mod}_{s-1} A$) is artinian, and hence $\text{rank}_s(M)$ is finite.

Let M and N be two A -modules. An A -module homomorphism $f : M \rightarrow N$ is called an *isomorphism modulo* $\text{Kdim } s$ if the corresponding map in $\text{Mod } A / \text{Mod}_s A$ is an isomorphism. In other words, f is an isomorphism modulo $\text{Kdim } s$ if and only if the kernel and the cokernel of f have Kdim less than or equal to s . It is easy to check that the composition of two isomorphisms modulo $\text{Kdim } s$ is again an isomorphism modulo $\text{Kdim } s$. The definitions of surjective and injective maps modulo $\text{Kdim } s$ are similar. A complex of A -modules

$$\cdots \rightarrow M^{i-1} \rightarrow M^i \rightarrow M^{i+1} \rightarrow \cdots$$

is said to be *exact modulo* $\text{Kdim } s$ if the corresponding complex in the level of the quotient category $\text{Mod } A / \text{Mod}_s A$ is exact, or if, equivalently, every cohomology of the complex as an A -module has $\text{Kdim} \leq s$. The following lemma is [Po, 4.3.10].

Lemma 1.3. *Suppose $0 \rightarrow L \rightarrow M \xrightarrow{f} N \rightarrow 0$ is a complex of A -modules that is exact modulo $\text{Kdim } s$. Then there is a short exact sequence of A -modules $0 \rightarrow L' \rightarrow M' \rightarrow N' \rightarrow 0$ such that the following hold.*

- (1) All L' , M' and N' have no nonzero submodule of $\text{Kdim} \leq s$.

(2) The following is a commutative diagram of A -modules:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \uparrow = & & \uparrow = & & \uparrow & & \\
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & f(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0
 \end{array}$$

- (3) All vertical maps in part (2) are isomorphisms modulo $\text{Kdim } s$.
 (4) If M is a noetherian A -module, then L', M' and N' can be chosen to be noetherian.

Proof. Let $M' = M/\tau_s(M)$. Then $M \rightarrow M'$ is an isomorphism modulo $\text{Kdim } s$. There is a commutative diagram

$$\begin{array}{ccc}
 M & \longrightarrow & N \\
 \downarrow & & \downarrow \\
 M' & \longrightarrow & N/\tau_s(N)
 \end{array}$$

Let N' be the image of $M \rightarrow N \rightarrow N/\tau_s(N)$, which is also the image of $M' \rightarrow N/\tau_s(N)$, and let L' be the kernel of the surjective map $M' \rightarrow N'$. The assertions are easy to check now. \square

Let M be an A -module. The left annihilator of M in A is denoted by $\text{l. ann}_A(M)$.

Lemma 1.4. *Let A and B be noetherian PI algebras. Let M be a noetherian A -module. Let j be an integer.*

- (1) *If M is a noetherian A - B -bimodule, then $\text{Ext}_A^j(M, A)$ is a noetherian B - A -bimodule.*
 (2) $\text{Kdim Ext}_A^j(M, A)_A \leq \text{Kdim}_A M$.
 (3) *If $M \rightarrow M'$ is an isomorphism modulo $\text{Kdim } s$, then*

$$\text{Ext}_A^j(M', A) \rightarrow \text{Ext}_A^j(M, A),$$

viewed as a map of A^0 -modules, is an isomorphism modulo $\text{Kdim } s$.

Proof. (1) This follows from an ungraded analogue of [SZ, 3.5].

(2) We use induction on $s = \text{Kdim}_A M$.

By the long exact sequence, the exactness of Kdim [Lemma 1.2(1)], and the noetherian property of M , it suffices to show that the assertion holds for a nonzero submodule of M . Hence we may assume that M is critical and that $\mathfrak{p} := \text{l. ann}_A(M)$ is a prime ideal of A . In this case there is a short exact sequence

$$(1.4.1) \quad 0 \rightarrow A/\mathfrak{p} \rightarrow M^{\oplus r} \rightarrow N \rightarrow 0$$

with $\text{Kdim } N < s$ by an ungraded analogue of [SZ, 2.1(ii)]. By the long exact sequence of $\text{Ext}_A^*(-, A)$, it suffices to show that the Krull dimension of $\text{Ext}_A^j(A/\mathfrak{p}, A)$ and $\text{Ext}_A^j(N, A)$ are no more than s . By induction hypothesis, $\text{Kdim Ext}_A^j(N, A) < s$. By (1), $\text{Ext}_A^j(A/\mathfrak{p}, A)$ is noetherian on both sides and we can compute the Kdim on either sides by the symmetry of Kdim [Lemma 1.2(2)]. Since $s = \text{Kdim } M = \text{Kdim } A/\mathfrak{p}$ and $\text{Ext}_A^j(A/\mathfrak{p}, A)$ is a (left) A/\mathfrak{p} -module, we obtain that $\text{Kdim Ext}_A^j(A/\mathfrak{p}, A) \leq s$. Thus the assertion follows.

(3) Let M'' be the image of $M \rightarrow M'$. Then we have two short exact sequences

$$0 \rightarrow L \rightarrow M \rightarrow M'' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow M'' \rightarrow M' \rightarrow P \rightarrow 0,$$

where L and P are A -modules of $\text{Kdim} \leq s$. By (2), $\text{Ext}_A^i(L, A)$ has $\text{Kdim} \leq s$ for all i . Hence the long exact sequence implies that $\text{Ext}_A^j(M'', A) \rightarrow \text{Ext}_A^j(M, A)$ is an isomorphism modulo $\text{Kdim} s$. Similarly, $\text{Ext}_A^j(M', A) \rightarrow \text{Ext}_A^j(M'', A)$ is an isomorphism modulo $\text{Kdim} s$. The assertion follows by composing these two maps. \square

A nonzero A -module M is called s -pure if $\tau_s(M) = M$ and $\tau_t(M) = 0$ for $t < s$, or equivalently, $s = \text{Kdim} M = \text{Kdim} N$ for all nonzero submodules $N \subset M$. The pure module is the same as the *homogeneous* module defined in [MR, 6.8.8] for Kdim .

Lemma 1.5. *Let A and B be noetherian PI algebras.*

- (1) *If M is a noetherian A - B -bimodule, then $\tau_s({}_A M) = \tau_s(M_B)$ for all s . As a consequence, ${}_A M$ is s -pure if and only if M_B is s -pure.*
- (2) *If M is an A -module and N is a noetherian B - A -module, then $\text{Kdim} N \otimes_A M \leq \text{Kdim} M$.*
- (3) *Let $L \subset M$ be s -pure noetherian A -modules such that $\text{Kdim} M/L < s$. If $\text{l.ann}_A(L) = \mathfrak{p}$ is a prime ideal of A , then $\text{l.ann}_A(M) = \text{l.ann}_A(L) = \mathfrak{p}$.*

Proof. (1) It is easy to check that $\tau_s(M_B)$ is a subbimodule. Since M is noetherian on both sides, $\tau_s(M_B)$ is noetherian on both sides. By the symmetry of Kdim [Lemma 1.2(2)], we have $\text{Kdim}_A(\tau_s(M_B)) = \text{Kdim}(\tau_s(M_B))_B \leq s$. Hence $\tau_s(M_B) \subset \tau_s({}_A M)$. Similarly, we have $\tau_s({}_A M) \subset \tau_s(M_B)$. Thus we have proved the first assertion. The second assertion is clear by the first one and the definition.

(2) As in the proof of Lemma 1.4(2), we may assume $M = A/\mathfrak{p}$ for some prime ideal \mathfrak{p} of A . It is clear that

$$\text{Kdim}_B(N \otimes_A M) = \text{Kdim}(N \otimes_A A/\mathfrak{p})_A \leq \text{Kdim} A/\mathfrak{p} = \text{Kdim} M.$$

(3) Tensoring $\mathfrak{p} \rightarrow A$ with $0 \rightarrow L \rightarrow M \rightarrow M/L \rightarrow 0$, we obtain

$$\begin{array}{ccccccc} \mathfrak{p} \otimes_A L & \longrightarrow & \mathfrak{p} \otimes_A M & \longrightarrow & \mathfrak{p} \otimes M/L & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ L & \longrightarrow & M & \longrightarrow & M/L & \longrightarrow & 0 \end{array}$$

The image of the first vertical map is zero by the definition of \mathfrak{p} . By (2), we have $\text{Kdim}(\mathfrak{p} \otimes_A M/L) < s$. Hence the image of the middle vertical map has $\text{Kdim} < s$. Since M is s -pure, the image is zero and $\mathfrak{p} = \text{l.ann}_A(M)$. \square

Lemma 1.5(3) will be used implicitly several times. For example, Lemma 1.5(3) implies that if M is a noetherian A - B -bimodule which is critical as a bimodule, then $\text{l.ann}_A(M)$ is a prime ideal of A and $\text{r.ann}_B(M)$ is a prime ideal of B . From now on, when we have a noetherian bimodule M we will just use $\tau_s(M)$ for either $\tau_s({}_A M)$ or $\tau_s(M_B)$. For our convenience in the next section, we state the next two lemmas for a ring B instead of A . The next lemma is a special case of [MR, 6.8.16].

