

HYPERBOLIC 2-SPHERES WITH CONICAL SINGULARITIES, ACCESSORY PARAMETERS AND KÄHLER METRICS ON $\mathcal{M}_{0,n}$

LEON TAKHTAJAN AND PETER ZOGRAF

ABSTRACT. We show that the real-valued function S_α on the moduli space $\mathcal{M}_{0,n}$ of pointed rational curves, defined as the critical value of the Liouville action functional on a hyperbolic 2-sphere with $n \geq 3$ conical singularities of arbitrary orders $\alpha = \{\alpha_1, \dots, \alpha_n\}$, generates accessory parameters of the associated Fuchsian differential equation as their common antiderivative. We introduce a family of Kähler metrics on $\mathcal{M}_{0,n}$ parameterized by the set of orders α , explicitly relate accessory parameters to these metrics, and prove that the functions S_α are their Kähler potentials.

1. INTRODUCTION

The existence and uniqueness of a hyperbolic metric (a conformal metric of constant negative curvature -1) with prescribed singularities at a finite number of points on a Riemann surface is a classical problem that is closely related (and in special cases is equivalent) to the famous Uniformization Problem of Klein and Poincaré. Actually, in 1898 Poincaré [11] solved this problem for the simplest case of *parabolic* singularities. Below we formulate his result for the particular case of the standard 2-sphere realized as the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Consider the punctured surface $X = \widehat{\mathbb{C}} \setminus \{z_1, \dots, z_n\}$ with $n \geq 3$ (by applying an appropriate Möbius transformation we can always assume that $z_{n-2} = 0, z_{n-1} = 1, z_n = \infty$). Then the Liouville equation

$$\varphi_{z\bar{z}} = \frac{1}{2} e^\varphi$$

(where subscripts stand for the corresponding partial derivatives) has a unique (real-valued) solution φ on X with the following asymptotics:

$$\varphi(z) = \begin{cases} -2 \log |z - z_i| - 2 \log |\log |z - z_i|| + O(1) & \text{as } z \rightarrow z_i, i \neq n, \\ -2 \log |z| - 2 \log \log |z| + O(1) & \text{as } z \rightarrow \infty \end{cases}$$

(such a singularity is called parabolic). Geometrically, the Liouville equation means that the conformal metric $ds^2 = e^\varphi |dz|^2$ on X has constant negative curvature -1 (that is, hyperbolic), and the above asymptotics of φ guarantee that ds^2 is complete and the area of X is $2\pi(n-2)$.

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Poincaré used this result to prove the uniformization theorem, i.e., to show that there exists a complex-analytic covering of the Riemann surface X by the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$. He introduced the quantity

$$T_\varphi = \varphi_{zz} - \frac{1}{2} \varphi_z^2$$

and showed that when φ satisfies the Liouville equation with parabolic singularities, then T_φ is a meromorphic function on $\widehat{\mathbb{C}}$ of the form

$$T_\varphi(z) = \sum_{i=1}^{n-1} \left(\frac{1}{2(z-z_i)^2} + \frac{c_i}{z-z_i} \right),$$

with the asymptotics

$$T_\varphi(z) = \frac{1}{2z^2} + \frac{c_n}{z^3} + O\left(\frac{1}{z^4}\right) \text{ as } z \rightarrow \infty.$$

The coefficients c_i are the famous accessory parameters. They satisfy three obvious linear relations imposed by the asymptotic behaviour of T_φ at ∞ . The coefficients c_1, \dots, c_n are uniquely characterized by the fact that the monodromy group of the Fuchsian differential equation

$$\frac{d^2 u}{dz^2} + \frac{1}{2} T_\varphi(z) u = 0$$

is conjugate in $\text{PSL}(2, \mathbb{C})$ to the group of deck transformations of a covering $\mathbb{H} \rightarrow X$.

These ideas of Poincaré were in the spotlight once again about 20 years ago due to Polyakov's path integral formulation of the bosonic string [12] and the conformal field theory of Belavin-Polyakov-Zamolodchikov [2]. Briefly, in the quantum Liouville theory the quantity T_φ plays the role of the $(2, 0)$ -component of the stress-energy tensor that satisfies conformal Ward identities reflecting conformal symmetry of the theory. At the semi-classical level, as it was first observed by Polyakov, the Ward identity establishes (at the physical level of rigor) a non-trivial relation between the accessory parameters and the critical value of the Liouville action functional (see [13] for details).

