

## MAXIMAL FUNCTIONS WITH POLYNOMIAL DENSITIES IN LACUNARY DIRECTIONS

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ABSTRACT. Given a real polynomial  $p(t)$  in one variable such that  $p(0) = 0$ , we consider the maximal operator in  $\mathbb{R}^2$ ,

$$M_p f(x_1, x_2) = \sup_{h>0, i, j \in \mathbb{Z}} \frac{1}{h} \int_0^h |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt .$$

We prove that  $M_p$  is bounded on  $L^q(\mathbb{R}^2)$  for  $q > 1$  with bounds that only depend on the degree of  $p$ .

### 1. INTRODUCTION

Maximal operators on the real line of the form

$$(1.1) \quad f(x) \mapsto \sup_{h>0} \frac{1}{h} \int_0^h |f(x - p(t))| dt ,$$

where  $p$  is a real polynomial with  $p(0) = 0$ , were considered in [CRW1], and it was shown that they satisfy weak-type 1-1 estimates that are uniform over all polynomials of fixed degree. Natural extensions of these operators to higher dimensions are discussed in [CRW2], in connection with  $\mathbb{R}^n$ -valued polynomials defined on  $\mathbb{R}^m$ .

We consider here a different kind of multi-dimensional analogue of (1.1), which is modelled on the maximal function in lacunary directions introduced in [NSW]. For simplicity, we restrict ourselves to two dimensions and to dyadic lacunary directions, i.e., determined by the vectors  $v_k = (1, 2^k)$  with  $k \in \mathbb{Z}$ . In addition, we allow dyadic scaling along each of these directions.

To be precise, given a real polynomial  $p(t)$  in one variable such that  $p(0) = 0$ , we define

$$(1.2) \quad \begin{aligned} M_p f(x_1, x_2) &= \sup_{h>0, i, k \in \mathbb{Z}} \frac{1}{h} \int_0^h |f(x - 2^i p(t)v_k)| dt \\ &= \sup_{h>0, i, j \in \mathbb{Z}} \frac{1}{h} \int_0^h |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt . \end{aligned}$$

We prove the following result.

**Theorem 1.**  *$M_p$  is bounded on  $L^q(\mathbb{R}^2)$  for  $q > 1$  with bounds that only depend on the degree of  $p$ .*

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It is easy to check that  $M_p$  cannot satisfy a weak-type 1-1 estimate.

The proof of Theorem 1 is based on the analysis of a general class of two-parameter maximal operators in the plane defined by compactly supported measures, subject to a decay assumption on their Fourier transforms. This result is in the spirit of [DR] and [RS], but here we consider the possibility that the Fourier transform of the measure has no decay within an angle that does not contain the coordinate axes.

**Theorem 2.** *For a probability measure  $\mu$  supported on the unit square, let  $\mu_{i,j}$  be the measure such that*

$$\int f d\mu_{i,j} = \int f(2^i x_1, 2^j x_2) d\mu(x_1, x_2) .$$

Assume that

(i) *there are constants  $C, \delta > 0$  and  $s > 1$  such that*

$$(1.3) \quad |\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\delta}$$

*away from the set where  $s^{-1} < \frac{|\xi_1|}{|\xi_2|} < s$ ;*

(ii) *the one-parameter maximal operator*

$$(1.4) \quad M_\mu^0 f(x) = \sup_{i \in \mathbb{Z}} |f * \mu_{i,i}(x)|$$

*is bounded on  $L^q(\mathbb{R}^2)$  for  $q > 1$ .*

*Then also, the two-parameter maximal operator,*

$$(1.5) \quad M_\mu f(x) = \sup_{i,j \in \mathbb{Z}} |f * \mu_{i,j}(x)| ,$$

*is bounded on  $L^q(\mathbb{R}^2)$  for  $q > 1$ , with bounds that only depend on  $s$ , the constants  $C, \delta$  in (1.3) and the norm of  $M_\mu^0$ .*

We start with the proof of Theorem 2, which combines methods from [NSW], [C] and [RS]. This is done in Section 2. Theorem 1 is proved in Section 3.

