

ON A MEASURE IN WIENER SPACE AND APPLICATIONS

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ABSTRACT. In this article, we consider a measure in Wiener space, induced by the sum of measures associated with an uncountable set of positive real numbers, and investigate the basic properties of this measure. We apply this measure to the various theories related to Wiener space. In particular, we can obtain a partial answer to Johnson and Skoug's open problems, raised in their 1979 paper. Moreover, we can improve and clarify some theories related to Wiener space.

1. INTRODUCTION

In 1923, Wiener showed that one can define a reasonable measure on the space $C_0[a, b]$ of real-valued continuous functions on a closed interval $[a, b]$ that vanish at a , the so-called Wiener space [14]. Since then, the theory of this measure has been investigated extensively and applied to various subjects by many mathematicians and mathematical physicists. So, the various theories related to Wiener space have been widely and deeply developed, and many papers and books related to this space have been published. But in the process of studying Wiener space, one frequently encounters the translation problem, which is one of the difficult problems in the theory of Wiener space. In 1979, Johnson and Skoug raised some open problems concerning translation in Wiener space [8, pp. 164-165].

In this article, we will introduce a measure $\bar{\tau}$ in Wiener space, induced by the sum of measures associated with an uncountable set of positive real numbers, and investigate the basic properties of this measure. Using the concept of $\bar{\tau}$, we will obtain a partial answer to Johnson and Skoug's open problems, raised in their 1979 paper. Furthermore, we can improve and clarify some theories related to Wiener space, for example, the relation between the Lebesgue integral and the Wiener integral, the converse measurability theorem and the theory of the Fourier-Feynman transform.

This article consists of five sections. In the first section, we will introduce some notation, definitions and basic facts which are needed to understand the contents of the next sections. In the second section, we will define a new measure $\bar{\tau}$ in Wiener space and investigate its basic properties. In the third section, we will treat the translation problems for $\bar{\tau}$ -measurable sets. Indeed, we will show that if N is

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$\bar{\tau}$ -null, then $N + y$ is $\bar{\tau}$ -null for $\bar{\tau}$ -a.e. y . In the fourth section, we will improve and clarify the converse measurability theorem and the Wiener integration formulas. In the last section, we will apply $\bar{\tau}$ -measurability to the theory of Fourier-Feynman transforms, and improve the results in [4], [9].

2. PRELIMINARIES

The real number system will be denoted by \mathbb{R} . For a natural number n , let \mathbb{R}^n denote the set of ordered n -tuples of real numbers, and let m_L denote Lebesgue measure on \mathbb{R}^n . Let \mathbb{C}_+ be the set of all complex numbers z with $\operatorname{Re} z > 0$.

For two real numbers a and b with $a < b$, let $C_0[a, b]$ denote Wiener space, that is, the space of all real-valued continuous functions on the closed interval $[a, b]$ that vanish at a . Given x in $C_0[a, b]$, we define the supremum norm $\|\cdot\|$ given by $\|x\| = \sup_{t \in [a, b]} |x(t)|$. Let \mathcal{B} denote the set of all Borel subsets of $C_0[a, b]$ with respect to the supremum norm topology and let m_1 denote Wiener measure on \mathcal{B} . For a positive real number λ , let m_λ be the Borel measure on \mathcal{B} given by $m_\lambda(B) = m_1(\lambda^{-1}B)$ for B in \mathcal{B} . For a positive real number λ , let $(C_0[a, b], \bar{\mathcal{B}}_\lambda, \bar{m}_\lambda)$ be the completion of $(C_0[a, b], \mathcal{B}, m_\lambda)$. A subset E of $C_0[a, b]$ is said to be scale-invariant measurable provided E belongs to $\bar{\mathcal{B}}_\lambda$ for all positive real numbers λ . Let \mathcal{S} be the set of all scale-invariant measurable subsets. A scale-invariant measurable set N is said to be scale-invariant null provided $\bar{m}_\lambda(N) = 0$ for all positive real numbers λ ; let \mathcal{N} be the set of all scale-invariant null subsets. A property which holds for all x in $C_0[a, b]$ except for a scale-invariant null subset is said to hold scale-invariant almost everywhere (briefly, s-a.e.). For more detail about scale-invariant measurability and scale change in Wiener space, see [7, pp. 67-78]. If two functions F and G on $C_0[a, b]$ are equal s-a.e., we write $F \approx G$.