In our paper [16], we rigorously proved Polyakov's conjecture using the Ahlfors-Bers theory of quasiconformal mappings and derived simple explicit formulas connecting the Liouville equation with accessory parameters and the Weil-Petersson metric on Teichmüller space. More specifically, let

$$\mathcal{Z}_n = \{(z_1, \dots, z_{n-3}) \in \mathbb{C}^{n-3} \mid z_i \neq 0, 1 \text{ and } z_i \neq z_k \text{ for } i \neq k\}$$

be the configuration space of singular points (\mathcal{Z}_n is isomorphic to the moduli space $\mathcal{M}_{0,n}$ of n -pointed rational curves over \mathbb{C}). Then there exists a smooth function $S : \mathcal{Z}_n \rightarrow \mathbb{R}$ (critical value of the Liouville action functional; cf. Section 3) such that

$$(I) \quad c_i = -\frac{1}{2\pi} \frac{\partial S}{\partial z_i}, \quad i = 1, \dots, n-3,$$

and

$$(II) \quad \frac{\partial c_i}{\partial \bar{z}_k} = \frac{1}{2\pi} \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k} \right\rangle_{WP}, \quad i, k = 1, \dots, n-3,$$

where $\langle \cdot, \cdot \rangle_{WP}$ denotes the Weil-Petersson metric on $\mathcal{Z}_n \cong \mathcal{M}_{0,n}$.¹ An immediate corollary of (I) and (II) is that the critical value S of the Liouville action is a potential for the Weil-Petersson metric.²

Although our methods generalize verbatim to hyperbolic 2-spheres with *elliptic* singularities of finite order (in which case there exists a ramified covering $\mathbb{H} \rightarrow \widehat{\mathbb{C}}$ branched over singular points z_1, \dots, z_n), they no longer work for conical singularities of general type (see Section 2 for precise definitions). However, exact analogs of formulas (I) and (II) hold in this general case as well, provided the orders $\{\alpha_1, \dots, \alpha_n\}$ of singularities z_1, \dots, z_n satisfy some rather mild natural conditions. Physical consideration based on semi-classical limits of conformal Ward identities also suggests the validity of these formulas in a general situation.

The objective of this paper is to give straightforward proofs of (I)-(II) in the case of hyperbolic 2-spheres with conical singularities of general type. Section 2 contains the definitions and background material about the classical Liouville equation, including detailed asymptotics of its solution. In Section 3 we present the action functional for the Liouville equation, introduced in [14], and prove an analogue of formula (I), Theorem 1.³ In Section 4 we prove an analogue of formula (II) that relates accessory parameters to certain Kähler metrics on the moduli space $\mathcal{M}_{0,n}$ similar to the Weil-Petersson metric — Theorem 2. It is worth noting that the proofs are considerably simpler than those in [16] and do not use Teichmüller theory.

2. BACKGROUND MATERIAL

Consider the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $n \geq 3$ distinct marked points z_1, \dots, z_n . As in the Introduction, we normalize the last three points to be 0, 1 and ∞ respectively; so in the sequel we will always assume that $z_{n-2} = 0, z_{n-1} = 1, z_n = \infty$. Let $\alpha = \{\alpha_1, \dots, \alpha_n\}$ be a set of real numbers such that $\alpha_i < 1, i = 1, \dots, n$, and

$$(1) \quad \sum_{i=1}^n \alpha_i > 2.$$

According to the classical result of Picard [9], [10] (see also [8] and, for a modern proof, [15])⁴ there exists a unique conformal metric of constant curvature -1 , or the *hyperbolic metric*, on $\widehat{\mathbb{C}}$ with conical singularities of order α_i at $z_i, i = 1, \dots, n$. Precisely, it means that such a metric has the form $ds^2 = e^\varphi |dz|^2$, where φ is a smooth function on $X = \mathbb{C} \setminus \{z_1, \dots, z_{n-1}\}$ satisfying the Liouville equation

$$(2) \quad \varphi_{z\bar{z}} = \frac{1}{2} e^\varphi$$

and having the following asymptotics near the singular points:

$$(3) \quad \varphi(z) = \begin{cases} -2\alpha_i \log |z - z_i| + O(1) & \text{as } z \rightarrow z_i, i \neq n, \\ -2(2 - \alpha_n) \log |z| + O(1) & \text{as } z \rightarrow \infty. \end{cases}$$

¹In [17] we formulated and proved analogs of (I)-(II) for compact Riemann surfaces of arbitrary genus.

²These results were used by the second author in the study of the asymptotic behaviour of accessory parameters for degenerating Riemann surfaces [18].

³A recent physicists' paper [4] gives a different, computationally more involved proof of Theorem 1.

⁴It is very instructive to compare the approaches of [9], [10], [8] and [15].