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## 2. PROOF OF THEOREM 2

Let  $\sigma_1$  and  $\sigma_2$  be the measures on the line defined by

$$\int_{\mathbb{R}} f(t) d\sigma_j(t) = \int_{\mathbb{R}^2} f(x_j) d\mu(x) .$$

Then  $\hat{\sigma}_1(\tau) = \hat{\mu}(\tau, 0)$  and  $\hat{\sigma}_2(\tau) = \hat{\mu}(0, \tau)$ , so that

$$(2.1) \quad |\hat{\sigma}_j(\tau)| \leq C(1 + |\tau|)^{-\delta} .$$

Let  $\varphi$  be a nonnegative smooth function on the line, supported on  $[-1, 1]$  and with integral equal to 1. Define

$$\nu = \mu - \sigma_1 \otimes \varphi - \varphi \otimes \sigma_2 + \varphi \otimes \varphi .$$

Clearly,  $\hat{\nu}$  satisfies (1.3), is supported on the unit square and

$$(2.2) \quad \hat{\nu}(\xi_1, 0) = \hat{\nu}(0, \xi_2) = 0 .$$

Since

$$M_\mu f \leq M_\nu f + M_{\sigma_1 \otimes \varphi} f + M_{\varphi \otimes \sigma_2} f + M_{\varphi \otimes \varphi} ,$$

we can discuss each of the maximal functions on the right-hand side separately.

The last term is controlled by the two-parameter strong maximal operator of Jessen, Marcinkiewicz and Zygmund. The  $L^q$ -boundedness of the two intermediate terms follows from Theorem 3.2 in [RS], once we observe that, by (2.1),

$$|\widehat{\sigma_1 \otimes \varphi}(\xi)| \leq C'(1 + |\xi|)^{-\delta} ,$$

and similarly for  $\varphi \otimes \sigma_2$ . (Alternatively, one can argue that  $M_{\sigma_1 \otimes \varphi}$  is controlled by the composition of the Hardy-Littlewood maximal operator in the  $x_2$ -variable with the one-parameter operator  $M_{\sigma_1}$  in the  $x_1$ -variable; to this operator one can apply Theorem A in [DR].)

Thus it remains to estimate  $M_\nu f$ . Due to the cancellations of  $\nu$  that are implicit in (2.2), it is convenient to introduce appropriate square functions. Given a measure  $\sigma$ , we shall need two types of such functions:

$$(2.3.a) \quad S_\sigma f(x) = \left( \sum_{i,j \in \mathbb{Z}} |f * \sigma_{i,j}(x)|^2 \right)^{\frac{1}{2}} ,$$

$$(2.3.b) \quad \tilde{S}_\sigma f(x) = \left( \sum_{k \in \mathbb{Z}} \left( \sup_{i \in \mathbb{Z}} |f * \sigma_{i,i+k}(x)| \right)^2 \right)^{\frac{1}{2}} .$$

Clearly,  $M_\sigma f \leq \tilde{S}_\sigma f \leq S_\sigma f$ . We shall also assume that  $q$  is finite, because there is nothing to prove for  $q = \infty$ .

Let  $\eta_\ell(x) = 2^{2\ell} \eta(2^\ell x)$ ,  $\ell \geq 0$ , be a smooth approximate identity in  $\mathbb{R}^2$ , with  $\eta$  supported on the unit disk. We set  $\psi_0 = \eta_0$ , and  $\psi_\ell = \eta_\ell - \eta_{\ell-1}$  for  $\ell \geq 1$ . Then

$$\nu = \sum_{\ell=0}^{\infty} \nu * \psi_\ell$$

and

$$S_\nu f \leq \sum_{\ell=0}^{\infty} S_{\nu * \psi_\ell} f .$$

**Lemma 2.1.** *For every  $\varepsilon > 0$  and  $1 < q < \infty$ ,  $\|S_{\nu * \psi_\ell} f\|_q \leq A2^{2\ell\varepsilon} \|f\|_q$ , where the constant  $A$  depends only on  $\varepsilon$  and  $q$ .*