Given a natural number n , let σ_n be the partition $a = t_0 < t_1 < t_2 < \cdots < t_{2^n} \leq b$ where $t_k = \frac{k}{2^n}(b - a) + a$ for $k = 0, 1, 2, \dots, 2^n$. Given x in $C_0[a, b]$, let

$$S_{\sigma_n}(x) = \sum_{k=1}^{2^n} (x(t_k) - x(t_{k-1}))^2.$$

For a nonnegative real number λ , let $C_\lambda = \{x \text{ in } C_0[a, b] \mid \text{the limit } \lim_{n \rightarrow +\infty} S_{\sigma_n}(x) \text{ exists and equals } \lambda^2(b - a)\}$ and $D = \{x \text{ in } C_0[a, b] \mid \text{the limit } \lim_{n \rightarrow +\infty} S_{\sigma_n}(x) \text{ does not exist}\}$. Note that $\lambda C_\mu = C_{\lambda\mu}$ for all positive real numbers λ and μ , and D and the C_λ ($\lambda \geq 0$) are all Borel subsets of $C_0[a, b]$. Moreover, we know that $C_0[a, b]$ is the disjoint union of the family $\{C_\lambda \mid \lambda \geq 0\}$ and D , $m_\lambda(C_\lambda) = 1$ for any positive real number λ , and $m_\lambda(C_\mu) = 0$ for any distinct positive real numbers λ and μ [7], [8].

In [7] and [8] we can find the following theorem.

Theorem 2.1. *Let λ and μ be positive real numbers and let E be in $\bar{\mathcal{B}}_{\sqrt{\lambda^2 + \mu^2}}$. Then $E + y$ and $E - y$ are in $\bar{\mathcal{B}}_\lambda$ for \bar{m}_μ -a.e. y , and $\bar{m}_\lambda(E + y)$ and $\bar{m}_\lambda(E - y)$ are \bar{m}_μ -measurable functions of y . Similarly, $E + x$ and $E - x$ are in $\bar{\mathcal{B}}_\mu$ for \bar{m}_λ -a.e. x ,*

and $\bar{m}_\mu(E + x)$ and $\bar{m}_\mu(E - x)$ are \bar{m}_λ -measurable functions of x . Furthermore,

$$\begin{aligned}
 & \int_{C_0[a,b]} \bar{m}_\lambda(E + y) \, d\bar{m}_\mu(y) \\
 &= \int_{C_0[a,b]} \bar{m}_\lambda(E - y) \, d\bar{m}_\mu(y) \\
 &= \bar{m}_\lambda \times \bar{m}_\mu (\{(x, y) \mid x + y \text{ is in } E\}) \\
 &= \bar{m}_{\sqrt{\lambda^2 + \mu^2}} (E) \\
 &= \int_{C_0[a,b]} \bar{m}_\mu(E + x) \, d\bar{m}_\lambda(x) \\
 (2.1) \quad &= \int_{C_0[a,b]} \bar{m}_\mu(E - x) \, d\bar{m}_\lambda(x).
 \end{aligned}$$

Given a number p such that $1 \leq p \leq +\infty$, p and p' will always be related by $\frac{1}{p} + \frac{1}{p'} = 1$.

Let F be a functional on $C_0[a, b]$ such that the integral

$$(2.2) \quad J(\lambda) = \int_{C_0[a,b]} F(\lambda^{-\frac{1}{2}}x) \, d\bar{m}_1(x)$$

exists for almost all positive real λ . If there exists an analytic function $J^*(\lambda)$ in \mathbb{C}_+ such that $J(\lambda) = J^*(\lambda)$ for almost all positive real λ , then we define this essential analytic extension J^* of J to be the analytic Wiener integral of F over $C_0[a, b]$ with parameter λ , and we write

$$(2.3) \quad \int_{C_0[a,b]}^{\text{anw}\lambda} F(x) \, d\bar{m}_1(x) = J^*(\lambda)$$

for λ in \mathbb{C}_+ . For λ in \mathbb{C}_+ and for y in $C_0[a, b]$, let

$$(2.4) \quad [T_\lambda(F)](y) \equiv \int_{C_0[a,b]}^{\text{anw}\lambda} F(x + y) \, d\bar{m}_1(x).$$

We finish this section by stating a lemma and then using this lemma to prove a theorem. The following lemma follows from [6, Exercise (19.70), p. 339].

