TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 355, Number 5, Pages 1987–2009
S 0002-9947(03)03237-9
Article electronically published on January 8, 2003

METRIC CHARACTER OF HAMILTON–JACOBI EQUATIONS

ANTONIO SICONOLFI

ABSTRACT. We deal with the metrics related to Hamilton–Jacobi equations of eikonal type. If no convexity conditions are assumed on the Hamiltonian, these metrics are expressed by an inf–sup formula involving certain level sets of the Hamiltonian. In the case where these level sets are star–shaped with respect to 0, we study the induced length metric and show that it coincides with the Finsler metric related to a suitable convexification of the equation.

INTRODUCTION

This paper is devoted to the (nonsymmetric) distances of $\mathbb{R}^N$ related to the Hamilton–Jacobi equations of eikonal type

$$H(x, Du) = 0 \tag{I}$$

with continuous Hamiltonian H satisfying the inequality $H(x, 0) < 0$ for any x.

For H convex in the second variable, the recognition of the metric character of equation (I) is a central point in Hamilton–Jacobi theory, see [10]. This issue has been revised by Kruzkov [13] and P. L. Lions [15] and other authors using weak notions of solutions.

In the first part of the paper we study the convex case, which for us means (see [8], [9]) that we just assume the convexity of the level sets

$$Z(x) = \{p : H(x, p) \le 0\}.$$

This is of course different from requiring the convexity of H as previously done. We also weaken slightly the regularity assumption on the Hamiltonian (see section 2). In this case the related distance L is then the value function of the variational problem

$$\inf \int_0^1 \delta(\xi, \dot{\xi}) \, \mathrm{d}t, \tag{II}$$

where $\delta(x, q)$ is the support function of $Z(x)$ at q and the infimum is taken with respect to all Lipschitz continuous curves defined in $[0, 1]$ with fixed endpoints.

In section 2 we prove that for any fixed y_0, $L(y_0, \cdot)$ is a viscosity solution of (I) in $\mathbb{R}^N \setminus \{y_0\}$. Although this fact is generally known, at least when H is convex, our proof is based on a different point of view. In fact it comes directly from the analysis of the local behaviour of L and from its convexity property.

Received by the editors May 9, 2000 and, in revised form, May 18, 2001.
2000 *Mathematics Subject Classification.* Primary 35F20, 49L25.
Key words and phrases. Hamilton–Jacobi equations, viscosity solutions, distance functions.
Research partially supported by the TMR Network "Viscosity Solutions and Applications".

We recall that in the literature a metric defined as in (II) is called a Finsler metric, and $Z(x)$ can be viewed as its dual tangential ball at x, see section 1.

One important property is that for each pair of points x, y, $L(x,y)$ is the infimum of the intrinsic lengths of the connecting curves, where the intrinsic length is the total variation of the curve with respect to the distance.

We emphasize that this situation is quite special, since for a general distance D the infimum of the lengths gives a new distance, denoted by D_l and called the length distance induced by D. The passage from D to D_l can be viewed as a sort of metric convexification. If $D = D_l$, we call it a path metric (see [14] for this terminology).

The main achievement of the present paper is to show that the metric character of (I) is preserved, even if with different features, also in the case where any convexity assumption on H and Z is removed.

Starting from the set–valued map Z, we give an inf–sup integral formula, involving the trajectories joining two given points, which defines a distance S on $\mathbb{R}^N$. Then we generalize the result previously quoted in the convex case and show that for any fixed y_0, $S(y_0,\cdot)$ is a solution of (I) in $\mathbb{R}^N \setminus \{y_0\}$.

Although it is expected that viscosity solutions of nonconvex Hamilton–Jacobi equations are represented by game theory–type formulae (see [1], [12]), ours is new because it can be written starting from a general Hamiltonian satisfying rather weak regularity assumptions.

We study some properties of S and compare it to the Finsler metric related to a "convexified" equation whose Hamiltonian has as level sets the convex hulls of $Z(x)$.

This can be seen as a problem dual to the one treated by Busemann and Mayer in [7]. They studied the variational problem (II) without the usual convexity assumptions on δ, obtaining again a distance as value function.

They proved that such a metric coincides with the value function of the problem defined by replacing δ in (II) by the gauge function of the convex hull of the set $\{q : \delta(x,q) \le 1\}$.

In our case the situation is different, for S is not in general a Finsler metric and not even a path metric.

Under the assumption that $Z(x)$ is star–shaped with respect to 0 for any x, we are able to show that S_l (the length metric induced by S) is equal to the Finsler metric related to the "convexified" equation we mentioned before.

This result establishes a connection between two sorts of convexifications obtained using quite different procedures. It is based on the analysis of the local behaviour of S (see section 5) done through PDE methods exploiting the relation between metrics and Hamilton–Jacobi equations that we have outlined before.

The paper is divided into 5 sections. Section 1 is devoted to illustrating some basic properties of Finsler metrics as well as the notion of length for a general metric. In section 2 we make precise the relation between Finsler distances and convex Hamilton–Jacobi equations of eikonal type.

The metric S is defined in section 3, and some of its properties are deduced, while in section 4 it is connected to Hamilton–Jacobi equations not satisfying any convexity assumptions.

Finally, in Section 5 the length distance induced by S is studied.

1. Finsler metrics

This section covers some introductory material about Finsler metrics on $\mathbb{R}^N$.

By the term metric (or, equivalently, distance) we mean a nonnegative function, say D, defined in $\mathbb{R}^N \times \mathbb{R}^N$, such that

$$D(x,y) = 0 \quad \text{if and only if } x = y,$$

$$D(x,y) \le D(x,z) + D(z,y) \quad \text{for any } x,y,z \in \mathbb{R}^N.$$

Note that the symmetry condition is not required.

We denote by E the Euclidean metric and write $|x-y|$ for the corresponding distance between two points x and y, and we write $B_E(x,\varepsilon)$ for the Euclidean (open) ball centered at x with radius a positive constant ε.

We put for any x, $A \subset \mathbb{R}^N$

$$d_E(x,A) = \inf_A |x-y|,$$
$$d_E^{\#}(x,A) = d_E(x,A) - d_E(x,\mathbb{R}^N \setminus A).$$

The latter is called the (Euclidean) signed distance of x from A.

The Hausdorff metric d_H is defined for any couple K_1, K_2 of compact subsets of $\mathbb{R}^N$ by

$$d_H(K_1,K_2) = \max\left\{ \max_{K_1} d_E(p_1,K_2), \max_{K_2} d_E(p_2,K_1) \right\}.$$

A Finsler metric, see [7], [5], [6], is defined to be the value function of the following variational problem:

Given $x,y \in \mathbb{R}^N$, find

$$\inf\left\{ \int_0^1 \delta(\xi,\dot{\xi})\,\mathrm{d}t : \xi \in \mathcal{A}_{x,y} \right\}, \tag{1.1}$$

where $\mathcal{A}_{x,y}$ is the set of Lipschitz continuous trajectories (with respect to E) defined in $[0,1]$ joining x and y, and δ is a continuous function on $\mathbb{R}^N \times \mathbb{R}^N$ such that

$$\delta(x,q) > 0 \qquad \text{for any } x, q \ne 0, \tag{1.2}$$

$$\delta(x,\lambda q) = \lambda\delta(x,q) \qquad \text{for any } x,q,\lambda \ge 0, \tag{1.3}$$

$$\delta(x,q+\overline{q}) \le \delta(x,q) + \delta(x,\overline{q}) \qquad \text{for any } x,q,\overline{q}. \tag{1.4}$$

Such a value function, denoted by L, assigns to any couple (x,y) the infimum in formula (1.1).

It is indeed a metric, because of properties (1.2), (1.3) and since for any x,y,z the juxtaposition of trajectories in $\mathcal{A}_{x,z}$ and $\mathcal{A}_{z,y}$ gives, up to a change of parameter, a curve in $\mathcal{A}_{x,y}$.

Note that the convexity does not play any role in this issue, but it is crucial for analyzing the local behaviour of L, see [7], [5].

Before doing it, we observe that L is locally equivalent to E in the sense that for any compact subset K of $\mathbb{R}^N$ there are positive constants $R > r$ such that

$$r|x-y| \le L(x,y) \le R|x-y|$$

for any $x,y \in K$.

Proposition 1.1. *For any x_0,*

$$\lim_{\substack{x,y\to x_0\\ x\neq y}} \frac{L(x,y)}{\delta(x_0,y-x)} = 1. \tag{1.5}$$

Proof. The first step is to show that if ξ_n is a sequence of Lipschitz continuous curves defined in $[0,1]$ satisfying, for any n, $t \in [0,1]$,

$$|\xi_n(t) - x_0| < \varepsilon_n$$

with $\varepsilon_n > 0$ converging to 0, then

$$\lim_n \frac{\int_0^1 \delta(x_0,\dot{\xi}_n)\,\mathrm{d}t}{\int_0^1 \delta(\xi_n,\dot{\xi}_n)\,\mathrm{d}t} = 1. \tag{1.6}$$

This is obtained through the estimate

$$\frac{|\int_0^1 (\delta(x_0,\dot{\xi}_n) - \delta(\xi_n,\dot{\xi}_n))\,\mathrm{d}t|}{|\int_0^1 \delta(\xi_n,\dot{\xi}_n)\,\mathrm{d}t|} \le \frac{\omega(\varepsilon_n)}{r},$$

which holds for n large enough, with ω a continuity modulus for $(x,q) \mapsto \delta(x,q)$ in $B_E(x_0,1) \times \partial B_E(0,1)$ and

$$r := \inf\{\delta(x,q) : (x,q) \in B_E(x_0,1) \times \partial B_E(0,1)\}.$$

At this point take x_n and y_n converging to x_0 with $x_n \neq y_n$ and $\xi_n \in \mathcal{A}_{x_n,y_n}$ such that

$$\int_0^1 \delta(\xi_n,\dot{\xi}_n)\,\mathrm{d}t \le \left(1 + \frac{1}{n}\right) L(x_n,y_n)$$

for any n.