The point z_i is then called a *conical singularity* of order α_i , or of angle $\theta_i = 2\pi(1 - \alpha_i)$ (we have $\theta_i > 2\pi$ when $\alpha_i < 0$).

Remark 1. If $\alpha_i = 1$, then z_i is a parabolic point, or *cusp* (conical singularity of zero angle), and the asymptotics (3) should be replaced by the one mentioned in the Introduction.

The configuration space \mathcal{Z}_n of singular points is an open subset in \mathbb{C}^{n-3} :

$$\mathcal{Z}_n = \{(z_1, \dots, z_{n-3}) \in \mathbb{C}^{n-3} \mid z_i \neq 0, 1 \text{ and } z_i \neq z_k \text{ for } i \neq k\}$$

and is isomorphic to the moduli space $\mathcal{M}_{0,n}$ of n -pointed rational curves over \mathbb{C} . For any fixed set of orders α the solution φ to the Liouville equation makes sense as a function of $n - 2$ complex variables z, z_1, \dots, z_{n-3} , defined on the space

$$\mathcal{Z}_{n+1} = \{(z, z_1, \dots, z_{n-3}) \in \mathbb{C}^{n-2} \mid z, z_i \neq 0, 1; z \neq z_i; z_i \neq z_k \text{ for } i \neq k\}.$$

The space \mathcal{Z}_{n+1} is fibered over \mathcal{Z}_n by “forgetting” the first coordinate z : the fiber over a point $(z_1, \dots, z_{n-3}) \in \mathcal{Z}_n$ is the surface $\mathbb{C} \setminus \{z_1, \dots, z_{n-3}, 0, 1\}$. It follows from the results of [10], [8], [15] that φ is a real-analytic function on \mathcal{Z}_{n+1} .

The $(2, 0)$ -component of the stress-energy tensor in the Liouville theory is given by the expression

$$(4) \quad T_\varphi = \varphi_{zz} - \frac{1}{2} \varphi_z^2.$$

The following result is classical.

Lemma 1. *Let φ be the solution to the Liouville equation with conical singularities (3). Then T_φ is a meromorphic function on $\hat{\mathbb{C}}$ with second-order poles at z_1, \dots, z_n . Explicitly,*

$$(5) \quad T_\varphi(z) = \sum_{i=1}^{n-1} \left(\frac{h_i}{2(z - z_i)^2} + \frac{c_i}{z - z_i} \right)$$

and

$$(6) \quad T_\varphi(z) = \frac{h_n}{2z^2} + \frac{c_n}{z^3} + O\left(\frac{1}{z^4}\right) \text{ as } z \rightarrow \infty,$$

where $h_i = \alpha_i(2 - \alpha_i)$, $i = 1, \dots, n$.⁵

Complex numbers c_i are called *accessory parameters*. They are uniquely determined by the singular points z_1, \dots, z_n and the set of orders α . Formula (6) imposes three linear equations on the parameters c_1, \dots, c_n :

$$\sum_{i=1}^{n-1} c_i = 0, \quad \sum_{i=1}^{n-1} (h_i + 2c_i z_i) = h_n, \quad \sum_{i=1}^{n-1} (h_i z_i + c_i z_i^2) = c_n,$$

so that c_{n-2}, c_{n-1} and c_n are explicit linear combinations of c_1, \dots, c_{n-3} with coefficients depending on z_i and α_i . Real analyticity of φ implies that the accessory parameters are also real-analytic functions on \mathcal{Z}_n .

To study the behaviour of φ near the singular points more thoroughly, consider the Fuchsian differential equation

$$(7) \quad \frac{d^2 u}{dz^2} + \frac{1}{2} T_\varphi(z) u = 0,$$

⁵The coefficients h_i are conformal weights in quantum Liouville theory [14].

with regular singular points at z_1, \dots, z_n . A classical result (see, e.g. [11]), which follows from the fact that $e^{-\varphi/2}$ is a solution to (7), asserts that the monodromy group Γ of the differential equation (7) is, up to a conjugation in $\text{PSL}(2, \mathbb{C})$, a subgroup of $\text{PSL}(2, \mathbb{R})$ (see, e.g., [6], [3], or [7]).⁶ Such a group Γ is discrete in $\text{PSL}(2, \mathbb{R})$ if and only if $\alpha_i = 1 - 1/l_i$ for all $i = 1, \dots, n$, where l_i is a positive integer or ∞ .

