

## HOMOLOGICAL PROPERTIES OF BALANCED COHEN-MACAULAY ALGEBRAS

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ABSTRACT. A balanced Cohen-Macaulay algebra is a connected algebra  $A$  having a balanced dualizing complex  $\omega_A[d]$  in the sense of Yekutieli (1992) for some integer  $d$  and some graded  $A$ - $A$  bimodule  $\omega_A$ . We study some homological properties of a balanced Cohen-Macaulay algebra. In particular, we will prove the following theorem:

**Theorem 0.1.** *Let  $A$  be a Noetherian balanced Cohen-Macaulay algebra, and  $M$  a nonzero finitely generated graded left  $A$ -module. Then:*

1.  $M$  has a finite resolution of the form

$$0 \rightarrow \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \rightarrow H \rightarrow M \rightarrow 0,$$

where  $H$  is a finitely generated maximal Cohen-Macaulay graded left  $A$ -module.

2.  $M$  has finite injective dimension if and only if  $M$  has a finite resolution of the form

$$\begin{aligned} 0 \rightarrow \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \\ \rightarrow \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \rightarrow M \rightarrow 0. \end{aligned}$$

As a corollary, we will have the following characterizations of AS Gorenstein algebras and AS regular algebras:

**Corollary 0.2.** *Let  $A$  be a Noetherian balanced Cohen-Macaulay algebra.*

1.  $A$  is AS Gorenstein if and only if  $\omega_A$  has finite projective dimension as a graded left  $A$ -module.
2.  $A$  is AS regular if and only if every finitely generated maximal Cohen-Macaulay graded left  $A$ -module is free.

### 1. HYPERHOMOLOGICAL ALGEBRAS

Throughout this paper, we fix a field  $k$ . A connected algebra is a graded algebra of the form  $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ . The augmentation ideal of  $A$  is defined by  $\mathfrak{m} = A_1 \oplus A_2 \oplus \cdots$ . In this first section, we will fix terminology and notation, and collect some elementary results on hyperhomological algebras over connected algebras.

Let  $A, B, C$  be connected algebras. The category of graded left  $A$ -modules and graded left  $A$ -module homomorphisms of degree 0 is denoted by  $\text{GrMod } A$ . For

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$M, N \in \text{GrMod } A$ , the set of graded left  $A$ -module homomorphisms  $M \rightarrow N$  of degree 0 is denoted by  $\text{Hom}_A(M, N)$ , which has a natural  $k$ -vector space structure. The full subcategory of  $\text{GrMod } A$  consisting of finitely generated graded left  $A$ -modules is denoted by  $\text{grmod } A$ . The category of graded right  $A$ -modules is denoted by  $\text{GrMod } A^o$ , where  $A^o$  is the opposite algebra of  $A$ . The category of graded  $A$ - $B$  bimodules is denoted by  $\text{GrMod}(A \otimes B^o)$ . In particular, the category of graded  $A$ - $A$  bimodules is denoted by  $\text{GrMod } A^e$ , where  $A^e = A \otimes A^o$ . The natural restriction functors are denoted by

$$\text{res}_A : \text{GrMod}(A \otimes B^o) \rightarrow \text{GrMod } A$$

and

$$\text{res}_{B^o} : \text{GrMod}(A \otimes B^o) \rightarrow \text{GrMod } B^o.$$

We write  $k = A/\mathfrak{m}$ , viewed as an object in  $\text{GrMod } A$ ,  $\text{GrMod } A^o$ , or  $\text{GrMod } A^e$ , depending on the context.

A graded left  $A$ -module  $M \in \text{GrMod } A$  is right bounded (resp. left bounded) if  $M_i = 0$  for all  $i \gg 0$  (resp.  $i \ll 0$ ), and bounded if it is both right bounded and left bounded. We say that  $M$  is locally finite if the  $M_i$  are finite dimensional over  $k$  for all  $i$ . For each integer  $n$ , the shift of  $M$  is denoted by  $M(n) \in \text{GrMod } A$ , so that  $M(n)_i = M_{n+i}$ . For  $M \in \text{GrMod}(A \otimes B^o)$  and  $N \in \text{GrMod}(A \otimes C^o)$ , we define

$$\underline{\text{Ext}}_A^i(M, N) = \bigoplus_{n=-\infty}^{\infty} \text{Ext}_A^i(M, N(n)),$$

which has a natural graded  $B$ - $C$  bimodule structure for each  $i$ . Similarly, for  $M \in \text{GrMod}(B \otimes A^o)$  and  $N \in \text{GrMod}(A \otimes C^o)$ ,  $\text{Tor}_i^A(M, N)$  has a natural graded  $B$ - $C$  bimodule structure for each  $i$ . For  $M \in \text{GrMod}(A \otimes B^o)$ , the Matlis dual of  $M$  is defined by  $M' = \underline{\text{Hom}}_k(M, k)$ , which has a natural graded  $B$ - $A$  bimodule structure. If  $M$  is locally finite, then  $M'' \cong M$  in  $\text{GrMod}(A \otimes B^o)$ .

Let  $X, Y$  be cochain complexes of graded left  $A$ -modules. The  $i$ th cohomology of  $X$  is denoted by  $h^i(X)$ . We say that a cochain map  $f : X \rightarrow Y$  is a quasi-isomorphism if the induced maps  $h^i(f) : h^i(X) \rightarrow h^i(Y)$  are isomorphisms in  $\text{GrMod } A$  for all  $i$ . The derived category of graded left  $A$ -modules is denoted by  $\mathcal{D}(A)$ , so that a cochain map  $f : X \rightarrow Y$  is a quasi-isomorphism if and only if it induces an isomorphism  $f : X \rightarrow Y$  in  $\mathcal{D}(A)$ . We define  $\mathcal{D}_{fg}(A)$  (resp.  $\mathcal{D}_{lf}(A)$ ) to be the full subcategory of  $\mathcal{D}(A)$  consisting of complexes whose cohomologies are all finitely generated (resp. locally finite) graded left  $A$ -modules.

For  $X \in \mathcal{D}(A)$ , we define

$$\sup X = \sup\{i \mid h^i(X) \neq 0\}$$

and

$$\inf X = \inf\{i \mid h^i(X) \neq 0\}.$$

If  $X \cong 0$  in  $\mathcal{D}(A)$ , then we define  $\sup X = -\infty$  and  $\inf X = \infty$ .

A complex  $X \in \mathcal{D}(A)$  is bounded above (resp. bounded below) if  $\sup X < \infty$  (resp.  $\inf X > -\infty$ ), and bounded if it is both bounded above and bounded below. The full subcategory of  $\mathcal{D}(A)$  consisting of bounded (resp. bounded above, resp. bounded below) complexes is denoted by  $\mathcal{D}^b(A)$  (resp.  $\mathcal{D}^-(A)$ , resp.  $\mathcal{D}^+(A)$ ).

The right derived functor of

$$\underline{\text{Hom}}_A(-, -) : \mathcal{D}^-(A \otimes B^o) \times \mathcal{D}^+(A \otimes C^o) \rightarrow \mathcal{D}(B \otimes C^o)$$

is denoted by  $R\underline{\text{Hom}}_A(-, -)$ , and its cohomologies are denoted by

$$\underline{\text{Ext}}_A^i(-, -) = h^i(R\underline{\text{Hom}}_A(-, -)).$$

The left derived functor of

$$- \otimes_A - : \mathcal{D}^-(B \otimes A^o) \times \mathcal{D}^-(A \otimes C^o) \rightarrow \mathcal{D}(B \otimes C^o)$$

is denoted by  $- \otimes_A^L -$ , and its cohomologies are denoted by

$$\text{Tor}_{-i}^A(-, -) = h^i(- \otimes_A^L -).$$

Let  $X \in \mathcal{D}(A)$ . For each integer  $n$ , the twist of  $X$  is denoted by  $X[n] \in \mathcal{D}(A)$ , so that  $(X[n])^i = X^{n+i}$ . Note that  $h^i(X) = 0$  for all  $i \neq n$  if and only if  $X \cong h^n(X)[-n]$  in  $\mathcal{D}(A)$ . If  $X \in \mathcal{D}^-(A \otimes B^o)$  and  $Y \in \mathcal{D}^+(A \otimes C^o)$ , then

$$R\underline{\text{Hom}}_A(X[n], Y) \cong R\underline{\text{Hom}}_A(X, Y[-n]) \cong R\underline{\text{Hom}}_A(X, Y)[-n]$$

in  $\mathcal{D}(B \otimes C^o)$  for each  $n$ . If  $X \in \mathcal{D}^-(B \otimes A^o)$  and  $Y \in \mathcal{D}^-(A \otimes C^o)$ , then

$$(X[n]) \otimes_A^L Y \cong X \otimes_A^L (Y[n]) \cong (X \otimes_A^L Y)[n]$$

in  $\mathcal{D}(B \otimes C^o)$  for each  $n$ .