Lemma 1.6. *Let B be a noetherian PI algebra. If B_B and ${}_B B$ are pure, then B has an artinian ring of fractions.*

Let $\mathcal{C}_B(0)$ denote the set of regular elements of B . If I is an ideal of B , let $\mathcal{C}_B(I) \subset B$ be the pre-image of $\mathcal{C}_{B/I}(0)$. If B has an artinian ring of fractions (e.g., in Lemma 1.6), then by Small’s theorem [MR, 4.1.4], $\mathcal{C}_B(0) = \mathcal{C}_B(N)$, where N is the prime radical of B . It is clear that $\mathcal{C}_B(N) = \bigcap_j \mathcal{C}_B(\mathfrak{p}_j)$, where $\{\mathfrak{p}_j\}$ is the set of all minimal primes of B . For the purpose of this paper, we say that $x \in \mathcal{C}_B(0)$ is a *special* regular element if for every minimal prime \mathfrak{p}_i the following hold:

- (1) The image $\bar{x} \in B/\mathfrak{p}_i$ is in $(\bigcap_{j \neq i} \mathfrak{p}_j)/\mathfrak{p}_i$.
- (2) \bar{x} is a central regular element in B/\mathfrak{p}_i .
- (3) If $\text{Kdim } B/\mathfrak{p}_i > 0$, then \bar{x} is not a unit.

Recall that if D is a noetherian prime PI ring, then every noetherian D° -module M such that $\text{Kdim } M < \text{Kdim } D$ has a nonzero right annihilator $\text{r. ann}_D(M)$ [GW, 8.9]. Since D is prime and PI, there is a central regular element c in $\text{r. ann}_D(M)$ such that $Mc = 0$ [Row, 6.1.28]. If $\text{Kdim } D > 0$, we can choose c to be a non-unit.

Lemma 1.7. *Let B be a noetherian PI algebra with an artinian ring of fractions. Suppose that \mathfrak{p}_t is a minimal prime ideal of B such that $\text{Kdim } B/\mathfrak{p}_t = s > 0$. Let L be a noetherian B° -module with $\text{r. ann}_B(L) = \mathfrak{p}_t$. Then there is a special regular element $x \in \mathcal{C}_B(0)$ such that $\text{Kdim } Lx/Lx^2 = \text{Kdim } L - 1$ and $\tau_{s-1}(L)x = 0$.*

Proof. For every minimal prime \mathfrak{p}_i of B , \mathfrak{p}_i does not contain $\bigcap_{j \neq i} \mathfrak{p}_j$. Thus there is a central regular element $\bar{c}_i \in B/\mathfrak{p}_i$ such that (i) a pre-image c_i is in $\bigcap_{j \neq i} \mathfrak{p}_j$ and (ii) \bar{c}_i is not a unit when $\text{Kdim } B/\mathfrak{p}_i > 0$. Let $x = \sum_i c_i$. For each i , $\bar{x} = \bar{c}_i$ in B/\mathfrak{p}_i . The properties of c_i imply that x is a special regular element.

When $i = t$ we need to choose c_i more finely to satisfy other requirements. First we note that $Lc_j = 0$ for all $j \neq t$. Let $D = B/\mathfrak{p}_t$. Since $\text{Kdim } \tau_{s-1}(L) < \text{Kdim } D$, there is a central regular element $\bar{c}' \in D$ such that $\tau_{s-1}(L)c' = 0$. Let $M = L/\tau_{s-1}(L)$. Pick a submodule $N \subset M$ such that $\text{Kdim } M/N = s - 1$ [GW, 13D and 13G]. Then there is a central regular element $\bar{c}'' \in D$ such that $(M/N)c'' = 0$. Replace c_t by $c_t c' c''$ and let x be constructed as in the previous paragraph by using the new c_t . Then $\text{Kdim } M/Mx = s - 1$ [GW, 13.7] and $\tau_{s-1}(L)x = 0$. Note that $Lx/Lx^2 \cong M/Mx$, and hence x has the required properties. \square

2. AN INDUCTIVE STEP

This section contains a key step of the proof of Theorem 0.1. The material presented here is technical. Throughout this section we assume A and B are noetherian PI algebras.

Hypothesis 2.1. Suppose there is a subset $\Phi \subset \mathbb{Z}$ such that the following hold.

- (1) If $i \in \Phi$, then $\text{Ext}_A^i(S, A) \neq 0$ for all simple A -modules S , and $\text{Ext}_A^i(-, A)$ is an exact functor on A -modules of finite length.
- (2) If $i \notin \Phi$, then $\text{Ext}_A^i(S, A) = 0$ for all simple A -modules S .

We will see in the next section that noetherian affine PI Hopf algebras satisfy Hypothesis 2.1. Starting from Hypothesis 2.1, we want to prove a higher dimensional version of it. To do so, we need to add a right B -module structure to carry on the inductive step.

Hypothesis 2.2. Let s be an integer.

(1s). For every i , $\text{Ext}_A^i(-, A)$ is an exact functor modulo $\text{Kdim}(s - 1)$ when applied to noetherian A - B -modules of $\text{Kdim} \leq s$.

(2s). For every $i \in \Phi$ (where Φ is given in Hypothesis 2.1), $\text{Kdim Ext}_A^{i-s}(M, A) = s$ for all noetherian A - B -modules M of $\text{Kdim } s$.

When $s = 0$, Hypothesis 2.2 follows from Hypothesis 2.1. We need a few lemmas to show that Hypothesis 2.2(1s,2s) follows from Hypothesis 2.1 for all s .

Lemma 2.3. *Let i be a fixed integer.*

- (1) *If $\text{Ext}_A^i(S, A) = 0$ for all simple A -modules S , then $\text{Kdim Ext}_A^{i-s}(M, A) < s$ for all noetherian A -modules M of $\text{Kdim} \leq s$.*
- (2) *If $\text{Ext}_A^j(S, A) = 0$ for all $i - s \leq j \leq i$ and all simple A -modules S , then $\text{Ext}_A^{i-s}(M, A) = 0$ for all noetherian A -modules M of $\text{Kdim} \leq s$.*
- (3) *If $\text{Ext}_A^j(S, A) = 0$ for all $j \leq i$ and all simple A -modules S , then $\text{Ext}_A^{j-s}(M, A) = 0$ for all $j \leq i$ and all noetherian A -modules M such that $\text{Kdim } M \leq s$.*
- (4) *If $\text{Ext}_A^j(S, A) = 0$ for all $j \geq i$ and all simple A -modules S , then $\text{Ext}_A^j(M, A) = 0$ for all $j \geq i$ and all noetherian A -modules M . In particular, $\text{id}_A A < i$.*
- (5) *Let M be a noetherian A -module. If $\text{Tor}_j^A(S, M) = 0$ for all $j \geq i$ and for all simple A -modules S , then $\text{pd } M < i$.*