In case of general conical singularities the monodromy group Γ is no longer discrete in $\text{PSL}(2, \mathbb{R})$. It is generated by local monodromies around regular singular points z_i , which, in general, are elliptic elements γ_i of infinite order. If we denote the fixed points of γ_i by w_i, \bar{w}_i , then

$$\frac{\gamma_i(z) - w_i}{\gamma_i(z) - \bar{w}_i} = \lambda_i \frac{z - w_i}{z - \bar{w}_i}, \quad i = 1, \dots, n,$$

where $\lambda_i = e^{2\pi\sqrt{-1}(1-\alpha_i)}$ is called the multiplier of γ_i .

Remark 2. It is an outstanding problem to find a geometric meaning of the monodromy group Γ in the case of general conical singularities, thus providing another interpretation for the accessory parameters. Perhaps this problem should be considered in the context of A. Connes’s [5] non-commutative differential geometry, where such group actions naturally appear.

Let $w = u_1/u_2$ be the ratio of two linearly independent solutions u_1, u_2 of the differential equation (7). It is a multi-valued meromorphic function on $\widehat{\mathbb{C}}$ with ramification points z_1, \dots, z_n , and it is single-valued on the universal cover of $X = \widehat{\mathbb{C}} \setminus \{z_1, \dots, z_n\}$. It is a classical result of Schwarz that

$$(8) \quad T_\varphi = \mathcal{S}(w)$$

on X , where $\mathcal{S}(w)$ denotes the Schwarzian derivative of w :

$$\mathcal{S}(w) = \frac{w'''}{w'} - \frac{3}{2} \left(\frac{w''}{w'} \right)^2.$$

Next, normalize u_1, u_2 in such a way that the monodromy group Γ of (7) is a subgroup of $\text{PSU}(1, 1)$. The multi-valued function w admits the following expansion in the neighborhood of each singular point z_i :

$$(9) \quad \sigma_i(w(z)) = \zeta_i^{1-\alpha_i} \sum_{k=0}^{\infty} a_i^{(k)} \zeta_i^k \quad \text{as } \zeta_i \rightarrow 0, \quad i = 1, \dots, n.$$

Here ζ_i is a local uniformizer: $\zeta_i = z - z_i$ for $i = 1, \dots, n - 1$, and $\zeta_n = 1/z$, and $\sigma_i \in \text{PSU}(1, 1)$ diagonalizes local monodromy γ_i around z_i , $i = 1, \dots, n$. Moreover, the coefficients $a_i^{(k)}$ are (locally) real-analytic on \mathcal{Z}_n , as follows from the analytic dependence on parameters of solutions to ordinary differential equations.

Lemma 2. *The solution φ to the Liouville equation (2) with conical singularities (3) is given by the formula*

$$e^\varphi = \frac{4|w'|^2}{(1 - |w|^2)^2},$$

where $w = u_1/u_2$, and u_1, u_2 are two linearly independent solutions of the Fuchsian differential equation (7) with monodromy in $\text{PSU}(1, 1)$.

⁶Among many available references, [6] is classical, [3] gives a detailed exposition of Fuchsian differential equations, and [7] is a modern introduction to the subject.

Proof. Since the monodromy is in $\text{PSU}(1, 1)$, the function $\log\left(\frac{4|w'|^2}{(1-|w|^2)^2}\right)$ is real and single-valued on X . Moreover, it is easy to check that this function satisfies the Liouville equation, and by (9) it has the same asymptotics (3) as φ . Therefore, it must be equal to φ . \square

Remark 3. When $\alpha_i = 1, i = 1, \dots, n$, it is more convenient to normalize solutions u_1, u_2 so that $\Gamma \subset \text{PSL}(2, \mathbb{R})$ (see [16]).

From the equality (8) and expansions (9) we readily get the following formula for the accessory parameters (cf. Lemma 1 in [16]).

Lemma 3.

$$c_i = \frac{h_i}{1 - \alpha_i} \cdot \frac{a_i^{(1)}}{a_i^{(0)}}, \quad i = 1, \dots, n,$$

where $h_i = \alpha_i(2 - \alpha_i)$.

Finally, we summarize all the necessary facts about the asymptotic behaviour of φ and its derivatives in the next statement (cf. Lemma 2 in [16]).