**Definition 1.1.** Let  $A$  be a connected algebra.

1. A free resolution of  $X \in \mathcal{D}^-(A)$  is a complex  $F$  of free graded left  $A$ -modules such that  $F \cong X$  in  $\mathcal{D}(A)$ . A complex  $F$  of free graded left  $A$ -modules is called minimal if the differentials in  $\underline{\text{Hom}}_A(F, k)$  are all zero.
2. A projective resolution of  $X \in \mathcal{D}^-(A)$  is a complex  $P$  of projective graded left  $A$ -modules such that  $P \cong X$  in  $\mathcal{D}(A)$ . We define the projective dimension of  $X$  by

$$\text{pd}_A(X) = \inf_P(-\inf\{i \mid P^i \neq 0\}),$$

where the infimum is taken over all projective resolutions  $P$  of  $X$ .

3. An injective resolution of  $X \in \mathcal{D}^+(A)$  is a complex  $E$  of injective graded left  $A$ -modules such that  $E \cong X$  in  $\mathcal{D}(A)$ . We define the injective dimension of  $X$  by

$$\text{id}_A(X) = \inf_E(\sup\{i \mid E^i \neq 0\}),$$

where the infimum is taken over all injective resolutions  $E$  of  $X$ .

4. A flat resolution of  $X \in \mathcal{D}^-(A)$  is a complex  $F$  of flat graded left  $A$ -modules such that  $F \cong X$  in  $\mathcal{D}(A)$ . We define the flat dimension of  $X$  by

$$\text{fd}_A(X) = \inf_F(-\inf\{i \mid F^i \neq 0\}),$$

where the infimum is taken over all flat resolutions  $F$  of  $X$ .

**Lemma 1.2.** Let  $A$  be a connected algebra.

1. For  $X \in \mathcal{D}^+(A)$ ,

$$\begin{aligned} \text{id}_A(X) &= \sup(\{\sup R\underline{\text{Hom}}_A(M, X) \mid M \in \text{GrMod } A\}) \\ &= \sup(\{\sup R\underline{\text{Hom}}_A(M, X) \mid M \in \text{grmod } A\}). \end{aligned}$$

2. For  $X \in \mathcal{D}^-(A)$ ,

$$\begin{aligned} \text{fd}_A(X) &= \sup(\{-\inf(N \otimes_A^L X) \mid N \in \text{GrMod } A^o\}) \\ &= \sup(\{-\inf(N \otimes_A^L X) \mid N \in \text{grmod } A^o\}). \end{aligned}$$

3. If  $X \in \mathcal{D}^-(A)$  has a minimal free resolution, then

$$\text{pd}_A(X) = \sup \text{RHom}(X, k) = -\inf(k \otimes_A^L X) = \text{fd}_A(X).$$

*Proof.* These are direct consequences of [6, Propositions 1.7, 1.8, 1.9]. □

The following lemma is standard (cf. [8, Lemma 1.8]).

**Lemma 1.3.** *Let  $A$  be a connected algebra.*

1. If  $X \in \mathcal{D}^-(A \otimes B^\circ)$  and  $Y \in \mathcal{D}^+(A \otimes C^\circ)$ , then

$$\inf \text{RHom}_A(X, Y) \geq \inf Y - \sup X.$$

2. If  $X \in \mathcal{D}^-(B \otimes A^\circ)$  and  $Y \in \mathcal{D}^-(A \otimes C^\circ)$ , then

$$\sup(X \otimes_A^L Y) \leq \sup X + \sup Y.$$

Moreover, if  $h^{\sup X}(X), h^{\sup Y}(Y) \in \text{GrMod } A$  are left bounded, then

$$\sup(X \otimes_A^L Y) = \sup X + \sup Y.$$

## 2. FOXBY EQUIVALENCE

**Definition 2.1.** Let  $A, B$  be connected algebras, and let  $\mathfrak{m} = A_{\geq 1}$  be the augmentation ideal of  $A$ . We define the functor  $\Gamma_{\mathfrak{m}} : \mathcal{D}(A \otimes B^\circ) \rightarrow \mathcal{D}(A \otimes B^\circ)$  by

$$\Gamma_{\mathfrak{m}}(-) = \lim_{n \rightarrow \infty} \text{Hom}_A(A/A_{\geq n}, -).$$

The right derived functor of  $\Gamma_{\mathfrak{m}}$  is denoted by  $R\Gamma_{\mathfrak{m}}$ , and its cohomologies are denoted by

$$H_{\mathfrak{m}}^i(-) = h^i(R\Gamma_{\mathfrak{m}}(-)) = \lim_{n \rightarrow \infty} \text{Ext}_A^i(A/A_{\geq n}, -).$$

Similarly, we define the functor  $\Gamma_{\mathfrak{m}^\circ} : \mathcal{D}(B \otimes A^\circ) \rightarrow \mathcal{D}(B \otimes A^\circ)$  by

$$\Gamma_{\mathfrak{m}^\circ}(-) = \lim_{n \rightarrow \infty} \text{Hom}_{A^\circ}(A/A_{\geq n}, -).$$

The right derived functor of  $\Gamma_{\mathfrak{m}^\circ}$  is denoted by  $R\Gamma_{\mathfrak{m}^\circ}$ , and its cohomologies are denoted by

$$H_{\mathfrak{m}^\circ}^i(-) = h^i(R\Gamma_{\mathfrak{m}^\circ}(-)) = \lim_{n \rightarrow \infty} \text{Ext}_{A^\circ}^i(A/A_{\geq n}, -).$$

Let us recall the following definition from [13].

**Definition 2.2.** Let  $A$  be a Noetherian connected algebra. A complex  $D \in \mathcal{D}^b(A^e)$  is called dualizing if

- $\text{res}_A D \in \mathcal{D}_{fg}^b(A), \text{res}_{A^\circ} D \in \mathcal{D}_{fg}^b(A^\circ)$ ,
- $\text{id}_A(D) < \infty, \text{id}_{A^\circ}(D) < \infty$ , and
- the natural morphisms  $A \rightarrow \text{RHom}_A(D, D)$  and  $A \rightarrow \text{RHom}_{A^\circ}(D, D)$  are isomorphisms in  $\mathcal{D}(A^e)$ .

A dualizing complex  $D \in \mathcal{D}(A^e)$  is called balanced if

- $R\Gamma_{\mathfrak{m}}(D) \cong R\Gamma_{\mathfrak{m}^\circ}(D) \cong A'$  in  $\mathcal{D}(A^e)$ .

By [13, Proposition 3.5], if  $D$  is a dualizing complex, then the functor

$$\text{RHom}_A(-, D) : \mathcal{D}(A) \rightarrow \mathcal{D}(A^\circ)$$

and the functor

$$\text{RHom}_{A^\circ}(-, D) : \mathcal{D}(A^\circ) \rightarrow \mathcal{D}(A)$$

define a duality between  $\mathcal{D}_{fg}^b(A)$  and  $\mathcal{D}_{fg}^b(A^\circ)$ , that is,

$$\text{RHom}_A(X, D) \in \mathcal{D}_{fg}^b(A^\circ) \text{ and } \text{RHom}_{A^\circ}(\text{RHom}_A(X, D), D) \cong X \text{ in } \mathcal{D}(A)$$

for all  $X \in \mathcal{D}_{fg}^b(A)$ , and

$$R\mathbf{Hom}_{A^\circ}(Y, D) \in \mathcal{D}_{fg}^b(A) \text{ and } R\mathbf{Hom}_A(R\mathbf{Hom}_{A^\circ}(Y, D), D) \cong Y \text{ in } \mathcal{D}(A^\circ)$$

for all  $Y \in \mathcal{D}_{fg}^b(A^\circ)$ . In this section, we study another type of equivalence, known as Foxby equivalence.

**Proposition 2.3.** *Let  $A, B$  be Noetherian connected algebras.*

1. *Let*

$$X \in \mathcal{D}^-(B \otimes A^\circ), \quad Y \in \mathcal{D}^b(A^e), \quad Z \in \mathcal{D}^+(A)$$

*be such that  $\text{res}_{A^\circ} X \in \mathcal{D}_{fg}^-(A^\circ)$ . If either  $\text{pd}_{A^\circ}(X) < \infty$  or  $\text{id}_A(Z) < \infty$ , then there is a natural isomorphism*

$$X \otimes_A^L R\mathbf{Hom}_A(Y, Z) \cong R\mathbf{Hom}_A(R\mathbf{Hom}_{A^\circ}(X, Y), Z)$$

*in  $\mathcal{D}(B)$ .*

2. *Let*

$$X \in \mathcal{D}^-(A \otimes B^\circ), \quad Y \in \mathcal{D}^b(A^e), \quad Z \in \mathcal{D}^-(A)$$

*be such that  $\text{res}_A X \in \mathcal{D}_{fg}^-(A)$ . If either  $\text{pd}_A(X) < \infty$  or  $\text{id}_A(Z) < \infty$ , then there is a natural isomorphism*

$$R\mathbf{Hom}_A(X, Y) \otimes_A^L Z \cong R\mathbf{Hom}_A(X, Y \otimes_A^L Z)$$

*in  $\mathcal{D}(B)$ .*

*Proof.* By [8, Theorem 1.4] and [6, Proposition 2.1], if  $X, Y, Z$  are as above, then the evaluation morphisms

$$\theta_{XYZ} : X \otimes_A^L R\mathbf{Hom}_A(Y, Z) \rightarrow R\mathbf{Hom}_A(R\mathbf{Hom}_{A^\circ}(X, Y), Z)$$

and

$$\omega_{XYZ} : R\mathbf{Hom}_A(X, Y) \otimes_A^L Z \rightarrow R\mathbf{Hom}_A(X, Y \otimes_A^L Z)$$

defined in [3, Notation 4.3] are isomorphisms in  $\mathcal{D}(k)$ . We will leave it to the reader to check that  $\theta_{XYZ}$  and  $\omega_{XYZ}$  are in fact induced by maps of complexes of graded left  $B$ -modules. □

**Definition 2.4.** Let  $A$  be a connected algebra. We define  $\widehat{\mathcal{I}}(A)$  to be the full subcategory of  $\mathcal{D}^b(A)$  consisting of complexes having finite injective dimension, and  $\widehat{\mathcal{F}}(A)$  to be the full subcategory of  $\mathcal{D}^b(A)$  consisting of complexes having finite flat dimension.