*Proof.* By the standard randomization argument, we can estimate the  $L^q$ -operator norm of the singular integral operators

$$f \mapsto \sum_{i,j} \pm (\nu * \psi_\ell)_{i,j} * f .$$

We apply Lemma 2.3 in [RS]. Thus, it is necessary to prove that

$$\sup_{0 < |h_2| < 2} |h_2|^{-\varepsilon} \int \left( \sup_{0 < |h_1| < 2} |h_1|^{-\varepsilon} \int |\Delta_{h_1}^1 \Delta_{h_2}^2 (\nu * \psi_\ell)(x)| dx_1 \right) dx_2 \leq C2^{2\ell\varepsilon} ,$$

where

$$\begin{aligned} \Delta_{h_1}^1 f(x_1, x_2) &= f(x_1 + h_1, x_2) - f(x_1, x_2) , \\ \Delta_{h_2}^2 f(x_1, x_2) &= f(x_1, x_2 + h_2) - f(x_1, x_2) . \end{aligned}$$

We observe that

$$\Delta_{h_1}^1 \Delta_{h_2}^2 (\nu * \psi_\ell) = \nu * (\Delta_{h_1}^1 \Delta_{h_2}^2 \psi_\ell)$$

and that  $\Delta_{h_1}^1 \Delta_{h_2}^2 \psi_\ell(x)$  is smaller than a constant times  $2^{(2+2\varepsilon)\ell} |h_1|^\varepsilon |h_2|^\varepsilon$ , and it is supported, for each  $x, h_1, h_2$ , on a set that is the union of four disks of radius  $2^{-\ell}$ . Therefore,

$$\begin{aligned} \int |\Delta_{h_1}^1 \Delta_{h_2}^2 (\nu * \psi_\ell)(x)| dx_1 &\leq \int_{\mathbb{R}^2} \left( \int |\Delta_{h_1}^1 \Delta_{h_2}^2 \psi_\ell(x-y)| dx_1 \right) d|\nu|(y) \\ &\leq C 2^{(1+2\varepsilon)\ell} |h_1|^\varepsilon |h_2|^\varepsilon \int_{\mathbb{R}^2} \chi_{y,h_2}(x_2) d|\nu|(y) , \end{aligned}$$

where  $\chi_{y,h_2}$  is the characteristic function of a set of measure  $2^{-\ell}$  depending on  $y$  and  $h_2$ .

This concludes the proof. □

In order to obtain better estimates, we introduce a spectral decomposition of  $\nu$ . Let  $\Phi(\xi)$  be homogeneous of degree 0, smooth away from the origin, identically equal to 1 inside the angle  $\Gamma_1 = \{ \xi : s^{-1} < |\xi_1|/|\xi_2| < s \}$ , and identically equal to 0 outside of the angle  $\Gamma_2 = \{ \xi : (2s)^{-1} < |\xi_1|/|\xi_2| < 2s \}$ .

We then define the “bad part”  $\nu_b$  of  $\nu$  as the distribution such that

$$\hat{\nu}_b(\xi) = \hat{\nu}(\xi)\Phi(\xi) ,$$

and the “good part”  $\nu_g$  as  $\nu_g = \nu - \nu_b$ .

The square functions  $S_{\nu_b} f, S_{\nu_b * \psi_\ell} f$ , etc. are defined as in (2.3.a) and (2.3.b) for Schwartz functions  $f$ .

We show first that each part of  $\nu$  shares the good properties of  $\nu$  given by Lemma 2.1.

**Lemma 2.2.** *The conclusion of Lemma 2.1 remains valid if we replace  $\nu$  by  $\nu_b$  or  $\nu_g$ .*

*Proof.* For  $k \in \mathbb{Z}$ , let  $P_k f = \mathcal{F}^{-1}(\Phi(\xi_1, 2^{-k}\xi_2)\hat{f}(\xi))$ . Because of the finite overlapping of the supports of the multipliers  $\Phi(\xi_1, 2^{-k}\xi_2)$ , we have the Littlewood-Paley estimate

$$(2.4) \quad \left\| \left( \sum_{k \in \mathbb{Z}} |P_k f|^2 \right)^{\frac{1}{2}} \right\|_q \sim \|f\|_q ,$$

for  $1 < q < \infty$ . Also, observe that

$$(\nu_b)_{i,j} * f = \nu_{i,j} * (P_{i-j} f) , \quad (\nu_b * \psi_\ell)_{i,j} * f = (\nu * \psi_\ell)_{i,j} * (P_{i-j} f) .$$