Lemma 4. *The solution φ to the Liouville equation (2) with conical singularities (3) has the following asymptotic expansions near the singular points $z = z_i$, uniform in a neighborhood of (z_1, \dots, z_{n-3}) in \mathcal{Z}_n :*

(i)

$$\varphi_z(z) = \begin{cases} -\frac{\alpha_i}{\zeta_i} + \frac{c_i}{\alpha_i} + \frac{f_i(|\zeta_i|)}{\zeta_i} + o(1) & \text{as } z \rightarrow z_i, i \neq n, \\ -(2 - \alpha_n)\zeta_n - \frac{c_n}{\alpha_n} \cdot \zeta_n^2 + f_n(|\zeta_n|)\zeta_n + o(|\zeta_n|^2) & \text{as } z \rightarrow \infty, \end{cases}$$

where $\zeta_i = z - z_i$ ($i \neq n$) and $\zeta_n = 1/z$ are local coordinates near the singular points, and

$$f_i(t) = O\left(t^{2(1-\alpha_i)}\right) \quad \text{as } t \rightarrow 0, i = 1, \dots, n.$$

(ii) For $i = 1, \dots, n - 3$

$$\varphi_{zz}(z) = \frac{\alpha_i + g_i^{(0)}(\zeta_i) + \zeta_i g_i^{(1)}(\zeta_i)}{\zeta_i^2} + O(1),$$

where

$$g_i^{(0)}(t), g_i^{(1)}(t) = O\left(t^{2(1-\alpha_i)}\right) \quad \text{as } t \rightarrow 0.$$

(iii) For $i = 1, \dots, n - 3$ and $k = 1, \dots, n$, there exist constants d_{ik} such that

$$\varphi_{z_i}(z) = \begin{cases} -\delta_{ik}\varphi_z(z) + d_{ik} + o(1) & \text{as } z \rightarrow z_k, k \neq n, \\ d_{in} + o(1) & \text{as } z \rightarrow \infty. \end{cases}$$

(iv) If $\alpha_k > 0$ for each $k = 1, \dots, n$, then for $i = 1, \dots, n - 3$,

$$-2e^{-\varphi}\varphi_{z_i\bar{z}} = \begin{cases} \delta_{ik} + O(|z - z_k|^{\min\{1, 2\alpha_k\}}) & \text{as } z \rightarrow z_k, k \neq n, \\ O(|z|^{\max\{1, 2(1-\alpha_n)\}}) & \text{as } z \rightarrow \infty. \end{cases}$$

Proof. Parts (i)-(iii) follow from (9) and Lemmas 2 and 3; part (iv) follows from (i), (iii), the Liouville equation (2) and asymptotics (3). Uniform estimates for the remainder terms follow from the real analyticity of the coefficients $a_i^{(k)}$ as functions of z_1, \dots, z_{n-3} . One can also prove (i)-(iv) directly from the Liouville equation and asymptotics (3) by observing that the solution φ admits the following expansion in a neighborhood of each z_i :

$$\varphi(z) = -2\alpha_i \log |z - z_i| + \xi^{(0)}(z) + \sum_{k=1}^{\infty} |z - z_i|^{2k(1-\alpha_i)} \xi^{(k)}(z), \quad i = 1, \dots, n - 1,$$

and a similar expansion at ∞ , where $\xi^{(k)}(z)$ are real-analytic as functions on the fibered space \mathcal{Z}_{n+1} (real-analytic dependence on z_1, \dots, z_{n-3} follows from the analysis in [9], [10], [8], [15]). □

3. LIOUVILLE ACTION AND ACCESSORY PARAMETERS

For a given set of orders $\alpha = \{\alpha_1, \dots, \alpha_n\}$ the action functional for the Liouville equation (2) is defined in [14] by the formula

$$(10) \quad S_\alpha[\psi] = \lim_{\varepsilon \rightarrow 0} S_\alpha^\varepsilon[\psi],$$

where

$$(11) \quad \begin{aligned} S_\alpha^\varepsilon[\psi] = & \iint_{X^\varepsilon} (|\psi_z|^2 + e^\psi) \left| \frac{dz \wedge d\bar{z}}{2} \right| \\ & + \frac{\sqrt{-1}}{2} \sum_{i=1}^{n-1} \alpha_i \oint_{C_i^\varepsilon} \psi \left(\frac{d\bar{z}}{\bar{z} - \bar{z}_i} - \frac{dz}{z - z_i} \right) \\ & + \frac{\sqrt{-1}}{2} (2 - \alpha_n) \oint_{C_n^\varepsilon} \psi \left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z} \right) \\ & - 2\pi \sum_{i=1}^{n-1} \alpha_i^2 \log \varepsilon - 2\pi(2 - \alpha_n)^2 \log \varepsilon. \end{aligned}$$

Here $X^\varepsilon = \mathbb{C} \setminus \left(\bigcup_{i=1}^{n-1} \{|z - z_i| < \varepsilon\} \cup \{|z| > 1/\varepsilon\} \right)$, and the circles $C_i^\varepsilon = \{|z - z_i| = \varepsilon\}$, $i = 1, \dots, n-1$, and $C_n^\varepsilon = \{|z| = 1/\varepsilon\}$ are oriented as the boundary components of X^ε .