Now Foxby equivalence is stated as follows:

**Theorem 2.5.** *Let  $A$  be a Noetherian connected algebra. If  $D \in \mathcal{D}^b(A^e)$  is a dualizing complex, then the functors  $D \otimes_A^L - : \mathcal{D}^b(A) \rightarrow \mathcal{D}^-(A)$  and  $R\mathbf{Hom}_A(D, -) : \mathcal{D}^b(A) \rightarrow \mathcal{D}^+(A)$  define inverse equivalences between  $\widehat{\mathcal{F}}(A)$  and  $\widehat{\mathcal{I}}(A)$ . They also define inverse equivalences between  $\widehat{\mathcal{F}}_{fg}(A)$  and  $\widehat{\mathcal{I}}_{fg}(A)$ .*

*Proof.* If  $X \in \widehat{\mathcal{F}}(A)$ , then

$$R\mathbf{Hom}_A(M, D \otimes_A^L X) \cong R\mathbf{Hom}_A(M, D) \otimes_A^L X$$

in  $\mathcal{D}(k)$  for all  $M \in \text{grmod } A$  by Proposition 2.3(2). By Lemma 1.2(1) and Lemma 1.3(2),

$$\begin{aligned} \text{id}_A(D \otimes_A^L X) &= \sup\{\sup \text{RHom}_A(M, D \otimes_A^L X) \mid M \in \text{grmod } A\} \\ &= \sup\{\sup(\text{RHom}_A(M, D) \otimes_A^L X) \mid M \in \text{grmod } A\} \\ &\leq \sup\{\sup \text{RHom}_A(M, D) + \sup X \mid M \in \text{grmod } A\} \\ &= \sup\{\sup \text{RHom}_A(M, D) \mid M \in \text{grmod } A\} + \sup X \\ &= \text{id}_A(D) + \sup X < \infty, \end{aligned}$$

and hence  $D \otimes_A^L X \in \widehat{\mathcal{I}}(A)$ . Moreover,

$$\text{RHom}_A(D, D \otimes_A^L X) \cong \text{RHom}_A(D, D) \otimes_A^L X \cong X$$

in  $\mathcal{D}(A)$  by Proposition 2.3(2).

If  $X \in \widehat{\mathcal{I}}(A)$ , then

$$N \otimes_A^L \text{RHom}_A(D, X) \cong \text{RHom}_A(\text{RHom}_{A^\circ}(N, D), X)$$

in  $\mathcal{D}(k)$  for all  $N \in \text{grmod } A^\circ$  by Proposition 2.3(1). By Lemma 1.2(2) and Lemma 1.3(1),

$$\begin{aligned} \text{fd}_A(\text{RHom}_A(D, X)) &= \sup\{-\inf(N \otimes_A \text{RHom}_A(D, X)) \mid N \in \text{grmod } A^\circ\} \\ &= \sup\{-\inf(\text{RHom}_A(\text{RHom}_{A^\circ}(N, D), X)) \mid N \in \text{grmod } A^\circ\} \\ &\leq \sup\{\sup(\text{RHom}_{A^\circ}(N, D)) - \inf X \mid N \in \text{grmod } A^\circ\} \\ &= \sup\{\sup(\text{RHom}_{A^\circ}(N, D)) \mid N \in \text{grmod } A^\circ\} - \inf X \\ &= \text{id}_{A^\circ}(D) - \inf X < \infty, \end{aligned}$$

and hence  $\text{RHom}_A(D, X) \in \widehat{\mathcal{F}}(A)$ . Moreover,

$$D \otimes_A^L \text{RHom}_A(D, X) \cong \text{RHom}_A(\text{RHom}_{A^\circ}(D, D), X) \cong X$$

in  $\mathcal{D}(A)$  by Proposition 2.3(1); hence the functors  $D \otimes_A^L -$  and  $\text{RHom}_A(D, -)$  define inverse equivalences between  $\widehat{\mathcal{F}}(A)$  and  $\widehat{\mathcal{I}}(A)$ .

If  $X \in \mathcal{D}_{fg}^b(A)$ , then

$$\begin{aligned} \text{RHom}_A(D, X) &\cong \text{RHom}_A(D, \text{RHom}_{A^\circ}(\text{RHom}_A(X, D), D)) \\ &\cong \text{RHom}_{A^\circ}(\text{RHom}_A(X, D), \text{RHom}_A(D, D)) \\ &\cong \text{RHom}_{A^\circ}(\text{RHom}_A(X, D), A) \end{aligned}$$

in  $\mathcal{D}(A)$  by [8, Theorem 1.2]. Since  $\text{RHom}_A(X, D) \in \mathcal{D}_{fg}^b(A^\circ)$ , it follows that  $\text{RHom}_A(D, X) \in \mathcal{D}_{fg}^+(A)$ , and hence the functors  $D \otimes_A^L -$  and  $\text{RHom}_A(D, -)$  define inverse equivalences between  $\widehat{\mathcal{F}}_{fg}(A)$  and  $\widehat{\mathcal{I}}_{fg}(A)$ . □

I thank the referee for comments on how to improve the above theorem.

### 3. COHEN-MACAULAY ALGEBRAS

In this section, we define a Cohen-Macaulay algebra and list some elementary properties of such an algebra.

**Definition 3.1.** Let  $A$  be a connected algebra, and  $M \in \text{GrMod } A$ . We define

- $\text{depth } M = \inf \text{RHom}_A(k, M) = \inf\{i \mid \text{Ext}_A^i(k, M) \neq 0\}$ ,
- $\text{ldim } M = \sup \text{R}\Gamma_{\mathfrak{m}}(M) = \sup\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}$ , and

- $\text{lcd}(A) = \sup\{\text{ldim } M \mid M \in \text{GrMod } A\}$ .

We say that  $M$  is Cohen-Macaulay if  $\text{depth } M = \text{ldim } M < \infty$ , and maximal Cohen-Macaulay if  $\text{depth } M = \text{ldim } M = \text{ldim } A < \infty$ . We say that  $A$  is Cohen-Macaulay on the left if  $A$  is Cohen-Macaulay as a graded left  $A$ -module.

If  $A$  is a connected algebra and  $M \in \text{GrMod } A$ , then

$$\text{depth } M = \inf R\Gamma_{\mathfrak{m}}(M) = \inf\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}$$

by [11, Chapter 11, Lemma 4.1]. So  $M$  is Cohen-Macaulay with  $\text{depth } M = m$  if and only if  $R\Gamma_{\mathfrak{m}}(M) \cong H_{\mathfrak{m}}^m(M)[-m] \neq 0$  in  $\mathcal{D}(A)$ .

**Definition 3.2.** Let  $A$  be a connected algebra with  $\text{ldim } A = d < \infty$ , and let  $\mathfrak{m} = A_{\geq 1}$  be the augmentation ideal. A left canonical module  $\omega_A$  is a graded  $A$ - $A$  bimodule such that for every  $M \in \text{GrMod } A$ ,

$$R\underline{\text{Hom}}_A(M, \omega_A) \cong R\Gamma_{\mathfrak{m}}(M)'[-d]$$

in  $\mathcal{D}(A^\circ)$ , that is, there are functorial isomorphisms

$$\underline{\text{Ext}}_A^i(M, \omega_A) \cong H_{\mathfrak{m}}^{d-i}(M)'$$

in  $\text{GrMod } A^\circ$  for all  $i$ .

If  $A$  has a left canonical module  $\omega_A$ , then  $M \in \text{GrMod } A$  is Cohen-Macaulay with  $\text{depth } M = m$  if and only if  $R\underline{\text{Hom}}_A(M, \omega_A) \cong \underline{\text{Ext}}_A^{d-m}(M, \omega_A)[m-d] \neq 0$  in  $\mathcal{D}(A^\circ)$ . In particular,  $M \in \text{GrMod } A$  is maximal Cohen-Macaulay if and only if  $R\underline{\text{Hom}}_A(M, \omega_A) \cong \underline{\text{Hom}}_A(M, \omega_A) \neq 0$  in  $\mathcal{D}(A^\circ)$ , that is,  $\underline{\text{Ext}}_A^i(M, \omega_A) = 0$  for all  $i \neq 0$  and  $\underline{\text{Hom}}_A(M, \omega_A) \neq 0$ .

**Definition 3.3** ([12]). Let  $A$  be a connected algebra. We say that  $A$  is Ext-finite if the  $\underline{\text{Ext}}_A^i(k, k)$  are finite dimensional over  $k$  for all  $i$ .