Lemma 2.2. *Let I be an uncountable set. Let (Ω, \mathcal{M}) be a measurable space and let $\{\mu_\lambda \mid \lambda \text{ is in } I\}$ be a collection of nonnegative measures on (Ω, \mathcal{M}) . For B in \mathcal{M} , let*

$$(2.5) \quad \mu(B) = \sup_A \sum_{\lambda \in A} \mu_\lambda(B)$$

where A ranges over all finite subsets of I . Then μ is a measure on (Ω, \mathcal{M}) .

Theorem 2.3. *Under the assumptions of Lemma 2.2, if $f : \Omega \rightarrow \mathbb{R}$ is nonnegative, bounded, and \mathcal{M} -measurable, then*

$$(2.6) \quad \int_\Omega f(x) \, d\mu(x) = \sup_A \sum_{\lambda \in A} \int_\Omega f(x) \, d\mu_\lambda(x)$$

where A ranges over all finite subsets of I .

Proof. For B in \mathcal{M} , clearly $\mu(B) \geq \sum_{\lambda \in A} \mu_\lambda(B)$ for any finite subset A of I . So the inequality

$$(2.7) \quad \int_{\Omega} f(x) d\mu(x) \geq \sup_A \sum_{\lambda \in A} \int_{\Omega} f(x) d\mu_\lambda(x)$$

holds where A ranges over all finite subsets of I . Hence, we must prove that

$$(2.8) \quad \int_{\Omega} f(x) d\mu(x) \leq \sup_A \sum_{\lambda \in A} \int_{\Omega} f(x) d\mu_\lambda(x)$$

where A ranges over all finite subsets of I . Now, let C be in \mathcal{M} with $\mu(C)$ finite. Then there is a sequence $\langle \lambda_{i,C} \rangle$ in I such that

$$(2.9) \quad \mu(C) = \sum_{i=1}^{\infty} \mu_{\lambda_{i,C}}(C) \quad \text{and}$$

$$(2.10) \quad \mu_\lambda(C) = 0 \quad \text{for } \lambda \neq \lambda_{i,C} \quad (i = 1, 2, \dots).$$

Let $K = \{\lambda_{i,C} \mid i \text{ is a natural number}\}$. For B in \mathcal{M} , we let

$$(2.11) \quad \mu_C^1(B) = \sum_{\lambda \in K} \mu_\lambda(B) \quad \text{and}$$

$$(2.12) \quad \mu_C^2(B) = \sup_A \sum_{\lambda \in A} \mu_\lambda(B)$$

where A ranges over all finite subsets of $I \setminus K$. Then by Lemma 2.2, μ_C^1 and μ_C^2 are both measures on (Ω, \mathcal{M}) such that $\mu_C^1 + \mu_C^2 = \mu$, $\mu_C^1(C) = \mu(C)$ and $\mu_C^2(C) = 0$. Hence, we have

$$(2.13) \quad \begin{aligned} \int_C f(x) d\mu(x) &= \int_C f(x) d\mu_C^1(x) + \int_C f(x) d\mu_C^2(x) \\ &= \int_C f(x) d\mu_C^1(x). \end{aligned}$$

Since f is bounded measurable, there is an increasing sequence $\langle \varphi_n \rangle$ of nonnegative simple functions on C such that $\langle \varphi_n \rangle$ converges uniformly to f on C μ_C^1 -a.e. Then we can easily check that

$$(2.14) \quad \int_C \varphi_n(x) d\mu_C^1(x) = \sum_{i=1}^{\infty} \int_C \varphi_n(x) d\mu_{\lambda_{i,C}}(x).$$