Proof. (1) We use induction on s . If $s = 0$, the statement follows from the long exact sequence of $\text{Ext}^i(-, A)$ and the noetherian property of M . Now we assume the statement holds for s . Let M be a noetherian A -module with $\text{Kdim } M = s + 1$. By the argument in the proof of Lemma 1.4(2), we may assume M is critical and $\text{l. ann}(M) = \mathfrak{p}$ is a prime of A . Then there is a short exact sequence (see (1.4.1))

$$0 \rightarrow A/\mathfrak{p} \rightarrow M^{\oplus r} \rightarrow N \rightarrow 0,$$

where N is A/\mathfrak{p} -torsion. Hence $\text{Kdim } N < s + 1$. By Lemma 1.4(2),

$$\text{Kdim Ext}_A^j(N, A) \leq s \quad \text{for all } j.$$

It follows from the long exact sequence that we only need to show that

$$\text{Kdim Ext}_A^{i-s-1}(A/\mathfrak{p}, A) < s + 1.$$

Let $B = A/\mathfrak{p}$. Since $\text{Ext}_A^{i-s-1}(B, A)$ is a noetherian (left) B -module, it has Kdim at most $s + 1$. If $\text{Kdim Ext}_A^{i-s-1}(B, A) = s + 1$, then there is a regular central element $c \in B$ such that the map $l_c : \text{Ext}_A^{i-s-1}(B, A) \rightarrow \text{Ext}_A^{i-s-1}(B, A)$ has cokernel of $\text{Kdim } s$. Applying $\text{Ext}_A^i(-, A)$ to the short exact sequence

$$(2.3.1) \quad 0 \rightarrow B \xrightarrow{r_c} B \rightarrow N_c \rightarrow 0,$$

where $N_c = B/(c)$, we obtain an exact sequence

$$(2.3.2) \quad \cdots \rightarrow \text{Ext}_A^{i-s-1}(B, A) \xrightarrow{l_c} \text{Ext}_A^{i-s-1}(B, A) \rightarrow \text{Ext}_A^{i-s}(N_c, A) \rightarrow \cdots$$

Since $\text{Kdim } N_c \leq s$, by the induction hypothesis, $\text{Kdim Ext}_A^{i-s}(N_c, A) < s$. This contradicts the fact that the cokernel of $\text{Ext}_A^{i-s-1}(B, A) \rightarrow \text{Ext}_A^{i-s-1}(B, A)$ has $\text{Kdim } s$. Therefore

$$\text{Kdim Ext}_A^{i-s-1}(B, A) < s + 1.$$

(2) We use the induction similar to (1) with some extra care. Assume that $\text{Ext}_A^j(S, A) = 0$ for all $i - s - 1 \leq j \leq i$ and for all simple A -modules S . By

the induction hypothesis, $\text{Ext}_A^{i-s}(N, A) = 0$ for all N with $\text{Kdim } N \leq s$. By (1), $\text{Kdim } \text{Ext}_A^{i-s-1}(B, A) \leq s$; so there is a regular element $c \in B$ such that

$$c \text{Ext}_A^{i-s-1}(B, A) = 0,$$

or in other words $l_c = 0$. Hence by (2.3.2) the second $\text{Ext}_A^{i-s-1}(B, A)$ in (2.3.2) is a submodule of $\text{Ext}_A^{i-s}(N_c, A)$, which is zero.

(3) This is a special case of (2).

(4) This is a special case of [WZ1, 1.4] (note that the assumption $\text{inj dim } X < \infty$ is not necessary).

(5) We use induction to show that $\text{Tor}_j^A(Z, M) = 0$ for all $j \geq i$ and all noetherian A° -modules Z . The induction process in the proof of Lemma 1.4(2) implies that we only need to consider $Z = A/\mathfrak{p}$ for some prime \mathfrak{p} . If $\text{Kdim } A/\mathfrak{p} = 0$, this is clear; so we assume $\text{Kdim } A/\mathfrak{p} > 0$.

Let $B = A/\mathfrak{p}$. Consider $T := \text{Tor}_j^A(A/\mathfrak{p}, M)$ for some $j \geq i$, which is a noetherian B -module. For every regular central element c in B , we have a short exact sequence (2.3.1). The corresponding long exact sequence of Tor is

$$(2.3.3) \quad \cdots \rightarrow \text{Tor}_j^A(B, M) \xrightarrow{l_c} \text{Tor}_j^A(B, M) \rightarrow \text{Tor}_j^A(N_c, M) \rightarrow \cdots$$

By the induction hypothesis, $\text{Tor}_j^A(N_c, M) = 0$. Hence the map l_c is always surjective. Let T_0 be the B -torsion submodule of T . Then l_c becomes bijective on T/T_0 . Thus $T/T_0 \cong Q(B) \otimes_B (T/T_0)$, where $Q(B)$ is the ring of fractions of B . But any nonzero $Q(B)$ -module is not noetherian over B when $\text{Kdim } B > 0$. Hence $T/T_0 = 0$ and $T = T_0$. Pick c such that $cT_0 = 0$. Then (2.3.3) shows that the second $\text{Tor}_j^A(B, M)$ is a submodule of $\text{Tor}_j^A(N_c, M)$, which is zero.

The above argument shows that the flat dimension of M is less than i . Since M and A are noetherian, the projective dimension of M is equal to its flat dimension. \square

It follows from Lemma 2.3(1) that if $i \notin \Phi$, then $\text{Kdim } \text{Ext}_A^{i-s}(M, A) < s$ for all noetherian A -modules M with $\text{Kdim } M \leq s$.

Lemma 2.4. *Suppose $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is a short exact sequence of A -modules. Then the sequence*

$$0 \rightarrow \text{Ext}_A^i(M_3, A) \rightarrow \text{Ext}_A^i(M_2, A) \rightarrow \text{Ext}_A^i(M_1, A) \rightarrow 0$$

is exact modulo $\text{Kdim } s$ for all i if and only if the map

$$\text{Ext}_A^i(M_2, A) \rightarrow \text{Ext}_A^i(M_1, A)$$

is surjective modulo $\text{Kdim } s$ for all i .

Proof. By the long exact sequence, the surjectivity of $\text{Ext}_A^i(M_2, A) \rightarrow \text{Ext}_A^i(M_1, A)$ implies the injectivity of $\text{Ext}_A^{i+1}(M_3, A) \rightarrow \text{Ext}_A^{i+1}(M_2, A)$ for all i . \square

Lemma 2.5. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of noetherian $(s + 1)$ -pure A - B -bimodules. Suppose that B is $(s + 1)$ -pure and that $\text{r. ann}_B(L)$ is a minimal prime ideal of B . Let $x \in \mathcal{C}_B(0)$ be a special regular element.*

(1) *The sequence*

$$0 \rightarrow L/Lx \rightarrow M/Mx \rightarrow N/Nx \rightarrow 0$$

is exact.

(2) Assume Hypothesis 2.2(1s). Then

$$0 \rightarrow \text{Ext}_A^i(N/Nx, A) \rightarrow \text{Ext}_A^i(M/Mx, A) \rightarrow \text{Ext}_A^i(L/Lx, A) \rightarrow 0$$

is exact modulo $\text{Kdim}(s - 1)$.

Proof. (1) Since $M/Mx \cong M \otimes_B B/Bx$, it suffices to show that the map $L/Lx \rightarrow M/Mx$ is injective, or to show that $\text{Tor}_1^B(N, B/Bx) = 0$. But

$$\text{Tor}_1^B(N, B/Bx) = \{n \in N \mid nx = 0\} = 0,$$

since N_B is $(s + 1)$ -pure.

(2) First of all, since L is $(s + 1)$ -pure, x is regular on L and $\text{Kdim } L/Lx \leq \text{Kdim } L - 1 = s$. The same holds for M/Mx and N/Nx . By Lemma 2.4, it suffices to show the surjectivity of $\text{Ext}_A^i(M/Mx, A) \rightarrow \text{Ext}_A^i(L/Lx, A)$ for all i .

We claim that there is a noetherian A -module Ω such that (i) Ω contains M/Mx as a submodule, and (ii) the map $f : \text{Ext}_A^i(\Omega, A) \rightarrow \text{Ext}_A^i(L/Lx, A)$ is surjective modulo $\text{Kdim}(s - 1)$. If this is true, then the map $g : \text{Ext}_A^i(M/Mx, A) \rightarrow \text{Ext}_A^i(L/Lx, A)$ is surjective modulo $\text{Kdim}(s - 1)$, since f factors through g .

We now construct Ω .

By Lemma 1.6, B has an artinian ring of fractions, denoted by $Q(B)$. Since M is $(s + 1)$ -pure, it follows that M is B° -torsion-free and M is a submodule of $\overline{M} := M \otimes_B Q(B)$. It is clear that the inclusion $M \rightarrow \overline{M}$ is an A - B -bimodule map. In the following we show that the A - B -subbimodule $Mx^{-1}B$ of \overline{M} is noetherian on both sides. Since $Mx^{-1}B = MBx^{-1}B$ is a factor of $M \otimes_B Bx^{-1}B$, it suffices to show that the latter is noetherian on both sides. By the noetherian property of M and induction reduction, we may assume that M is a critical bimodule and $\mathfrak{p} := \text{r. ann}_B(M)$ is a minimal prime of B . Let c be the image of x in $D := B/\mathfrak{p}$. Then

$$M \otimes_B Bx^{-1}B = M \otimes_D Dc^{-1}D = Mc^{-1},$$

which is isomorphic to M as an A - B -bimodule. Hence $M \otimes_B Bx^{-1}B$ is noetherian on both sides.