Therefore,

$$\begin{aligned} S_{\nu_b * \psi_\ell} f &= \left( \sum_{i,j \in \mathbb{Z}} |(\nu * \psi_\ell)_{i,j} * (P_{i-j} f)(x)|^2 \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{i,j,k \in \mathbb{Z}} |(\nu * \psi_\ell)_{i,j} * (P_k f)(x)|^2 \right)^{\frac{1}{2}} . \end{aligned}$$

The last quantity equals the  $L^2$ -norm on  $[0, 1]^3$  of the function

$$(t, u, v) \longmapsto \sum_{i,j,k \in \mathbb{Z}} (\nu * \psi_\ell)_{i,j} * (P_k f)(x) r_i(t) r_j(u) r_k(v) ,$$

where  $r_n$  is the  $n$ th Rademacher function. By the properties of Rademacher functions, the  $L^2$ -norm is equivalent to the  $L^q$ -norm. Therefore,

$$\|S_{\nu_b * \psi_\ell} f\|_q^q \leq C \int_{\mathbb{R}^2} \int_{[0,1]^3} \left| \sum_{i,j,k \in \mathbb{Z}} (\nu * \psi_\ell)_{i,j} * (P_k f)(x) r_i(t) r_j(u) r_k(v) \right|^q dt du dv dx .$$

We denote

$$K_{t,u} = \sum_{i,j} r_i(t) r_j(u) (\nu * \psi_\ell)_{i,j} , \quad f_v = \sum_k r_k(v) P_k f .$$

Changing the order of integration, we have

$$\|S_{\nu_b * \psi_\ell} f\|_q^q \leq C \int_{[0,1]^3} \|K_{t,u} * f_v\|_q^q dt du dv .$$

The proof of Lemma 2.1 shows that the  $L^q$ -operator norms of the  $K_{t,u}$  are uniformly bounded by a constant times  $2^{2\ell\varepsilon}$ . Hence,

$$\|S_{\nu_b * \psi_\ell} f\|_q^q \leq C 2^{2\ell\varepsilon} \int_{[0,1]} \|f_v\|_q^q dv .$$

Changing the order of integration again, replacing the  $L^q$ -norm on  $[0, 1]$  with the  $L^2$ -norm, and using (2.4), we obtain the conclusion for  $\nu_b$ .

For  $\nu_g$  it is sufficient to observe that  $S_{\nu_g * \psi_\ell} f \leq S_{\nu * \psi_\ell} f + S_{\nu_b * \psi_\ell} f$ . □

We shall now improve the estimate on  $S_{\nu_g * \psi_\ell}$ , using the uniform decay of  $\hat{\nu}_g(\xi)$  as  $\xi$  goes to infinity. In fact, as we already observed,  $\hat{\nu}$  satisfies (1.3); hence,

$$(2.5) \quad |\hat{\nu}_g(\xi)| \leq C(1 + |\xi|)^{-\delta} .$$

We shall assume, w.l.o.g., that  $\delta < 1$ .

**Lemma 2.3.**  $\|S_{\nu_g * \psi_\ell} f\|_2 \leq A 2^{-\ell\delta/4} \|f\|_2$ , with  $A$  depending only on  $\delta$  and  $C$ .

*Proof.* By the Plancherel formula, we have to prove that

$$(2.6) \quad \sum_{i,j \in \mathbb{Z}} |\hat{\nu}_g(2^i \xi_1, 2^j \xi_2)|^2 |\hat{\psi}_\ell(2^i \xi_1, 2^j \xi_2)|^2 \leq A 2^{-\ell\delta/2} .$$

By (2.2),

$$\hat{\nu}(\xi) = \int (e^{-ix_1 \xi_1} - 1)(e^{-ix_2 \xi_2} - 1) d\nu(\xi) .$$

Since  $\nu$  is supported on the unit square,

$$|\hat{\nu}(\xi)| \leq C |\xi_1| |\xi_2| .$$

Combining this with (2.5), we obtain that, if  $0 < \varepsilon < 1$ ,

$$|\hat{\nu}_g(\xi)| \leq C \frac{|\xi_1|^\varepsilon |\xi_2|^\varepsilon}{(1 + |\xi|)^{\delta(1-\varepsilon)}} .$$

