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SUPERCONGRUENCES BETWEEN TRUNCATED $_2F_1$ HYPERGEOMETRIC FUNCTIONS AND THEIR GAUSSIAN ANALOGS

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ABSTRACT. Fernando Rodriguez-Villegas has conjectured a number of supercongruences for hypergeometric Calabi-Yau manifolds of dimension $d \leq 3$. For manifolds of dimension $d = 1$, he observed four potential supercongruences. Later the author proved one of the four. Motivated by Rodriguez-Villegas's work, in the present paper we prove a general result on supercongruences between values of truncated $_2F_1$ hypergeometric functions and Gaussian hypergeometric functions. As a corollary to that result, we prove the three remaining supercongruences.

1. INTRODUCTION

In [RV1], Rodriguez-Villegas studied special families of hypergeometric functions, and then, via toric geometry, he associated to them Calabi-Yau manifolds. He observed (numerically) supercongruences between the values of the truncated hypergeometric functions and expressions derived from the number of $\mathbb{F}_p$-points of the associated Calabi-Yau manifolds. This had been motivated by his joint work with Candelas and de la Ossa [COV], where they studied Calabi-Yau manifolds defined over finite fields. Supercongruences of this type were first observed by Beukers [B] in connection with the Apéry numbers used in the proof of the irrationality of $\zeta(3)$. Ishikawa [I] proved Beukers' conjecture relating Apéry numbers to the coefficients of a certain weight-four newform for "ordinary primes"; later, Ahlgren and Ono [A-O] proved Beukers' supercongruence completely.

In [RV1, (36)], Rodriguez-Villegas identified four elliptic curves with potential supercongruences. We give the conjectured supercongruences in all four cases below.

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Conjecture 1. *If $p \geq 5$ is a prime and if $(\frac{\cdot}{p})$ is the Legendre symbol modulo p, then*

$$\sum_{n=0}^{p-1} \frac{(2n)!^2}{(n!)^4} 2^{-4n} \equiv \left(\frac{-4}{p}\right) \pmod{p^2}, \tag{1.1}$$

$$\sum_{n=0}^{p-1} \frac{(3n)!}{(n!)^3} 3^{-3n} \equiv \left(\frac{-3}{p}\right) \pmod{p^2}, \tag{1.2}$$

$$\sum_{n=0}^{p-1} \frac{(4n)!}{(n!)^2(2n)!} 2^{-6n} \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \tag{1.3}$$

$$\sum_{n=0}^{p-1} \frac{(6n)!}{(n!)(2n)!(3n)!} 2^{-4n}3^{-3n} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}. \tag{1.4}$$

Rodriguez-Villegas [RV2] proved these congruences modulo p. In [M], the author proved (1.1). Here we prove the remaining three conjectures by finding and developing the connection between classical hypergeometric series and their finite field counterparts. Moreover, we prove general results on the values of certain truncated ${}_2F_1$ hypergeometric functions.

To state our two main theorems, we recall basic facts about characters and Jacobi sums, and define notation. We denote by $\mathbb{F}_q$ the finite field with $q = p^f$ elements, where p is a prime. We extend all multiplicative characters χ on $\mathbb{F}_q^\times$ to $\mathbb{F}_q$ by setting $\chi(0) := 0$. If A and B are two characters on $\mathbb{F}_q$, then we define $\binom{A}{B}$ in terms of the Jacobi sum by

$$\binom{A}{B} := \frac{B(-1)}{q} J(A, \bar{B}) = \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x)\bar{B}(1-x). \tag{1.5}$$

If A, B, and C are characters on $\mathbb{F}_q$ and if $x \in \mathbb{F}_q$, then J. Greene [G] defines ${}_2F_1$ *Gaussian hypergeometric series* by

$${}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \mid x\right)_q := \epsilon_q(x) \frac{BC(-1)}{q} \sum_{y \in \mathbb{F}_q} B(y)\bar{B}C(1-y)\bar{A}(1-xy). \tag{1.6}$$

Here $\epsilon_q(x)$ is the trivial character on $\mathbb{F}_q$. He investigates properties of these functions in analogy with properties of classical hypergeometric functions. Among other results, Greene [G, 3.6] proves that

$${}_2F_1\left(\begin{matrix} A, & B \\ & C \end{matrix} \mid x\right)_q = \frac{q}{q-1} \sum_{\chi} \binom{A\chi}{\chi}\binom{B\chi}{C\chi}\chi(x), \tag{1.7}$$

where the sum runs over all characters χ on $\mathbb{F}_q$. If m is a positive integer, then we define the truncated hypergeometric series by

$${}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \mid x\right)_{\mathrm{tr}(m)} := \sum_{k=0}^{m-1} \frac{(a)_k(b)_k}{k!(c)_k} x^k, \tag{1.8}$$

where

$$(a)_k := a(a+1)\cdots(a+k-1). \tag{1.9}$$

For special a, b, and c we are able to relate these series by supercongruences.

Theorem 1. *Let m and d be integers with $1 \leq m < d$. If $p \equiv 1 \pmod{d}$ is prime and ρ is a character of order d on $\mathbb{F}_p$, then*

$$ {}_2F_1\left(\begin{matrix} \frac{m}{d}, & \frac{d-m}{d} \\ & 1 \end{matrix} \;\middle|\; 1\right)_{tr(p)} \equiv -p \cdot {}_2F_1\left(\begin{matrix} \rho^m, & \overline{\rho}^m \\ & \epsilon_p \end{matrix} \;\middle|\; 1\right)_p \pmod{p^2}. $$

Theorem 2. *Let m and d be integers with $1 \leq m < d$. If $p \equiv -1 \pmod{d}$ is prime and ρ is a character of order d on $\mathbb{F}_{p^2}$, then*

$$ {}_2F_1\left(\begin{matrix} \frac{m}{d}, & \frac{d-m}{d} \\ & 1 \end{matrix} \;\middle|\; 1\right)^2_{tr(p)} \equiv -p^2 \cdot {}_2F_1\left(\begin{matrix} \rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2} \end{matrix} \;\middle|\; 1\right)_{p^2} \pmod{p^2}. $$

Corollary 3. *Conjecture* 1 *is true.*

To prove Theorems 1 and 2, we derive an explicit relationship between the truncated classical hypergeometric series and their Gaussian analogs. For the proof of Theorem 1, we note that if $p \equiv 1 \pmod{d}$, then there is a character ρ of order d on $\mathbb{F}_p$. We form a Gaussian hypergeometric series over $\mathbb{F}_p$ by selecting its parameters in a natural way in terms of ρ. To relate the Gaussian hypergeometric series to the truncated hypergeometric series, we use basic character theory, the Gross-Koblitz formula, properties of the p-adic Γ-function, and Wilf-Zeilberger Theory. To prove Theorem 2, we apply a similar analysis; however, we form a Gaussian hypergeometric series over $\mathbb{F}_{p^2}$.

It should be noted that some of the ideas in the proofs of Theorems 1 and 2 are borrowed from Ahlgren and Ono [A-O]. Other papers, [A] and [M], have also used ideas from [A-O] to prove certain explicit supercongruences. However, there are important differences in this work. Here we develop a general framework for studying relationships between classical hypergeometric series and their Gaussian analogs. This includes investigating Gaussian hypergeometric series over $\mathbb{F}_{p^2}$. As in the previous works, we encounter strange combinatorial identities involving harmonic numbers. The earlier papers required computer software to prove their identities. We bypass this requirement by explicitly developing fundamental identities from which we derive all the identities needed in this paper. This is the observation that allows us to obtain infinite families of supercongruences.

The organization of this paper is as follows. In Section 2, we prove Corollary 3 using Theorems 1, 2 and [RV2]. In Section 3 we introduce our fundamental identities, prove them with Wilf-Zeilberger Theory, and give an example as to how they are used. In Section 4 we give some p-adic preliminaries. We prove Theorem 1 in Section 5 and prove Theorem 2 in Section 6.