For a natural number n , we let $\psi_n = \varphi_n - \varphi_{n-1}$ where φ_0 is the zero function. Also, for two natural numbers i and j , we let

$$(2.15) \quad a_{i,j} = \int_C \psi_j(x) d\mu_{\lambda_{i,C}}(x).$$

Let ϵ be any positive real number and let $M \equiv \sup_{\omega \in \Omega} |f(\omega)|$. Since $\mu(C) = \sum_{i=1}^{\infty} \mu_{\lambda_{i,C}}(C)$

is finite, there is a natural number n_0 such that, for $n \geq n_0$,

$$\sum_{i=n+1}^{\infty} \mu_{\lambda_i, C}(C) < \frac{\epsilon}{2(M+1)},$$

and there is a natural number m_0 such that, for $m \geq m_0$,

$$|f(x) - \varphi_m(x)| < \frac{\epsilon}{2(\mu(C) + 1)}$$

for μ_C^1 - a.e. x . Hence, for $n \geq n_0$ and $m \geq m_0$,

$$\begin{aligned} & \left| \sum_{i=1}^n \sum_{j=1}^m a_{i,j} - \sum_{i=1}^{\infty} \int_C f(x) d\mu_{\lambda_i, C}(x) \right| \\ & \leq \left| \sum_{i=1}^n \int_C (\varphi_m(x) - f(x)) d\mu_{\lambda_i, C}(x) \right| \\ & \quad + \left| \sum_{i=n+1}^{\infty} \int_C f(x) d\mu_{\lambda_i, C}(x) \right| \\ & \leq \frac{\epsilon}{2(\mu(C) + 1)} \sum_{i=1}^n \mu_{\lambda_i, C}(C) + \frac{\epsilon}{2(M+1)} M \\ (2.16) \quad & < \epsilon. \end{aligned}$$

That is, the double series

$$\left\langle \sum_{i=1}^n \sum_{j=1}^m a_{i,j} \right\rangle = \left\langle \sum_{i=1}^n \int_C \varphi_m(x) d\mu_{\lambda_i, C}(x) \right\rangle$$

converges absolutely to $\sum_{i=1}^{\infty} \int_C f(x) d\mu_{\lambda_i, C}(x)$. Thus, we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_C \varphi_m(x) d\mu_{\lambda_i, C}(x) \\ & = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{i=1}^n \int_C \varphi_m(x) d\mu_{\lambda_i, C}(x) \\ (2.17) \quad & = \sum_{i=1}^{\infty} \int_C f(x) d\mu_{\lambda_i, C}(x). \end{aligned}$$

Thus, we have

$$\begin{aligned}
 & \int_C f(x) \, d\mu(x) \\
 & \stackrel{(1)}{=} \int_C f(x) \, d\mu_C^1(x) \\
 & \stackrel{(2)}{=} \lim_{m \rightarrow \infty} \int_C \varphi_m(x) \, d\mu_C^1(x) \\
 & \stackrel{(3)}{=} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_C \varphi_m(x) \, d\mu_{\lambda_{i,C}}(x) \\
 & \stackrel{(4)}{=} \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{i=1}^n \int_C \varphi_m(x) \, d\mu_{\lambda_{i,C}}(x) \\
 & \stackrel{(5)}{=} \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_C f(x) \, d\mu_{\lambda_{i,C}}(x) \\
 (2.18) \quad & \stackrel{(6)}{=} \sum_{i=1}^{\infty} \int_C f(x) \, d\mu_{\lambda_{i,C}}(x).
 \end{aligned}$$

Step (1) follows from the equality (2.13). Steps (2) and (5) result from the dominated convergence theorem. By the equality (2.14), we obtain step (3). From the equality (2.17), we have step (4). By the definition of series, we have step (6). Therefore,

$$\begin{aligned}
 & \int_{\Omega} f(x) \, d\mu(x) \\
 & = \sup_{\mu(C) < +\infty} \int_C f(x) \, d\mu(x) \\
 & = \sup_{\mu(C) < +\infty} \sum_{i=1}^{\infty} \int_C f(x) \, d\mu_{\lambda_{i,C}}(x) \\
 & \leq \sup_{\mu(C) < +\infty} \sum_{i=1}^{\infty} \int_{\Omega} f(x) \, d\mu_{\lambda_{i,C}}(x) \\
 (2.19) \quad & \leq \sup_A \sum_{\lambda \in A} \int_{\Omega} f(x) \, d\mu_{\lambda}(x)
 \end{aligned}$$

where A ranges over all finite subsets of I . Hence, the theorem is proved. \square

3. THE MEASURE $\bar{\tau}$ AND ITS BASIC PROPERTIES

In this section, we will introduce a complete measure $\bar{\tau}$ on the measurable space $(C_0[a, b], \mathcal{T})$, and we will investigate its basic properties.