Then observe that $\max_{[0,1]} |\xi_n(t) - x_0|$ converges to 0 for n going to infinity. So we can use (1.6) and Jensen's inequality to get

$$\limsup_n \frac{\delta(x_0,y_n - x_n)}{L(x_n,y_n)} \le \lim_n \frac{\int_0^1 \delta(x_0,\dot{\xi}_n)\,\mathrm{d}t}{\frac{n}{n+1}\int_0^1 \delta(\xi_n,\dot{\xi}_n)\,\mathrm{d}t} = 1.$$

Finally, for any n set $\eta_n(t) = (1-t)x_n + ty_n$ and use (1.6) again to obtain

$$\lim_n \frac{\delta(x_0,y_n - x_n)}{L(x_n,y_n)} \ge \lim_n \frac{\int_0^1 \delta(x_0,\dot{\eta}_n)\,\mathrm{d}t}{\int_0^1 \delta(\eta_n,\dot{\eta}_n)\,\mathrm{d}t} = 1.$$

□

Menger's definition of metric convexity (see [6], [4]) requires a distance D to satisfy for any couple x,y the equality

$$D(x,z) + D(z,y) = D(x,y)$$

for a suitable point z depending on (x,y).

L satisfies a strengthened form of such a condition.

Proposition 1.2. *Given two points x,y, for any $s \in]0,|x-y|[$ there is z such that*

i) $|z-x| = s$ ($|z-y| = s$),
ii) $L(x,z) + L(z,y) = L(x,y)$.

Proof. Let ξ_n be a sequence in $\mathcal{A}_{x,y}$ satisfying

$$L(x,y) \geq \int_0^1 \delta(\xi_n, \dot{\xi}_n)\,\mathrm{d}t - \frac{1}{n}\ .$$

Take $t_n \in [0,1]$ such that

$$|\xi_n(t_n) - x| = s$$

for a certain $s \in \,]0, |x-y|[$.

Any limit point z of $\xi_n(t_n)$ satisfies i) and ii). □

We assume

$$\delta(x,q) \geq \frac{a}{|x|+b}|q| \tag{1.7}$$

for any x, q and certain positive constants a, b.

Proposition 1.3. *Under the hypothesis* (1.7) L *is complete, and*

$$\lim_{|x|\to+\infty} L(y_0, x) = +\infty,$$

for any y_0.

Proof. For any fixed $x, y \in \mathbb{R}^N$ and $\xi \in \mathcal{A}_{x,y}$ one has

$$\int_0^1 \delta(\xi,\dot{\xi})\,\mathrm{d}t \geq \int_0^1 |\dot{\xi}|\frac{a}{|\xi|+b}\,\mathrm{d}t \geq a\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\ln(|\xi|+b)\,\mathrm{d}t$$

and so

$$e^{L(x,y)/a}(|x|+b) - b \geq |y|,$$

which implies that any ball of L has compact closure, and then the thesis. □

We proceed to introduce the notion of intrinsic length of a curve. See [6], [4], [14].

In the remainder of the section, D will denote a metric locally equivalent to E.

Definition 1.1. The *D–length* l_D of a continuous curve ξ (i.e., a continuous mapping of a compact interval, say $[0,1]$, in $\mathbb{R}^N$) is defined by the formula

$$l_D(\xi) = \sup \sum_i D(\xi(t_{i-1}), \xi(t_i)),$$

where the supremum is taken with respect to all finite increasing sequences $\{t_1, \ldots, t_n\}$ with $t_1 = 0$ and $t_n = 1$.

Note that in this definition the orientation of the curve is relevant, due to the lack of symmetry of D.

If ξ is Lipschitz–continuous, then (see [14])

$$l_D(\xi) = \int_0^1 g\,\mathrm{d}t, \tag{1.8}$$

where

$$g(t) = \sup\left\{\limsup_n \frac{D(\xi(t_n),\xi(s_n))}{s_n - t_n} : s_n, t_n \to t, s_n > t_n\right\}. \tag{1.9}$$

From (1.8), (1.9) and Proposition 1.1 we deduce

Proposition 1.4. *For any Lipschitz–continuous curve* ξ *defined in* $[0,1]$,

$$l_L(\xi) = \int_0^1 \delta(\xi,\dot\xi)\,\mathrm{d}t \ .$$

Proof. Let g be defined as in (1.9) with D replaced by L. Thanks to Proposition 1.1,

$$g(t) = \sup\left\{\limsup_n \delta\left(\xi(t), \frac{\xi(s_n)-\xi(t_n)}{s_n-t_n}\right) : s_n, t_n \to t, \quad s_n > t_n\right\}$$

for any t. If t is a point where ξ is differentiable, then $g(t) \geq \delta(\xi(t),\dot\xi(t))$, and so $l_L(\xi) \geq \int_0^1 \delta(\xi,\dot\xi)\,\mathrm{d}t$. The reverse inequality comes directly from the definition of length. □

Definition 1.2. For any two points x, y we define a metric via the formula

$$D_l(x,y) = \inf\{l_D\xi : \xi \text{ continuous curve joining } x \text{ and } y\},$$

and we call it *the length metric induced by* D. D is called a *path metric* provided the equality $D = D_l$ holds.

It is not difficult to see that D_l is indeed a metric. Moreover, it can be proved by exploiting the semicontinuity properties of the length that it is a path metric, is locally equivalent to E, and complete provided D is so.

Proposition 1.4, formula (1.1) and the local equivalence of L and E imply that L is a path metric.

Remark 1.1. The passage from D to D_l can be viewed as a sort of metric convexification. If indeed a distance is complete, then the properties of being a path metric, being convex in Menger's sense, and of having any two points joined by a curve whose length realizes the distance, are equivalent.

We introduce for any x the set

$$C(x) = \{q \in \mathbb{R}^N : \delta(x,q) \leq 1\}$$

and its polar

$$C^0(x) = \{p \in \mathbb{R}^N : pq \leq 1 \quad \forall\, q \in C(x)\}.$$

Thanks to (1.2), (1.3), (1.4) these sets are convex, compact and contain 0 as an interior point; moreover, the set–valued maps $x \mapsto C(x)$ and $x \mapsto C^0(x)$ are continuous with respect to the Hausdorff distance. If (1.7) is assumed, then

$$C(x) \subset B_E\left(0, \frac{|x|+b}{a}\right), \quad C^0(x) \supset B_E\left(0, \frac{a}{|x|+b}\right).$$

δ can be recovered from C and C^0 using the gauge and support functions. We have for any x, q,

$$\delta(x,q) = \gamma_{C(x)}(q) = \sigma_{C^0(x)}(q),$$

where

$$\gamma_{C(x)}(q) = \inf\left\{\lambda > 0 : \frac{q}{\lambda} \in C(x)\right\}$$

and

$$\sigma_{C^0(x)}(q) = \max\{pq : p \in C^0(x)\}.$$

We can think of $C(x)$ ($C^0(x)$) as a closed (dual) unit tangential ball of L at x; a Finsler metric can then be viewed as a generalization of a Riemannian one having general convex compact sets as tangential balls instead of ellipsoids.

Remark 1.2. The previous discussion shows that a Finsler metric can be defined starting from any continuous set–valued map Z with convex compact values and with $0 \in \operatorname{int} Z(x)$, for any x.

One simply makes use of formula (1.1), setting $\delta(x,q) = \sigma_{Z(x)}(q)$ for any (x,y).

If the condition $Z(x) \supset B_E\left(0, \frac{a}{|x|+b}\right)$ holds for any x and certain positive constants a, b, then such a metric is complete and its balls have compact closure.

2. Convex Hamilton–Jacobi equations

We consider a Finsler metric L and use the notation of the previous section. Our aim is to relate it to a suitable Hamilton–Jacobi equation

$$H(x, Du) = 0 \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

This will be possible using the Crandall–Lions theory of viscosity solutions.

We start by recalling some basic definitions. See [1], [3] for a complete treatment on viscosity solutions.

Definition 2.1. Given an l.s.c. (u.s.c.) function u, a continuous function φ is called *subtangent* (*supertangent*) to u at x if x is a local minimizer (maximizer) of $(u - \varphi)$.

Definition 2.2. We say that an l.s.c. (u.s.c.) function u is a (*viscosity*) *super* (*sub*) *solution* of (2.1) if for any x and φ C^1–subtangent (supertangent) to u at x one has

$$H(x, D\varphi(x)) \geq 0 \qquad (\leq 0)\ .$$

A continuous function that is a super- and subsolution at the same time, is called a (*viscosity*) *solution.*

For a locally Lipschitz continuous function v we define the (Clarke) generalized gradient at a point x by

$$\partial v(x) = co\{\lim_n Dv(x_n) : x_n \in \operatorname{dom}(Du)\ ,\ x = \lim_n x_n\},$$

where *co* indicates the convex hull and $\operatorname{dom}(Du)$ is the set of points where u is differentiable.

It is well known that the gradients of C^1–supertangents or subtangents to v at x belong to $\partial v(x)$.