Let $\Lambda = MBx^{-1}B$. Then Λ is a noetherian A - B -subbimodule of \overline{M} that contains M . Since x is central in $B/\text{r. ann}_B(L)$, $Lx^{-1} = Lx^{-1}B$ is an A - B -subbimodule of Λ . Hence there is an exact sequence

$$0 \rightarrow Lx^{-1}/(Lx^{-1} \cap M) \rightarrow \Lambda/M \rightarrow P \rightarrow 0$$

for some P . By (1), $Mx \cap L = Lx$, which is equivalent to $M \cap Lx^{-1} = L$. Hence we have an exact sequence of noetherian A - B -bimodules

$$(2.5.1) \quad 0 \rightarrow Lx^{-1}/L \rightarrow \Lambda/M \rightarrow P \rightarrow 0.$$

Since \overline{M}/M is right B -torsion, $\text{Kdim } \Lambda/M \leq s$. By Hypothesis 2.2(1s), the map $\text{Ext}_A^i(\Lambda/M, A) \rightarrow \text{Ext}_A^i(Lx^{-1}/L, A)$ is surjective modulo $\text{Kdim}(s - 1)$ for all i . Tensoring (2.5.1) with $- \otimes_B Bx$, we have a short exact sequence

$$0 \rightarrow L/Lx \rightarrow \Lambda x/Mx \rightarrow P \otimes_B Bx \rightarrow 0,$$

which is isomorphic to (2.5.1) as complexes of A -modules. Hence the map

$$\text{Ext}_A^i(\Lambda x/Mx, A) \rightarrow \text{Ext}_A^i(L/Lx, A)$$

is surjective modulo $\text{Kdim}(s - 1)$ for all i . Let $\Omega = \Lambda x/Mx$. It is clear that Ω contains M/Mx . □

Lemma 2.6. *Hypothesis 2.2(1(s+1)) is equivalent to the following:
Hypothesis 2.6(1(s+1)): for every short exact sequence of noetherian (s + 1)-pure A-B-bimodules*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

with $r. \text{ann}_B(L)$ being a minimal prime of B , the sequence

$$0 \rightarrow \text{Ext}_A^i(N, A) \rightarrow \text{Ext}_A^i(M, A) \rightarrow \text{Ext}_A^i(L, A) \rightarrow 0$$

is exact modulo $\text{Kdim } s$ for all i .

Proof. It is clear that Hypothesis 2.6(1(s+1)) is weaker than 2.2(1(s+1)). We now prove that Hypothesis 2.6(1(s+1)) implies Hypothesis 2.2(1(s+1)). Without loss of generality, we may assume $r. \text{ann}_B(M) = 0$.

For every short exact sequence of noetherian A - B -bimodules of $\text{Kdim} \leq s + 1$,

$$(2.6.1) \quad 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

and for every i , we need to show that

$$0 \rightarrow \text{Ext}_A^i(N, A) \rightarrow \text{Ext}_A^i(M, A) \rightarrow \text{Ext}_A^i(L, A) \rightarrow 0$$

is exact modulo $\text{Kdim } s$. By Lemmas 1.3 and 1.4 we may assume that all modules in (2.6.1) are $(s + 1)$ -pure. We use induction on the $(s + 1)$ -rank of the middle term M . (The definition of the rank is given in Section 1.) The statement is trivial when the $(s + 1)$ -rank of M is 1, since in that case either L or N is zero. Similarly we may assume the $(s + 1)$ -ranks of L and N are nonzero and strictly less than the $(s + 1)$ -rank of M . Pick a subbimodule $L_1 \subset L$ such that $r. \text{ann}_B(L_1)$ is a minimal prime of B and M/L_1 is also $(s + 1)$ -pure (this can be achieved by using Lemma 1.5(3)). Then we have a diagram of short exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L_1 & \longrightarrow & L_1 & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L/L_1 & \longrightarrow & M/L_1 & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Applying $\text{Ext}_A^i(-, A)$ to this diagram, we obtain a diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longleftarrow & \text{Ext}_A^i(L_1, A) & \longleftarrow & \text{Ext}_A^i(L_1, A) & \longleftarrow & 0 \longleftarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longleftarrow & \text{Ext}_A^i(L, A) & \longleftarrow & \text{Ext}_A^i(M, A) & \longleftarrow & \text{Ext}_A^i(N, A) \longleftarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longleftarrow & \text{Ext}_A^i(L/L_1, A) & \longleftarrow & \text{Ext}_A^i(M/L_1, A) & \longleftarrow & \text{Ext}_A^i(N, A) \longleftarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

The first column and the third row are exact modulo $\text{Kdim } s$ because the $(s + 1)$ -rank of L and the $(s + 1)$ -rank of M/L_1 are less than the $(s + 1)$ -rank of M . The third column and the first row are trivially exact. The middle column is exact modulo $\text{Kdim } s$ because $\text{r. ann}_B(L_1)$ is a minimal prime of B and by Hypothesis 2.6(1(s+1)). By a version of the five lemma or an easy computation, the middle row is exact modulo $\text{Kdim } s$. □

Lemma 2.7. *Assume Hypothesis 2.2(1s). Then Hypothesis 2.2(1(s+1)) holds.*

Proof. By Lemma 2.6, it suffices to show Hypothesis 2.6(1(s+1)).

Suppose $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of noetherian $(s+1)$ -pure A - B -bimodules such that $\text{r. ann}_B(L) = \mathfrak{p}$ is a minimal prime of B . Then $\text{Kdim } B/\mathfrak{p} = s + 1 > 0$. Without loss of generality, we may assume that $\text{r. ann}_B(M) = 0$. Since M is $(s + 1)$ -pure, B is also $(s + 1)$ -pure by [GW, 8.9] and Lemma 1.5(1).

Since B is $(s + 1)$ -pure, it has an artinian ring of fractions [Lemma 1.6]. By Lemma 2.4 it suffices to show that the map $\phi : \text{Ext}_A^{i-1}(M, A) \rightarrow \text{Ext}_A^{i-1}(L, A)$ is surjective modulo $\text{Kdim } s$ for all i . We prove it by contradiction. Assume that ϕ is not surjective modulo $\text{Kdim } s$. Let C be the cokernel of ϕ . Hence C is a noetherian B/\mathfrak{p} - A -bimodule of $\text{Kdim } (s + 1)$. By Lemma 1.7, there is a special regular element $x \in \mathcal{C}_B(0)$ such that $\text{Kdim } xC/x^2C = s$ and $x\tau_s(C) = 0$. In particular, the image \bar{x} is central in B/\mathfrak{p} .

We now consider the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow x^n & & \downarrow x^n & & \downarrow x^n \\
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L/Lx^n & \longrightarrow & M/Mx^n & \longrightarrow & N/Nx^n \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

By Lemma 2.5(1) the last row and hence all short exact sequences are exact. For the rest of the proof we write $E^i(M)$ for $\text{Ext}_A^i(M, A)$. Applying $E^i(-)$ to the above, we have

$$\begin{array}{ccccccc}
 \longleftarrow & E^i(L) & \longleftarrow & E^i(M) & \longleftarrow & E^i(N) & \longleftarrow \\
 & \uparrow & & \uparrow & & \uparrow & \\
 0 & \longleftarrow & E^i(L/Lx^n) & \longleftarrow & E^i(M/Mx^n) & \longleftarrow & E^i(N/Nx^n) \longleftarrow 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
 \longleftarrow & E^{i-1}(L) & \longleftarrow & E^{i-1}(M) & \longleftarrow & E^{i-1}(N) & \longleftarrow \\
 & \uparrow l_{x^n} & & \uparrow l_{x^n} & & \uparrow l_{x^n} & \\
 \longleftarrow & E^{i-1}(L) & \longleftarrow & E^{i-1}(M) & \longleftarrow & E^{i-1}(N) & \longleftarrow
 \end{array}$$

where the second row is exact modulo $\text{Kdim}(s - 1)$ by Lemma 2.5(2). Note that $E^{i-1}(M)$ is a B - A -bimodule and the left multiplication by x^n , denoted by l_{x^n} , is the map given in the above diagram. The cokernel of the middle l_{x^n} is mapped into $E^i(M/Mx^n)$. In other words, the induced map

$$E^{i-1}(M)/x^n E^{i-1}(M) \rightarrow E^i(M/Mx^n)$$

is injective. The same holds for L and N . We now have a diagram (2.7.1)