Since  $\underline{\text{Ext}}_A^i(k, k)' \cong \text{Tor}_A^i(k, k) \cong \underline{\text{Ext}}_{A^\circ}^i(k, k)'$  as graded  $k$ -vector spaces,  $A$  is Ext-finite if and only if  $A^\circ$  is Ext-finite. In particular, if  $A$  is either left Noetherian or right Noetherian, then  $A$  is Ext-finite.

**Theorem 3.4.** *Let  $A$  be a connected algebra. If  $A$  has a left canonical module, then  $A$  is Cohen-Macaulay on the left. Conversely, if  $A$  is an Ext-finite Cohen-Macaulay algebra on the left such that  $\text{lcd}(A) < \infty$ , then  $A$  has a left canonical module  $\omega_A = H_{\mathfrak{m}}^d(A)'$ , where  $d = \text{ldim } A < \infty$ .*

*Proof.* If  $A$  has a left canonical module  $\omega_A$ , then

$$H_{\mathfrak{m}}^i(A)' \cong \underline{\text{Ext}}_A^{d-i}(A, \omega_A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ \omega_A & \text{if } i = d, \end{cases}$$

in  $\text{GrMod } A^\circ$ . So  $A$  is Cohen-Macaulay on the left with  $\text{depth } A = \text{ldim } A = d < \infty$ .

Conversely, suppose that  $A$  is an Ext-finite Cohen-Macaulay algebra on the left with  $\text{depth } A = \text{ldim } A = d < \infty$ . Since  $R\Gamma_{\mathfrak{m}}(A) \cong H_{\mathfrak{m}}^d(A)[-d]$  in  $\mathcal{D}(A^\circ)$ , we have

$$\begin{aligned} R\underline{\text{Hom}}_A(M, H_{\mathfrak{m}}^d(A)') &\cong \underline{\text{Hom}}_A(M, R\Gamma_{\mathfrak{m}}(A)'[-d]) \\ &\cong R\underline{\text{Hom}}_A(M, R\Gamma_{\mathfrak{m}}(A)'[-d]) \\ &\cong R\Gamma_{\mathfrak{m}}(M)'[-d] \end{aligned}$$

in  $\mathcal{D}(A^\circ)$  for every  $M \in \text{GrMod } A$  by [12, Theorem 5.1]. So  $A$  has a left canonical module  $\omega_A = H_{\mathfrak{m}}^d(A)'$ . □

If  $A$  is an Ext-finite connected algebra, then  $k$  has a finitely generated minimal free resolution  $F$  in  $\text{GrMod } A^o$ . Since  $F'$  is an injective resolution of  $k$  in  $\text{GrMod } A$  and  $(F')^i \cong (F^{-i})'$  is a finite direct sum of shifts of  $A'$  that is torsion for each  $i$ , it follows that

$$H_m^i(k) = h^i(R\Gamma_m(k)) \cong h^i(\Gamma_m(F')) \cong h^i(F') \cong \begin{cases} k & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases}$$

that is,  $R\Gamma_m(k) \cong k$  in  $\mathcal{D}(A)$ . Using this fact, we can prove the following lemma (cf. [11, Chapter 11, Lemma 5.6]):

**Lemma 3.5.** *Let  $A$  be a Cohen-Macaulay algebra on the left with  $\text{ldim } A = d < \infty$ , and let  $\omega_A$  be a left canonical module. Then:*

1.  $\text{lcd}(A) = d < \infty$ . In particular,  $M \in \text{GrMod } A$  is maximal Cohen-Macaulay if and only if  $\text{depth } M = d$ .
2.  $\text{id}_A(\omega_A) = d < \infty$ .
3. If  $A$  is Ext-finite, then  $R\Gamma_m(\omega_A) \cong A'[-d]$  in  $\mathcal{D}(A)$ , that is,

$$H_m^i(\omega_A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ A' & \text{if } i = d, \end{cases}$$

in  $\text{GrMod } A$ . In particular,  $\omega_A$  is maximal Cohen-Macaulay as a graded left  $A$ -module.

4. If  $A$  is Ext-finite, then  $R\underline{\text{Hom}}_A(\omega_A, \omega_A) \cong A$  in  $\mathcal{D}(A^o)$ , that is,

$$\underline{\text{Ext}}_A^i(\omega_A, \omega_A) \cong \begin{cases} 0 & \text{if } i \neq 0, \\ A & \text{if } i = 0, \end{cases}$$

in  $\text{GrMod } A^o$ .

**Theorem 3.6.** *Let  $A$  be a Cohen-Macaulay algebra on the left, and let  $\omega_A$  be a left canonical module. If  $M \in \text{GrMod } A$  with  $0 \leq m = \text{depth } A - \text{depth } M < \infty$ , then  $M$  has a finite resolution of the form*

$$0 \rightarrow H \rightarrow F^{-m-1} \rightarrow \dots \rightarrow F^0 \rightarrow M \rightarrow 0$$

in  $\text{GrMod } A$ , where  $F^{-i} \in \text{GrMod } A$  are free, and  $H \in \text{GrMod } A$  is maximal Cohen-Macaulay.

*Proof.* Let  $F$  be a free resolution of  $M$  and let  $H^{-i}$  be the  $i$ -th syzygy of  $M$ , that is,

$$0 \rightarrow H^{-i-1} \rightarrow F^{-i} \rightarrow H^{-i} \rightarrow 0$$

is an exact sequence in  $\text{GrMod } A$  for each  $i \geq 0$ , where  $H^0 = M$ . By the long exact sequence of  $\underline{\text{Ext}}_A^i(k, -)$ ,

$$\begin{aligned} \text{depth } H^{-m} &= \inf R\underline{\text{Hom}}_A(k, H^{-m}) \\ &= \inf R\underline{\text{Hom}}_A(k, H^{-m+1}) + 1 \\ &= \dots \\ &= \inf R\underline{\text{Hom}}_A(k, H^0) + m \\ &= \text{depth } M + m = d. \end{aligned}$$

So  $H^{-m} \in \text{GrMod } A$  is maximal Cohen-Macaulay by Lemma 3.5(1). □

4. BALANCED COHEN-MACAULAY ALGEBRAS

**Definition 4.1.** Let  $A$  be a connected algebra. A graded  $A$ - $A$  bimodule  $\omega_A$  is called a dualizing module if

- $\text{res}_A \omega_A \in \text{grmod } A, \text{res}_{A^\circ} \omega_A \in \text{grmod } A^\circ,$
- $\text{id}_A(\omega_A) < \infty, \text{id}_{A^\circ}(\omega_A) < \infty,$  and
- the natural morphisms  $A \rightarrow \underline{R\text{Hom}}_A(\omega_A, \omega_A)$  and  $A \rightarrow \underline{R\text{Hom}}_{A^\circ}(\omega_A, \omega_A)$  are isomorphisms in  $\mathcal{D}(A^e).$

A dualizing module  $\omega_A$  is called balanced if  $R\Gamma_{\mathfrak{m}}(\omega_A) \cong R\Gamma_{\mathfrak{m}^\circ}(\omega_A) \cong A'[-d]$  in  $\mathcal{D}(A^e)$  for some integer  $d.$

A connected algebra  $A$  is called balanced Cohen-Macaulay if  $A$  has a balanced dualizing module.

Let  $A$  be a Noetherian connected algebra. Then a graded  $A$ - $A$  bimodule  $\omega_A$  is a dualizing module if and only if  $\omega_A$  is a dualizing complex, viewed as an object in  $\mathcal{D}(A^e).$  Moreover,  $\omega_A$  is a balanced dualizing module if and only if  $\omega_A[d] \in \mathcal{D}(A^e)$  is a balanced dualizing complex for some integer  $d.$  In particular, a balanced dualizing module  $\omega_A$  is unique up to isomorphisms in  $\text{GrMod } A^e$  by [13].

**Definition 4.2.** Let  $A$  be a connected algebra and  $M \in \text{GrMod } A.$  We say that  $\chi$  holds for  $M$  if  $\underline{\text{Ext}}_A^i(k, M)$  are bounded for all  $i.$  We say that  $A$  satisfies  $\chi$  on the left if  $\chi$  holds for all  $M \in \text{grmod } A.$

We say that  $A$  is AS Gorenstein if  $A$  satisfies  $\chi$  on both sides and  $\text{id}_A(A) = \text{id}_{A^\circ}(A) < \infty.$  We say that  $A$  is AS regular if  $A$  satisfies  $\chi$  on both sides and  $\text{gldim } A < \infty.$

Clearly, every AS regular algebra is AS Gorenstein. By [5, Theorem 1.2], if  $A$  is a Noetherian AS Gorenstein algebra, then  $A$  has a balanced dualizing complex  $A_\alpha(-l)[d]$  for some graded algebra automorphism  $\alpha$  of  $A,$  some integer  $l,$  and  $d = \text{id}_A(A) = \text{id}_{A^\circ}(A).$  So  $A_\alpha(-l)$  is a balanced dualizing module and  $A$  is balanced Cohen-Macaulay. In fact, let  $A$  be a Noetherian Cohen-Macaulay algebra on the left. Then  $A$  is balanced Cohen-Macaulay if and only if  $A$  is a graded quotient algebra of a Noetherian AS Gorenstein algebra, by [7, Theorem 1.6].