We take any continuous Hamiltonian H, for instance $H(x,p) = d_E^{\#}(x,p)$ or $H(x,p) = \gamma_{C^0(x)}(p) - 1$, satisfying

$$H(x,p) < 0 \quad \text{if } p \in \operatorname{int} C^0(x), \tag{2.2}$$

$$H(x,p) > 0 \quad \text{if } p \in \mathbb{R}^N \smallsetminus C^0(x), \tag{2.3}$$

where $C^0(x)$ is the dual unit tangential ball of L at x (see Section 1).

We can prove the following fact.

Theorem 2.1. *For any $y_0 \in \mathbb{R}^N$ the function $u(x) = L(y_0, x)$ is a supersolution of* (2.1) *in $\mathbb{R}^N \smallsetminus \{y_0\}$ and a subsolution in $\mathbb{R}^N$.*

Note that by the local equivalence of L and E, u is locally Lipschitz continuous.

Proof. Let ψ be a C^1–supertangent to u at a point x_0 and $q \in \mathbb{R}^N$. Exploit (1.5) to get

$$\begin{aligned} D\psi(x_0)q &= \lim_{h\to 0^+} \frac{\psi(x_0)-\psi(x_0-hq)}{h} \\ &\le \liminf_{h\to 0^+} \frac{L(y_0,x_0)-L(y_0,x_0-hq)}{h} \\ &\le \lim_{h\to 0^+} \frac{L(x_0-hq,x_0)}{h} = \delta(x_0,q)\ . \end{aligned}$$

By (2.2), (2.3) and the convexity of $C^0(x_0)$, this implies $H(x_0, D\psi(x_0)) \le 0$.

Let φ be subtangent to u at x_0. Thanks to Proposition 1.2 there is a sequence x_n converging to x_0 such that $L(y_0,x_n)+L(x_n,x_0) = L(x_0,y_0)$. The sequence $\frac{x_0-x_n}{|x_0-x_n|}$ can be assumed convergent to an element q.

Calculate using (1.5)

$$\begin{aligned} D\varphi(x_0)q &= \lim_n \frac{\varphi(x_0)-\varphi(x_n)}{|x_n-x_0|} \ge \lim_n \frac{L(y_0,x_0)-L(y_0,x_n)}{|x_n-x_0|} \\ &= \lim_n \frac{L(x_n,x_0)}{|x_n-x_0|} = \delta(x_0,q)\ . \end{aligned}$$

This implies

$$H(x_0, D\varphi(x_0)) \ge 0\ .$$

□

The previous construction relating a Hamilton–Jacobi equation to a given Finsler metric can be reversed. We can start from an equation of type (2.1) with suitable properties and define a Finsler metric in such a way that Theorem 3.1 is still valid.

For this, on H and the level sets

$$Z(x) := \{p : H(x,p) \le 0\},$$

we need the following assumptions:

$$H \text{ is continuous in } (x,p), \tag{2.4}$$

and, for any x,

$$H(x,0) < 0, \tag{2.5}$$

$$\liminf_{|p|\to+\infty} H(x,p) > 0, \tag{2.6}$$

$$Z(x) \text{ is convex}, \tag{2.7}$$

$$\partial \operatorname{int}(Z(x)) = \{p : H(x,p) = 0\}. \tag{2.8}$$

Therefore $Z(x)$ is convex, compact, and contains 0 as an interior point for any x. Moreover, we have

Proposition 2.1. *The set–valued map $x \mapsto Z(x)$ is continuous with respect to the Hausdorff metric.*

Proof [See [9]]. Let x_n be a sequence converging to a point x_0, observe that by (2.4)–(2.7) any sequence p_n with $p_n \in Z(x_n)$ is bounded, and deduce using (2.4) again the relation

$$\lim_n \max_{Z(x_n)} d_E(p, Z(x_0)) = 0 \ .$$

To show that $\lim_n \max_{Z(x_0)} d_E(p, Z(x_n)) = 0$, assume for a contradiction that there exist $p \in Z(x_0)$ and $\varepsilon > 0$ such that $B_E(p, \varepsilon) \cap Z(x_n) = \phi$ up to a subsequence. This cannot be, by (2.4) and (2.8). $\square$

Consequently a Finsler metric, again denoted by L, can be obtained starting from Z as described in Remark 1.1. Theorem 2.1 still holds if in the proof we replace C^0 by Z.

If we strengthen (2.5) by assuming that there are positive constants a, b for which

$$H(x,p) < 0 \quad \text{for any } x,\ p \in B_E\left(0, \frac{a}{|x|+b}\right), \tag{2.9}$$

then L is complete and $\lim_{|x|\to+\infty} L(y_0, x) = +\infty$.

We get the following uniqueness result using straightforward techniques.

Proposition 2.2. *Under assumptions* (2.4), (2.9), (2.6), (2.7), (2.8)*, L is the unique metric with* $\lim_{|x|\to+\infty} L(y_0, x) = +\infty$ *for any y_0 for which the statement of Theorem 2.1 holds.*

Proof. Assume for purposes of contradiction that there is another distance L' satisfying the assumptions and

$$L'(y_0, x_0) > L(y_0, x_0) \tag{2.10}$$

for certain x_0, y_0.

Apply the Kruzkov transform to $L(y_0, \cdot)$ and $L'(y_0, \cdot)$, obtaining $v(\cdot) := 1 - e^{-L(y_0,\cdot)}$ and $w(\cdot) := 1 - e^{-L'(y_0,\cdot)}$.

These two functions are Lipschitz continuous and solve the problem

$$u + \gamma_{Z(x)}(Du) = 1, \tag{2.11}$$

$$u(y_0) = 0, \tag{2.12}$$

$$\lim_{|x|\to+\infty} u(x) = 1. \tag{2.13}$$

By (2.10) $\sup_{\mathbb{R}^N} w - v > 0$, and by (2.12), (2.13) this supremum is attained at a certain point $\overline{x} \neq y_0$. Exploiting the convex character of Z and the fact that w is a subsolution of (2.11), one finds that $\gamma_{Z(\overline{x})}(p) \leq 1 - w(\overline{x})$ for any $p \in \partial w(\overline{x})$.

On the other hand, w is a Lipschitz continuous subtangent to v at $\overline{x}$, and v is a supersolution of (2.11). Therefore (see [8]) there exists $p_0 \in \partial w(\overline{x})$ such that $u(\overline{x}) + \gamma_{Z(\overline{x})}(p_0) \geq 1$. This is impossible, since $w(\overline{x}) > v(\overline{x})$. $\square$

The results of this section establish in a more general setting the usual equivalence between Finsler metrics and Hamilton–Jacobi equations of eikonal type.

3. An inf–sup formula

Here we consider a Hamiltonian H and its level sets $Z(x)$ without any convexity assumptions. We require (2.4), (2.5), and (2.8). Moreover, due to the lack of connectedness of $Z(x)$ we need a strengthened form of the coercivity condition (2.6) to obtain continuity properties on Z:

$$\liminf_{|p|\to+\infty} H(x,p) > 0 \quad \text{locally uniformly on } x, \tag{3.1}$$

i.e., for any compact set K there exists $R > 0$ such that $\inf\{H(x,p) : |p| > R,\ x \in K\} > 0$.

Our scope is to define a metric that we will relate in the next section to equation (2.1), generalizing the previous results in this nonconvex setting.

It will be expressed by an inf–sup formula involving the set–valued map Z. See [2], [12], [11] for representation formulae of a similar type.

We shall denote again by L the Finsler distance given by (1.1) with $\delta(x,q) = \sigma_{Z(x)}(q)$ for any x, q.

Remark 3.1. The metric L can be obtained through the procedure described in Remark 1.2, starting from the set–valued map $x \mapsto co\,Z(x)$ which satisfies all the required assumptions. So it comes in a sense from a convexification of the Hamiltonian H.

Proposition 3.1. *The set-valued maps $x \mapsto Z(x)$, $x \mapsto \partial Z(x)$ are continuous and consequently $(x,p) \mapsto d_E^{\#}(p, Z(x))$ is so.*

Proof. The continuity of Z can be obtained as in Proposition 2.1, using the local boundedness, which comes from (3.1).

$x \mapsto \partial Z(x)$ is upper semicontinuous, for it is locally bounded, (2.8) holds and H is continuous. If it was not lower semicontinuous, there should be $x, p \in \partial Z(x)$, a sequence x_n converging to x and $\varepsilon > 0$ such that $B_E(p,\varepsilon) \cap \partial Z(x_n) = \emptyset$. But the continuity of Z and (2.8) could be used to select, for n sufficiently large, p_n', $p_n'' \in B_E(p,\varepsilon)$ with $p_n' \in Z(x_n)$, $p_n'' \notin Z(x_n)$, and so to obtain a contradiction.

Therefore $(x,p) \mapsto d_E(p, \partial Z(x))$ is continuous.

The same holds for $(x,p) \mapsto d_E^{\#}(p, Z(x))$ since the sets $\Gamma_+ = \{(x,p) : d_E^{\#}(p, Z(x)) > 0\}$ and $\Gamma_- = \{(x,p) : d_E^{\#}(p, Z(x)) < 0\}$ are open. □

We proceed to fix some notation and to recall the definition of (nonanticipating) strategy, see [1], [12].

For any $T > 0$ we shall denote by B^T the space of measurable essentially bounded functions defined in $]0,T[$ with values in $\mathbb{R}^N$.

For simplicity we shall write B instead of B^1.

Definition 3.1. Given $T > 0$ and two subsets B_1, B_2 of B^T, we define a *strategy* to be a mapping $\gamma : B_1 \to B_2$ such that if $t \in]0,T[$ and $\eta_1, \eta_2 \in B_1$ with $\eta_1(s) = \eta_2(s)$ a.e. in $]0,t[$, then $\gamma[\eta_1](s) = \gamma[\eta_2](s)$ for a.e. $s \in]0,t[$.