Definition 3.1. For B in \mathcal{B} , we let

$$(3.1) \quad \tau(B) = \sup_A \sum_{\lambda \in A} m_{\lambda}(B)$$

where A ranges over all finite subsets of positive real numbers.

From Lemma 2.2, we know that τ is a Borel measure on Wiener space.

Remark 3.2. (1) For B in \mathcal{B} and for a positive real number λ ,

$$(3.2) \quad \tau(B) = \tau(\lambda B).$$

(2) For a Borel subset B of C_λ ,

$$(3.3) \quad \tau(B) = m_\lambda(B).$$

(3) τ is not σ -finite.

(4) If E is a Borel subset of \mathbb{R}^n and $a = t_0 < t_1 < t_2 < \cdots < t_n \leq b$, letting $B = \{x \text{ in } C_0[a, b] \mid (x(t_1), x(t_2), \dots, x(t_n)) \text{ belongs to } E\}$, by the classical Wiener integration formula [7, Theorem 3.3.5, p. 44], B is a Borel subset of $C_0[a, b]$ and $\tau(B)$ is either zero or infinite.

(5) Let $f \in L_2[a, b]$ and let $PWZ(f) = \{x \text{ in } C_0[a, b] \mid \text{the Paley-Wiener-Zygmund integral } \int_a^b f(t) \tilde{d}x(t) \text{ exists}\}$. Then $\lambda PWZ(f)^c$ is a Borel subset of $C_0[a, b]$ and $m_1(\lambda PWZ(f)^c) = 0$ for nonnegative real numbers λ . So, we have $\bar{\tau}(PWZ(f)^c) = 0$.

Definition 3.3. Let $(C_0[a, b], \mathcal{T}, \bar{\tau})$ be the completion of the measure space $(C_0[a, b], \mathcal{B}, \tau)$. A property that holds for all x in $C_0[a, b]$ except for a $\bar{\tau}$ -null set is said to hold $\bar{\tau}$ -almost everywhere (briefly, $\bar{\tau}$ -a.e.). If two functionals F and G on $C_0[a, b]$ are equal $\bar{\tau}$ -a.e., we write $F \bar{\approx} G$.

Remark 3.4. (1) The relation $\bar{\approx}$ is an equivalence relation on the class of all $\bar{\tau}$ -measurable Wiener functionals.

(2) $\bar{\tau}$ is not σ -finite.

(3) If N is $\bar{\tau}$ -null, then N is in \mathcal{N} .

(4) The set \mathcal{T} is a subset of \mathcal{S} .

(5) For E in \mathcal{S} , let $\mu(E) = \sup \sum_{\lambda \in A} \bar{m}_\lambda(E)$ where A ranges over all finite subsets of positive real numbers. Then by Lemma 2.2, μ is a measure on $(C_0[a, b], \mathcal{S})$ and the measure space $(C_0[a, b], \mathcal{S}, \mu)$ is a complete measure space.

Now we will show the relationships between the σ -algebras $\mathcal{B}, \mathcal{T}, \mathcal{S}$ and $\bar{\mathcal{B}}_\lambda$ where λ is a positive real number.

The next lemma follows from [3], [7], [8].

Lemma 3.5. For an arbitrary function $f : (0, +\infty) \rightarrow [0, 1]$, there is a set E in \mathcal{S} such that $f(\lambda) = \bar{m}_\lambda(E)$ for positive real numbers λ .

The following lemma was established in [13].

Lemma 3.6. For B in \mathcal{B} , $m_\lambda(B)$ is a Borel measurable function of λ .

By Definition 3.3, for E in \mathcal{T} , we can write $E = B \cup N$ such that B is in \mathcal{B} , N is $\bar{\tau}$ -null and $B \cap N = \emptyset$. Then, by Lemma 3.6, $\bar{m}_\lambda(E) = m_\lambda(B)$ is a Borel measurable function of λ . Hence, we have the following.

Corollary 3.7. For E in \mathcal{T} , $\bar{m}_\lambda(E)$ is a Borel measurable function of λ .