2. Proof of Corollary 3

We begin the paper with the proof of Corollary 3 assuming Theorems 1 and 2. In terms of truncated hypergeometric series, Conjecture 1 is as follows.

Conjecture 1. *If $p \geq 5$ is a prime, then*

$$ {}_2F_1\left(\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ & 1\end{matrix} \mid 1\right)_{tr(p)} \equiv \left(\frac{-4}{p}\right) \pmod{p^2}, \tag{2.1}$$

$$ {}_2F_1\left(\begin{matrix}\frac{1}{3}, & \frac{2}{3} \\ & 1\end{matrix} \mid 1\right)_{tr(p)} \equiv \left(\frac{-3}{p}\right) \pmod{p^2}, \tag{2.2}$$

$$ {}_2F_1\left(\begin{matrix}\frac{1}{4}, & \frac{3}{4} \\ & 1\end{matrix} \mid 1\right)_{tr(p)} \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \tag{2.3}$$

$$ {}_2F_1\left(\begin{matrix}\frac{1}{6}, & \frac{5}{6} \\ & 1\end{matrix} \mid 1\right)_{tr(p)} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}. \tag{2.4}$$

From definition (1.6), we obtain the simple fact

$$ -q \cdot {}_2F_1\left(\begin{matrix}\chi, & \overline{\chi} \\ & \epsilon_q\end{matrix} \mid 1\right)_q = \overline{\chi}(-1). \tag{2.5}$$

Proof of Corollary 3. For each part, we let $m = 1$ and let $d = 2, 3, 4$, or 6. For primes $p \equiv 1 \pmod d$ we use Theorem 1. By (2.5), the Gaussian hypergeometric series evaluates to

$$ -p \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix} \mid 1\right)_p = \overline{\rho}^m(-1) = (-1)^{\frac{m}{d}(p-1)}. \tag{2.6}$$

Quadratic reciprocity shows that this equals the appropriate Legendre symbol. For primes $p \equiv -1 \pmod d$ we use Theorem 2. By (2.5), the Gaussian hypergeometric series evaluates to 1. Since Rodriguez-Villegas [RV2] proved these conjectured supercongruences modulo p, the ambiguity of the choice in sign is resolved. □

3. Strange Combinatorial Identities and Wilf-Zeilberger Theory

We require the following combinatorial identities in the proofs of Theorems 1 and 2. We begin with Lemma 3.1, a result which to our knowledge is new.

Lemma 3.1. *If n, $r \geq 1$ are integers, then*

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{k}\binom{n}{k}\frac{1}{k+r} = \frac{(-1)^n}{r}\prod_{j=1}^{n}\left(\frac{r-j}{r+j}\right).$$

Proof. To prove this identity we use a method from Wilf-Zeilberger Theory [PWZ], known as creative telescoping. If we define $S(n)$ by

$$ S(n) := \sum_{k=0}^{n} (-1)^k \binom{n+k}{k}\binom{n}{k}\frac{1}{k+r}, \tag{3.1}$$

then it suffices to prove the recurrence

$$ (r+n)S(n) + (r-n)S(n-1) = 0. \tag{3.2}$$

Choosing this recurrence comes naturally, since the right-hand side of the proposed identity easily satisfies it. Given an $n \geq 1$, we write

$$ S(n) = \sum_{k=-\infty}^{\infty} F(n,k), \tag{3.3}$$

where $F(n,k)$ is zero for all but finitely many k. The objective is to find a function $G(n,k)$, again zero for all but finitely many k, so that

$$(r+n)F(n,k)+(r-n)F(n-1,k)=G(n,k)-G(n,k-1). \tag{3.4}$$

Recurrence (3.2) then follows, since

$$\begin{aligned}(r+n)S(n)+(r-n)S(n-1) &= \sum_{k=-\infty}^{\infty}\left((r+n)F(n,k)+(r-n)F(n-1,k)\right)\\ &= \sum_{k=-\infty}^{\infty}\left(G(n,k)-G(n,k-1)\right)=0.\end{aligned} \tag{3.5}$$

To implement this strategy, we define

$$F(n,k):=\begin{cases}(-1)^k\binom{n+k}{k}\binom{n}{k}\frac{1}{k+r} & \text{if } \ 0\le k\le n,\\ 0 & \text{else.}\end{cases} \tag{3.6}$$

We construct the $G(n,k)$. If we let $G(n,-1)=0$, then we see that

$$\begin{aligned}G(n,0) &= (r+n)F(n,0)+(r-n)F(n-1,0)\\ &= (r+n)\frac{1}{r}+(r-n)\frac{1}{r}\\ &= 2 = 2rF(n-1,0).\end{aligned} \tag{3.7}$$

Setting $k=1$, we obtain

$$\begin{aligned}G(n,1) &= G(n,0)+(r+n)F(n,1)+(r-n)F(n-1,1)\\ &= 2+(r+n)(-1)\frac{(n+1)n}{r+1}+(r-n)(-1)\frac{n(n-1)}{r+1}\\ &= 2-2n^2=\frac{2(r+1)(n+1)}{n}F(n-1,1).\end{aligned} \tag{3.8}$$

Continuing in this manner suggests that we define

$$G(n,k):=\begin{cases}\frac{2(r+k)(n+k)}{n}F(n-1,k) & \text{if } \ 0\le k\le n,\\ 0 & \text{else.}\end{cases} \tag{3.9}$$

The verification that this is the correct definition follows from checking (3.4):

$$\begin{aligned}&(r+n)F(n,k)+(r-n)F(n-1,k)\\ &\quad=\frac{r+n}{k+r}\frac{(-1)^k}{k!^2}\frac{(n+k)!}{(n-k)!}+\frac{r-n}{k+r}\frac{(-1)^k}{k!^2}\frac{(n-1+k)!}{(n-1-k)!}\\ &\quad=(r+n)\frac{(-1)^k}{k!^2}\frac{\Pi_{j=-k+1}^{k}(n+j)}{k+r}+(r-n)\frac{(-1)^k}{k!^2}\frac{\Pi_{j=-k}^{k-1}(n+j)}{k+r}\\ &\quad=\frac{(-1)^k}{k!^2}\left(\prod_{j=-k+1}^{k-1}(n+j)\right)\left((r+n)(n+k)+(r-n)(n-k)\right)\frac{1}{k+r}\\ &\quad=2n\frac{(-1)^k}{k!^2}\left(\prod_{j=-k+1}^{k-1}(n+j)\right),\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&G(n,k)-G(n,k-1)\\
&\quad=\frac{2(r+k)(n+k)}{n}F(n-1,k)-\frac{2(r+k-1)(n+k-1)}{n}F(n-1,k-1)\\
&\quad=\frac{2}{n}(n+k)\frac{(-1)^k}{k!^2}\frac{(n-1+k)!}{(n-1-k)!}-\frac{2}{n}(n+k-1)\frac{(-1)^{k-1}}{(k-1)!^2}\frac{(n-1+k-1)!}{(n-1-k+1)!}\\
&\quad=\frac{2}{n}(n+k)\frac{(-1)^k}{k!^2}\prod_{j=-k}^{k-1}(n+j)-\frac{2}{n}(n+k-1)\frac{(-1)^{k-1}}{(k-1)!^2}\prod_{j=-k+1}^{k-2}(n+j) \qquad (3.11)\\
&\quad=\frac{2}{n}\frac{(-1)^k}{k!^2}\left(\prod_{j=-k+1}^{k-1}(n+j)\right)\left((n+k)(n-k)+k^2\right)\\
&\quad=2n\frac{(-1)^k}{k!^2}\left(\prod_{j=-k+1}^{k-1}(n+j)\right).
\end{aligned}$$

□

Remark. In the proof of Theorem 2, we will use the fact that the identity in Lemma 3.1 evaluates to zero when $1 \leq r \leq n$.