The following characterization of a balanced Cohen-Macaulay algebra is immediate from [12, Theorem 6.3].

**Theorem 4.3.** *Let  $A$  be a Noetherian connected algebra. Then  $A$  is balanced Cohen-Macaulay if and only if  $A$  is Cohen-Macaulay satisfying  $\chi$  on both sides. If  $A$  is a Noetherian balanced Cohen-Macaulay algebra, then a balanced dualizing module is given by  $\omega_A \cong H_{\mathfrak{m}}^d(A)' \cong H_{\mathfrak{m}^\circ}^d(A)'$  in  $\text{GrMod } A^e,$  where  $d = \text{ldim}_A A = \text{ldim}_{A^\circ} A < \infty.$  In particular,  $\omega_A$  is a left and right canonical module.*

**Definition 4.4.** Let  $A$  be a balanced Cohen-Macaulay algebra and  $\omega_A$  a balanced dualizing module. If  $M \in \text{GrMod } A,$  then we define  $M^\dagger = \underline{\text{Hom}}_A(M, \omega_A) \in \text{GrMod } A^\circ.$  Similarly, if  $N \in \text{GrMod } A^\circ,$  then we define  $N^\dagger = \underline{\text{Hom}}_{A^\circ}(N, \omega_A) \in \text{GrMod } A.$  We say that  $M \in \text{GrMod } A$  is totally  $\omega_A$ -reflexive if  $\underline{\text{Ext}}_A^i(M, \omega_A) = \underline{\text{Ext}}_A^i(M^\dagger, \omega_A) = 0$  for all  $i \neq 0$  and  $M^{\dagger\dagger} \cong M$  in  $\text{GrMod } A.$

Let  $\mathcal{A}$  be an abelian category and  $\mathcal{B} \subset \mathcal{A}$  a full subcategory. A  $\mathcal{B}$ -resolution of an object  $M \in \mathcal{A}$  is an exact sequence

$$\dots \rightarrow B^{-i} \rightarrow \dots \rightarrow B^{-1} \rightarrow B^0 \rightarrow M \rightarrow 0$$

in  $\mathcal{A},$  where  $B^{-i} \in \mathcal{B}$  for all  $i \geq 0.$

**Definition 4.5.** We define  $\mathcal{H}$  to be the full subcategory of  $\text{GrMod } A$  consisting of totally  $\omega_A$ -reflexive modules, and  $\mathcal{H}_{fg}$  to be the full subcategory of  $\text{grmod } A$  consisting of totally  $\omega_A$ -reflexive modules.

For  $M \in \text{GrMod } A$ , we define

$$\text{Hdim } M = \inf_H (-\inf\{i \mid H^i \neq 0\}),$$

where the infimum is taken over all  $\mathcal{H}$ -resolutions  $H$  of  $X$ , and

$$\text{hdim } M = \inf_H (-\inf\{i \mid H^i \neq 0\}),$$

where the infimum is taken over all  $\mathcal{H}_{fg}$ -resolutions  $H$  of  $X$ .

**Lemma 4.6.** *Let  $A$  be a Noetherian balanced Cohen-Macaulay algebra and  $\omega_A$  a balanced dualizing module. For  $M \in \text{grmod } A$ , the following are equivalent:*

1.  $M \in \mathcal{H}_{fg}$ ;
2.  $\underline{\text{Ext}}_A^i(M, \omega_A) = 0$  for all  $i \neq 0$ ;
3.  $M$  is either a maximal Cohen-Macaulay module or the zero module.

*Proof.* (1)  $\Rightarrow$  (2): By definition.

(2)  $\Rightarrow$  (3): Suppose that  $\underline{\text{Ext}}_A^i(M, \omega_A) = 0$  for all  $i \neq 0$ . If  $\underline{\text{Hom}}_A(M, \omega_A) \neq 0$ , then  $M \in \text{grmod } A$  is maximal Cohen-Macaulay. Since  $\omega_A \in \mathcal{D}^b(A^e)$  is a dualizing complex, if  $\underline{\text{Hom}}_A(M, \omega_A) = 0$ , then  $M \cong \underline{\text{RHom}}_{A^o}(\underline{\text{RHom}}_A(M, \omega_A), \omega_A) = 0$ .

(3)  $\Rightarrow$  (1): If  $M \in \text{grmod } A$  is maximal Cohen-Macaulay, then  $\underline{\text{Ext}}_A^i(M, \omega_A) = 0$  for all  $i \neq 0$ , that is,  $\underline{\text{RHom}}_A(M, \omega_A) \cong \underline{\text{Hom}}_A(M, \omega_A) = M^\dagger$  in  $\mathcal{D}(A^o)$ . Since  $\omega_A \in \mathcal{D}(A^e)$  is a dualizing complex,

$$\underline{\text{RHom}}_{A^o}(M^\dagger, \omega_A) \cong \underline{\text{RHom}}_{A^o}(\underline{\text{RHom}}_A(M, \omega_A), \omega_A) \cong M$$

in  $\mathcal{D}(A)$ . It follows that  $\underline{\text{Ext}}_{A^o}^i(M^\dagger, \omega_A) = 0$  for all  $i \neq 0$ , and

$$M^{\dagger\dagger} \cong \underline{\text{Hom}}_{A^o}(M^\dagger, \omega_A) \cong M$$

in  $\text{GrMod } A$ . Clearly,  $0 \in \mathcal{H}_{fg}$ . □

In particular, if  $A$  is a Noetherian balanced Cohen-Macaulay algebra and  $\omega_A$  is a balanced dualizing module, then  $\text{hdim } A = \text{hdim } \omega_A = 0$ .

Let  $A$  be a left Noetherian connected algebra and  $M \in \text{grmod } A$ . If  $A$  satisfies  $\chi$  on the left and  $\text{pd}(M) < \infty$ , then Jørgensen proved the Auslander-Buchsbaum formula

$$\text{pd}(M) + \text{depth } M = \text{depth } A$$

in [6, Theorem 3.2] (he actually proved the formula for  $X \in \mathcal{D}_{fg}^b(A)$ , using the obvious definition of  $\text{depth } X$ ). The following theorem can be regarded as a version of the Auslander-Buchsbaum formula for  $\text{hdim}$ .

**Theorem 4.7.** *Let  $A$  be a Noetherian balanced Cohen-Macaulay algebra and  $\omega_A$  a balanced dualizing module. If  $0 \neq M \in \text{grmod } A$ , then*

$$\text{hdim } M = \sup \underline{\text{RHom}}_A(M, \omega_A) = \text{depth } A - \text{depth } M \leq \text{depth } A < \infty.$$

*In particular, if  $\text{pd}(M) < \infty$ , then  $\text{hdim } M = \text{pd}(M)$ .*

*Proof.* Let  $d = \text{depth } A = \text{ldim } A < \infty$ . For  $M \in \text{GrMod } A$ ,

$$\sup \underline{\text{RHom}}_A(M, \omega_A) = \sup \text{R}\Gamma_m(M)'[-d] = d - \inf \text{R}\Gamma_m(M) = d - \text{depth } M \leq d.$$

We will now prove that  $\text{hdim } M = \sup \underline{\text{RHom}}_A(M, \omega_A)$  using induction on  $m = \sup \underline{\text{RHom}}_A(M, \omega_A)$ . If  $m = 0$ , then  $M$  is maximal Cohen-Macaulay; so  $\text{hdim } M =$

0 by Lemma 4.6. Suppose that  $m \geq 1$ . Let  $F \rightarrow M \rightarrow 0$  be a finitely generated minimal free resolution, and  $0 \rightarrow N \rightarrow F^0 \rightarrow M \rightarrow 0$  be an exact sequence in  $\text{GrMod } A$ . Since  $\underline{\text{Ext}}_A^i(F^0, \omega_A) = 0$  for all  $i \geq 1$ , from the long exact sequence of  $\underline{\text{Ext}}_A^i(-, \omega_A)$ , we have an epimorphism  $\underline{\text{Hom}}_A(N, \omega_A) \rightarrow \underline{\text{Ext}}_A^1(M, \omega_A)$  and isomorphisms  $\underline{\text{Ext}}_A^i(N, \omega_A) \cong \underline{\text{Ext}}_A^{i+1}(M, \omega_A)$  for all  $i \geq 1$ . So  $\sup \text{RHom}_A(N, \omega_A) = m - 1$ . By induction,  $\text{hdim } M \leq \text{hdim } N + 1 = \sup \text{RHom}_A(N, \omega_A) + 1 = m$ .