For any two points $x, y \in \mathbb{R}^n$ and any $T > 0$ we set

$$B_{x,y}^T = \{\zeta \in B^T : x + \int_0^T \zeta\,\mathrm{d}t = y\}$$

and denote by $\Gamma^T, \Gamma_{x,y}^T$ the sets of strategies from B^T to B^T and from B^T to $B_{x,y}^T$ respectively.

If $\eta \in B^T$, $\gamma \in \Gamma^T$, we write $\xi(\eta,\gamma,x,\cdot)$ for the integral curve of $\gamma[\eta]$ defined in $[0,T]$ that is equal to x at 0. It is apparent that if $\gamma \in \Gamma_{x,y}^T$, such a curve joins x and y.

If $T = 1$ we shall omit it in the notation of the previous sets; so for instance Γ will stand for Γ^1.

For $\eta \in B^T$, $\gamma \in \Gamma^T$ we define

$$\Im_x^T(\eta,\gamma) := \int_0^T \gamma[\eta]\eta - |\gamma[\eta]| d_E^{\#}(\eta, Z(\xi(\eta,\gamma,x,\cdot)))\,\mathrm{d}t \tag{3.2}$$

and

$$S(x,y) = \inf_{\gamma\in\Gamma_{x,y}} \sup_{\eta\in B} \Im_x(\eta,\gamma). \tag{3.3}$$

Proposition 3.2. *For any $x, y \in \mathbb{R}^N$,*

$$0 \le S(x,y) \le L(x,y). \tag{3.4}$$

For the proof we need a preliminary lemma.

Lemma 3.1. *Let K be a compact subset and p_0, q_0 elements of $\mathbb{R}^N$. Then*

$$q_0 p_0 - |q_0| d_E^{\#}(p_0, K) \le \sigma_K(q_0). \tag{3.5}$$

Proof. It can be assumed that $|q_0| = 1$ without losing generality.

If $d_E^{\#}(p_0,K) \ge 0$, denote by p_0' an element of K satisfying $|p_0' - p_0| = d_E^{\#}(p_0,K)$ and calculate

$$\begin{aligned} q_0p_0 - d_E^{\#}(p_0,K) &= q_0p_0' + q_0(p_0 - p_0') - d_E^{\#}(p_0,K) \\ &\le \sigma_K(q_0) + |p_0 - p_0'| - d_E^{\#}(p_0,K) = \sigma_K(q_0)\ . \end{aligned}$$

If $d_E^{\#}(p_0,K) < 0$, set $C = \{p : pq_0 = \sigma_K(q_0)\}$ and observe that $C \subset cl(\mathbb{R}^N \setminus K)$.

Therefore the projection point $\overline{p}_0$ of p_0 on C satisfies $|p_0 - \overline{p}_0| \ge -d_E^{\#}(p_0,K)$; on the other hand, $\overline{p}_0 = p_0 + (\sigma_k(q_0) - p_0q_0)q_0$, and so (3.5) is obtained. □

Proof of Proposition 3.2. Let ξ be any trajectory of $\mathcal{A}_{x,y}$.

The assignment $\gamma[\eta] = \dot\xi$ for any $\eta \in B$ defines a strategy of $\Gamma_{x,y}$, and using the previous lemma one has

$$\begin{aligned} S(x,y) &\le \sup_{\eta\in B}\int_0^1 \dot\xi\eta - |\dot\xi| d_E^{\#}(\eta, Z(\xi))\,\mathrm{d}t \\ &\le \int_0^1 \sigma_{Z(\xi)}(\dot\xi)\,\mathrm{d}t \end{aligned}$$

and

$$S(x,y) \le L(x,y)\ .$$

Denote by $\overline{\eta}$ the null function of B. Then use (2.5) to get

$$\begin{aligned} S(x,y) &\ge \inf_{\Gamma_{x,y}} \Im_x(\overline{\eta},\gamma) \\ &= \inf_{\Gamma_{x,y}} \int_0^1 -|\gamma[\overline{\eta}]| d_E^{\#}(0,\ Z(\xi(\overline{\eta},\gamma,x,\cdot))\,\mathrm{d}t \ge 0\ . \end{aligned}$$

□

The next result shows that the definition of S is invariant for changes of parametrization.

Proposition 3.3. *For any $T > 0$ and $x, y \in \mathbb{R}^N$,*

$$S(x,y) = \inf_{\gamma \in \Gamma^T_{x,y}} \sup_{\eta \in B^T} \Im^T_x(\eta,\gamma). \tag{3.6}$$

Proof. Define bijective mappings

$$\begin{aligned} \phi_1 &: B \to B^T, \\ \phi_2 &: B_{x,y} \to B^T_{x,y}, \\ \phi_3 &: \Gamma_{x,y} \to \Gamma^T_{x,y}, \end{aligned}$$

setting

$$\begin{array}{ll} \phi_1(\eta)(s) = \eta\left(\frac{s}{T}\right) & \text{for } \eta \in B, \text{ a.e. } s \in]0,T[, \\ \phi_2(\zeta)(s) = \frac{1}{T}\zeta\left(\frac{s}{T}\right) & \text{for } \zeta \in B_{x,y}, \text{ a.e. } s \in]0,T[, \\ \phi_3(\gamma)[\phi_1(\eta)](s) = \phi_2(\gamma[\eta])(s) & \text{for } \gamma \in \Gamma_{x,y}, \eta \in B, \text{ a.e. } s \in]0,T[\ . \end{array}$$

Given $\eta \in B$ and $\gamma \in \Gamma_{x,y}$, note the relation

$$\xi(\phi_1(\eta), \phi_3(\gamma), x, s) = \xi\left(\eta, \gamma, x, \frac{s}{T}\right)$$

for any $s \in [0,T]$, and calculate

$$\Im^T_x(\phi_1(\eta), \phi_3(\gamma)) = \int_0^T \frac{1}{T}\gamma[\eta]\left(\frac{s}{T}\right)\eta\left(\frac{s}{T}\right) - \frac{1}{T}\left|\gamma[\eta]\left(\frac{s}{T}\right)\right|$$

$$\times\, d^{\#}_E\left(\eta\left(\frac{s}{T}\right)\right), Z\left(\xi\left(\eta,\gamma,\frac{s}{T}\right)\right)\,\mathrm{d}s = \Im_x(\eta,\gamma)\ .$$

Then the assertion is obtained by taking into account that the maps ϕ_1 and ϕ_2 are bijective and (3.3). □

We establish a dynamical programming principle that will be used to prove a triangle inequality for S and to relate it in the next section to the equation (2.1).

Proposition 3.4. *For any $x, y \in \mathbb{R}^N$ and $T > 0$,*

$$S(x,y) = \inf_{\gamma \in \Gamma^T} \sup_{\eta \in B^T} \left\{ \Im^T_x(\eta,\gamma) + S(\xi(\eta,\gamma,x,T),y) \right\}. \tag{3.7}$$

Proof. Given $\varepsilon > 0$ and $z \in \mathbb{R}^N$, denote by γ_z a strategy from B^T to $B^T_{z,y}$ such that

$$S(z,y) \geq \sup_{B^T} \Im^T_z(\eta,\gamma_z) - \varepsilon; \tag{3.8}$$

then denote by γ_0 a strategy from B^T to B^T satisfying

$$\begin{aligned} &\inf_{\gamma \in \Gamma^T} \sup_{B^T} \left\{ \Im^T_x(\eta,\gamma) + S(\xi(\eta,\gamma,x,T),y) \right\} \\ &\qquad \geq \sup_{B^T} \left\{ \Im^T_x(\eta,\gamma_0) + S(\xi(\eta,\gamma_0,x,T),y) \right\} - \varepsilon\ . \end{aligned} \tag{3.9}$$

For any $\eta \in B^T$, put $z_\eta = \xi(\eta,\gamma_0,x,T)$ and define a strategy $\overline{\gamma}$ from B^{2T} to $B^{2T}_{x,y}$ by setting, for any $\vartheta \in B^{2T}$,

$$\begin{array}{ll} \overline{\gamma}[\vartheta](s) = \gamma_0[\vartheta_1](s) & \text{for a.e. } s \in]0,T[, \\ \overline{\gamma}[\vartheta](s) = \gamma_{z_{\vartheta_1}}[\vartheta_2](s-T) & \text{for a.e. } s \in]T,2T[, \end{array}$$

where ϑ_1, ϑ_2 are the restrictions to $]0,T[$ of ϑ and $\vartheta(\cdot + T)$ respectively.

Claim.

$$\sup_{B^T}\left\{\Im_x^T(\eta,\gamma_0)+\sup_{\zeta\in B^T}\Im_{z_\eta}^T(\zeta,\gamma_{z_\eta})\right\}=\sup_{B^{2T}}\Im_x^{2T}(\vartheta,\overline{\gamma})\ . \tag{3.10}$$

Take $\delta>0$ and η_0,ζ_0 in B^T such that

$$\sup_{B^T}\left\{\Im_x^T(\eta,\gamma_0)+\sup_{\zeta\in B^T}\Im_{z_\eta}^T(\zeta,\gamma_{z_\eta})\right\}\le\Im_x^T(\eta_0,\gamma_0)+\Im_{z_{\eta_0}}^T(\zeta_0,\gamma_{z_{\eta_0}})+\delta.$$

The right-hand side of the previous formula in turn equals

$$\Im_x^{2T}(\overline{\vartheta},\overline{\gamma})+\delta,$$

where $\overline{\vartheta}\in B^{2T}$ is defined by

$$\begin{array}{ll}\overline{\vartheta}(s)=\eta_0(s) & \text{for a.e. } s\in\,]0,T[,\\ \overline{\vartheta}(s)=\zeta_0(s-T) & \text{for a.e. } s\in\,]T,2T[,\end{array}$$

and so the "$\le$" part of (3.10) is proved.