The next lemma is well-known; however, we include a sketch of a proof.

Lemma 3.2. *If $n \geq 0$ is an integer, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}\binom{n}{k}=(-1)^n.$$

Proof. If we define the left-hand side of the above equation to be $T(n)$, it suffices to prove that $T(n)+T(n-1)=0$. This can be shown by arguing as in the proof of Lemma 3.1. □

Corollary 3.3. *If n, $r \geq 1$ are integers, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}\binom{n}{k}\frac{k}{k+r}=(-1)^n-(-1)^n\prod_{j=1}^{n}\left(\frac{r-j}{r+j}\right).$$

Proof. This is just the identity in Lemma 3.2 minus r times the identity in Lemma 3.1. □

Remark. In the proof of Theorem 1, we will use the fact that the identity in Corollary 3.3 evaluates to $(-1)^n$ when $1 \leq r \leq n$.

Example 3.4. We demonstrate how to use these identities. Let

$$H_n := 1+\tfrac{1}{2}+\cdots+\tfrac{1}{n}. \qquad (3.12)$$

Let p be an odd prime. Using Lemma 3.2 and Corollary 3.3, it is easy to verify the following:

$$\begin{aligned}(3.13)\qquad &\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}\Big(1+2k(H_{n+k}-H_k)\Big)\\ &=\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{\frac{p-1}{2}+k}{k}\binom{\frac{p-1}{2}}{k}\Big(1+2k\sum_{j=1}^{\frac{p-1}{2}}\frac{1}{k+j}\Big)\\ &=(-1)^{\frac{p-1}{2}}(1+2\cdot\tfrac{p-1}{2})=(-1)^{\frac{p-1}{2}}p\equiv 0 \pmod p.\end{aligned}$$

4. p-Adic Preliminaries

Before we prove Theorems 1 and 2, we require some preliminaries on Gauss sums, Jacobi sums, the p-adic Γ-function, and the Gross-Koblitz formula. Throughout this section we let p be an odd prime. These facts can be found in, for example, [A-O], [COV], and [I-R]. Let $\pi\in\mathbb{C}_p$ be a fixed root of $x^{p-1}+p=0$, and let ζ_p be the unique p-th root of unity in $\mathbb{C}_p$ such that $\zeta_p\equiv 1+\pi \pmod{\pi^2}$. We let $q=p^f$ and let $\mathrm{Tr}(x)$ be the standard trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$, defined as

$$(4.1)\qquad \mathrm{Tr}(x):=x+x^p+x^{p^2}+\cdots+x^{p^{f-1}}.$$

We then define the Gauss sum to be

$$(4.2)\qquad g(\chi):=\sum_{x=0}^{p^f-1}\chi(x)\zeta_p^{\mathrm{Tr}(x)}.$$

Proposition 4.1. *Let χ and ψ be characters on $\mathbb{F}_q$, where $q=p^f$. The following are true.*

(1) *If $\overline{\chi}$ denotes the inverse of χ, then $g(\chi)g(\overline{\chi})=\chi(-1)p^f$.*
(2) *If χ is not trivial, then $J(\chi,\overline{\chi})=-\chi(-1)$.*
(3) *If $\chi\psi\neq\epsilon$, then*

$$J(\chi,\psi)=\frac{g(\chi)g(\psi)}{g(\chi\psi)}.$$

We define the p-adic Γ-function on the ring $\mathbb{Z}_p$ of p-adic integers by

$$(4.3)\qquad \Gamma_p(n):=(-1)^n\prod_{j<n,\,p\nmid j}j,\quad \text{for } n\in\mathbb{N},$$

and then by

$$(4.4)\qquad \Gamma_p(x):=\lim_{n\to x}\Gamma_p(n),\ \text{for } x\in\mathbb{Z}_p,$$

where in the limit we take any sequence of integers that approaches x in the p-adic sense.

Proposition 4.2. *If $p\geq 5$ is prime, $x,y\in\mathbb{Z}_p$, and $z\in p\mathbb{Z}_p$, then the following are true.*

(1) $\Gamma_p(x+1)=\begin{cases}-x\Gamma_p(x) & \text{if } x\in\mathbb{Z}_p^*,\\ -\Gamma_p(x) & \text{if } x\in p\mathbb{Z}_p.\end{cases}$
(2) *If $n\geq 1$ and $x\equiv y \pmod{p^n}$, then $\Gamma_p(x)\equiv\Gamma_p(y) \pmod{p^n}$.*
(3) $\Gamma_p'(x+z)\equiv\Gamma_p'(x) \pmod p$.

(4) $\Gamma_p(x+z) \equiv \Gamma_p(x) + z\Gamma_p'(x) \pmod{p^2}$.
(5) *If $R(x)$ denotes the reduction of x modulo p to the range $\{1,\dots,p\}$, then* $\Gamma_p(x)\Gamma_p(1-x) = (-1)^{R(x)}$.

We define the logarithmic derivative of the p-adic Γ-function to be

$$G(x) := \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{4.5}$$

If $x \in \mathbb{Z}_p$, then $G(x) \in \mathbb{Z}_p$. Using Proposition 4.2 (1), we have

$$G(x+1) - G(x) = \frac{1}{x}, \quad \text{if } x \in \mathbb{Z}_p^*. \tag{4.6}$$

For $x \in \mathbb{Z}_p$ and $z \in p\mathbb{Z}_p$, we combine Proposition 4.2 (4) and (4.5) to obtain

$$\Gamma_p(x+z) \equiv \Gamma_p(x)\big(1 + zG(x)\big) \pmod{p^2}. \tag{4.7}$$

We are now able to state a relationship between Gauss sums and the p-adic Γ-function. Let ω denote the Teichmüller character on $\mathbb{F}_q$, where $q = p^f$. This is the primitive character defined uniquely by the property that $\omega(x) \equiv x \pmod q$ for $x = 0,\dots,q-1$. We let $\langle a\rangle$ denote the fractional part of a, with $0 \leq \langle a\rangle < 1$. The Gross-Koblitz formula [Gr-Ko] for characters on $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ states that

$$g(\overline{\omega}^j) = -\pi^{(p-1)\left(\sum_{k=0}^{f-1}\left\langle\frac{jp^k}{p^f-1}\right\rangle\right)} \prod_{k=0}^{f-1} \Gamma_p\left(\left\langle\frac{jp^k}{p^f-1}\right\rangle\right),\ 0 \leq j \leq p^f - 2. \tag{4.8}$$

5. Proof of Theorem 1

We give an overview of the proof. We begin by using (1.7) to write the Gaussian ${}_2F_1$ as an expression in terms of Jacobi sums. This expression is converted into one involving Gauss sums. Next, we use the Gross-Koblitz formula to rewrite this expression in terms of the p-adic Γ-function. Then, we proceed with a lengthy calculation, using properties of the p-adic Γ-function, to derive a combinatorial expression. Finally, we use the results of Section 3 to obtain the desired truncated classical hypergeometric function modulo p^2.

Before we begin the proof, we introduce a technical proposition.