Let  $H \rightarrow M \rightarrow 0$  be an  $\mathcal{H}_{fg}$ -resolution of minimal length, and  $0 \rightarrow N \rightarrow H^0 \rightarrow M \rightarrow 0$  an exact sequence in  $\text{GrMod } A$ . By a similar argument, we can show that  $\sup \text{RHom}_A(N, \omega_A) = m - 1$ , and

$$\text{hdim } M = \text{hdim } N + 1 = \sup \text{RHom}_A(N, \omega_A) + 1 = m.$$

Since  $A$  satisfies  $\chi$  on the left by Theorem 4.3, if  $\text{pd}(M) < \infty$ , then

$$\text{pd}(M) = \text{depth } A - \text{depth } M = \text{hdim } A$$

by the classical Auslander-Buchsbaum formula. □

**Corollary 4.8.** *Let  $A$  be a Noetherian balanced Cohen-Macaulay algebra. Then every finitely generated maximal Cohen-Macaulay module having finite projective dimension is free.*

*Proof.* If  $M \in \text{grmod } A$  is maximal Cohen-Macaulay having finite projective dimension, then  $\text{pd}(M) = \text{hdim } M = 0$  by Lemma 4.6 and Theorem 4.7; so  $M$  is free. □

### 5. MAXIMAL COHEN-MACAULAY APPROXIMATIONS

In this section, we will prove the theorem proposed in the abstract, using maximal Cohen-Macaulay approximations and Foxby equivalence.

Throughout this section, we assume that  $A$  is a Noetherian balanced Cohen-Macaulay algebra with  $\text{ldim}_A A = \text{ldim}_{A^\circ} A = d < \infty$ , and that  $\omega_A$  is a balanced dualizing module.

Let  $\mathcal{A}$  be an abelian category and  $\mathcal{B} \subset \mathcal{A}$  a full subcategory. We define  $\widehat{\mathcal{B}}$  to be the full subcategory of  $\mathcal{A}$  consisting of objects  $M \in \mathcal{A}$  having  $\mathcal{B}$ -resolutions of finite length.

A full additive subcategory  $\mathcal{B} \subset \mathcal{A}$  is called additively closed if  $\mathcal{B}$  is closed under finite direct sums in  $\mathcal{A}$  and closed under direct summands in  $\mathcal{A}$ . A full additive subcategory  $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$  is called a cogenerator for  $\mathcal{B}$  if for every object  $M \in \mathcal{B}$ , there is an exact sequence

$$0 \rightarrow M \rightarrow C \rightarrow B \rightarrow 0$$

in  $\mathcal{A}$ , where  $C \in \mathcal{C}$  and  $B \in \mathcal{B}$ .

**Theorem 5.1** ([2, Theorem 1.1]). *Let  $\mathcal{A}$  be an abelian category,  $\mathcal{B} \subset \mathcal{A}$  an additively closed subcategory that is closed under extensions in  $\mathcal{A}$ , and  $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$  an additively closed subcategory that is a cogenerator for  $\mathcal{B}$ . For every  $M \in \widehat{\mathcal{B}}$ , there are exact sequences*

$$0 \rightarrow \widehat{C}_M \rightarrow B_M \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow \widehat{C}^M \rightarrow B^M \rightarrow 0$$

in  $\mathcal{A}$ , where  $B_M, B^M \in \mathcal{B}$  and  $\widehat{C}_M, \widehat{C}^M \in \widehat{\mathcal{C}}$ .

Using Theorem 5.1, we will prove maximal Cohen-Macaulay approximations for Noetherian balanced Cohen-Macaulay algebras (cf. [2, Example 3]).

**Definition 5.2.** We define  $\mathcal{J}$  to be the full subcategory of  $\mathcal{H}$  consisting of all modules  $M \in \mathcal{H}$  having finite injective dimension.

**Proposition 5.3.** For every  $M \in \text{grmod } A$ , there are exact sequences

$$0 \rightarrow \widehat{J}_M \rightarrow H_M \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow M \rightarrow \widehat{J}^M \rightarrow H^M \rightarrow 0$$

in  $\text{GrMod } A$ , where  $H_M, H^M \in \mathcal{H}_{fg}$  and  $\widehat{J}_M, \widehat{J}^M \in \widehat{\mathcal{J}}_{fg}$ .

*Proof.* We will apply Theorem 5.1 to  $\mathcal{A} = \text{grmod } A, \mathcal{B} = \mathcal{H}_{fg}$ , and  $\mathcal{C} = \mathcal{J}_{fg}$ . By Theorem 4.7,  $\widehat{\mathcal{H}}_{fg} = \text{grmod } A$ .

Clearly,  $\mathcal{H}_{fg}$  and  $\mathcal{J}_{fg}$  are closed under finite direct sums in  $\text{grmod } A$ . Let  $M \in \text{grmod } A$ , and let  $N \in \text{grmod } A$  be a direct summand of  $M$ . If  $M \in \mathcal{H}_{fg}$ , then  $\underline{\text{Ext}}_A^i(N, \omega_A) \subset \underline{\text{Ext}}_A^i(M, \omega_A) = 0$  for all  $i \neq 0$ ; so  $N \in \mathcal{H}_{fg}$  by Lemma 4.6. If  $M \in \mathcal{J}_{fg}$ , then

$$\begin{aligned} \text{id}(N) &= \sup\{\sup \text{RHom}_A(L, N) \mid L \in \text{grmod } A\} \\ &\leq \sup\{\sup \text{RHom}_A(L, M) \mid L \in \text{grmod } A\} \\ &= \text{id}(M) < \infty \end{aligned}$$

by Lemma 1.2(1); so  $N \in \mathcal{J}_{fg}$ . It follows that  $\mathcal{H}_{fg}$  and  $\mathcal{J}_{fg}$  are additively closed in  $\text{grmod } A$ . Let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be an exact sequence in  $\text{grmod } A$ . If  $L, N \in \mathcal{H}_{fg}$ , then  $\underline{\text{Ext}}_A^i(M, \omega_A) = 0$  for all  $i \neq 0$  from the long exact sequence of  $\underline{\text{Ext}}_A^i(-, \omega_A)$ , and so  $M \in \mathcal{H}_{fg}$  by Lemma 4.6; hence  $\mathcal{H}_{fg}$  is closed under extensions.

Finally, we will prove that  $\mathcal{J}_{fg}$  is a cogenerator for  $\mathcal{H}_{fg}$ . Let  $0 \neq M \in \mathcal{H}_{fg}$ . Since  $M^\dagger = \underline{\text{Hom}}_A(M, \omega_A) \in \text{grmod } A^\circ$ , there is an exact sequence

$$0 \rightarrow N \rightarrow F \rightarrow M^\dagger \rightarrow 0$$

in  $\text{grmod } A^\circ$ , where  $F \in \text{grmod } A^\circ$  is free. Since  $F, M^\dagger \in \text{grmod } A^\circ$  are maximal Cohen-Macaulay, it follows that  $M^{\dagger\dagger} \cong M$  in  $\text{grmod } A$  by Lemma 4.6. So there is an exact sequence

$$0 \rightarrow M^{\dagger\dagger} \cong M \rightarrow F^\dagger \rightarrow N^\dagger \rightarrow \underline{\text{Ext}}_{A^\circ}^1(M^\dagger, \omega_A) = 0$$

in  $\text{grmod } A$ . Since  $F^\dagger = \underline{\text{Hom}}_{A^\circ}(F, \omega_A)$  is a finite direct sum of shifts of  $\omega_A$  in  $\text{grmod } A$ , it follows that  $F^\dagger \in \mathcal{J}_{fg}$ . Since  $M, F^\dagger \in \mathcal{H}_{fg}$ , it follows that  $\underline{\text{Ext}}_A^i(N^\dagger, \omega_A) = 0$  for all  $i \geq 2$  from the long exact sequence of  $\underline{\text{Ext}}_A^i(-, \omega_A)$ . Since  $F \in \text{grmod } A^\circ$  is maximal Cohen-Macaulay, we have  $F^{\dagger\dagger} \cong F$  in  $\text{grmod } A^\circ$  by Lemma 4.6. So there is an exact sequence

$$0 \rightarrow \underline{\text{Hom}}_A(N^\dagger, \omega_A) \rightarrow F^{\dagger\dagger} \cong F \rightarrow M^\dagger \rightarrow \underline{\text{Ext}}_A^1(N^\dagger, \omega_A) \rightarrow \underline{\text{Ext}}_A^1(F^\dagger, \omega_A) = 0$$

in  $\text{grmod } A^\circ$ . It follows that  $\underline{\text{Hom}}_A(N^\dagger, \omega_A) \cong N$  in  $\text{grmod } A^\circ$  and  $\underline{\text{Ext}}_A^1(N^\dagger, \omega_A) = 0$ ; so  $N^\dagger \in \mathcal{H}_{fg}$  by Lemma 4.6. This shows that  $\mathcal{J}_{fg}$  is a cogenerator for  $\mathcal{H}_{fg}$ .  $\square$

Now we will return to Foxby equivalence discussed in section 2, and apply it to Noetherian balanced Cohen-Macaulay algebras.