The functional S_α is well-defined on the space \mathcal{CM}_α of all conformal metrics $e^\psi |dz|^2$ on $\widehat{\mathbb{C}}$ with conical singularities at z_1, \dots, z_n of orders $\alpha_1, \dots, \alpha_n$, satisfying

$$(12) \quad \psi_z(z) = \begin{cases} -\frac{\alpha_i}{z - z_i} (1 + O(|z - z_i|^{\min\{1, 2(1-\alpha_i)\}})) & \text{as } z \rightarrow z_i, \quad i \neq n, \\ -(2 - \alpha_n) \frac{1}{z} (1 + O(|z|^{-\min\{1, 2(1-\alpha_n)\}})) & \text{as } z \rightarrow \infty. \end{cases}$$

Remark 4. The Liouville equation is the Euler-Lagrange equation for the functional S_α . Indeed, the contour integrals in (11) ensure that for any $e^\psi |dz|^2 \in \mathcal{CM}_\alpha$ and $u \in C^\infty(\widehat{\mathbb{C}}, \mathbb{R})$,

$$\lim_{t \rightarrow 0} \frac{S[\psi + tu] - S[\psi]}{t} = \iint_{\mathbb{C}} (-2\psi_{z\bar{z}} + e^\psi) u \frac{|dz \wedge d\bar{z}|}{2},$$

where the integral on the right-hand side is convergent. Thus the functional S_α has a non-degenerate critical point given by the hyperbolic metric.

The Liouville action evaluated on the solution φ to the Liouville equation is a real-valued function $S_\alpha[\varphi] = S_\alpha(z_1, \dots, z_{n-3})$ on the configuration space \mathcal{Z}_n depending on $\alpha_1, \dots, \alpha_n$ as parameters.

Theorem 1. *For any fixed set of orders $\alpha = \{\alpha_1, \dots, \alpha_n\}$ such that $\alpha_i < 1$ and $\sum_{i=1}^n \alpha_i > 2$, the function $S_\alpha : \mathcal{Z}_n \rightarrow \mathbb{R}$ is differentiable and*

$$(13) \quad c_i = -\frac{1}{2\pi} \frac{\partial S_\alpha}{\partial z_i}, \quad i = 1, \dots, n - 3,$$

where c_i are the accessory parameters defined by (5).

Proof. First we show that

$$(14) \quad \lim_{\varepsilon \rightarrow 0} \frac{\partial S_\alpha^\varepsilon}{\partial z_i} = -2\pi c_i$$

pointwise on the configuration space \mathcal{Z}_n . We have

$$(15) \quad \begin{aligned} \frac{\partial S_\alpha^\varepsilon}{\partial z_i} &= \frac{\sqrt{-1}}{2} \left(\iint_{X^\varepsilon} \frac{\partial}{\partial z_i} (|\varphi_z|^2 + e^\varphi) dz \wedge d\bar{z} + \oint_{C_i^\varepsilon} (|\varphi_z|^2 + e^\varphi) d\bar{z} \right) \\ &\quad + \frac{\sqrt{-1}}{2} \sum_{k=1}^{n-1} \alpha_k \oint_{C_k^\varepsilon} (\varphi_{z_i} + \delta_{ik} \varphi_z) \left(\frac{d\bar{z}}{\bar{z} - \bar{z}_k} - \frac{dz}{z - z_k} \right) \\ &\quad + \frac{\sqrt{-1}}{2} (2 - \alpha_n) \oint_{C_n^\varepsilon} \varphi_{z_i} \left(\frac{d\bar{z}}{\bar{z}} - \frac{dz}{z} \right). \end{aligned}$$

Using part (i) of Lemma 4, we see that

$$\frac{\sqrt{-1}}{2} \oint_{C_i^\varepsilon} |\varphi_z|^2 d\bar{z} \rightarrow \pi c_i \quad \text{as } \varepsilon \rightarrow 0.$$

From the Liouville equation we get

$$\oint_{C_i^\varepsilon} e^\varphi d\bar{z} = -\frac{1}{2} \oint_{C_i^\varepsilon} \varphi_{zz} dz,$$

which tends to 0 as $\varepsilon \rightarrow 0$ because of part (ii) of Lemma 4. It follows from part (iii) of Lemma 4 that the contour integrals in the second and third lines of (15) tend to

$$-2\pi c_i - 2\pi \sum_{k=1}^{n-1} \alpha_k d_{ik} - 2\pi(\alpha_n - 2)d_{in}$$

as $\varepsilon \rightarrow 0$. An obvious identity,

$$\frac{\partial}{\partial z_i} |\varphi_z|^2 dz \wedge d\bar{z} = d(\varphi_{z_i} \varphi_{\bar{z}} d\bar{z} - \varphi_{z_i} \varphi_z dz) - 2\varphi_{z_i} \varphi_{z\bar{z}} dz \wedge d\bar{z},$$

combined with the Liouville equation yields the following simple formula:

$$(16) \quad \frac{\partial}{\partial z_i} (|\varphi_z|^2 + e^\varphi) dz \wedge d\bar{z} = d(\varphi_{z_i} \varphi_{\bar{z}} d\bar{z} - \varphi_{z_i} \varphi_z dz).$$