A suitable choice of $\tilde{\vartheta}$ in B^{2T} can be done to obtain

$$\sup_{B^{2T}}\Im_x^{2T}(\vartheta,\overline{\gamma})\le\Im_x^{2T}(\tilde{\vartheta},\overline{\gamma})+\delta=\Im_x^T(\tilde{\vartheta}_1,\gamma_0)+\Im_{z_{\tilde{\vartheta}_1}}^T(\tilde{\vartheta}_2,\gamma_{z_{\tilde{\vartheta}_1}}),$$

where $\tilde{\vartheta},\tilde{\vartheta}_2$ are the restrictions to $]0,T[$ of $\tilde{\vartheta}$ and $\tilde{\vartheta}(\cdot+T)$ respectively.

This completes the proof of the claim.

From (3.8), (3.9), (3.10) and Proposition 3.3 we get

$$\begin{aligned}&\inf_{\Gamma^T}\sup_{B^T}\left\{\Im_x^T(\eta,\gamma)+S(\xi(\eta,\gamma,x,T),y)\right\}\\ &\qquad\ge\sup_{B^T}\left\{\Im_x^T(\eta,\gamma_0)+\sup_{\zeta\in B^T}\Im_{z_\eta}(\zeta,\gamma_{z_\eta})\right\}-2\varepsilon\\ &\qquad=\sup_{B^{2T}}\Im_x^{2T}(\vartheta,\overline{\gamma})-2\varepsilon\ge S(x,y)-2\varepsilon\ .\end{aligned}$$

On the other hand, use (3.6) to get

$$\begin{aligned}&\inf_{\Gamma^T}\sup_{B^T}\left\{\Im_x^T(\eta,\gamma)+S(\xi(\eta,\gamma,x,T),y)\right\}\\ &\qquad\le\inf_{\Gamma^T_{x,y}}\sup_{B^T}\left\{\Im_x^T(\eta,\gamma)+S(y,y)\right\}=S(x,y)\ .\end{aligned}$$

□

Proposition 3.5. *S is a distance.*

Proof. In Proposition 3.2 it has already been shown that S is nonnegative.

Given $x\in\mathbb{R}^N$, define a strategy $\overline{\gamma}\in\Gamma_{x,x}$ by setting

$$\overline{\gamma}[\eta](t)=0\quad\text{a.e. } t\in\,]0,1[$$

for any $\eta\in B$; then

$$S(x,x)\le\sup_B\Im_x(\eta,\overline{\gamma})=0\ .$$

Now take two points x_0,y_0 with $x_0\ne y_0$, and set

$$r_0=\inf\left\{-d_E^{\#}(0,Z(x)):x\in B_E(x_0,|y_0-x_0|)\right\};$$

by (2.5) and the continuity of $d_E^{\#}(x, Z(\cdot))$, r_0 is positive. Denote by $\overline{\eta}$ the null function of B, and write

$$\begin{aligned} S(x_0, y_0) &\geq \inf_{\Gamma_{x_0,y_0}} \Im_{x_0}(\overline{\eta}, \gamma) \\ &= \inf_{\Gamma_{x_0,y_0}} \int_0^1 -|\gamma(\overline{\eta})| d_E^{\#}(0, Z(\xi(\overline{\eta}, \gamma, x_0, \cdot))) \, \mathrm{d}t \geq r_0 |x_0 - y_0| > 0. \end{aligned}$$

Finally, the triangle inequality comes from (3.7). In fact, for any $z_0 \in \mathbb{R}^N$,

$$S(x_0, y_0) \leq \inf_{\Gamma_{x_0,z_0}} \sup_B \{\Im_{x_0}(\eta, \gamma) + S(z_0, y_0)\} = S(x_0, z_0) + S(z_0, y_0) \ .$$

□

Proposition 3.6. *S is locally equivalent to E. Moreover, if* (2.9) *holds, then it is complete and* $\lim_{|x| \to +\infty} S(y_0, x) = +\infty$, *for any* y_0.

Proof. Let K be a compact set and $z_0 \in K$. Set $\tilde{K} = cl B_E \, (z_0, 2\text{diam}_E K)$ ($\text{diam}_E K$ is the Euclidean diameter of K), and observe that $r \leq |p| \leq R$ for $p \in \partial Z(x)$, $x \in \tilde{K}$ and suitable positive constants r, R because Z and $x \mapsto -d_E^{\#}(0, Z(x))$ are continuous and by the assumption (2.5).

Take $x_0 \neq y_0 \in K$ and define $\overline{\gamma} \in \Gamma_{x_0,y_0}$ by

$$\overline{\gamma}[\eta](t) = y_0 - x_0 \quad \text{a.e. } t \in \,]0, 1[$$

for any $\eta \in B$. Consequently,

$$\xi(\eta, \overline{\gamma}, x_0, t) = x_0 + t(y_0 - x_0) =: \overline{\xi}(t), \quad t \in [0, 1],$$

for any η. Then use (3.5) to get

$$\begin{aligned} S(x_0, y_0) &\leq \sup_B \Im_{x_0}(\eta, \overline{\gamma}) \\ &= \sup_B \int_0^1 (y_0 - x_0)\eta - |y_0 - x_0| d_E^{\#}(\eta, Z(\overline{\xi})) \, \mathrm{d}t \\ &\leq \int_0^1 \sigma_{Z(\overline{\xi})}(y_0 - x_0) \, \mathrm{d}t \leq R|x_0 - y_0| \ . \end{aligned}$$

Observe that $B_E(x_0, |y_0 - x_0|) \subset \tilde{K}$, and so

$$r_0 := \inf\{-d_E^{\#}(0, Z(x)) : x \in B_E(x_0, |y_0, x_0|)\} \geq r;$$

then argue as in Proposition 3.5 to get

$$S(x_0, y_0) \geq r|x - y| \ .$$

If (2.9) holds, then, arguing as in Proposition 1.3, one sees that for any $x, y, \gamma \in \Gamma_{x,y}$,

$$\Im_{x_0}(\overline{\eta}, \gamma) \geq a \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \ln(|\xi(\overline{\eta}, \gamma, x, \cdot)| + b) \, \mathrm{d}t$$

where $\overline{\eta}$ is the null function of B. This implies that any ball of S has compact closure and the thesis. □

From the previous result we immediately derive

Corollary 3.1. *For any fixed $y_0 \in \mathbb{R}^N$ the function $S(y_0, x)$ is locally Lipschitz continuous.*

4. Nonconvex Hamilton–Jacobi equations

Here we relate S to the equation (2.1) when H satisfies (2.4), (2.5), (3.1), (2.8) and show that Theorem 2.1 is still valid when S replaces L.

We first prove the following:

Lemma 4.1. *Let $\overline{S}$ be the distance obtained via formulae* (3.2) *and* (3.3) *when we replace Z by $-Z$. Then*

$$S(x,y) = \overline{S}(y,x)$$

for any $x, y \in \mathbb{R}^N$.

Proof. Define bijective mappings

$$\begin{aligned}&\phi_1 : B \to B,\\ &\phi_2 : B_{x,y} \to B_{y,x},\\ &\phi_3 : \Gamma_{x,y} \to \Gamma_{y,x},\end{aligned}$$

setting

$$\begin{array}{ll}\phi_1(\eta)(s) = -\eta(1-s) & \text{for } \eta \in B, \text{ a.e. } s \in]0,1[,\\ \phi_2(\zeta)(s) = -\zeta(1-s) & \text{for } \zeta \in B_{x,y}, \text{ a.e. } s \in]0,1[,\\ \phi_3(\gamma)[\phi_1(\eta)](s) = \phi_2(\gamma[\eta])(s) & \text{for } \gamma \in \Gamma_{x,y}, \eta \in B, \text{ a.e. } s \in]0,1[.\end{array}$$

Moreover, for any $\eta \in B$, $\gamma \in \Gamma$, define the quantity $\overline{\Im}_y(\eta,\gamma)$ as in (3.2) with $-Z$ in place of Z.

One has, for any $\eta \in B$, $\gamma \in \Gamma_{x,y}$,

$$\begin{aligned}&\overline{\Im}_y(\phi_1(\eta), \phi_3(\gamma))\\ &\quad = \int_0^1 -\eta(1-s)(-\gamma[\eta](1-s) - |\gamma[\eta](1-s)|\\ &\qquad\quad \times d_E^{\#}(-\eta(1-s), -Z(\xi(\eta,\gamma,x,1-s))\,\mathrm{d}s\\ &\quad = \Im(\eta,\gamma).\end{aligned}$$

Now we recall the definitions of S and $\overline{S}$, and use the bijectivity of ϕ_1 and ϕ_3. This completes the proof. □

Theorem 4.1. *For any fixed $y_0 \in \mathbb{R}^N$ the function $x \mapsto S(y_0, x)$ is a subsolution of* (2.1) *in $\mathbb{R}^N$.*

Proof. In view of Lemma 4.1 we will prove the equivalent assertion that $v(x) := S(x, y_0)$ satisfies $H(x, -Dv) \leq 0$ in $\mathbb{R}^N$ in the viscosity sense.

If this is not the case, there exists a C^1–function ψ supertangent to v at a certain point x_0 with $\psi(x_0) = v(x_0)$ such that $H(x_0, -D\psi(x_0)) > 0$. This implies $x_0 \neq y_0$, since otherwise x_0 would be the minimum point of v and so $-D\psi(x_0) = 0$ should belong to $Z(x_0)$.