Proposition 5.1. *Let m and d be integers such that $1 \leq m \leq d-m < d$. If $p \equiv 1 \pmod d$ is a prime, then define n such that $p = dn+1$. Also, define $M_1 := mn$ and $M_2 := (d-m)n$.*

(1) *If $0 \leq j \leq M_2$, then*

$$\frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p(1+j)^2} = \begin{cases} (-1)^{mn+1}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2} & \text{if } 0 \leq j \leq M_1, \\ (-1)^{mn+1}\left(\frac{d}{mp}\right)\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2} & \text{if } M_1+1 \leq j \leq M_2. \end{cases}$$

(2) *If $0 \leq j \leq M_1$, then*

$$(-1)^j\binom{M_2+j}{j}\binom{M_2}{j} \equiv \frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2} \pmod p.$$

(3) *If* $M_1+1\le j\le M_2$, *then*

$$(-1)^j\binom{M_2+j}{j}\binom{M_2}{j}\equiv\left(\frac{d}{m}\right)\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\pmod{p^2}.$$

Proof of Proposition 5.1. For (1), we determine when the two rising factorials in the numerator acquire factors that are multiples of p. This will tell which factors to omit when we use Proposition 4.2 (1) to expand the p-adic Γ-functions. For example, we have

$$\left(\frac{m}{d}\right)_j=\left(\frac{m}{d}\right)\left(\frac{m+d}{d}\right)\cdots\left(\frac{m+dj-d}{d}\right) \tag{5.1}$$

and note that the numerator of each factor is congruent to m modulo d. Hence the first factor which has p dividing the numerator is $\frac{mp}{d}$, and the next factor which has p dividing the numerator is $\frac{(d+m)p}{d}$. The first occurs when $j=M_1+1$, and the next occurs when $j=p+M_1+1$. For $(\frac{d-m}{d})_j$, the numerator of each factor is congruent to $(d-m)$ modulo d. The first factor with numerator divisible by p is $\frac{(d-m)p}{d}$. This occurs when $j=M_2+1$. Using Proposition 4.2 (1), it follows that

$$\Gamma_p\left(\frac{d-m}{d}+j\right)=(-1)^j\left(\frac{d-m}{d}\right)_j\Gamma_p\left(\frac{d-m}{d}\right)\ \text{if } 0\le j\le M_2 \tag{5.2}$$

and

$$\Gamma_p\left(\frac{m}{d}+j\right)=\begin{cases}(-1)^j\left(\frac{m}{d}\right)_j\Gamma_p\left(\frac{m}{d}\right) & \text{if } 0\le j\le M_1,\\ (-1)^j\left(\frac{d}{mp}\right)\left(\frac{m}{d}\right)_j\Gamma_p\left(\frac{m}{d}\right) & \text{if } M_1+1\le j\le M_2.\end{cases} \tag{5.3}$$

We use Proposition 4.2 (5) to obtain

$$\begin{aligned}\Gamma_p\left(\frac{m}{d}\right)\Gamma_p\left(\frac{d-m}{d}\right)&=(-1)^{R(\frac{d-m}{d})}=(-1)^{R(\frac{mp+d-m}{d})}\\&=(-1)^{\frac{mp+d-m}{d}}=(-1)^{mn+1}.\end{aligned} \tag{5.4}$$

Noting that $\Gamma_p(1+j)^2=j!^2$ completes the proof of part (1).

For part (2), given $j\le M_1$, we have $M_2+j\le p-1$. Hence no factor of p occurs in the numerator of the first binomial coefficient. We use (4.3) and get

$$\begin{aligned}(-1)^j\binom{M_2+j}{j}\binom{M_2}{j}&=\frac{(-1)^j}{(j!)^2}\frac{(\frac{d-m}{d}(p-1)+j)!}{(\frac{d-m}{d}(p-1)-j)!}\\&=\frac{(-1)^j}{(j!)^2}\frac{\Gamma_p(\frac{d-m}{d}(p-1)+1+j)}{\Gamma_p(\frac{d-m}{d}(p-1)+1-j)}.\end{aligned} \tag{5.5}$$

Using Proposition 4.2 (2) yields

$$(-1)^j\binom{M_2+j}{j}\binom{M_2}{j}\equiv(-1)^j\frac{\Gamma_p(\frac{m}{d}+j)}{\Gamma_p(\frac{m}{d}-j)}\frac{1}{(j!)^2}\pmod{p}. \tag{5.6}$$

We use Proposition 4.2 (5) to obtain

$$\Gamma_p(\tfrac{m}{d}-j)\Gamma_p(\tfrac{d-m}{d}+j)=(-1)^{R(\frac{d-m}{d}+j)}=(-1)^{\frac{mp+d-m}{d}+j}=(-1)^{mn+1+j}. \tag{5.7}$$

We combine (5.6) with (5.7) to find that

$$(5.8)\qquad (-1)^j\binom{M_2+j}{j}\binom{M_2}{j}\equiv(-1)^{mn+1}\frac{\Gamma_p(\frac{m}{d}+j)\Gamma_p(\frac{d-m}{d}+j)}{(j!)^2}\pmod{p}.$$

Applying part (1) completes the proof of part (2).

For part (3), given $M_1+1\le j\le M_2$, we have $p\le M_2+j\le 2M_2<2p$. We see that we have a single factor of p present in the numerator of the first binomial coefficient. This p is not included in the expansion of Γ_p using (4.3):

$$(5.9)\qquad \begin{aligned}(-1)^j\binom{M_2+j}{j}\binom{M_2}{j}&=\frac{(-1)^j}{(j!)^2}\frac{(\frac{d-m}{d}(p-1)+j)!}{(\frac{d-m}{d}(p-1)-j)!}\\&=\frac{(-1)^j}{(j!)^2}\frac{p\Gamma_p(\frac{d-m}{d}(p-1)+1+j)}{\Gamma_p(\frac{d-m}{d}(p-1)+1-j)}.\end{aligned}$$

Using Proposition 4.2 (2) and then using Proposition 4.2 (5), we have

$$(5.10)\qquad (-1)^j\binom{M_2+j}{j}\binom{M_2}{j}\equiv(-1)^{mn+1}\frac{p\Gamma_p(\frac{m}{d}+j)\Gamma_p(\frac{d-m}{d}+j)}{(j!)^2}\pmod{p^2}.$$

Applying part (1) completes the proof of part (3). □

Proof of Theorem 1. We have $p\equiv 1\pmod{d}$. We let n, M_1, and M_2 be as in Proposition 5.1. Since $d\mid(p-1)$, there is a character ρ of order d on $\mathbb{F}_p$. We use (1.7) to obtain

$$(5.11)\qquad -p\cdot{}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m\\ & \epsilon_p\end{matrix}\ \Big|\ 1\right)_p=\frac{p^2}{1-p}\sum_{\chi}\binom{\rho^m\chi}{\chi}\binom{\overline{\rho}^m\chi}{\epsilon_p\chi}.$$

We use (1.5) to write this expression in terms of Jacobi sums and Proposition 4.1 (3) to rewrite this in terms of Gauss sums. Noting that $\rho^m(-1)=(-1)^{mn}$, we use Proposition 4.1 (1) to evaluate the product of Gauss sums in the denominator. We then have

$$(5.12)\qquad \begin{aligned}-p\cdot{}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m\\ & \epsilon_p\end{matrix}\ \Big|\ 1\right)_p&=\frac{1}{1-p}\sum_{\chi}J(\rho^m\chi,\overline{\chi})J(\overline{\rho}^m\chi,\overline{\chi})\\&=\frac{1}{1-p}\left\{\sum_{\chi}\frac{g(\rho^m\chi)g(\overline{\rho}^m\chi)g(\overline{\chi})^2}{g(\rho^m)g(\overline{\rho}^m)}\right\}\\&=\frac{1}{1-p}\left\{\sum_{\chi}\frac{g(\rho^m\chi)g(\overline{\rho}^m\chi)g(\overline{\chi})^2}{(-1)^{mn}p}\right\}.\end{aligned}$$

Next, we rewrite this expression using the Teichmüller character ω to obtain

$$
\begin{aligned}
&- p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix} \mid 1\right)_p \\
&= \frac{(-1)^{mn}}{1-p}\cdot\frac{1}{p}\Big\{\sum_{j=0}^{\frac{m}{d}(p-1)} g(\overline{\omega}^{\frac{m}{d}(p-1)-j})g(\overline{\omega}^{\frac{d-m}{d}(p-1)-j})g(\overline{\omega}^j)^2 \\
&\quad + \sum_{j=\frac{m}{d}(p-1)+1}^{\frac{d-m}{d}(p-1)} g(\overline{\omega}^{\frac{d+m}{d}(p-1)-j})g(\overline{\omega}^{\frac{d-m}{d}(p-1)-j})g(\overline{\omega}^j)^2 \\
&\quad + \sum_{j=\frac{d-m}{d}(p-1)+1}^{p-2} g(\overline{\omega}^{\frac{d+m}{d}(p-1)-j})g(\overline{\omega}^{\frac{2d-m}{d}(p-1)-j})g(\overline{\omega}^j)^2\Big\}.
\end{aligned} \tag{5.13}
$$