If  $\ell \geq 1$ , then

$$|\hat{\psi}_\ell(\xi)| = |\hat{\psi}_1(2^{-(\ell-1)} \xi)| \leq C 2^{-\ell\varepsilon} |\xi|^\varepsilon ,$$

because  $\hat{\psi}_1(0) = 0$ . Hence,

$$|\hat{\nu}_g(\xi) \hat{\psi}_\ell(\xi)| \leq C 2^{-\ell\varepsilon} \frac{|\xi_1|^\varepsilon |\xi_2|^\varepsilon}{(1 + |\xi|)^{\delta(1-\varepsilon)-\varepsilon}} .$$

We can assume that  $|\xi_1| \sim |\xi_2| \sim 1$  in (2.6). Then we simply have to observe that, taking  $\varepsilon = \delta/4$ , the exponent in the denominator is bigger than  $\delta/2 = 2\varepsilon$ , and that the series

$$\sum_{i,j \in \mathbb{Z}} \frac{2^{2\varepsilon i} 2^{2\varepsilon j}}{(1 + 2^i + 2^j)^{2\alpha}}$$

is convergent for  $\alpha > 2\varepsilon$ . □

Interpolating between the  $L^2$ -estimate in Lemma 2.3 and the  $L^q$ -estimate in Lemma 2.2 for  $S_{\nu_g * \psi_\ell}$ , we obtain that for every  $q \in (1, \infty)$  there is an  $\varepsilon_q > 0$  such that  $\|S_{\nu_g * \psi_\ell} f\|_q \leq A 2^{-\ell \varepsilon_q} \|f\|_q$ . Therefore,

**Proposition 2.4.**  *$S_{\nu_g}$  is bounded on  $L^q$  for  $1 < q \leq 2$ .*

In order to complete the proof of Theorem 2, we may just observe that we are in the hypotheses of Theorem B in [C] (attributed to M. Christ). We give, however, an independent proof, based on the extrapolation argument in [NSW], adapted to  $\tilde{S}_{\nu_b}$ .

*End of the proof of Theorem 2.* The starting point is that  $\tilde{S}_{\nu_b}$  is bounded on  $L^2$ . In fact, assumption (ii) implies that  $M_{\nu_{0,k}}^0$  is uniformly bounded on  $L^q$  independently of  $k$ . Therefore,

$$\begin{aligned} \int \tilde{S}_{\nu_b} f(x)^2 dx &= \sum_{k \in \mathbb{Z}} \int \sup_{i \in \mathbb{Z}} |\nu_{i,i+k} * P_k f(x)|^2 dx \\ &= \sum_{k \in \mathbb{Z}} \int (M_{\nu_{0,k}}^0 P_k f(x))^2 dx \\ &\leq C \sum_{k \in \mathbb{Z}} \int (P_k f(x))^2 dx \\ &= C \|f\|_2^2. \end{aligned}$$

In general, the boundedness of  $\tilde{S}_{\nu_b}$  on some  $L^q$  implies, by Proposition 2.4, the boundedness of  $M_\nu$  on the same  $L^q$ , and hence that of  $M_\mu$ .

Assume now that  $M_\mu$  is bounded on some  $L^q$ , and consider the inequality

$$(2.7) \quad \left\| \left( \sum_{k \in \mathbb{Z}} M_{\mu_{0,k}}^0 f_k(x)^r \right)^{1/r} \right\|_s \leq C \left\| \left( \sum_{k \in \mathbb{Z}} |f_k(x)|^r \right)^{1/r} \right\|_s.$$

This is equivalent to saying that the linear operator

$$T : \{f_k\} \mapsto \{\mu_{i,i+k} * f_k\}$$

is bounded from  $L^s(\ell^r)$  to  $L^s(\ell^r(\ell^\infty))$ .