This reduces the area integral in (15) to a sum of contour integrals. These contour integrals are again easy to evaluate using parts (i) and (iii) of Lemma 4, and all together they tend to

$$-\pi c_i + 2\pi \sum_{k=1}^{n-1} \alpha_k d_{ik} + 2\pi(\alpha_n - 2)d_{in}$$

as $\varepsilon \rightarrow 0$. Adding all the terms on the right-hand side of (15), we get $-2\pi c_i$ in the limit as $\varepsilon \rightarrow 0$. Finally, we observe that the convergence of (14) is uniform on compact subsets of \mathcal{Z}_n because so are the estimates in Lemma 4. \square

Remark 5. The same method works for $\alpha_i = 1, i = 1, \dots, n$. In this case, formula (11) for the functional $S^\varepsilon[\varphi]$ contains an additional regularizing term $4\pi(n - 2) \log |\log \varepsilon|$. By part 2) of Lemma 2 in [16], no contour integrals contribute to the classical action $S[\varphi]$. This gives a much simpler proof of Theorem 1 in [16] along the lines of this paper without using either the uniformization theorem or the quasiconformal mappings.

4. ACCESSORY PARAMETERS AND KÄHLER METRICS ON $\mathcal{M}_{0,n}$

Throughout this section we assume, in addition, that the orders $\alpha_1, \dots, \alpha_n$ are all positive,⁷ i.e., $\alpha_i \in (0, 1)$ for each $i = 1, \dots, n$, and $\sum_{i=1}^n \alpha_i > 2$. To every such set of orders $\alpha = \{\alpha_1, \dots, \alpha_n\}$ we associate a Hermitian metric on the configuration space $\mathcal{Z}_n \cong \mathcal{M}_{0,n}$ as follows.

Consider the kernel

$$(17) \quad R(\zeta, z) = -\frac{1}{\pi} \left(\frac{1}{\zeta - z} + \frac{z - 1}{\zeta} - \frac{z}{\zeta - 1} \right), \quad (\zeta, z) \in \mathbb{C} \times \mathbb{C},$$

and put

$$(18) \quad Q_i(z) = R(z, z_i), \quad i = 1, \dots, n - 3.$$

Clearly, the functions Q_i are linearly independent. It follows from the positivity of orders α_i and (3) that the functions Q_i are square-integrable on $\widehat{\mathbb{C}}$ with respect to the measure $e^{-\varphi} \frac{|dz \wedge d\bar{z}|}{2}$. We define the scalar products of the basis of 1-forms on \mathcal{Z}_n over the point $(z_1, \dots, z_{n-3}) \in \mathcal{Z}_n$ by the formula

$$(19) \quad (dz_i, dz_k)_\alpha = \iint_{\widehat{\mathbb{C}}} Q_i \bar{Q}_k e^{-\varphi} \frac{|dz \wedge d\bar{z}|}{2}, \quad i, k = 1, \dots, n - 3.$$

The scalar products $\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k} \rangle_\alpha$ are given by the elements of the inverse matrix to $\{(dz_i, dz_k)_\alpha\}_{i,k=1}^{n-3}$. Since the matrix $\{\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k} \rangle_\alpha\}_{i,k=1}^{n-3}$ is non-degenerate and depends real analytically on z_i , it gives rise to a Hermitian metric on \mathcal{Z}_n which we denote by $\langle \cdot, \cdot \rangle_\alpha$. This metric is analogous to the celebrated Weil-Petersson metric on the moduli space $\mathcal{M}_{0,n}$.⁸

Remark 6. In Teichmüller theory, when all $\alpha_i = 1$, the holomorphic cotangent space to \mathcal{Z}_n at the point (z_1, \dots, z_{n-3}) is identified by means of quasiconformal mappings with the space of rational functions on $\widehat{\mathbb{C}}$ with only simple poles at $z_1, \dots, z_{n-3}, 0, 1, \infty$, and dz_i then corresponds to Q_i (see, e.g., [16] and references therein). Here we use the same identification directly.

⁷This is equivalent to the condition that all conformal weights h_i are positive.