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 E^{i-1}(M)/x^n E^{i-1}(M) & \xrightarrow{f} & E^{i-1}(L)/x^n E^{i-1}(L) & \xrightarrow{g} & C/x^n C & \longrightarrow & 0 \\
 \alpha \downarrow & & \beta \downarrow & & & & \\
 E^i(M/Mx^n) & \xrightarrow{h} & E^i(L/Lx^n) & \longrightarrow & 0 & & \\
 \downarrow & & \downarrow & & & & \\
 \tau_s(E^i(M)) & & \tau_s(E^i(L)) & & & &
 \end{array}$$

The first row of (2.7.1) is exact since it is obtained by applying $B/x^n B \otimes_B -$ to an exact sequence. The second row of (2.7.1) is exact modulo $\text{Kdim}(s - 1)$ by Lemma 2.5(2). The columns are clearly exact. The s -ranks of all modules in (2.7.1) are finite, and some of them might be dependent on n . The s -rank of $\tau_s(E^i(M))$, say r , is finite and independent of n . Let K be the kernel of h . Then the kernel of f is isomorphic to $(\text{im } \alpha) \cap K$. Since the s -rank is additive with respect to short exact sequences, an easy computation shows that

$$\begin{aligned} \text{rank}_s(C/x^n C) &= \text{rank}_s(\ker f) - \text{rank}_s(E^{i-1}(M)/x^n E^{i-1}(M)) \\ &\quad + \text{rank}_s(E^{i-1}(L)/x^n E^{i-1}(L)) \\ &= \text{rank}_s(\ker f) - (\text{rank}_s(E^i(M/Mx^n)) - \text{rank}_s(\text{coker } \alpha)) \\ &\quad + (\text{rank}_s(E^i(L/Lx^n)) - \text{rank}_s(\text{coker } \beta)) \\ &= \text{rank}_s(\ker f) - \text{rank}_s(\ker h) + \text{rank}_s(\text{coker } \alpha) - \text{rank}_s(\text{coker } \beta) \\ &= \text{rank}_s(\text{im } \alpha \cap K) - \text{rank}_s(K) + \text{rank}_s(\text{coker } \alpha) - \text{rank}_s(\text{coker } \beta) \\ &\leq \text{rank}_s(\text{coker } \alpha) \leq r. \end{aligned}$$

By the choice of x given in Lemma 1.7, $\text{Kdim } xC/x^2 C = s$ and $x^n C/x^{n+1} C \cong xC/x^2 C$ for all i . Thus the s -rank of $C/x^n C$ cannot be bounded by r for large n . This yields a contradiction, as desired. \square

Lemma 2.8. *Suppose Hypothesis 2.2(1s,2s) holds. Let L be a noetherian A - B -bimodule with $\text{Kdim } L = s + 1$ such that $\text{r. ann}_B(L)$ is a minimal prime of B . Then for every $i \in \Phi$,*

$$\text{Kdim Ext}_A^{i-s-1}(L, A) = s + 1.$$

Proof. By Lemma 1.4(3) we may assume that L is $(s + 1)$ -pure. By Lemma 1.7, there is a special regular element x such that $\text{Kdim } L/Lx = s$. Applying $\text{Ext}_A^i(-, A)$ to the short exact sequence

$$0 \rightarrow L \xrightarrow{x^n} L \rightarrow L/Lx^n \rightarrow 0,$$

we obtain an exact sequence

$$\rightarrow \text{Ext}_A^{i-s-1}(L, A) \rightarrow \text{Ext}_A^{i-s-1}(L, A) \rightarrow \text{Ext}_A^{i-s}(L/Lx^n, A) \rightarrow \text{Ext}_A^{i-s}(L, A) \rightarrow .$$

If $\text{Kdim Ext}_A^{i-s-1}(L, A) < s + 1$, we may pick x such that $x \text{Ext}_A^{i-s-1}(L, A) = 0$. Thus we have an exact sequence of modules of $\text{Kdim} \leq s$,

$$0 \rightarrow \text{Ext}_A^{i-s-1}(L, A) \rightarrow \text{Ext}_A^{i-s}(L/Lx^n, A) \rightarrow \tau_s \rightarrow ,$$

where $\tau_s = \tau_s(\text{Ext}_A^{i-s}(L, A))$. This shows that the s -rank of $\text{Ext}_A^{i-s}(L/Lx^n, A)$ is bounded for all n . By Hypothesis 2.2(2s) the s -rank of $\text{Ext}_A^{i-s}(L/Lx, A)$ is nonzero. Applying Hypothesis 2.2(1s) to the short exact sequence

$$0 \rightarrow Lx/Lx^n \rightarrow L/Lx^n \rightarrow L/Lx \rightarrow 0,$$

we see that the s -rank of $\text{Ext}_A^{i-s}(L/Lx^n, A)$ is equal to n times the s -rank of $\text{Ext}_A^{i-s}(L/Lx, A)$. This yields a contradiction, and therefore $\text{Kdim Ext}_A^{i-s-1}(L, A) = s + 1$. \square

Theorem 2.9. *Assume Hypothesis 2.1. Then Hypothesis 2.2(1s,2s) holds for all s .*

Proof. When $s = 0$ the assertion is clear. By induction and Lemma 2.7, Hypothesis 2.2(1s) holds for all s .

Suppose now Hypothesis 2.2(2s) holds. We prove Hypothesis 2.2(2(s+1)) by induction on the $(s + 1)$ -rank of the noetherian A - B -bimodule M . By the first paragraph, Hypothesis 2.2(1w) holds for all w . By Lemma 1.4(3), it suffices to show that $\text{Kdim Ext}_A^{i-s-1}(M/\tau_s(M), A) = s + 1$. So we may assume M is $(s + 1)$ -pure. If the $(s + 1)$ -rank of M is 1, then M is critical and $\text{r.ann}_B(M)$ is prime. Replacing B by $B/\text{r.ann}_B(M)$, we may assume B is prime. By Lemma 2.8, $\text{Kdim Ext}_A^{i-s-1}(M, A) = s + 1$ for all $i \in \Phi$, and Hypothesis 2.2(2(s+1)) holds for this special case. Since $\text{Ext}_A^{i-s-1}(-, A)$ is exact [Hypothesis 2.2(1(s+1))], induction on the $(s + 1)$ -rank of M implies that Hypothesis 2.2(2(s+1)) holds for all M .

The induction on s completes the proof. □

Corollary 2.10. *Suppose Hypothesis 2.1 holds.*

- (1) Φ is non-empty and consists of a single element $\{d\}$.
- (2) $\text{id}_A A = d = \text{Kdim } A < \infty$.

Proof. By Theorem 2.9, Hypothesis 2.2(1s,2s) holds for all s .

(1) If Φ is empty, by Lemma 2.3(4), $\text{Hom}_A(A, A) = 0$. This is impossible. Let d' be any element in Φ . Let M be a noetherian A - B -bimodule with $\text{Kdim } M = s$. So by Hypothesis 2.2(2s), $\text{Ext}_A^{d'-s}(M, A) \neq 0$. Since $\text{Ext}^i(-, A) = 0$ for all $i < 0$, it follows that $d' - s \geq 0$. This means that s is bounded by d' , and hence every noetherian A - B -bimodule of finite Krull dimension has Krull dimension no more than d' . It follows from the inductive definition of the Krull dimension that every noetherian A - B -bimodule has Krull dimension no more than d' . Since A itself is a noetherian A - A -bimodule (where $B = A$), we get $d := \text{Kdim } A \leq d'$. Now Hypothesis 2.2(2s) implies that

$$\Phi \subseteq \{i \mid \text{Kdim Ext}_A^{i-d}(A, A) = d\}.$$

Since $\text{Ext}_A^i(A, A) = 0$ for all $i \neq 0$, $\Phi = \{d\}$.

(2) Lemma 2.3(4), Hypothesis 2.1, and the fact that $\Phi = \{d\}$ imply that $\text{id}_A A = d$. □

If A is not noetherian, then Φ in Hypothesis 2.1 could be empty, as the next example shows.

Example 2.11. Let $A = \mathbb{C}[x_1, x_2, x_3, \dots]$ be the commutative polynomial ring of countably many variables over the field of complex numbers. Then A is a Hopf algebra, and every simple A -module is 1-dimensional over \mathbb{C} . It is not noetherian. One can verify that $\text{Ext}_A^i(S, A) = 0$ for all i and all simple A -modules S . Thus Hypothesis 2.1 holds when Φ is empty.