Since  $\mu$  is a positive measure and we are assuming that  $M_\mu$  is bounded on  $L^q$ , (2.7) is verified for  $r = \infty$  and  $s = q$ . In addition, it is verified for  $r = s > 1$  by the uniform boundedness of  $M_{\mu_{0,k}}^0$ . Hence,  $T$  is bounded from  $L^q(\ell^\infty)$  to  $L^q(\ell^\infty(\ell^\infty))$  and from  $L^r(\ell^r)$  to  $L^r(\ell^r(\ell^\infty))$  for  $r > 1$ . By interpolation, (2.7) holds for  $r = 2$  and  $\frac{1}{q} < \frac{1}{s} < \frac{1}{2}(1 + \frac{1}{q})$ .

The same inequality holds with  $\mu$  replaced by  $\sigma \otimes \varphi$ ,  $\varphi \otimes \sigma$  and  $\varphi \otimes \varphi$ , and hence with  $\mu$  replaced by  $\nu$ .

Taking  $f_k = P_k f$ , this implies that  $\tilde{S}_{\nu_b}$  is bounded on the same spaces  $L^s$ . Since each  $q \in (1, 2)$  can be reached by iteration in a finite number of steps, we conclude that  $M_\mu$  is bounded on  $L^q$  for every  $q > 1$ . □

### 3. PROOF OF THEOREM 1

The starting point for the proof of Theorem 1 is Lemma 2.5 in [CRW1]. We give a slightly different (and less complete) formulation of it.

**Lemma 3.1.** *For every  $n$  there are constants  $A(n) \geq 1$  and  $B = B(n)$  with the following property: if  $p(t)$  is a monic real polynomial of degree  $n$  such that  $p(0) = 0$ ,  $A \geq A(n)$ , and  $m \in \mathbb{Z}$  is such that no complex zero of  $p$  lies in the strip*

$$\{z : A^{m-1} \leq |z| \leq A^{m+2}\},$$

then the following properties hold:

- (i)  $p$  has constant sign and is strictly monotonic on  $I_m = [A^m, A^{m+1}]$ ;
- (ii)  $|p(t)| \leq Bt|p'(t)|$  for  $t \in I_m$ ;
- (iii)  $\max_{t \in I_m} |p(t)| \leq B \min_{t \in I_m} |p(t)|$ .

Observe that we are allowed to replace the polynomial  $p(t)$  in (1.2), when convenient, by  $\tilde{p}(t) = bp(at)$ , with  $a, b > 0$ . In fact, the identity

$$M_{\tilde{p}}f(x) = M_p f_b \left( \frac{x}{b} \right),$$

where  $f_b(x) = f(bx)$ , implies that  $M_p$  and  $M_{\tilde{p}}$  have the same operator norm. In particular, we can assume that  $p$  is monic.

Also, the maximal function  $M_p$  can be replaced by

$$\tilde{M}_p f(x_1, x_2) = \sup_{m \in \mathbb{Z}} \tilde{M}_{p,m} f(x_1, x_2)$$

where

$$(3.1) \quad \tilde{M}_{p,m} f(x_1, x_2) = \sup_{i,j \in \mathbb{Z}} A^{-m} \int_{I_m} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt.$$

Let  $I_m$  be one of the “good” dyadic intervals satisfying properties (i)–(iii) in Lemma 3.1. Making the change of variable  $u = p(t)$ , we have

$$\begin{aligned} A^{-m} \int_{I_m} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt & \\ & \leq A \int_{A^m}^{A^{m+1}} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| \frac{dt}{t} \\ & \leq AB \int_{I_m} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| \frac{|p'(t)|}{|p(t)|} dt \\ & = AB \int_{p(I_m)} |f(x_1 - 2^i u, x_2 - 2^j u)| \frac{du}{|u|}. \end{aligned}$$