The exponents of the $\overline{\omega}$'s have been chosen to be between 0 and $p-2$. Using (4.8) yields

$$
\begin{aligned}
-p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix} \mid 1\right)_p &\equiv \frac{(-1)^{mn}}{1-p}\Big\{-\sum_{j=0}^{M_1}\Gamma_p(\tfrac{m}{d}-\tfrac{j}{p-1})\Gamma_p(\tfrac{d-m}{d}-\tfrac{j}{p-1})\Gamma_p(\tfrac{j}{p-1})^2 \\
&\quad + p\sum_{j=M_1+1}^{M_2}\Gamma_p(\tfrac{d+m}{d}-\tfrac{j}{p-1})\Gamma_p(\tfrac{d-m}{d}-\tfrac{j}{p-1})\Gamma_p(\tfrac{j}{p-1})^2\Big\} \pmod{p^2}.
\end{aligned} \tag{5.14}
$$

We convert (5.14) into a combinatorial expression. Since $\frac{1}{1-p}\equiv 1+p \pmod{p^2}$, Proposition 4.2 (2) allows us to rewrite (5.14) as

$$
\begin{aligned}
&- p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix} \mid 1\right)_p \\
&\equiv (-1)^{mn}(1+p)\Big\{p\sum_{j=M_1+1}^{M_2}\Gamma_p(\tfrac{m}{d}+1+j)\Gamma_p(\tfrac{d-m}{d}+j)\Gamma_p(-j)^2 \\
&\quad -\sum_{j=0}^{M_1}\Gamma_p(\frac{m}{d}+j+jp)\Gamma_p(\frac{d-m}{d}+j+jp)\Gamma_p(-j-jp)^2\Big\} \pmod{p^2}.
\end{aligned} \tag{5.15}
$$

Using Proposition 4.2 (5) produces

$$
\begin{aligned}
&- p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix} \mid 1\right)_p \\
&\equiv (-1)^{mn}(1+p)\Big\{-\sum_{j=0}^{M_1}\frac{\Gamma_p\left(\frac{m}{d}+j+jp\right)\Gamma_p\left(\frac{d-m}{d}+j+jp\right)}{\Gamma_p\left(1+j+jp\right)^2} \\
&\quad + p\sum_{j=M_1+1}^{M_2}\frac{\Gamma_p\left(\frac{m}{d}+1+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p\left(1+j\right)^2}\Big\} \pmod{p^2}.
\end{aligned} \tag{5.16}
$$

Using (4.7), we obtain

$$\begin{aligned} &-p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix}\ \Big|\ 1\right)_p \equiv (-1)^{mn}(1+p) \\ (5.17)\quad &\times\Big\{-\sum_{j=0}^{M_1}\frac{\Gamma_p\left(\frac{m}{d}+j\right)\left(1+jpG(\frac{m}{d}+j)\right)\Gamma_p\left(\frac{d-m}{d}+j\right)\left(1+jpG(\frac{d-m}{d}+j)\right)}{\left(\Gamma_p\left(1+j\right)\left(1+jpG(1+j)\right)\right)^2} \\ &+p\sum_{j=M_1+1}^{M_2}\frac{\Gamma_p\left(\frac{m}{d}+1+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p\left(1+j\right)^2}\Big\} \pmod{p^2}. \end{aligned}$$

Multiplying the numerator and denominator of the first sum by $1-2jpG(1+j)$, we have

$$\begin{aligned} &-p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix}\ \Big|\ 1\right)_p \\ &\equiv (-1)^{mn}(1+p)\Big\{p\sum_{j=M_1+1}^{M_2}\frac{\Gamma_p\left(\frac{m}{d}+1+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p\left(1+j\right)^2} \\ (5.18)\quad &-\sum_{j=0}^{M_1}\frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p\left(1+j\right)^2} \\ &\times\left(1+jp(G(\tfrac{m}{d}+j)+G(\tfrac{d-m}{d}+j)-2G(1+j))\right)\Big\} \pmod{p^2}. \end{aligned}$$

We rewrite the logarithmic derivatives as combinatorial expressions. For $0\le j\le M_1$, we use Proposition 4.2 (2)-(3) to find that

$$(5.19)\qquad G\left(\tfrac{m}{d}+j\right)-G\left(1+j\right)\equiv G\left(\tfrac{(d-m)p+m}{d}+j\right)-G\left(1+j\right) \pmod{p},$$

and

$$(5.20)\qquad G\left(\tfrac{d-m}{d}+j\right)-G\left(1+j\right)\equiv G\left(\tfrac{mp+d-m}{d}+j\right)-G\left(1+j\right) \pmod{p}.$$

Applying (4.6) repeatedly, we obtain

$$(5.21)\qquad G\left(\tfrac{m}{d}+j\right)-G\left(1+j\right)\equiv H_{\frac{d-m}{d}(p-1)+j}-H_j \pmod{p}$$

and

$$(5.22)\qquad G\left(\tfrac{d-m}{d}+j\right)-G\left(1+j\right)\equiv H_{\frac{m}{d}(p-1)+j}-H_j \pmod{p}.$$

Consequently, this yields

$$(5.23)\qquad -p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix}\;\middle|\;1\right)_p$$

$$\equiv (-1)^{mn}(1+p)\Big\{p\sum_{j=M_1+1}^{M_2}\frac{\Gamma_p\left(\frac{m}{d}+1+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p\left(1+j\right)^2}$$

$$-\sum_{j=0}^{M_1}\frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p\left(1+j\right)^2}$$

$$\times\left(1+jp(H_{M_2+j}+H_{M_1+j}-2H_j)\right)\Big\}\pmod{p^2}.$$

In the first sum, we use Proposition 4.2 (1) to obtain

$$(5.24)\qquad \Gamma_p(\tfrac{m}{d}+1+j)=-(\tfrac{m}{d}+j)\Gamma_p(\tfrac{m}{d}+j).$$

Using Proposition 5.1 (1) and then gathering terms yields

$$-p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix}\;\middle|\;1\right)_p$$

$$\equiv (-1)^{mn}(1+p)\Big\{\sum_{j=M_1+1}^{M_2}(-1)^{mn}\left(\frac{dj+m}{m}\right)\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}$$

$$+\sum_{j=0}^{M_1}(-1)^{mn}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\left(1+jp(H_{M_2+j}+H_{M_1+j}-2H_j)\right)\Big\}$$

$$(5.25)\qquad \equiv\sum_{j=0}^{M_2}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}$$

$$+p\Big\{\sum_{j=0}^{M_1}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\left(1+j(H_{M_2+j}+H_{M_1+j}-2H_j)\right)$$

$$+\sum_{j=M_1+1}^{M_2}\left(\frac{dj}{mp}\right)\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\Big\}\pmod{p^2}.$$

In the third sum, we allow the factor $(\frac{dj}{mp})$ since $(\frac{m}{d})_j$ contains the factor $(\frac{mp}{d})$. Instances similar to this will occur again. Using Proposition 5.1 (2)-(3), we have

$$-p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix}\mid 1\right)_p \tag{5.26}$$
$$\equiv \sum_{j=0}^{M_2}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}+p\Big\{\sum_{j=M_1+1}^{M_2}(-1)^j\binom{M_2+j}{j}\binom{M_2}{j}\frac{j}{p}$$
$$+\sum_{j=0}^{M_1}(-1)^j\binom{M_2+j}{j}\binom{M_2}{j}$$
$$\times\left(1+j(H_{M_2+j}+H_{M_1+j}-2H_j)\right)\Big\} \pmod{p^2}.$$