3. THE MAIN RESULTS

We prove Theorems 0.1 and 0.2 in this section. Recall that an algebra is *Gorenstein* if it has finite left and right injective dimension over itself. There are two stronger variants of the Gorenstein property for noncommutative rings (see the discussions in [Le]). Here is the first one.

Definition 3.1. Let A be a noetherian algebra. We say A is *left AS-Gorenstein* (where AS stands for Artin and Schelter) if

- (1) ${}_A A$ has finite injective dimension, say d , and
- (2) for every simple A -module S , $\text{Ext}_A^i(S, A) = 0$ for all $i \neq d$ and $\text{Ext}_A^d(S, A)$ is a simple A° -module.

Right AS-Gorensteinness is defined similarly. We say A is *AS-Gorenstein* if it is both left and right AS-Gorenstein.

Proposition 3.2. *Let A be a noetherian PI algebra. Then the following are equivalent.*

- (1) A is AS-Gorenstein.
- (2) A is left AS-Gorenstein.
- (3) A is right AS-Gorenstein.
- (4) Hypothesis 2.1 holds.

We need a slightly more general version of the double-Ext spectral sequence. This was first proved by Ischebeck [Is, 1.8]; so it is sometimes called *Ischebeck's spectral sequence*.

Lemma 3.3. *Let A and B be algebras. Let M be an A° -module and N be an A - B -bimodule. Suppose that M_A has a resolution by finitely generated projectives, and that ${}_A N$ has finite injective dimension. Then there is a convergent spectral sequence of B° -modules*

$$E_2^{p,q} := \text{Ext}_A^p(\text{Ext}_{A^\circ}^{-q}(M, A), N) \Rightarrow \text{Tor}_{-p-q}^A(M, N).$$

Proof of Proposition 3.2. (1) \Rightarrow (2) and (2) \Rightarrow (4) are easy. (2) and (3) are symmetric. So it remains to show (4) \Rightarrow (1).

Assume Hypothesis 2.1 holds. By Corollary 2.10, ${}_A A$ has finite injective dimension, say d , and $\Phi = \{d\}$. By Hypothesis 2.1, we see that if S is an A -module of finite length, then $\text{Ext}_A^i(S, A) = 0$ for all $i \neq d$. By Lemma 1.4(2), $\text{Ext}_A^d(S, A)$ is an A° -module of finite length.

The Ischebeck spectral sequence in Lemma 3.3 doesn't require that A_A has finite injective dimension. Set $N = A$ and let M be a nonzero finite length A° -module in Lemma 3.3. Since every $\text{Ext}_{A^\circ}^i(M, A)$ is an A -module of finite length, $\text{Ext}_A^p(\text{Ext}_{A^\circ}^i(M, A), A) = 0$ for all $p \neq d$. Hence the spectral sequence collapses to

$$\text{Ext}_A^d(\text{Ext}_{A^\circ}^q(M, A), A) \cong \text{Tor}_{-d+q}^A(M, A) = \begin{cases} M & \text{if } q = d, \\ 0 & \text{if } q \neq d. \end{cases}$$

Thus Hypothesis 2.1 (when $\Phi = \{d\}$) implies that $\text{Ext}_{A^\circ}^d(M, A) \neq 0$ and that $\text{Ext}_{A^\circ}^q(M, A) = 0$ for all $q \neq d$. The long exact sequence shows that $\text{Ext}_{A^\circ}^q(-, A)$ is exact on finite length A° -modules for all q . This means that the right-handed version of Hypothesis 2.1 holds. By the right-handed version of Corollary 2.10, $\text{id } A_A = d$.

By Ischebeck's spectral sequence (when $N = A$), we see that the functors $\text{Ext}_A^d(-, A)$ and $\text{Ext}_{A^\circ}^d(-, A)$ induce a duality between the category of finite length A -modules and that of A° -modules. In particular, if S is a simple A -module, $\text{Ext}_A^d(S, A)$ is a simple A° -module. This statement holds when A and A° are exchanged. Therefore A is left and right AS-Gorenstein. \square

We now come to Hopf algebras. We refer to [Mon] for basic definitions and properties about general Hopf algebras, and to [Br], [BG] for noetherian PI Hopf algebras. Let H be a Hopf algebra over k and let k also denote the trivial H -module

$H/\epsilon^{-1}(0)$, where $\epsilon : H \rightarrow k$ is the counit. The following lemma is [BG, 1.11] and [WZ1, 4.8]. The proof of [WZ1, 4.8] was given for one n , but works for all n . Recall that an H -module means a left H -module.

Lemma 3.4. *Let H be a Hopf algebra. Let n be an integer.*

- (1) *The following conditions are equivalent:*
 - (a) $\text{Ext}_H^n({}_H k, H) \neq 0$.
 - (b) $\text{Ext}_H^n(V, H) \neq 0$ for some finite dimensional H -module V .
 - (c) $\text{Ext}_H^n(V, H) \neq 0$ for all nonzero finite dimensional H -modules V .
- (2) $\text{Ext}_H^n(-, H)$ is exact on finite dimensional H -modules.

Lemma 3.4 is basically saying that Hypothesis 2.1 holds for H if all simple H -modules are finite dimensional. We now prove Theorem 0.1.

Theorem 3.5. *Let H be a noetherian PI Hopf algebra over k such that every simple H -module is finite dimensional over k . Then H is AS-Gorenstein.*

Proof. By assumption, every simple H -module is finite dimensional. Let Φ be the set of integers n such that $\text{Ext}_H^n(k, H) \neq 0$. By Lemma 3.4, Hypothesis 2.1 holds for H . By Proposition 3.2, H is left and right AS-Gorenstein. □

Proof of Theorem 0.1. When H is affine, every simple H -module is finite dimensional [MR, 13.10.3(i)]. The assertion follows from Theorem 3.5. □

Theorem 0.2 is known for all AS-Gorenstein noetherian PI algebras. Next we recall the second stronger version of the Gorenstein property.

Definition 3.6. We say a noetherian ring A is *Auslander-Gorenstein* if

- (1) A is Gorenstein, i.e., A has finite left and right injective dimension,
- (2) for every noetherian A -module M and every A° -submodule $N \subset \text{Ext}_A^j(M, A)$, one has $\text{Ext}_{A^\circ}^i(N, A) = 0$ for all $i < j$, and
- (3) part (2) holds when A and A° are exchanged.

The *grade* (or the *j -number*) of M is defined to be

$$j(M) = \min\{i \mid \text{Ext}_A^i(M, A) \neq 0\}.$$

The same is defined for A° -modules. Let δ be a dimension function. We say A is *δ -Macaulay* if

$$j(M) + \delta M = \delta A$$

for all noetherian A -modules (and A° -modules) M . In this paper δ is either Kdim or GKdim . When $\delta = \text{GKdim}$, GKdim -Macaulay is also called *Cohen-Macaulay*.

A version of *depth* is defined for non-local algebras in [WZ1, p. 521]. Let A be a noetherian ring and let M be a noetherian A -module. The *depth* of M is defined to be

$$\text{depth } M = \min\{i \mid \text{Ext}_A^i(S, M) \neq 0 \text{ for some simple } A\text{-module } S\}.$$

We refer to [WZ1] for the Auslander-Buchsbaum formula, Bass's theorem, and the no-holes theorem for AS-Gorenstein rings. The catenary property is defined in [MR, 13.10.13]. A good reference for the catenarity of an Auslander-Gorenstein algebra is [GL].

The following is a collection of some known results.

Proposition 3.7. *Let A be an AS-Gorenstein noetherian PI algebra of injective dimension d .*

- (1) A is Auslander-Gorenstein and Kdim -Macaulay.
- (2) A has a quasi-Frobenius artinian ring of fractions.
- (3) For every minimal prime \mathfrak{p} of A , $\text{Kdim } A/\mathfrak{p} = d$.
- (4) A has a resolution of A -bimodules

$$(3.7.1) \quad 0 \rightarrow A \rightarrow I^{-d} \rightarrow I^{-d+1} \rightarrow \dots \rightarrow I^0 \rightarrow 0$$

such that when restricted to the left-hand side (respectively, the right-hand side), (3.7.1) is a minimal injective resolution of ${}_A A$ (respectively, A_A), and each I^{-i} is pure of $\text{Kdim } i$ on the left and the right respectively.