By (i) and (iii), the interval  $p(I_m)$  is contained in an interval of the form  $\pm[\alpha_m, B\alpha_m]$ , with  $\alpha_m > 0$ . Therefore, assuming w.l.o.g. that  $p$  is positive on  $I_m$ ,

$$\begin{aligned}
 A^{-m} \int_{I_m} |f(x_1 - 2^i p(t), x_2 - 2^j p(t))| dt &\leq AB \int_{\alpha_m}^{B\alpha_m} |f(x_1 - 2^i u, x_2 - 2^j u)| \frac{du}{u} \\
 &\leq \frac{AB}{\alpha_m} \int_{\alpha_m}^{B\alpha_m} |f(x_1 - 2^i u, x_2 - 2^j u)| du \\
 &\leq \frac{AB^2}{B\alpha_m} \int_0^{B\alpha_m} |f(x_1 - 2^i u, x_2 - 2^j u)| du .
 \end{aligned}$$

This shows that the contribution to  $\tilde{M}_p f$  given by the “good” intervals is controlled by the maximal function in lacunary directions

$$\mathcal{M}f(x_1, x_2) = \sup_{h>0, k \in \mathbb{Z}} \frac{1}{h} \int_0^h |f(x_1 - t, x_2 - 2^k t)| dt$$

of [NSW]. Since  $\mathcal{M}$  is bounded on  $L^q$  for  $q > 1$  [NSW], it remains to consider the contribution from the “bad” intervals. Since there are at most  $3n$  of these intervals, it is enough to prove that  $\tilde{M}_{p,m}$  acts on  $L^q$  for  $q > 1$ , with operator norm bounded independently of the polynomial  $p$  and integer  $m$ .

We claim it suffices to show that there exists a constant  $C_{q,n}$  such that

$$\sup_{m \in \mathbb{Z}} \|\tilde{M}_{p,m} f\|_q \leq C_{q,n} \|f\|_q$$

for every  $f \in L^q$  and monic polynomial  $p$  of degree  $n$  satisfying  $p(0) = 0$  and

$$(3.2) \quad A^{-n} \leq \max_{t \in I_m} |p(t)| \leq 1.$$

To see this, suppose  $p$  is an arbitrary monic polynomial with  $p(0) = 0$ , and choose  $k \in \mathbb{Z}$  such that

$$A^{-n} \leq \max_{t \in I_m} A^{-kn} |p(t)| \leq 1.$$

Let  $\tilde{p}(t) = A^{-kn} p(A^k t)$ . Since

$$A^{-n} \leq \max_{t \in I_{m-k}} |\tilde{p}(t)| \leq 1,$$

the  $(L^q, L^q)$  operator norm of  $\tilde{M}_{\tilde{p}, m-k}$  is at most  $C_{q,n}$ . Since

$$\tilde{M}_{p,m} f(x) = \tilde{M}_{\tilde{p}, m-k} f_{A^k}(A^{-k} x),$$

$\tilde{M}_{p,m}$  also acts on  $L^q$  with bounds that are independent of  $m$  and  $p$ .

Consequently, we need to investigate the measure  $\mu$  given by

$$\int f d\mu = A^{-m} \int_{I_m} f(p(t), p(t)) dt$$

where  $p$  satisfies (3.2). This measure is supported on the segment  $\{(u, u) : -1 \leq u \leq 1\}$  and, up to a factor depending on  $A$ , is a probability measure.

The proof of Theorem 1 will be complete once we show that the operator  $M_\mu$  is bounded on  $L^q$  for  $q > 1$  with bounds that depend only on  $n$  and  $q$ . We apply Theorem 2.

The Fourier transform of  $\mu$  is

$$(3.3) \quad \hat{\mu}(\xi_1, \xi_2) = A^{-m} \int_{I_m} e^{-i(\xi_1 + \xi_2)p(t)} dt .$$

**Lemma 3.2.** *There is an integer  $k \in \{1, 2, \dots, n\}$  such that, if  $A$  is large enough (depending on  $n$ ), then*

$$|\hat{\mu}(\xi_1, \xi_2)| \leq CA^n(1 + |\xi_1 + \xi_2|)^{-1/k} ,$$

with  $C$  independent of  $p$  and  $m$ .

*Proof.* Let  $t_1 = 0, t_2, \dots, t_n$  be the zeroes of  $p$ , ordered so that  $0 \leq |t_2| \leq \dots \leq |t_n|$ . Let  $m'$  be the smallest integer greater than  $m$  such that  $I_{m'}$  does not contain any of the  $|t_j|$ . Then  $m' \leq m + n$ , so that  $A^{m'}$  is comparable with  $A^m$ . Also let  $k$  be such that  $|t_j| < A^{m'}$  for  $j \leq k$  and  $|t_j| > A^{m'+1}$  for  $j > k$ .