Therefore, by the continuity of $d_E^{\#}(\cdot, Z(\cdot))$ and $D\psi$, there are positive constants T_0 and ϑ such that

$$d_E^{\#}(D\psi(x), -Z(x)) > \vartheta \tag{4.1}$$

for any $x \in B_E(x_0, T_0)$.

Set

$$R=\sup\{|p| : p\in -Z(x),\ x\in B_E(x_0,T_0)\},$$
$$M=\sup\{|\nabla\psi(x)| : x\in B_E(x_0,T_0)\},$$
$$\varepsilon=\min\left(\frac{\vartheta}{2}\frac{1}{M+\vartheta+R},\frac{1}{2}\right),$$

and select $T\in\,]0,T_0[$ satisfying

$$v(x)\le\psi(x), \tag{4.2}$$

$$|D\psi(x)-D\psi(y)|<\frac{\vartheta}{8}\varepsilon, \tag{4.3}$$

for any $x,y\in B_E(x_0,T)$.

Fix a unit vector q_0, and define a map $f:\mathbb{R}^N\to\mathbb{R}^N$ by setting

$$f(p)=\frac{p-D\psi(x_0)}{|p-D\psi(x_0)|}\quad \text{if } p\ne D\psi(x_0),$$

$$f(D\psi(x_0))=q_0.$$

Claim.

$$d_E^{\#}(p,-Z(x))+f(p)(p-D\psi(x))\ge\frac{\vartheta}{2} \tag{4.4}$$

for any $x\in B_E(x_0,T)$ *and* $p\in\mathbb{R}^N$.

First assume $|p-D\psi(x)|\le\frac{\vartheta}{4}$, then observe that by (4.1), $d_E^{\#}(p,-Z(x))>0$, denote by p' an element of $-Z(x)$ satisfying $|p-p'|=d_E^{\#}(p,-Z(x))$, and exploit (4.1) again, to get

$$d_E^{\#}(p,-Z(x))=|p-p'|\ge|p'-D\psi(x)|-|p-D\psi(x)|\ge\frac{3}{4}\vartheta$$

and

$$d_E^{\#}(p,-Z(x))+f(p)(p-D\psi(x))\ge d_E^{\#}(p,-Z(x))-|p-D\psi(x)|\ge\frac{\vartheta}{2}\ .$$

If instead $|p-D\psi(x)|>\frac{\vartheta}{4}$, then by (4.3),

$$|p-D\psi(x_0)|\ge|p-D\psi(x)|-|D\psi(x)-D\psi(x_0)|>\frac{\vartheta}{8},$$

and so

$$\begin{aligned}\left|\frac{p-D\psi(x)}{|p-D\psi(x)|}-\frac{p-D\psi(x_0)}{|p-D\psi(x_0)|}\right| &= \frac{8}{\vartheta}\left|\frac{\vartheta}{8}\frac{p-D\psi(x)}{|p-D\psi(x)|}-\frac{\vartheta}{8}\frac{p-D\psi(x_0)}{|p-D\psi(x_0)|}\right|\\ &\le \frac{8}{\vartheta}|D\psi(x)-D\psi(x_0)|\le\varepsilon\ .\end{aligned}$$

Since $\varepsilon<1$, this implies that $f(p)(p-D\psi(x))>0$.

Now divide the proof into two cases according to whether $|p|>R+\vartheta$ or $|p|\le R+\vartheta$.

If $|p|>R+\vartheta$, then $d_E^{\#}(p,-Z(x))>\vartheta$ and so

$$d_E^{\#}(p,-Z(x))+f(p)(p-D\psi(x))>d_E^{\#}(p,-Z(x))>\vartheta\ .$$

If $|p| \leq R + \vartheta$, calculate

$$\begin{aligned} \vartheta &\leq d_E^{\#}(D\psi(x), -Z(x)) \leq d_E^{\#}(p, -Z(x)) + |p - D\psi(x)| \\ &= d_E^{\#}(p, -Z(x)) + \left(\frac{p - D\psi(x)}{|p - D\psi(x)|} - \frac{p - D\psi(x_0)}{|p - D\psi(x_0)|} \right)(p - D\psi(x)) \\ &+ f(p)(p - D\psi(x)) \leq d_E^{\#}(p, -Z(x)) + \varepsilon(R + \vartheta + M) \\ &+ f(p)(p - D\psi(x)) \leq d_E^{\#}(p, -Z(x)) + \frac{\vartheta}{2} + f(p)(p - D\psi(x)), \end{aligned}$$

which ends the proof of the claim.

Define a strategy $\gamma_f \in \Gamma^T$ by

$$\gamma_f[\eta](t) = f(-\eta(t)) \quad \text{a.e. } t \in]0, T[$$

for any $\eta \in B^T$.

Write for simplicity $\xi(\eta, \cdot)$ instead of $\xi(\eta, \gamma_f, x_0, \cdot)$ for $\eta \in B^T$, and observe that since $|\gamma_f[\eta](t)| = 1$ a.e. t, it follows that

$$\xi(\eta, t) \in B_E(x_0, T) \tag{4.5}$$

for any $t \in [0, T]$, $\eta \in B^T$.

Then exploit (3.7), (4.2), (4.5) to get

$$\begin{aligned} \psi(x_0) = v(x_0) &\leq \sup_{B^T} \left\{ \Im_{x_0}^T(\eta, \gamma_f) + v(\xi(\eta, T)) \right\} \\ &\leq \sup_{B^T} \left\{ \Im_{x_0}^T(\eta, \gamma_f) + \psi(\xi(\eta, T)) \right\}, \end{aligned}$$

and consequently,

$$\begin{aligned} &\sup_{B^T} \left\{ \int_0^T f(-\eta)(D\psi(\xi(\eta, \cdot) + \eta) - d_E^{\#}(\eta, Z(\xi(\eta, \cdot)))\text{ţ} \right\} \\ &\quad = \sup_{B^T} \left\{ \int_0^T f(-\eta)(D\psi(\xi(\eta, \cdot) - (-\eta)) - d_E^{\#}(-\eta, -Z(\xi(\eta, \cdot))\text{ţ} \right\} \geq 0, \end{aligned}$$

which contradicts (4.4) if we take (4.5) into account. □

Theorem 4.2. *For any given $y_0 \in \mathbb{R}^N$ the function $x \mapsto S(y_0, x)$ is a supersolution of* (2.1) *in $\mathbb{R}^N \setminus \{y_0\}$.*

Proof. In view of Lemma 4.1 we just need to show that $v(x) = S(x, y_0)$ satisfies $H(x, -Dv) \geq 0$ in $\mathbb{R}^n \setminus \{y_0\}$ in the viscosity sense.

The argument is by contradiction. Let φ be a strict subtangent to v at a point $x_0 \neq y_0$ with $\varphi(x_0) = v(x_0)$ satisfying

$$H(x_0, -D\varphi(x_0)) < 0\ . \tag{4.6}$$

Then the function

$$g(x) := |D\varphi(x) - D\varphi(x_0)| + d_E^{\#}(D\varphi(x_0), -Z(x)) < 0 \tag{4.7}$$

for $x \in B_E(x_0, T)$, where T is a suitable positive constant, which can be taken so that $y_0 \notin cl B_E(x_0, T)$ and

$$0 < \delta < \min_{\partial B_E(x_0, T)} (v - \varphi)$$

for some $\delta > 0$.

Choose a strategy $\overline{\gamma} \in \Gamma_{x_0,y_0}$ such that

$$v(x_0) \geq \sup_{\eta \in B} \Im_{x_0}(\eta, \overline{\gamma}) - \delta/2 \ .$$

Put

$$\overline{\eta}(t) = -D\varphi(x_0) \quad \text{a.e. } t \in]0,1[$$

and denote by $\overline{\xi}(\cdot), \overline{t}$ and $\overline{x}$ the trajectory $\xi(\overline{\eta}, \overline{\gamma}, x_0, \cdot)$ and the exit time of $\xi(\overline{\eta}, \overline{\gamma}, x_0, \cdot)$ from $B_E(x_0, T)$ and $\xi(\overline{\eta}, \overline{\gamma}, x_0, \overline{t})$ respectively.

Then define a map $\phi : B^{1-\overline{t}} \to B$ by

$$\begin{array}{ll} \phi(\zeta)(t) = \overline{\eta}(t) & \text{a.e. } t \in]0, \overline{t}[, \\ \phi(\zeta)(t) = \zeta(t + \overline{t}) & \text{a.e. } t \in]\overline{t}, 1[, \end{array}$$

for any $\zeta \in B^{1-\overline{t}}$.