We convert the coefficient of p into a strange combinatorial identity similar to Example 3.4. For $M_1+1\leq j\leq M_2$, one sees that H_j and H_{M_1+j} contain no $\frac{1}{p}$ terms. For the same range of j, we note H_{M_2+j} contains precisely one $\frac{1}{p}$ term. We also note that the numerator of the first binomial coefficient in the first sum has a factor of p. We then find that

$$(-1)^j\binom{M_2+j}{j}\binom{M_2}{j}\frac{j}{p} \tag{5.27}$$
$$\equiv(-1)^j\binom{M_2+j}{j}\binom{M_2}{j}\left(1+j(H_{M_2+j}+H_{M_1+j}-2H_j)\right) \pmod{p}.$$

Thus we arrive at

$$-p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix}\mid 1\right)_p \equiv \sum_{j=0}^{M_2}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2} \tag{5.28}$$
$$+p\Big\{\sum_{j=0}^{M_2}(-1)^j\binom{M_2+j}{j}\binom{M_2}{j}$$
$$\times\left(1+j(H_{M_2+j}+H_{M_1+j}-2H_j)\right)\Big\} \pmod{p^2}.$$

Finally, we use Lemma 3.2 and Corollary 3.3 to see that the coefficient of p evaluates to $(-1)^{M_2}p$. We then have

$$-p\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_p\end{matrix}\mid 1\right)_p \equiv \sum_{j=0}^{M_2}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2} \pmod{p^2}. \tag{5.29}$$

We note that the difference between (5.29) and the expression in Theorem 1 is trivially a multiple of p^2. Specifically, each missing rising factorial has a factor of p in it. $\square$

6. Proof of Theorem 2

The method of the proof is the same as that of Theorem 1. However, we work with a Gaussian hypergeometric series over $\mathbb{F}_{p^2}$. We require two technical propositions, one of which is analogous to Proposition 5.1.

Before we begin the proof, we introduce the first technical proposition.

Proposition 6.1. *Let m and d be integers with $1 \leq m \leq d-m < d$. If $p \equiv -1 \pmod{d}$ is prime, then define n such that $p = dn + d - 1$. Also, define $N_1 := m(n+1) - 1$ and $N_2 := (d-m)(n+1) - 1$.*

(1) *If $0 \leq j \leq N_2$, then*

$$\frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p(1+j)^2} = \begin{cases} (-1)^{m(n+1)} \frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2} & \text{if } 0 \leq j \leq N_1, \\ (-1)^{m(n+1)} \left(\frac{d}{mp}\right) \frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2} & \text{if } N_1 + 1 \leq j \leq N_2. \end{cases}$$

(2) *If $0 \leq j \leq N_1$, then*

$$(-1)^j \binom{N_2+j}{j}\binom{N_2}{j} \equiv \frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2} \pmod{p}.$$

(3) *If $N_1 + 1 \leq j \leq N_2$, then*

$$(-1)^j \binom{N_2+j}{j}\binom{N_2}{j} \equiv \left(\frac{d}{m}\right) \frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2} \pmod{p^2}.$$

Proof of Proposition 6.1. The proof is analogous to the proof of Proposition 5.1 with one notable difference. In (1) the rising factorial $(\frac{m}{d})_j$ acquires the factor $(\frac{(d-m)p}{d})$ when $j = N_2 + 1$. The rising factorial $(\frac{d-m}{d})_j$ acquires the factor $(\frac{mp}{d})$ when $j = N_1 + 1$. □

Proof of Theorem 2. We have $p \equiv -1 \pmod{d}$. We let n, N_1, and N_2 be as in Proposition 6.1. Since $d \mid (p^2 - 1)$, there is a character ρ of order d on $\mathbb{F}_{p^2}$. By (1.7), we have

$$-p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \;\middle|\; 1\right)_{p^2} = \frac{p^4}{1-p^2}\sum_{\chi}\binom{\rho^m\chi}{\chi}\binom{\overline{\rho}^m\chi}{\epsilon_{p^2}\chi}. \tag{6.1}$$

Using (1.5), Proposition 4.1, and noting that $\rho^m(-1) = 1$, we obtain

$$\begin{aligned} -p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \;\middle|\; 1\right)_{p^2} &= \frac{1}{1-p^2}\sum_{\chi} J(\rho^m\chi, \overline{\chi})J(\overline{\rho}^m\chi, \bar{\chi}) \\ &= \frac{1}{1-p^2}\left\{\sum_{\chi}\frac{g(\rho^m\chi)g(\overline{\rho}^m\chi)g(\overline{\chi})^2}{g(\rho^m)g(\overline{\rho}^m)}\right\} \qquad (6.2) \\ &= \frac{1}{1-p^2}\left\{\sum_{\chi}\frac{g(\rho^m\chi)g(\overline{\rho}^m\chi)g(\overline{\chi})^2}{p^2}\right\}. \end{aligned}$$

Next, we convert our Gauss sum expression to p-adic Γ-function terms using the Gross-Koblitz formula. If ω is the Teichmüller character, then

$$\begin{aligned} &-p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \mid 1\right)_{p^2} \\ &= \frac{1}{1-p^2}\cdot\frac{1}{p^2}\Big\{\sum_{j=0}^{p^2-2} g(\overline{\omega}^{\frac{m}{d}(p^2-1)-j})g(\overline{\omega}^{\frac{d-m}{d}(p^2-1)-j})g(\overline{\omega}^j)^2\Big\}. \end{aligned} \tag{6.3}$$

By the Gross-Koblitz formula (4.8), we have

(6.4)

$$\begin{aligned} &-p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \mid 1\right)_{p^2} \\ &= \frac{1}{1-p^2}\cdot\frac{1}{p^2}\sum_{j=0}^{p^2-2}\pi^{(p-1)S(j)}\Big\{\Gamma_p\left(\left\langle\tfrac{m}{d}-\tfrac{j}{p^2-1}\right\rangle\right)\Gamma_p\left(\left\langle\tfrac{d-m}{d}-\tfrac{j}{p^2-1}\right\rangle\right)\Gamma_p\left(\left\langle\tfrac{j}{p^2-1}\right\rangle\right)^2 \\ &\quad \cdot\Gamma_p\left(\left\langle\tfrac{mp}{d}-\tfrac{jp}{p^2-1}\right\rangle\right)\Gamma_p\left(\left\langle\tfrac{(d-m)p}{d}-\tfrac{jp}{p^2-1}\right\rangle\right)\Gamma_p\left(\left\langle\tfrac{jp}{p^2-1}\right\rangle\right)^2\Big\}, \end{aligned}$$

where

$$\begin{aligned} S(j) := &\left\langle\tfrac{m}{d}-\tfrac{j}{p^2-1}\right\rangle+\left\langle\tfrac{d-m}{d}-\tfrac{j}{p^2-1}\right\rangle+2\left\langle\tfrac{j}{p^2-1}\right\rangle \\ &+\left\langle\tfrac{mp}{d}-\tfrac{jp}{p^2-1}\right\rangle+\left\langle\tfrac{(d-m)p}{d}-\tfrac{jp}{p^2-1}\right\rangle+2\left\langle\tfrac{jp}{p^2-1}\right\rangle. \end{aligned} \tag{6.5}$$

Since we are only concerned with the modulus p^2, we can disregard any j such that $S(j)\geq 4$. This motivates the following proposition.