Theorem 3.8. $\mathcal{B} \subsetneq \mathcal{T} \subsetneq \mathcal{S} \subsetneq \bar{\mathcal{B}}_\lambda$ holds for all positive real numbers λ .

Proof. From [7] and [8] one can find that $\mathcal{B} \subsetneq \mathcal{S} \subsetneq \bar{\mathcal{B}}_\lambda$ is true for all positive real numbers λ . Since it is trivial that $\mathcal{B} \subset \mathcal{T} \subset \mathcal{S}$, it remains to show that $\mathcal{B} \neq \mathcal{T}$ and $\mathcal{T} \neq \mathcal{S}$. Let t be fixed with $a < t \leq b$ and let $P : C_0[a, b] \rightarrow \mathbb{R}$ be a function with

$P(x) = x(t)$. Let N be a Lebesgue null subset of \mathbb{R} but not a Borel measurable subset. Then there is a Borel measurable subset N^* of \mathbb{R} such that $N \subset N^*$ and $m_L(N^*) = 0$. For any positive real number λ , $\lambda^{-1}N^*$ is a Borel measurable subset with $m_L(\lambda^{-1}N^*) = 0$. Hence, by the classical Wiener integration formula, we know that $m_\lambda(P^{-1}(N^*)) = m_1(\lambda^{-1}P^{-1}(N^*)) = m_1(P^{-1}(\lambda^{-1}N^*)) = 0$; that is, $P^{-1}(N^*)$ is τ -null. Since $P^{-1}(N) \subset P^{-1}(N^*)$, $P^{-1}(N)$ is in \mathcal{T} . But, by the converse measurability theorem [5], [7], we know that $P^{-1}(N)$ is not in \mathcal{B} , which implies that $\mathcal{T} \neq \mathcal{B}$. Now, if K is a non-measurable subset of $(0, +\infty)$, letting $F = \bigcup_{\lambda \in K} C_\lambda$, $\overline{m}_\lambda(F)$ is not a measurable function of λ . So by Corollary 2.7, F is not in \mathcal{T} . Trivially, F is in \mathcal{S} , which implies that $\mathcal{T} \neq \mathcal{S}$, as desired. \square

Remark 3.9 (The generalized Radon-Nikodým theorem [10], [11]). Let I be an arbitrary subset of positive real numbers. For B in \mathcal{B} , let

$$(3.4) \quad \tau_I(B) = \sup_A \sum_{\lambda \in A} m_\lambda(B)$$

where A ranges over all finite subsets of I . Then by Lemma 1.2, τ_I is a measure on $(C_0[a, b], \mathcal{B})$ and τ_I is absolutely continuous with respect to τ . Moreover, for B in \mathcal{B} with $\tau(B)$ finite,

$$(3.5) \quad \tau_I(B) = \int_B \chi_M(x) d\tau(x)$$

where $M = \bigcup_{\lambda \in I} C_\lambda$ and χ_M is a characteristic function associated with a set M . Here, χ_M may be not measurable, but for a set B in \mathcal{B} with $\tau(B)$ finite, the restriction of χ_M to B is measurable. Similarly, equation (3.5) also holds with τ and τ_I replaced by $\bar{\tau}$ and $\bar{\tau}_I$, respectively [11].

For E in \mathcal{T} with $\bar{\tau}(E)$ positive, there is a positive real number λ_0 such that $\overline{m}_{\lambda_0}(E)$ is positive. Letting $F = E \cap C_{\lambda_0}$, F is in \mathcal{T} with $F \subset E$ and $0 < \bar{\tau}(F) = \overline{m}_{\lambda_0}(E) \leq 1$; that is, $\bar{\tau}$ has the finite subset property [11, p. 68]. Let $\mathcal{R} = \{A \times B \mid A, B \text{ are in } \mathcal{T}\}$. For $A \times B$ in \mathcal{R} , we let $\alpha(A \times B) = \bar{\tau}(A) \bar{\tau}(B)$. Then α is countably additive on a semi-algebra \mathcal{R} . Since $\bar{\tau}$ has the finite subset property, by [11, Exercise 6, p. 323], there is a unique extension $\bar{\tau} \times \bar{\tau}$ of α on the smallest σ -algebra $\mathcal{T} \otimes \mathcal{T}$ containing \mathcal{R} . Here we will establish the integration formula for the product measure $\bar{\tau} \times \bar{\tau}$, which plays a very important role in this article.