The strategy $\overline{\gamma}$, thanks to its nonanticipating character, gives a $\overline{\sigma} \in \Gamma^{1-\overline{t}}_{\overline{x},y_0}$ by

$$\overline{\sigma}[\zeta](t) = \overline{\gamma}[\phi(\zeta)](t + \overline{t}) \quad \text{a.e. } t \in]0, 1 - \overline{t}[$$

for any $\zeta \in B^{1-\overline{t}}$. Therefore

$$\begin{aligned} v(x_0) \quad &\geq \quad \int_0^{\overline{t}} \overline{\eta\gamma}[\overline{\eta}] - |\overline{\gamma}[\overline{\eta}]| d_E^{\#}(\overline{\eta}, Z(\overline{\xi}))\, \mathrm{d}t + \sup_{\zeta \in B^{1-\overline{t}}} \Im^{1-\overline{t}}_{\overline{x}}(\zeta, \overline{\sigma}) - \delta/2 \\ &\geq \quad \int_0^{\overline{t}} \overline{\eta\gamma}[\overline{\eta}] - |\overline{\gamma}[\overline{\eta}]| d_E^{\#}(\overline{\eta}, Z(\overline{\xi}))\, \mathrm{d}t + \varphi(\overline{x}) + \delta/2, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_0^{\overline{t}} \overline{\gamma}[\overline{\eta}](D\varphi(x_0) - D\varphi(\overline{\xi})) + |\overline{\gamma}[\overline{\eta}]| d_E^{\#}(D\varphi(x_0), -Z(\overline{\xi}))\, \mathrm{d}t \\ &\qquad = \int_0^{\overline{t}} \overline{\gamma}[\overline{\eta}](-\overline{\eta} - D\varphi(\overline{\xi})) + |\gamma[\overline{\eta}]| d_E^{\#}(\overline{\eta}, -Z(\overline{\xi}))\, \mathrm{d}t \geq \delta/2 \end{aligned}$$

and

$$\int_0^{\overline{t}} |\overline{\gamma}[\overline{\eta}]| g(\overline{\xi})\, \mathrm{d}t \geq \delta/2 \ .$$

This contradicts (4.7), since $\overline{\xi}(t) \in B_E(x_0, T)$ for $t \in [0, \overline{t}[$. □

Remark 4.1. If we assume (2.9) and the convexity condition (2.7), then the previous results imply in the light of Propositions 2.2 and 3.6 that $S = L$.

The next example shows that if Z has nonconvex values, then L and S do not in general coincide. To prove this we will use Theorem 4.1.

Example 4.1. We start by defining a set–valued map Z_0; afterwards we will introduce a continuous Hamiltonian having $Z_0(x)$ as level set for any x.

We give for any $x \in \mathbb{R}^N$ an open cone K_x by

$$K_x = \left\{ \lambda p : \lambda \geq 0,\ |p| = 1 \quad \text{with } px > |x| - \frac{|x|^2}{2} \right\}.$$

It is clearly empty at 0, and equals $\mathbb{R}^N$ if $|x|$ is sufficiently large.

Then we set

$$Z_0(x) = (cl B_E(0,1) \setminus K_x) \cup cl B_E(0, g(x)),$$

where $g(x) = \max(1 - 2/3|x|, 1/3)$.

The set-valued maps Z_0 and ∂Z_0 are continuous, and $Z_0(x)$ contains 0 as an interior point for any x; moreover, it has compact values, is star-shaped with respect to 0, and is constant outside a certain compact.

Proposition 4.1. *Let x_0 be in $B_E(0,1)$, and let q_0 be a unit vector such that $x_0 q_0 \geq 0$. Then*

$$\sigma_{Z_0(x)}(q_0) \geq \left(1 - \frac{1}{2} q_0 x_0\right). \tag{4.8}$$

Proof. Assume $x_0 \neq 0$, and recall that $\sigma_{Z_0(x)}(q_0) = \sigma_{co\, Z_0(x)}(q)$.

Observe that

$$co\, Z_0(x_0) = B_E(0,1) \cap \left\{ p : p x_0 \leq |x_0| - \frac{|x_0|^2}{2} \right\}; \tag{4.9}$$

then $\lambda_0 := \frac{1}{|x_0|} - \frac{1}{2}$ satisfies $\lambda_0 x_0 \in \partial\, co\, Z_0(x_0)$.

Put $a = \lambda_0 |x_0| = 1 - \frac{1}{2}|x_0|$.

Express q_0 in the form $\frac{\lambda_0 x_0 + tq}{\sqrt{\lambda_0^2 |x_0|^2 + t^2}}$, where q is a suitable unit vector orthogonal to x_0, and $t \geq 0$. If $t \geq (1-a^2)^{1/2}$, then by (4.9) $\sigma_{Z(x_0)}(q_0) = 1$, while if $t = 0$, then $\sigma_{Z(x_0)}(q_0) = a$, and so (4.8) is verified.

If $t \in\,]0, (1-a^2)^{1/2}[$, then $\sigma_{Z_0(x_0)}(q_0) = q_0(\lambda_0 x_0 + (1-a^2)^{1/2} q)$. Then set

$$f(t) = \sigma_{Z_0(x_0)}(q_0) + \frac{1}{2} q_0 x_0 = \frac{a^2 + (1-a^2)^{1/2} t + 1/2\, a |x_0|}{(a^2 + t^2)^{1/2}}$$

and observe that f has no local minima in $]0, (1-a^2)^{1/2}[$. Since $f(0) = 1$ and $f(\sqrt{1-a^2}) \geq 1$, this implies $f(t) > 1$ in $]0, (1-a^2)^{1/2}[$, and so the assertion. □

Define a metric L_0 starting from $x \mapsto \mathrm{co}\, Z_0(x)$ as indicated in Remark 1.2.

Proposition 4.2. *For any $x \in B_E(0,1)$,*

$$L_0(0,x) = |x| - \frac{1}{4}|x|^2\ . \tag{4.10}$$

Proof. Take $x \in B_E(0,1)$, $x \neq 0$, and $\xi \in \mathcal{A}_{0,x}$. Let t_0 be the exit time of ξ from $B_E(0,|x|)$, and set $J = \left\{ t \in [0,t_0] : t \in \mathrm{dom}(\dot\xi),\ \xi(t)\dot\xi(t) \geq 0 \right\}$.

Claim.

$$\mathcal{H}^1(\xi(J)) \geq |x|, \tag{4.11}$$

where $\mathcal{H}^1$ is the one–dimensional Hausdorff measure relative to E.

To show this, define $f : \mathbb{R}^N \to \mathbb{R}^N$ by setting $f(y) = \frac{|y|}{|x|} x$ for any y, and observe that it maps $\xi(J)$ onto $\{\lambda \frac{x}{|x|} : \lambda \in [0,|x|]\}$. Therefore (4.11) is obtained, since f is Lipschitz continuous with Lipschitz constant 1.

Calculate, using (4.8), (4.11) and the fact that $|x| < 1$,

$$\begin{aligned}\int_0^1 \sigma_{Z_0(\xi)}(\dot\xi)\,\mathrm{d}t &\geq \int_0^{t_0} \sigma_{Z_0(\xi)}(\dot\xi)\,\mathrm{d}t = \int_{[0,t_0]\smallsetminus J} \sigma_{Z_0(\xi)}(\dot\xi)\,\mathrm{d}t \\ &+ \int_J \sigma_{Z_0(\xi)}(\dot\xi)\,\mathrm{d}t \geq \int_{[0,t_0]\smallsetminus J} |\dot\xi|\mathrm{t} + \int_J (|\dot\xi| - \frac{1}{2}\dot\xi\xi)\,\mathrm{d}t \\ &\geq \mathcal{H}^1(\xi(J)) - \frac{1}{2}\int_0^{t_0} \dot\xi\xi\,\mathrm{d}t \geq |x| - \frac{1}{4}|x|^2 .\end{aligned}$$

Finally, setting $\xi(t) = tx$, we get

$$\int_0^1 \sigma_{Z_0(\eta)}(\dot\eta)\,\mathrm{d}t = |x| - \frac{1}{4}|x|^2 .$$

□

At this point define the metric S_0 using (3.2) with Z replaced by Z_0 and (3.3), and take a continuous Hamiltonian H_0, for instance $H_0(x,p) = d_E^{\#}(p, Z_0(x))$ (see Proposition 3.1), representing Z_0 in the sense of formulae (2.2) and (2.3) with C^0 replaced by Z_0. In light of Theorem 4.1, $S_0(0,\cdot)$ is a subsolution of equation (2.1) with H_0 in place of H. This property does not hold for L_0—in fact from (4.10) we get

$$DL_0(0,x) = \frac{x}{|x|} - \frac{1}{2}x \notin Z(x)$$

for any $x \in B_E(0,1) \smallsetminus 0$. This shows that $S_0 \neq L_0$.

5. Convexifications

Here we compare two sorts of convexifications that can be done in the problem we are dealing with and that lead to two, a priori different, path metrics.

We can convexify the level sets $Z(x)$ of the Hamiltonian and then define the Finsler metric L as indicated in Remark 3.1. On the other hand, we can directly "convexify" S by passing to the induced length metric S_l, see Remark 1.1.

We require that

(5.1) $\qquad Z(x)$ is star–shaped with respect to 0

for any x.

Moreover we assume (2.4), (2.5), (2.9) and (2.6), which is enough to guarantee the continuity properties of Proposition 3.1, since $Z(x)$ is connected for any x (see the proof of Proposition 2.1).

Under these conditions we are able to establish

Theorem 5.1. *$S_l = L$.*

This result in particular implies in view of Example 4.1 that S is not in general a path metric. This is indeed the metric counterpart of the lack of convexity in H and Z.

The crucial point for proving Theorem 5.1 is to examine the local behaviour of S, as we did for a general Finsler metric in Proposition 1.1. It is worth noting that in the Finsler case this result was used to show the relation of such a metric to the Hamilton–Jacobi equation; here the procedure is inverted, and the local properties of S can be studied by exploiting its link with the equation (2.1).

We start by recording for later use the following fact.

Lemma 5.1. *Assume $K \subset \mathbb{R}^N$ is star–shaped with respect to 0. If $p_0 \in \operatorname{int} K$, then*

$$\lambda p_0 \in \operatorname{int} K \quad \text{for any } \lambda \in \,]0,1]\ .$$

Proof. Suppose $B_E(p_0,\varepsilon) \subset K$ for a certain $\varepsilon > 0$ and take $\lambda \in \,]0,1[$; then $B_E(\lambda p_0, \lambda\varepsilon) \subset K$, since $\frac{1}{\lambda} p \in B_E(p_0,\varepsilon)$ for $p \in B_E(\lambda p_0, \lambda\varepsilon)$. □

In the following lemma we essentially exploit hypothesis (5.1).