**Lemma 5.4.** *Let  $0 \neq X \in \widehat{\mathcal{I}}(A)$ . If  $R\mathbf{Hom}_A(\omega_A, X) \in \widehat{\mathcal{F}}(A)$  has a minimal free resolution, then*

$$\mathrm{pd}(R\mathbf{Hom}(\omega_A, X)) = \mathrm{fd}(R\mathbf{Hom}(\omega_A, X)) = \mathrm{depth} A - \mathrm{depth} X.$$

*Proof.* Since  $X \in \widehat{\mathcal{I}}(A)$ ,

$$\begin{aligned} k \otimes_A^L R\mathbf{Hom}_A(\omega_A, X) &\cong R\mathbf{Hom}_A(R\mathbf{Hom}_{A^e}(k, \omega_A), X) \\ &\cong R\mathbf{Hom}_A(k[-d], X) \\ &\cong R\mathbf{Hom}_A(k, X)[d] \end{aligned}$$

in  $\mathcal{D}(k)$  by Proposition 2.3(1). If  $R\mathbf{Hom}_A(\omega_A, X) \in \widehat{\mathcal{F}}(A)$  has a minimal free resolution, then, by Lemma 1.2(3),

$$\begin{aligned} \mathrm{pd}(R\mathbf{Hom}_A(\omega_A, X)) &= \mathrm{fd}(R\mathbf{Hom}_A(\omega_A, X)) \\ &= -\inf(k \otimes_A^L R\mathbf{Hom}_A(\omega_A, X)) \\ &= -\inf(R\mathbf{Hom}_A(k, X)[d]) \\ &= -\inf(R\mathbf{Hom}_A(k, X)) + d \\ &= \mathrm{depth} A - \mathrm{depth} X. \quad \square \end{aligned}$$

**Proposition 5.5.** *If  $M \in \mathrm{grmod} A$  has finite injective dimension, then*

$$R\mathbf{Hom}_A(\omega_A, M) \cong \mathbf{Hom}_A(\omega_A, M)$$

*in  $\mathcal{D}_{fg}(A)$ ; that is,  $\mathbf{Ext}_A^i(\omega_A, M) = 0$  for all  $i \neq 0$ , and  $\mathbf{Hom}_A(\omega_A, M) \in \mathrm{grmod} A$ . Moreover, if  $M \neq 0$ , then*

$$\mathrm{pd}(\mathbf{Hom}_A(\omega_A, M)) = \mathrm{depth} A - \mathrm{depth} M = \mathrm{hdim} M.$$

*Proof.* Suppose that  $M \in \mathrm{grmod} A$  has finite injective dimension. Since  $\omega_A$  and  $\mathbf{Ext}_A^i(\omega_A, M) \in \mathrm{GrMod} A$  are left bounded for all  $i$  by [1, Proposition 3.1(1)], it follows that

$$\begin{aligned} \sup(R\mathbf{Hom}_A(\omega_A, M)) &= \sup \omega_A + \sup(R\mathbf{Hom}_A(\omega_A, M)) \\ &= \sup(\omega_A \otimes_A^L R\mathbf{Hom}_A(\omega_A, M)) \\ &= \sup M = 0 \end{aligned}$$

by Lemma 1.3(2). So  $R\mathbf{Hom}_A(\omega_A, M) \cong \mathbf{Hom}_A(\omega_A, M)$  in  $\mathcal{D}(A)$ . Since  $M \in \widehat{\mathcal{I}}_{fg}(A)$ , by Theorem 2.5,  $R\mathbf{Hom}_A(\omega_A, M) \cong \mathbf{Hom}_A(\omega_A, M) \in \widehat{\mathcal{F}}_{fg}(A)$  has a minimal free resolution. By Lemma 5.4 and Theorem 4.7,

$$\mathrm{pd}(\mathbf{Hom}_A(\omega_A, M)) = \mathrm{depth} A - \mathrm{depth} M = \mathrm{hdim} M. \quad \square$$

**Corollary 5.6.** *Every finitely generated maximal Cohen-Macaulay module having finite injective dimension is a finite direct sum of shifts of  $\omega_A$ .*

*Proof.* Let  $M \in \mathrm{grmod} A$  be maximal Cohen-Macaulay having finite injective dimension. By Proposition 5.5,  $R\mathbf{Hom}_A(\omega_A, M) \cong \mathbf{Hom}_A(\omega_A, M)$  in  $\mathcal{D}(A)$ . Since

$$\omega_A \otimes_A^L \mathbf{Hom}_A(\omega_A, M) \cong \omega_A \otimes_A^L R\mathbf{Hom}_A(\omega_A, M) \cong M$$

in  $\mathcal{D}(A)$  by Theorem 2.5,  $\omega_A \otimes_A \mathbf{Hom}_A(\omega_A, M) \cong M$  in  $\mathrm{GrMod} A$ . By Proposition 5.5,  $\mathrm{pd}(\mathbf{Hom}_A(\omega_A, M)) = \mathrm{depth} A - \mathrm{depth} M = 0$ . So  $\mathbf{Hom}_A(\omega_A, M) \in \mathrm{grmod} A$  is finitely generated free; hence  $M \cong \omega_A \otimes_A \mathbf{Hom}_A(\omega_A, M) \in \mathrm{grmod} A$  is a finite direct sum of shifts of  $\omega_A$ .  $\square$

*Remark 5.7.* It would be much nicer if we could prove the dual statement of Proposition 5.5 as in the commutative case [4, Corollary 3.6], namely,  $A$  has the following property (P): “if  $M \in \text{grmod } A$  has finite flat dimension, then  $\omega_A \otimes_A^L M \cong \omega_A \otimes_A M$  in  $\mathcal{D}_{fg}(A)$ , so that  $\omega_A \otimes_A M \in \text{grmod } A$  has finite injective dimension.” This property (P) implies an important property of  $\omega_A$  discussed in [7] and [9], namely, “if  $x \in A$  is regular on  $A$ , then  $x$  is regular on  $\omega_A$  from both sides.” In fact, suppose that  $A$  has the property (P). Let  $x \in A$  be a homogeneous regular element on  $A$  of degree  $l$ , and let

$$0 \rightarrow A(-l) \xrightarrow{x} A \rightarrow M \rightarrow 0$$

be an exact sequence in  $\text{GrMod } A$ . Since  $M \in \text{grmod } A$  has finite flat dimension, it follows that  $\text{Tor}_1^A(\omega_A, M) = 0$  by the property (P). So

$$0 \rightarrow \omega_A(-l) \xrightarrow{x} \omega_A \rightarrow \omega_A \otimes_A M \rightarrow 0$$

is an exact sequence in  $\text{GrMod } A$ . It follows that  $x$  is regular on  $\omega_A$  from the right. By symmetry,  $x$  is regular on  $\omega_A$  from the left.

Let  $M \in \text{grmod } A$ . By Theorem 3.6,  $M$  has a finite resolution of the form

$$0 \rightarrow H \rightarrow \bigoplus_{j=1}^{r_{m-1}} A(-l_{m-1j}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{r_0} A(-l_{0j}) \rightarrow M \rightarrow 0$$

in  $\text{GrMod } A$ , where  $H \in \mathcal{H}_{fg}$  and  $m = \text{depth } A - \text{depth } M = \text{hdim } M < \infty$ . It is well known that  $M$  has finite projective dimension if and only if  $M$  has a finite resolution of the form

$$0 \rightarrow \bigoplus_{j=1}^{r_m} A(-l_{mj}) \rightarrow \bigoplus_{j=1}^{r_{m-1}} A(-l_{m-1j}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{r_0} A(-l_{0j}) \rightarrow M \rightarrow 0$$

in  $\text{GrMod } A$ , where  $m = \text{pd}(M) = \text{hdim } M < \infty$ . The following theorem can be compared with these facts.

**Theorem 5.8.** *Let  $M \in \text{grmod } A$ . Then:*

1.  $M$  has a finite resolution of the form

$$0 \rightarrow \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \rightarrow H \rightarrow M \rightarrow 0$$

in  $\text{GrMod } A$ , where  $H \in \mathcal{H}_{fg}$ ;

2.  $M$  has finite injective dimension if and only if  $M$  has a finite resolution of the form

$$(*) \quad 0 \rightarrow \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \rightarrow \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \rightarrow M \rightarrow 0$$

in  $\text{GrMod } A$ , where  $m = \text{hdim } M < \infty$ .

*Proof.* (1): By Proposition 5.3, there is an exact sequence

$$0 \rightarrow \widehat{J} \rightarrow H \rightarrow M \rightarrow 0$$

in  $\text{GrMod } A$ , where  $H \in \mathcal{H}_{fg}$  and  $\widehat{J} \in \widehat{\mathcal{J}}_{fg}$ . Since every  $J \in \mathcal{J}_{fg}$  is a finite direct sum of shifts of  $\omega_A$  or the zero module by Corollary 5.6,  $\widehat{J}$  has a finite resolution

of the form

$$0 \rightarrow \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{r_1} \omega_A(-l_{1j}) \rightarrow \widehat{J} \rightarrow 0,$$

in  $\text{GrMod } A$ , hence the result.

(2): Suppose that  $M \in \text{grmod } A$  has finite injective dimension. By Proposition 5.5,  $\underline{\text{Hom}}_A(\omega_A, M) \in \text{grmod } A$ , and  $\text{pd}(\underline{\text{Hom}}_A(\omega_A, M)) = \text{hdim } M = m < \infty$ . So  $R\underline{\text{Hom}}_A(\omega_A, M) \cong \underline{\text{Hom}}_A(\omega_A, M) \in \mathcal{D}_{fg}^b(A)$  has a finitely generated minimal free resolution  $F$  of length  $m$ . By Theorem 2.5,

$$\omega_A \otimes_A F \cong \omega_A \otimes_A^L R\underline{\text{Hom}}_A(\omega_A, M) \cong M$$

in  $\mathcal{D}(A)$ . So  $\omega_A \otimes_A F$  is a resolution of the form (\*).