Proposition 6.2. *Let k be a nonnegative integer. Let m, d, n, N_1, and N_2 be as in Proposition* 6.1. *For $0\leq j\leq p^2-2$ we have the following.*

1. *If $0\leq k\leq N_1$ and $kp\leq j\leq kp+N_1$, then $S(j)=2$.*
2. *If $0\leq k\leq N_1$ and $kp+N_1+1\leq j\leq kp+N_2$, then $S(j)=3$.*
3. *If $N_1+1\leq k\leq N_2$ and $kp\leq j\leq kp+N_1$, then $S(j)=3$.*
4. *For all other j, $S(j)\geq 4$.*

Proof of Proposition 6.2. We define the following:

$$S_A(j) := \left\langle\tfrac{m}{d}-\tfrac{j}{p^2-1}\right\rangle+\left\langle\tfrac{d-m}{d}-\tfrac{j}{p^2-1}\right\rangle+2\left\langle\tfrac{j}{p^2-1}\right\rangle \tag{6.6}$$

and

$$S_B(j) := \left\langle\tfrac{mp}{d}-\tfrac{jp}{p^2-1}\right\rangle+\left\langle\tfrac{(d-m)p}{d}-\tfrac{jp}{p^2-1}\right\rangle+2\left\langle\tfrac{jp}{p^2-1}\right\rangle. \tag{6.7}$$

For the first bracketed term in S_B, we have

$$\left\langle\tfrac{mp}{d}-\tfrac{jp}{p^2-1}\right\rangle=\left\langle\tfrac{m(dn+d-1)}{d}-\tfrac{jp}{p^2-1}\right\rangle=\left\langle-\tfrac{m}{d}-\tfrac{jp}{p^2-1}\right\rangle=\left\langle\tfrac{(d-m)}{d}-\tfrac{jp}{p^2-1}\right\rangle. \tag{6.8}$$

Applying a similar argument to the second bracketed term in S_B yields

$$S_B(j)=\left\langle\tfrac{d-m}{d}-\tfrac{jp}{p^2-1}\right\rangle+\left\langle\tfrac{m}{d}-\tfrac{jp}{p^2-1}\right\rangle+2\left\langle\tfrac{jp}{p^2-1}\right\rangle. \tag{6.9}$$

We write $j=rp+s$, where $0\leq r,s\leq p-1$. We form the $p\times p$ matrices M_A and M_B such that r denotes the row, s denotes the column, $(0,0)$ denotes the top left corner, and $(p-1,p-1)$ denotes the lower right corner. In each (r,s) entry we place the value of $S_A(rp+s)$ and $S_B(rp+s)$, respectively. In each $(p-1,p-1)$ entry we place an x. It is easy to check the following for values of S_A:

$$S_A(j)=\begin{cases}1 & \text{if } \ 0\le j\le \frac{m}{d}(p^2-1),\\ 2 & \text{if } \ \frac{m}{d}(p^2-1)+1\le j\le \frac{d-m}{d}(p^2-1),\\ 3 & \text{if } \ \frac{d-m}{d}(p^2-1)+1\le j\le p^2-2.\end{cases} \tag{6.10}$$

To see where these boundary values occur in the matrices, we note the following. For $\frac{m}{d}(p^2-1)=rp+s$, we have $r=N_1$ and $s=N_2$. In particular, in M_A, all entries in rows 0 through N_1-1 are 1. The entries in row N_1 from columns 0 to N_2 are also 1. For $\frac{d-m}{d}(p^2-1)=rp+s$, we have $r=N_2$ and $s=N_1$. Excluding the two $(p-1,p-1)$ entries, it is straightforward to check for $j=r_1p+r_2$ that $S_A(r_1p+r_2)=S_B(r_2p+r_1)$. This shows $M_A=M_B^{\mathrm{T}}$. Since $S(j)=S_A(j)+S_B(j)$, we now see how Proposition 6.2 follows. Specifically, part (1) corresponds to the values of $j=rp+s$ where $0\le r\le N_1$ and $0\le s\le N_1$. Part (2) corresponds to the values of j where $0\le r\le N_1$ and $N_1+1\le s\le N_2$, and part (3) to the values of j where $N_1+1\le r\le N_2$ and $0\le s\le N_1$. Part (4) is everything else. ☐

Using Proposition 6.2, we rewrite (6.4) using only the relevant j indices. By adding and subtracting appropriate integers to the terms in brackets, we may obtain purely fractional values and are then able to drop the brackets. For example, if $0\le k\le N_1$ and $kp\le j\le kp+N_1$, one easily checks that

$$\left\langle \tfrac{jp}{p^2-1}\right\rangle=\left\langle \tfrac{jp}{p^2-1}-k\right\rangle=\tfrac{jp}{p^2-1}-k. \tag{6.11}$$

We also use Proposition 4.2 (5). We then have

$$\begin{aligned}
&-p^2\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix}\ \Big|\ 1\right)_{p^2} \\
&\equiv \sum_{k=0}^{N_1}\sum_{j=kp}^{kp+N_1}\frac{\Gamma_p(\frac{m}{d}-\frac{j}{p^2-1})\Gamma_p(\frac{d-m}{d}-\frac{j}{p^2-1})}{\Gamma_p(1-\frac{j}{p^2-1})^2} \\
&\qquad\cdot\frac{\Gamma_p(\frac{m}{d}+k-\frac{jp}{p^2-1})\Gamma_p(\frac{d-m}{d}+k-\frac{jp}{p^2-1})}{\Gamma_p(1+k-\frac{jp}{p^2-1})^2} \\
&\quad -p\sum_{k=0}^{N_1}\sum_{j=kp+N_1+1}^{kp+N_2}\frac{\Gamma_p(\frac{m}{d}-\frac{j}{p^2-1})\Gamma_p(\frac{d-m}{d}-\frac{j}{p^2-1})}{\Gamma_p(1-\frac{j}{p^2-1})^2} \\
&\qquad\cdot\frac{\Gamma_p(\frac{m}{d}+k+1-\frac{jp}{p^2-1})\Gamma_p(\frac{d-m}{d}+k-\frac{jp}{p^2-1})}{\Gamma_p(1+k-\frac{jp}{p^2-1})^2} \\
&\quad -p\sum_{k=N_1+1}^{N_2}\sum_{j=kp}^{kp+N_1}\frac{\Gamma_p(\frac{m}{d}+1-\frac{j}{p^2-1})\Gamma_p(\frac{d-m}{d}-\frac{j}{p^2-1})}{\Gamma_p(1-\frac{j}{p^2-1})^2} \\
&\qquad\cdot\frac{\Gamma_p(\frac{m}{d}+k-\frac{jp}{p^2-1})\Gamma_p(\frac{d-m}{d}+k-\frac{jp}{p^2-1})}{\Gamma_p(1+k-\frac{jp}{p^2-1})^2} \pmod{p^2}.
\end{aligned} \tag{6.12}$$

Noting that $\frac{1}{1-p^2} \equiv 1 \pmod{p^2}$ and using Proposition 4.2 (2), we obtain

(6.13)

$$\begin{aligned}
&-p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \Big| \ 1\right)_{p^2} \\
&\equiv \sum_{k=0}^{N_1}\sum_{j=kp}^{kp+N_1} \frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p(1+j)^2}\frac{\Gamma_p\left(\frac{m}{d}+k+jp\right)\Gamma_p\left(\frac{d-m}{d}+k+jp\right)}{\Gamma_p(1+k+jp)^2} \\
&\quad - p\sum_{k=0}^{N_1}\sum_{j=kp+N_1+1}^{kp+N_2} \frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p(1+j)^2}\frac{\Gamma_p\left(\frac{m}{d}+1+k\right)\Gamma_p\left(\frac{d-m}{d}+k\right)}{\Gamma_p(1+k)^2} \\
&\quad - p\sum_{k=N_1+1}^{N_2}\sum_{j=kp}^{kp+N_1} \frac{\Gamma_p\left(\frac{m}{d}+1+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p(1+j)^2}\frac{\Gamma_p\left(\frac{m}{d}+k\right)\Gamma_p\left(\frac{d-m}{d}+k\right)}{\Gamma_p(1+k)^2} \\
&\qquad \pmod{p^2}.
\end{aligned}$$