The  $k$ th derivative of  $p$  equals

$$p^{(k)}(t) = \prod_{j=k+1}^n (t - t_j) + r(t) ,$$

where  $r(t)$  is a sum where each term is a product of  $n - k$  factors  $t - t_j$ , with at least one of the  $j$  less than or equal to  $k$ .

If  $t \in I_m$ ,  $|t - t_j| < 2A^{m'}$  for  $j \leq k$ , and  $|t - t_j| > (1 - A^{-1})|t_j| > (A - 1)A^{m'}$  for  $j > k$ . Therefore, if  $A$  is large enough,

$$|p^{(k)}(t)| \geq C \prod_{j=k+1}^n |t_j| ,$$

for  $t \in I_m$ .

By van der Corput's lemma,

$$A^{-m} \left| \int_{I_m} e^{-i\lambda p(t)} dt \right| \leq CA^{-m} \left( \prod_{j=k+1}^n |t_j| \right)^{-1/k} |\lambda|^{-1/k} .$$

If  $\bar{t} \in I_m$  is such that  $|p(\bar{t})| \geq A^{-n}$ , we have

$$A^{-n} \leq |p(\bar{t})| \leq 2^n A^{km'} \prod_{j=k+1}^n |t_j| .$$

Therefore,  $\prod_{j=k+1}^n |t_j| \geq CA^{-n} A^{-km'}$ , so that

$$A^{-m} \left| \int_{I_m} e^{-i\lambda p(t)} dt \right| \leq C|\lambda|^{-1/k} A^n ,$$

with  $C$  independent of  $p$  and  $m$ . Since the left-hand side is trivially bounded by 1, this concludes the proof.  $\square$

Thus,  $\hat{\mu}$  clearly satisfies hypothesis (i) of Theorem 2. It remains to prove that the one-parameter maximal operator  $M_\mu^0$  in (1.4) is bounded on  $L^q$  for  $q > 1$  with bounds that only depend on  $n$  and  $q$ . This follows from a transference argument: because  $\mu$  is supported on a line, it is sufficient to consider the maximal operator on  $\mathbb{R}$ ,

$$M_{\tilde{\mu}}g(x) = \sup_{i \in \mathbb{Z}} |g * \tilde{\mu}_i(x)| ,$$

where

$$\int_{\mathbb{R}} g d\tilde{\mu} = A^{-m} \int_{I_m} g(p(t)) dt .$$

By Lemma 3.2,  $|\hat{\mu}(\eta)| \leq C(1 + |\eta|)^{-1/k}$ , with  $1 \leq k \leq n$  and  $C$  depending only on  $n$ . The conclusion follows from Theorem A in [DR].

*Remark.* In [CRW1] the authors show that the “supermaximal function” on the real line

$$f(x) \longmapsto \sup_{\substack{h>0 \\ \deg p \leq k, p(0)=0}} \frac{1}{h} \int_0^h |f(x-p(t))| dt$$

is of restricted weak type  $k-k$  and hence of strong type  $q-q$  for  $q > k$ .

The proof can be adapted to show that the operator

$$\begin{aligned} \mathcal{M}_k f(x) &= \sup_{\deg p \leq k, p(0)=0} M_p f(x) \\ &= \sup_{\substack{h>0, j \in \mathbb{Z} \\ \deg p \leq k, p(0)=0}} \frac{1}{h} \int_0^h |f(x_1 - p(t), x_2 - 2^j p(t))| dt \end{aligned}$$

is bounded on  $L^q(\mathbb{R}^2)$  for  $q > k$ .

In fact, if  $f$  is the characteristic function of a measurable set in the plane, the same proof as in [CRW1] gives the pointwise domination

$$\mathcal{M}_k f(x) \leq C(M^* f^k(x))^{1/k},$$

where  $M^*$  is the maximal function in dyadic direction of [NSW]. This implies that  $\mathcal{M}_k$  is of restricted weak type  $q-q$  for  $q > k$ , and hence of strong type.

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