- (5) If M is a noetherian A -module of finite projective dimension, then $\text{pd } M + \text{depth } M = d$.
- (6) If M is a noetherian A -module of finite injective dimension, then $\text{id } M = d$.
- (7) A is catenary.

Proof. (1) is an ungraded analogue of [SZ, 3.10], and (2,3) follow from [AjSZ, 6.1].

(4) By [YZ2, 4.10], A has a residual complex. By the definition of residual complex [YZ2, 4.3], the resolution has the required properties.

(5,6) are [WZ1, 0.1], and (7) follows from [GL, 1.6] or [YZ1, 2.23]. \square

Corollary 3.8. *Let H be a noetherian PI Hopf algebra over k such that every simple H -module is finite dimensional. Then H is AS-Gorenstein, and the assertions in Proposition 3.7 hold.*

Proof. This follows from Theorem 3.5 and Proposition 3.7. \square

Proof of Theorem 0.2. By Lemma 1.2(3), $\text{Kdim} = \text{GKdim}$ when H is affine. Now Theorem 0.2 follows from Corollary 3.8 or Proposition 3.7. \square

Remark 3.9. We do not assume that a noetherian PI Hopf algebra has a bijective antipode. It is unknown whether every noetherian affine PI Hopf algebra has a bijective antipode.

4. PROJECTIVITY OVER SUBALGEBRAS

We prove Theorem 0.3 in this section. The key property we use is the AS-Gorenstein property of Hopf algebras H and H' . First we recall a standard spectral sequence for change of rings (see [Rot, 11.65]).

Lemma 4.1. *Let $A \rightarrow B$ be an algebra homomorphism. Let M be an A -module and N be a B -module. Then there is a convergent spectral sequence*

$$(4.1.1) \quad \text{Ext}_B^p(\text{Tor}_q^A(B, M), N) \Rightarrow \text{Ext}_A^{p+q}(M, N).$$

Lemma 4.2. *Let A be a noetherian Gorenstein algebra and let M be a noetherian A -module. Then $\text{pd } M < \infty$ if and only if $\text{id } M < \infty$.*

Proof. Here we need a little bit about complexes and derived functors. Since A has finite left and right injective dimension, A is a dualizing complex over A in the sense of [Ye] (or see [YZ1, 1.1]). Let F be the derived functor $\text{RHom}_A(-, A)$ and F° be the derived functor $\text{RHom}_{A^\circ}(-, A)$.

We prove that the assertion holds for bounded complexes M with noetherian cohomologies. First assume $\text{pd } M < \infty$. Then M has a bounded resolution by

noetherian projectives. Since A has finite injective dimension, each noetherian projective has finite injective dimension. Hence M has finite injective dimension. Conversely, if $\text{id } M < \infty$, then by [WZ1, 2.1] the bounded complex $F(M)_A$ has finite projective dimension. By the above argument, $F(M)_A$ has finite injective dimension. By [WZ1, 2.1] again, $M \cong F^\circ F(M)$ has finite projective dimension. \square

Proposition 4.3. *Let A and B be noetherian PI AS-Gorenstein algebras of the same injective dimension and let $A \rightarrow B$ be an algebra homomorphism such that B_A and ${}_A B$ are noetherian. Then the following are equivalent:*

- (1) $\text{id } B_A < \infty$.
- (2) $\text{id } {}_A B < \infty$.
- (3) $\text{pd } B_A < \infty$.
- (4) $\text{pd } {}_A B < \infty$.
- (5) B_A is projective.
- (6) ${}_A B$ is projective.

As a consequence, if A has finite global dimension, then B_A and ${}_A B$ are projective.

Proof. Let $d = \text{id } B = \text{id } A$.

Let S be a simple A -module. Then $\text{Tor}_q^A(B, S)$ is a B -module of finite length. By the AS-Gorenstein property of B , $\text{Ext}_B^p(\text{Tor}_q^A(B, S), B) = 0$ for all $p \neq d$. Let $M = S$ and $N = B$ in (4.1.1). Then (4.1.1) becomes

$$(4.3.1) \quad \text{Ext}_B^d(\text{Tor}_q^A(B, S), B) \cong \text{Ext}_A^{d+q}(S, B)$$

for all q .

(2) \Rightarrow (5) If $\text{id } {}_A B < \infty$, by Bass’s theorem in [WZ1, 0.1(2)], $\text{id } {}_A B = d$. Then $\text{Ext}_A^{d+q}(S, B) = 0$ for $q > 0$. By (4.3.1) and the AS-Gorenstein property of B , $\text{Tor}_q^A(B, S) = 0$ for all $q > 0$ and for all simple A -modules S . By Lemma 2.3(5), $\text{pd } B_A = 0$.

(1) \Rightarrow (6) is similar to (2) \Rightarrow (5).

(5) \Rightarrow (3) and (6) \Rightarrow (4) are trivial. By Lemma 4.2, (1) and (3) are equivalent and (2) and (4) are equivalent. Therefore these are all equivalent.

If A has finite global dimension, then (3) and (4) hold. Therefore (5) and (6) hold. \square

Immediately Proposition 4.3 holds for Hopf algebras in Theorem 3.5, and in particular, we have Theorem 0.3.

Proof of Theorem 0.3. By Theorem 0.2, H and H' are AS-Gorenstein. Hence the assertion follows from Proposition 4.3. \square

Corollary 4.4. *Let H be a Hopf algebra that is a finitely generated module over its affine center. Then there is a central subring $C \subset H$ such that $C \cong k[z_1, \dots, z_d]$ and H is a finitely generated free C -module.*

Proof. Let Z be the center of H , which is an affine commutative ring. By Noether normalization, there is a subring $C \subset Z$ such that $C \cong k[z_1, \dots, z_d]$ and Z_C is finitely generated. Here $d = \text{GKdim } Z$, which is equal to $\text{GKdim } H$. Since C is another Hopf algebra and regular, by Theorem 0.3, H_C is projective. Since C is a polynomial ring, projective C -modules are free. \square

Remark 4.5. There are noetherian commutative Hopf algebras $H \subset H'$ such that H'_H is finitely generated but not free (see [OS] and [Mon, 3.5.2]). By a result of Takeuchi [Ta1], if H is a sub-Hopf algebra of a commutative (or cocommutative) Hopf algebra H' , then H'_H is always faithfully flat.

5. EXTENDED NOETHERIAN CASE

In this section we study some noetherian PI Hopf algebras that are possibly not affine. If every noetherian PI Hopf algebra is affine, then this section is redundant. Molnar proved that every noetherian commutative Hopf algebra over a field is affine [Mol]. Molnar's proof was based on two facts about commutative Hopf algebras: (a) the antipode has finite order (the order is 2 in this case) and (b) there is a bijection between Hopf subalgebras and Hopf ideals. Neither of these has been established for noetherian PI Hopf algebras. At this point we could not solve the following question.

Question 5.1. Is every noetherian (PI) Hopf algebra affine?

We can extend Brown's question to the non-affine case.

Question 5.2. Is every noetherian PI Hopf algebra Gorenstein?

We don't have an answer to Question 5.2 either. A k -algebra A is called *extended noetherian* if for every field extension $K \supset k$, $A \otimes_k K$ is noetherian. We show that some versions of Theorems 0.1, 0.2 and 0.3 hold for extended noetherian PI Hopf algebras. If the answer to Question 5.1 is negative, then it is worth presenting this section. Another reason why we present this section is that some ideas here apply to general extended noetherian rings. There are many extended noetherian rings that are not affine (e.g., $A = k(x)$); however, we don't know of any extended noetherian PI Hopf algebra that is not affine.

Let K be a field extension of k . We write A_K for the algebra $A \otimes_k K$, and M_K for $M \otimes_k K$. We also use $\text{Kdim}_K M_K$ for the Krull dimension of M_K as an A_K -module.

Lemma 5.3. *Let A be a noetherian PI k -algebra.*

- (1) *If A is affine, then A is extended noetherian.*
- (2) *If A is extended noetherian, then $\text{GKdim } A < \infty$.*
- (3) *If A is extended noetherian, then there is a field extension K such that every simple A_K -module is finite dimensional.*
- (4) *Let $\text{GKdim } A = s < \infty$. Let K be a field extension of k such that (a) $A \otimes_k K$ is noetherian and (b) $K \supset k(t_1, \dots, t_s)$. Then $\text{Kdim}_K M_K = \text{GKdim } M$ for all noetherian A -modules M .*

Proof. (1) This is a special case of [Sm, Proposition 53]. A proof was also given in [ArSZ, 4.4].