Conversely, if  $M$  has a resolution of the form (\*), then clearly  $\text{id}(M) < \infty$ .  $\square$

As corollaries, we have the following characterizations of AS Gorenstein algebras and AS regular algebras.

**Corollary 5.9.** *Let  $A$  be a Noetherian balanced Cohen-Macaulay algebra. Then the following are equivalent:*

1.  $A$  is AS Gorenstein;
2.  $\text{id}_A(A) < \infty$ ;
3.  $\text{pd}_A(\omega_A) < \infty$ ;
4. for every  $M \in \text{grmod } A$ ,  $\text{id}(M) < \infty$  if and only if  $\text{pd}(M) < \infty$ .

*Proof.* (4)  $\Rightarrow$  (3): Suppose that  $A$  has the property (4). Since  $\text{id}_A(\omega_A) < \infty$ , it follows that  $\text{pd}_A(\omega_A) < \infty$ .

(3)  $\Rightarrow$  (2): If  $\text{pd}_A(\omega_A) < \infty$ , then  $\omega_A$  is free by Corollary 4.8. Since  $\text{id}_A(\omega_A) < \infty$ , it follows that  $\text{id}_A(A) < \infty$ .

(2)  $\Rightarrow$  (1): This follows from [8, Corollary 4.6].

(1)  $\Rightarrow$  (4): If  $A$  is AS Gorenstein, then  $\omega_A \cong A(-l)$  in  $\text{GrMod } A$  for some integer  $l$  by [5, Theorem 1.2]. The result follows from Theorem 5.8.  $\square$

*Remark 5.10.* The direction (1)  $\Rightarrow$  (4) of the above corollary was proved by Zhang, using a spectral sequence [11, Chapter 1, Proposition 6.7].

**Corollary 5.11.** *Let  $A$  be a Noetherian balanced Cohen-Macaulay algebra. Then the following are equivalent:*

1.  $A$  is AS regular;
2. every  $H \in \mathcal{H}_{fg}$  has finite projective dimension;
3. every nonzero  $H \in \mathcal{H}_{fg}$  is free;
4. every  $H \in \mathcal{H}_{fg}$  has finite injective dimension;
5. every nonzero  $H \in \mathcal{H}_{fg}$  is a finite direct sum of shifts of  $\omega_A$ .

*Proof.* (2)  $\Leftrightarrow$  (3) by Corollary 4.8, and (4)  $\Leftrightarrow$  (5) by Corollary 5.6.

If  $A$  is AS regular, then clearly every  $M \in \text{GrMod } A$  has finite projective dimension and finite injective dimension; so (1)  $\Rightarrow$  (2), (4).

If every  $H \in \mathcal{H}_{fg}$  has finite projective dimension, then every  $M \in \text{grmod } A$  has finite projective dimension by Theorem 5.8; so (2)  $\Rightarrow$  (1). Similarly, if every  $H \in \mathcal{H}_{fg}$  has finite injective dimension, then every  $M \in \text{grmod } A$  has finite injective dimension by Theorem 5.8; so (4)  $\Rightarrow$  (1).  $\square$

6. AN APPLICATION TO THE INTERSECTION MULTIPLICITY

We will end the paper by an application of Theorem 5.8 to the intersection multiplicity discussed in [10].

**Definition 6.1.** For  $V \in \text{GrMod } k$  locally finite, we define the Hilbert series of  $V$  by

$$H_V(t) = \sum_{i=-\infty}^{\infty} \dim_k V_i t^i \in \mathbb{Z}[[t, t^{-1}]].$$

If  $H_V(t)$  is a rational function over  $\mathbb{C}$ , then we define  $\text{GKdim } V$  to be the order of the pole of  $H_V(t)$  at  $t = 1$ , and we define the multiplicity of  $V$  by

$$e(V) = \lim_{t \rightarrow 1} (1 - t)^{\text{GKdim } V} H_V(t).$$

For  $X \in \mathcal{D}_{lf}^b(k)$ , we define the Hilbert series of  $X$  by

$$H_X(t) = \sum_{i=-\infty}^{\infty} (-1)^i H_{h^i(X)}(t) \in \mathbb{Z}[[t, t^{-1}]].$$

Let  $A$  be a Cohen-Macaulay algebra on the left and  $\omega_A$  a left canonical module. If  $M \in \text{grmod } A$  has a finite  $\omega_A$ -resolution of the form

$$(*) \quad 0 \rightarrow \bigoplus_{j=1}^{r_m} \omega_A(-l_{mj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \rightarrow M \rightarrow 0$$

in  $\text{GrMod } A$ , then the  $\omega_A$ -characteristic polynomial of  $M$  is defined by

$$r_M(t) := \sum_{i=0}^m (-1)^i \sum_{j=1}^{r_i} t^{l_{ij}} \in \mathbb{Z}[t, t^{-1}].$$

Note that if  $M$  has a finite  $\omega_A$ -resolution of the form  $(*)$ , then

$$H_M(t) = \sum_{i=0}^m (-1)^i \sum_{j=1}^{r_i} H_{\omega_A(-l_{ij})}(t) = \sum_{i=0}^m (-1)^i \sum_{j=1}^{r_i} t^{l_{ij}} H_{\omega_A}(t) = r_M(t) H_{\omega_A}(t).$$

**Definition 6.2** ([9], [10]). Let  $A$  be a connected algebra, and let  $M \in \text{GrMod } A$  be locally finite. We say that  $M$  is rational if

- $R\Gamma_m(M) \in \mathcal{D}_{lf}^b(A)$ ;
- $H_M(t)$  and  $H_{R\Gamma_m}(t)$  are both rational functions over  $\mathbb{C}$ ;
- $H_M(t) = H_{R\Gamma_m(M)}(t)$  as rational functions over  $\mathbb{C}$ .

We say that  $A$  is universally rational, if every  $M \in \text{grmod } A$  is rational.

**Lemma 6.3.** Let  $A$  be a universally rational, Noetherian balanced Cohen-Macaulay algebra, and  $M, N \in \text{grmod } A$ . If  $N$  has finite injective dimension, then

$$H_{R\text{Hom}_A(M, N)}(t) = H_M(t^{-1}) H_N(t) / H_A(t^{-1}).$$

*Proof.* Since  $N \in \text{grmod } A$  has finite injective dimension,  $N$  has a finite  $\omega_A$ -resolution of the form

$$0 \rightarrow \bigoplus_{j=1}^{r_n} \omega_A(-l_{nj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{r_0} \omega_A(-l_{0j}) \rightarrow N \rightarrow 0,$$

by Theorem 5.8.

Since  $A$  is universally rational, Noetherian balanced Cohen-Macaulay, we have

$$H_{\omega_A}(t) = H_{R\Gamma_m(A)'[-d]}(t) = (-1)^d H_{R\Gamma_m(A)}(t^{-1}) = (-1)^d H_A(t^{-1}).$$

So

$$\begin{aligned} H_{R\underline{\text{Hom}}_A(M,N)}(t) &= \sum_{i=0}^n (-1)^i \sum_{j=1}^{r_i} H_{R\underline{\text{Hom}}_A(M, \omega_A(-l_{ij}))}(t) \\ &= \sum_{i=0}^n (-1)^i \sum_{j=1}^{r_i} t^{l_{ij}} H_{R\underline{\text{Hom}}_A(M, \omega_A)}(t) \\ &= r_N(t) H_{R\Gamma_m(M)'[-d]}(t) \\ &= (-1)^d H_{R\Gamma_m(M)}(t^{-1}) r_N(t) \\ &= (-1)^d H_M(t^{-1}) H_N(t) / H_{\omega_A}(t) \\ &= H_M(t^{-1}) H_N(t) / H_A(t^{-1}). \quad \square \end{aligned}$$

Let  $A$  be a connected algebra. For  $M, N \in \text{grmod } A$ , we define the intersection multiplicity of  $M$  and  $N$  by

$$M \cdot N = (-1)^{\text{GKdim } N} \sum_{i=0}^{\infty} (-1)^i \dim_k \underline{\text{Ext}}_A^i(M, N).$$

It is well defined if  $\underline{\text{Ext}}_A^i(M, N) = 0$  for all  $i \gg 0$ , and  $\dim_k \underline{\text{Ext}}_A^i(M, N) < \infty$  for all  $i \geq 0$ . We can then prove a version of Serre's multiplicity conjectures as in [10].

**Theorem 6.4.** *Let  $A$  be a universally rational, Noetherian balanced Cohen-Macaulay algebra, and  $M, N \in \text{grmod } A$ . Suppose that  $N$  has finite injective dimension, and  $M \cdot N$  is well defined. Then*

1. (Dimension)  $\text{GKdim } M + \text{GKdim } N \leq \text{GKdim } A$ .
2. (Vanishing) *If  $\text{GKdim } M + \text{GKdim } N < \text{GKdim } A$ , then  $M \cdot N = 0$ .*
3. (Positivity) *If  $\text{GKdim } M + \text{GKdim } N = \text{GKdim } A$ , then*

$$M \cdot N = e(M)e(N)/e(A) > 0.$$

*Proof.* Using Lemma 6.3, exactly the same proof as in [10, Theorem 3.9] goes through.  $\square$

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