We replace j with $j+kp$ in the three double sums and notice that the last two double sums are in fact equal. This yields

$$\begin{aligned}
(6.14)\qquad &-p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \Big| \ 1\right)_{p^2} \\
&\equiv \sum_{k=0}^{N_1}\sum_{j=0}^{N_1} \frac{\Gamma_p\left(\frac{m}{d}+j+kp\right)\Gamma_p\left(\frac{d-m}{d}+j+kp\right)}{\Gamma_p(1+j+kp)^2} \\
&\qquad \cdot \frac{\Gamma_p\left(\frac{m}{d}+k+jp\right)\Gamma_p\left(\frac{d-m}{d}+k+jp\right)}{\Gamma_p(1+k+jp)^2} \\
&\quad - 2p\sum_{k=0}^{N_1}\sum_{j=N_1+1}^{N_2} \frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p(1+j)^2} \\
&\qquad \cdot \frac{\Gamma_p\left(\frac{m}{d}+1+k\right)\Gamma_p\left(\frac{d-m}{d}+k\right)}{\Gamma_p(1+k)^2} \pmod{p^2}.
\end{aligned}$$

Now we need to consider the case where $m=1$ and $d=p+1$. Since $N_1=0$ and $N_2=p-1$, (6.14) collapses. We then use Proposition 4.2 (1) and Proposition 6.1 (1) to find that

(6.15)

$$\begin{aligned}
&-p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \Big| \ 1\right)_{p^2} \\
&\equiv 1 - 2p\sum_{j=1}^{p-1} \frac{\Gamma_p\left(\frac{m}{d}+j\right)\Gamma_p\left(\frac{d-m}{d}+j\right)}{\Gamma_p(1+j)^2}\Gamma_p\left(\frac{m}{d}+1\right)\Gamma_p\left(\frac{d-m}{d}\right) \\
&\equiv 1 + 2p\sum_{j=1}^{p-1} \frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\frac{d}{mp}\frac{m}{d} \equiv \left(\sum_{j=0}^{p-1}\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j\right)^2 \pmod{p^2}.
\end{aligned}$$

Continuing with the main proof, we convert (6.14) into a combinatorial expression. We expand the p-adic Γ-functions in the first double sum using (4.7). We note that

$$\Gamma_p(\tfrac{m}{d}+1+k) = -(\tfrac{m}{d}+k)\Gamma_p(\tfrac{m}{d}+k) \tag{6.16}$$

in the second double sum. We then use Proposition 6.1 (1) to obtain

$$\begin{aligned}
&-p^2\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix}\ \Big|\ 1\right)_{p^2} \\
&\quad\equiv \sum_{k=0}^{N_1}\sum_{j=0}^{N_1}\Big\{\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\big(1+kp\big\{G(\tfrac{m}{d}+j)+G(\tfrac{d-m}{d}+j)-2G(1+j)\big\}\big) \\
&\qquad\cdot\frac{\left(\frac{m}{d}\right)_k\left(\frac{d-m}{d}\right)_k}{k!^2}\big(1+jp\big\{G(\tfrac{m}{d}+k)+G(\tfrac{d-m}{d}+k)-2G(1+k)\big\}\big)\Big\} \\
&\qquad+2p\sum_{k=0}^{N_1}\sum_{j=N_1+1}^{N_2}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\frac{\left(\frac{m}{d}\right)_k\left(\frac{d-m}{d}\right)_k}{k!^2}\left(\frac{dk+m}{mp}\right) \pmod{p^2}.
\end{aligned} \tag{6.17}$$

We expand the product in the first double sum and collect terms to have

$$\begin{aligned}
&-p^2\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix}\ \Big|\ 1\right)_{p^2} \equiv \left(\sum_{k=0}^{N_2}\frac{\left(\frac{m}{d}\right)_k\left(\frac{d-m}{d}\right)_k}{k!^2}\right)^2 \\
&+2p\Big\{\sum_{k=0}^{N_1}k\cdot\frac{\left(\frac{m}{d}\right)_k\left(\frac{d-m}{d}\right)_k}{k!^2}\Big\} \\
&\quad\cdot\Big\{\sum_{j=0}^{N_1}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\big\{G(\tfrac{m}{d}+j)+G(\tfrac{d-m}{d}+j)-2G(1+j)\big\} \\
&\qquad+\sum_{j=N_1+1}^{N_2}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\left(\frac{d}{mp}\right)\Big\} \pmod{p^2}.
\end{aligned} \tag{6.18}$$

Arguing as in the proofs of (5.21) and (5.22), we find that

$$\begin{aligned}
&-p^2\cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix}\ \Big|\ 1\right)_{p^2} \\
&\equiv \left(\sum_{k=0}^{N_2}\frac{\left(\frac{m}{d}\right)_k\left(\frac{d-m}{d}\right)_k}{k!^2}\right)^2+2p\Big\{\sum_{k=0}^{N_1}k\cdot\frac{\left(\frac{m}{d}\right)_k\left(\frac{d-m}{d}\right)_k}{k!^2}\Big\} \\
&\quad\cdot\Big\{\sum_{j=N_1+1}^{N_2}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\left(\frac{d}{mp}\right) \\
&\quad+\sum_{j=0}^{N_1}\frac{\left(\frac{m}{d}\right)_j\left(\frac{d-m}{d}\right)_j}{j!^2}\big\{H_{N_1+j}+H_{N_2+j}-2H_j\big\}\Big\} \pmod{p^2}.
\end{aligned} \tag{6.19}$$

Using Proposition 6.1 (2)-(3), we rewrite the rising factorials to obtain
(6.20)

$$-p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \mid 1\right)_{p^2} \equiv \left(\sum_{k=0}^{N_2} \frac{\left(\frac{m}{d}\right)_k \left(\frac{d-m}{d}\right)_k}{k!^2}\right)^2$$
$$+ 2p\Big\{\sum_{k=0}^{N_1} k \cdot \frac{\left(\frac{m}{d}\right)_k \left(\frac{d-m}{d}\right)_k}{k!^2}\Big\}\Big\{\sum_{j=N_1+1}^{N_2} (-1)^j \binom{N_2+j}{j}\binom{N_2}{j}\frac{1}{p}$$
$$+ \sum_{j=0}^{N_1} (-1)^j \binom{N_2+j}{j}\binom{N_2}{j}\big\{H_{N_1+j} + H_{N_2+j} - 2H_j\big\}\Big\} \pmod{p^2}.$$

Arguing as in the proof of (5.27) yields
(6.21)

$$-p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \mid 1\right)_{p^2} \equiv \left(\sum_{k=0}^{N_2} \frac{\left(\frac{m}{d}\right)_k \left(\frac{d-m}{d}\right)_k}{k!^2}\right)^2 + 2p\Big\{\sum_{k=0}^{N_1} k \cdot \frac{\left(\frac{m}{d}\right)_k \left(\frac{d-m}{d}\right)_k}{k!^2}\Big\}$$
$$\cdot \Big\{\sum_{j=0}^{N_2} (-1)^j \binom{N_2+j}{j}\binom{N_2}{j}\big\{H_{N_1+j} + H_{N_2+j} - 2H_j\big\}\Big\} \pmod{p^2}.$$

By Lemma 3.1, the sum in the last set of brackets evaluates to zero. Our end result is

$$-p^2 \cdot {}_2F_1\left(\begin{matrix}\rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}\end{matrix} \mid 1\right)_{p^2} \equiv \left(\sum_{j=0}^{N_2} \frac{\left(\frac{m}{d}\right)_j \left(\frac{d-m}{d}\right)_j}{j!^2}\right)^2 \pmod{p^2}. \tag{6.22}$$

The difference between (6.22) and the expression in Theorem 2 is trivially a multiple of p^2. □

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