(2) By Lemma 1.1(2) we may assume A is prime. Let Q be the ring of fractions of A . Then Q is extended noetherian. Let Z be the center of Q ; then Z is extended noetherian and Q is a finitely generated module over Z [Row, 6.1.25]. By a result of Vámos [Va, 11], $Z \otimes_k Z$ is noetherian if and only if Z is finitely generated as a field extension. Hence $\text{GKdim } Z < \infty$ [Row, 6.3.41]. This implies that Q and A have finite Gelfand-Kirillov dimension.

(3) Let K be a field extension of k such that (a) K is algebraically closed and (b) the cardinality of K is bigger than the number of k -algebra generators of A . Then by [Row, 6.1.24 and 6.1.15] every simple A_K -module is finite dimensional.

(4) Without loss of generality, we may assume that $K = k(t_1, \dots, t_s) := K_s$. We use induction on $\text{Kdim } M$. By the reduction in the proof of Lemma 1.4(2) we may assume $M = A/\mathfrak{p}$ for some prime \mathfrak{p} . Without loss of generality, we may assume $\mathfrak{p} = 0$ and A is prime with an artinian ring of fractions, say Q . Let n be $\text{GKdim } A$, which is no more than s . Since Q is obtained by inverting central elements in A [MR, 13.6.5], $\text{GKdim } Q = \text{GKdim } A$ [MR, 8.2.13]. Letting Z be the center of Q , we see by Noether normalization that Z contains a copy of K_n . Since $K_n \otimes_k K_n$ has Krull dimension n , the Krull dimension of $Q \otimes_k K_n$ is at least n [MR, 6.6.17]. Therefore $\text{Kdim}_K A_K \geq \text{Kdim}_{K_n} A_{K_n} \geq n$ [MR, 6.5.3]. By Lemma 1.1(5), $\text{Kdim}_K A_K \leq \text{GKdim } A \otimes_k K = n$ [Row, 6.2.26']. \square

Lemma 5.4. *Let A be a noetherian algebra. Let M and N be noetherian A -modules. Then for every i and for every field extension $K \supset k$,*

$$\text{Ext}_{A_K}^i(M_K, N_K) \cong \text{Ext}_A^i(M, N) \otimes_k K.$$

Proof. This is true for $M = A$. The assertion follows from the fact that M has a resolution by finite free A -modules. \square

Lemma 5.5. *Let A be a noetherian algebra and let M be a noetherian A -module. Let K be a field extension such that A_K is noetherian.*

- (1) $\text{pd } M_K = \text{pd } M$.
- (2) $\text{id } M_K \geq \text{id } M$.
- (3) *If A_K is Gorenstein, so is A .*
- (4) *If A_K is Auslander-Gorenstein, so is A .*
- (5) $j(M_K) = j(M)$.
- (6) *If A_K is Cohen-Macaulay, so is A .*

Proof. (1) Since the faithfully flat functor $- \otimes_k K$ preserves projective modules, it preserves pd .

(2) follows from Lemma 5.4, and (3) is a special case of (2).

(4,5) Both of these follow from Lemma 5.4 and the definitions.

(6) This follows from (5) and the fact that GKdim is preserved by field extensions [Row, 6.2.26']. \square

Theorem 5.6. *Let H be an extended noetherian PI Hopf algebra over a field k and let d be the injective dimension of ${}_H H$.*

- (1) *There is a field extension K such that H_K is AS-Gorenstein and the conclusions in Proposition 3.7 hold for the Hopf algebra H_K .*
- (2) *H is Gorenstein, and $\text{id } H = \text{id } H_K$. In particular, d is finite.*
- (3) $\text{GKdim } H = d$.
- (4) *H is Auslander-Gorenstein and Cohen-Macaulay.*
- (5) $\text{Kdim } H \leq d$.
- (6) *H has a quasi-Frobenius artinian ring of fractions.*
- (7) *For every minimal prime \mathfrak{p} of H , $\text{GKdim } H/\mathfrak{p} = d$.*
- (8) *H has a resolution of H -bimodules*

$$(5.6.1) \quad 0 \rightarrow H \rightarrow I^{-d} \rightarrow I^{-d+1} \rightarrow \dots \rightarrow I^0 \rightarrow 0$$

such that when restricted to the left-hand side (respectively, the right-hand side), (5.6.1) is a minimal injective resolution of ${}_H H$ (respectively, H_H), and each I^{-i} is pure of GKdim i on the left and the right respectively.

- (9) If M is a noetherian H -module of finite projective dimension, then $\text{pd } M \leq d$.
 (10) If M is a noetherian H -module of finite injective dimension, then $\text{id } M \leq d$.
 (11) H is catenary.

Proof. We choose an extension field $K \supset k$ such that the conclusions of Lemma 5.3(3,4) hold.

(1) This follows from Lemma 5.3(3) and Corollary 3.8.

(2) Let $n = \text{id } H_K$. Since H_K is AS-Gorenstein (see (1)), it follows that n is finite and $\text{Ext}_{H_K}^n(K, H_K) \neq 0$. By Lemma 5.4, $\text{Ext}_H^n(k, H) \neq 0$. Thus $\text{id } H \geq n$. By Lemma 5.5(2), $\text{id } H \leq \text{id } H_K = n$. Therefore $n = d$.

(3) By Proposition 3.7(3) for H_K ,

$$\text{Kdim } {}_K H_K = \max_{\mathfrak{p}} \{ \text{Kdim } H_K / \mathfrak{p} \} = d,$$

where \mathfrak{p} runs over all minimal prime ideals of H_K . By the choice of K , we have $\text{GKdim } H = \text{Kdim } {}_K H_K = d$.

(4) This follows from (1) and Lemma 5.5(4,6).

(5) This follows from (3) and Lemma 1.1(5).

(6,7) These follow from (4) and [AjSZ, 6.1].

(8) This follows from (4) and [YZ2, 4.10].

(9) By (1) and Proposition 3.7(5), $\text{pd } M_K = d - \text{depth } M_K \leq d$. The assertion follows from Lemma 5.5(1).

(10) Since H is Gorenstein, by Lemma 4.2, $\text{pd } M < \infty$. By Lemma 5.5(1), $\text{pd } M_K < \infty$. By Lemma 4.2 again, $\text{id } M_K < \infty$. By Proposition 3.7(6), $\text{id } M_K = d$. By Lemma 5.5(2), $\text{id } M \leq \text{id } M_K$.

(11) This follows from (4) and [GL, 1.6] or [YZ1, 2.23]. \square

Corollary 5.7. *If H is an extended noetherian PI Hopf algebra and every simple H -module is finite dimensional, then $\text{Kdim } M = \text{GKdim } M$ for all noetherian H -modules M .*

Proof. By Theorem 5.6(4), H is Cohen-Macaulay. By Corollary 3.8, H is Kdim-Macaulay. Hence $\text{Kdim} = \text{GKdim}$. \square

Corollary 5.8. *Let H and H' be extended noetherian PI Hopf algebras of the same injective dimension. Suppose that there is an algebra homomorphism $H \rightarrow H'$ such that H' is a noetherian H -module on both sides. If either $\text{id}_H H' < \infty$ or $\text{pd}_H H' < \infty$, then H'_H and ${}_H H'$ are projective.*

Proof. By Lemma 5.3(3), there is a field extension K such that the simple modules over $A := H \otimes_k K$ and over $B := H' \otimes_k K$ are finite dimensional over K . By Theorem 5.6(1), A and B are AS-Gorenstein. Hence the algebra map $A \rightarrow B$ satisfies the hypotheses in Proposition 4.3. If either $\text{id}_H H' < \infty$ or $\text{pd}_H H' < \infty$, then, by Lemmas 4.2 and 5.5(1), $\text{id}_A B < \infty$ and $\text{pd}_A B < \infty$. By Proposition 4.3, B is projective over A on both sides. By Lemma 5.5(1), H' is projective over H on both sides. \square

Remark 5.9. We don't know of any noetherian Hopf algebra that is not Gorenstein. So it is natural to extend Brown's question [Br, Question A] to the non-PI case:

is every noetherian Hopf algebra Gorenstein? We have shown in [WZ2] that this is true for graded rings with balanced dualizing complexes: *Let H be a noetherian locally finite \mathbb{N} -graded Hopf algebra. If H has a balanced dualizing complex, then it has finite left and right injective dimension.*

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