Lemma 5.2. *For any $x_0 \in \mathbb{R}^N$ and $p_0 \in \operatorname{int} Z(x_0)$, there exists a set–valued map Z_0 with compact convex values such that*

i. *$x \mapsto Z_0(x)$ is continuous,*
ii. *$p_0 \in Z_0(x_0)$,*
iii. *$B_E\left(0, \frac{a'}{|x|+b'}\right) \subset Z_0(x)$ for any x and suitable positive constants a', b',*
iv. *$Z_0(x) \subset Z(x)$ for any x.*

Proof. Using the previous lemma and the fact that p_0 is an interior point of $Z(x_0)$, it is clear that there is an $\varepsilon > 0$ such that

$$cl B_E(\lambda p_0, \varepsilon) \subset \operatorname{int} Z(x_0)$$

for any $\lambda \in [0,1]$.

Set

$$C_0 = \bigcup_{\lambda \in [0,1]} cl B_E(\lambda p_0, \varepsilon)$$

and observe that this set is convex, compact and contained in $\operatorname{int} Z(x_0)$. Therefore by the continuity of Z there exists $\delta > 0$ such that

$$C_0 \subset \operatorname{int} Z(x) \quad \text{for any } x \in B_E(x_0, \delta)\ .$$

Define Z_0 by setting

$$Z_0(x) = \lambda(x) C_0 + (1 - \lambda(x)) cl B_E(0, \varepsilon)$$

for $x \in B_E(x_0, \delta/2)$,

$$Z_0(x) = (1 - \lambda(x)) cl B_E(0,\varepsilon) + \lambda(x) cl B_E(0, -d_E^{\#}(0, Z(x))$$

for $x \in B_E(x_0,\delta) \smallsetminus B_E(x_0, \delta/2)$, and

$$Z_0(x) = cl B_E(0, -d_E^{\#}(0, Z(x))$$

for $x \in \mathbb{R}^N \smallsetminus B_E(x_0, \delta)$.

Here $\lambda : \mathbb{R}^N \to [0,1]$ is a continuous function satisfying

$$\begin{aligned} \lambda(x) &= 1 \quad \text{for } x \in \partial B_E(x_0,\delta) \cup \{x_0\}, \\ \lambda(x) &= 0 \quad \text{for } x \in \partial B_E(x_0, \delta/2)\ . \end{aligned}$$

It can easily be seen that Z_0 satisfies the statement. □

We proceed to prove the announced theorem on the local behaviour of S. From now on we set $\delta(x,p) = \sigma_{Z(x)}(p)$ for any x and p.

Theorem 5.2. *For any $x_0 \in \mathbb{R}^N$,*

$$\lim_{\substack{x,y \to x_0 \\ x \neq y}} \frac{S(x,y)}{\delta(x_0, y-x)} = 1. \tag{5.2}$$

Proof. Fix $\varepsilon > 0$ and denote by r a positive constant verifying

$$S(x,y) \geq r|x-y| \tag{5.3}$$

for $x, y \in B_E(x_0, 1)$. Such an r does exist, by the local equivalence of S and E shown in Proposition 3.6.

The compactness of $\partial Z(x_0)$ and the condition (2.8) enable us to select a finite set of elements $\{p_1, \ldots, p_n\}$ in $\operatorname{int} Z(x_0)$ such that for any $p \in \partial Z(x_0)$ there is an $i \in \{1, \ldots, n\}$ for which the inequality $|p - p_i| < \varepsilon$ holds.

For any i, denote by Z_i a set-valued map verifying the statement of Lemma 5.2 with p_0 replaced by p_i.

By the continuity and convexity properties of Z_i there is a continuous Hamiltonian H_i representing it in the sense that (2.2) and (2.3) hold with H and C^0 replaced by H_i and Z_i. Now recall that the Z_i have convex values, and deduce that the Finsler metric

$$L_i(x,y) = \inf \left\{ \int_0^1 \delta_i(\xi, \dot{\xi}) \, \mathrm{d}t : \xi \in \mathcal{A}_{x,y} \right\}$$

(with $\delta_i(x,p) = \sigma_{Z_i(x)}(p)$) and S_i obtained via formulae (3.2) and (3.3), with Z_i in place of Z, are related to the equation (2.1) as specified in Theorems 2.1, 4.1, and 4.2 and so must coincide in the light of Remark 4.1. From this and the relation $Z_i(x) \subset Z(x)$ for any x, it follows that

$$L_i(x,y) \leq S(x,y) \quad \text{for any } x, y, i. \tag{5.4}$$

Since by Proposition 1.1,

$$\lim_{\substack{x,y \to x_0 \\ x \neq y}} \frac{L_i(x,y)}{\delta_i(x_0, y-x)} = 1,$$

there exists $\delta_\varepsilon \in \,]0,1[$ such that

$$\delta_i(x_0, y-x) \leq (1+\varepsilon) L_i(x,y) \tag{5.5}$$

for any i and any $x, y \in B_E(x_0, \delta_\varepsilon)$.

Take $x, y \in B_E(x_0, \delta_\varepsilon)$ and use the density property of $\{p_1, \ldots, p_n\}$ to get

$$\delta(x_0, y-x) \leq p_i(y-x) + \varepsilon|y-x| \tag{5.6}$$

for a suitable i.

Then from (5.4), (5.5), (5.6) and the fact that $p_i \in Z_i(x_0)$ we get

$$\begin{aligned} \delta(x_0, y-x) &\leq \delta_i(x_0, y-x) + \varepsilon|y-x| \\ &\leq (1+\varepsilon) L_i(x,y) + \varepsilon|y-x| \\ &\leq (1+\varepsilon) S(x,y) + \varepsilon|y-x| \; . \end{aligned}$$

From this and (5.3) we see that

$$\frac{\delta(x_0, y-x)}{S(x,y)} \leq 1 + \varepsilon + \frac{\varepsilon}{r},$$

which implies

$$\limsup_{\substack{x,y \to x_0 \\ x \neq y}} \frac{\delta(x_0, y-x)}{S(x,y)} \leq 1,$$

since ε is arbitrary.

On the other hand, by (3.4) and Proposition 1.1,

$$\liminf_{\substack{x,y\to x_0\\ x\neq y}} \frac{\delta(x_0, y-x)}{S(x,y)} \geq \lim_{\substack{x,y\to x_0\\ x\neq y}} \frac{\delta(x_0, y-x)}{L(x,y)} = 1 \ .$$

□

From the previous theorem we conclude that the S-length and the L-length of a Lipschitz continuous curve coincide.

The proof goes as in Proposition 1.4.

Theorem 5.3. *For any Lipschitz continuous curve ξ defined in $[0,1]$,*

$$l_S(\xi) = l_L(\xi) = \int_0^1 \delta(\xi, \dot{\xi})\, \mathrm{d}t.$$

From this and the local equivalence of S with E we finally get Theorem 5.1.

References

[1] M. Bardi and I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of Hamilton–Jacobi–Bellman equations*, Birkhäuser, Boston, 1997. MR **99e:**49001

[2] M. Bardi and L. C. Evans, "On Hopf's formulas for solutions of Hamilton–Jacobi equations", *Nonlinear Anal. TMA* **8** (1984), 1373–1381. MR **85k:**35043

[3] G. Barles, *Solutions de viscosité des équations de Hamilton–Jacobi*, Springer, Paris, 1994. MR **2000b:**49054

[4] L. M. Blumenthal, *Theory and applications of distance geometry*, Oxford Univ. Press, London, 1953, reprinted Chelsea Publ., New York, 1979. MR **14:**1009a; MR **42:**3678

[5] H. Busemann, *Metric methods in Finsler spaces and in the foundations of geometry*, Ann. Math. Study, Princeton, 1942. MR **4:**109e

[6] H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955. MR **17:**779a

[7] H. Busemann and W. Mayer, "On the foundations of calculus of variations", *Trans. Amer. Math. Soc.* **49** (1941), 173–198. MR **2:**225d

[8] F. Camilli and A. Siconolfi, "Maximal subsolutions for a class of degenerate Hamilton–Jacobi equations", *Indiana Univ. Math. J.* **48** (1999), 1111–1131. MR **2001a:**49028

[9] F. Camilli and A. Siconolfi, "Nonconvex degenerate Hamilton-Jacobi equations", *Math. Z.* **242** (2002), 1–21.

[10] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volume II*, John Wiley , New York, 1962, reprinted 1989. MR **25:**4216; MR **90k:**35001

[11] L. C. Evans and H. Ishii, "Differential games and nonlinear first order PDE in bounded domains", *Manuscripta Math.* **49** (1984), 109–139. MR **86f:**35048

[12] L. C. Evans and P. Souganidis, "Differential games and representation formulas for solutions of Hamilton–Jacobi–Isaacs equations", *Indiana Univ. Math. J.* **33** (1984), 773–797. MR **86d:**90185

[13] S. N. Kruzkov, "Generalized solutions of the Hamilton–Jacobi equations of eikonal type", *Math. USSR Sbornik* **27** (1975), 406–446. MR **53:**8670

[14] M. Gromov, *Metric structures for Riemannian and non–Riemannian spaces*, Birkhäuser, Boston, 1998. MR **2000d:**53065

[15] P. L. Lions, *Generalized solutions of Hamilton–Jacobi equations*, Pitman, London, 1982. MR **84a:**49038

Dipartimento di Matematica, Università di Roma "La Sapienza", Piazzale Aldo Moro, 2, 00185 Roma, Italy

E-mail address: `siconolfi@mat.uniroma1.it`