⁸We get the Weil-Petersson metric if all the orders α_i are equal to 1.

The kernel R , roughly speaking, inverts the operator $\partial/\partial\bar{z}$ on \mathbb{C} . The precise statement (see, e.g., [1] for details) is essentially a version of the Pompeiu formula.

Lemma 5. *Let g be a locally integrable function on \mathbb{C} such that $g(z) = o(|z|)$ as $z \rightarrow \infty$. Then the equation*

$$f_{\bar{z}} = g$$

has a unique solution on \mathbb{C} satisfying $f(0) = f(1) = 0$ and $f(z) = o(|z|^2)$ as $z \rightarrow \infty$. This solution is explicitly given by the formula

$$(20) \quad f(z) = \iint_{\mathbb{C}} g(\zeta)R(\zeta, z) \frac{|d\zeta \wedge d\bar{\zeta}|}{2}.$$

Let us formulate the main result of this section.

Theorem 2. *For any set of orders $\alpha = \{\alpha_1, \dots, \alpha_n\}$ such that $\alpha_i \in (0, 1)$ for each $i = 1, \dots, n$ and $\sum_{i=1}^n \alpha_i > 2$, we have*

$$(21) \quad \frac{\partial c_i}{\partial \bar{z}_k} = \frac{1}{2\pi} \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k} \right\rangle_{\alpha}, \quad i, k = 1, \dots, n - 3.$$

Proof. As we mentioned in Section 2, the accessory parameters c_1, \dots, c_{n-3} are real-analytic functions on \mathcal{Z}_n . Now consider the functions

$$F^i = -2e^{-\varphi} \varphi_{z_i \bar{z}}, \quad i = 1, \dots, n - 3.$$

According to part (iv) of Lemma 4 we have

$$(22) \quad \begin{aligned} F^i(z_k) &= \delta_{ik}, & k = 1, \dots, n - 1, \\ F^i(z) &= O(|z|^{\max\{1, 2(1-\alpha_n)\}}), & z \rightarrow \infty. \end{aligned}$$

Moreover, as follows from (4) and (5),

$$\begin{aligned} F_{\bar{z}}^i &= 2e^{-\varphi} \varphi_{\bar{z}} \varphi_{z_i \bar{z}} - 2e^{-\varphi} \varphi_{z_i \bar{z} \bar{z}} = -2e^{-\varphi} \frac{\partial}{\partial z_i} \left(\varphi_{\bar{z} \bar{z}} - \frac{1}{2} \varphi_{\bar{z}}^2 \right) \\ &= -2e^{-\varphi} \sum_{k=1}^{n-1} \frac{1}{\bar{z} - \bar{z}_k} \cdot \frac{\partial \bar{c}_k}{\partial z_i} = 2\pi e^{-\varphi} \sum_{k=1}^{n-3} \frac{\partial \bar{c}_k}{\partial z_i} \bar{Q}_k. \end{aligned}$$

Lemma 5 applied to $g = F_{\bar{z}}^i$ yields

$$F^i(z) = \iint_{\mathbb{C}} F_{\bar{\zeta}}^i(\zeta)R(\zeta, z) \frac{|d\zeta \wedge d\bar{\zeta}|}{2}, \quad i = 1, \dots, n - 3.$$

Putting $z = z_j$ and using (22) we get that

$$\delta_{ij} = 2\pi \sum_{k=1}^{n-3} \frac{\partial \bar{c}_k}{\partial z_i} (dz_j, dz_k)_{\alpha}, \quad i, j = 1, \dots, n - 3,$$

which proves the theorem. □

Remark 7. The same arguments prove Theorem 2 in [16], making the uniformization theorem and quasiconformal mappings redundant also in the case when all $\alpha_i = 1$.

Corollary 1. *For any set α as in Theorem 2,*

$$\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_k} \right\rangle_{\alpha} = -\frac{\partial^2 S_{\alpha}}{\partial z_i \partial \bar{z}_k}, \quad i, k = 1, \dots, n - 3.$$

That is, the metric $\langle \cdot, \cdot \rangle_{\alpha}$ is Kähler and the function $-S_{\alpha}$ is its real-analytic Kähler potential on \mathcal{Z}_n .

Proof. Immediately follows from Theorems 1 and 2. □

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DEPARTMENT OF MATHEMATICS, SUNY AT STONY BROOK, STONY BROOK, NEW YORK 11794-3651

E-mail address: leontak@math.sunysb.edu

STEKLOV MATHEMATICAL INSTITUTE, ST. PETERSBURG, 191011 RUSSIA

E-mail address: zograf@pdmi.ras.ru