Theorem 3.10. *Let $f : C_0[a, b] \times C_0[a, b] \rightarrow \mathbb{R}$ be nonnegative $\bar{\tau} \times \bar{\tau}$ -measurable. Then*

$$(3.6) \quad \begin{aligned} & \int_{C_0[a, b] \times C_0[a, b]} f(x, y) d\bar{\tau} \times \bar{\tau}(x, y) \\ &= \sup_A \sup_B \sum_{\lambda \in A} \sum_{\mu \in B} \int_{C_0[a, b] \times C_0[a, b]} f(x, y) d\overline{m}_\lambda \times \overline{m}_\mu(x, y) \end{aligned}$$

where A and B range over all finite subsets of positive real numbers.

Proof. Trivially, we have $\bar{\tau} \times \bar{\tau}(E) \geq \sum_{\lambda \in A} \sum_{\mu \in B} \overline{m}_\lambda \times \overline{m}_\mu(E)$ for all $\bar{\tau} \times \bar{\tau}$ -measurable subsets E and for any finite subsets A and B of positive real numbers. Hence, we

have

$$(3.7) \quad \int_{C_0[a,b] \times C_0[a,b]} f(x, y) \, d\bar{\tau} \times \bar{\tau}(x, y) \geq \sup_A \sup_B \sum_{\lambda \in A} \sum_{\mu \in B} \int_{C_0[a,b] \times C_0[a,b]} f(x, y) \, d\bar{m}_\lambda \times \bar{m}_\mu(x, y).$$

If

$$\sup_A \sup_B \sum_{\lambda \in A} \sum_{\mu \in B} \int_{C_0[a,b] \times C_0[a,b]} f(x, y) \, d\bar{m}_\lambda \times \bar{m}_\mu(x, y) = +\infty,$$

then (3.6) clearly holds. Now, we suppose that

$$\sup_A \sup_B \sum_{\lambda \in A} \sum_{\mu \in B} \int_{C_0[a,b] \times C_0[a,b]} f(x, y) \, d\bar{m}_\lambda \times \bar{m}_\mu(x, y)$$

is finite. Letting

$$T = \{(\lambda, \mu) \mid \int_{C_0[a,b] \times C_0[a,b]} f(x, y) \, d\bar{m}_\lambda \times \bar{m}_\mu(x, y) \text{ is positive}\},$$

we see that T is countable, say $T = \{(\lambda_i, \mu_i) \mid i \text{ is a natural number}\}$. Let $L = \bigcup_{i=1}^\infty C_{\lambda_i} \times C_{\mu_i}$ and let $M = C_0[a, b] \times C_0[a, b] \setminus L$. Then L and M are all Borel subsets, and $L \cap M = \emptyset$. Hence, we have

$$(3.8) \quad \begin{aligned} & \int_{C_0[a,b] \times C_0[a,b]} f(x, y) \, d\bar{\tau} \times \bar{\tau}(x, y) \\ &= \int_L f(x, y) \, d\bar{\tau} \times \bar{\tau}(x, y) + \int_M f(x, y) \, d\bar{\tau} \times \bar{\tau}(x, y) \\ &= \sum_{i=1}^\infty \int_{C_{\lambda_i} \times C_{\mu_i}} f(x, y) \, d\bar{m}_{\lambda_i} \times \bar{m}_{\mu_i}(x, y) + \int_M f(x, y) \, d\bar{\tau} \times \bar{\tau}(x, y). \end{aligned}$$

If we prove that $\int_M f(x, y) \, d\bar{\tau} \times \bar{\tau}(x, y) = 0$, then equality (3.6) holds. Let B be a $\bar{\tau} \times \bar{\tau}$ -measurable subset of M with $\bar{\tau} \times \bar{\tau}(B)$ finite. Then there is a disjoint sequence $\langle G_n \rangle$ in \mathcal{R} such that $B \subset \bigcup_{n=1}^\infty G_n \subset M$ and $\sum_{n=1}^\infty \bar{\tau} \times \bar{\tau}(G_n) < \bar{\tau} \times \bar{\tau}(B) + 1$. Put $G_n = E_n \times F_n$ for each natural number n . Then there is a subsequence $\langle G_{n_i} \rangle$ of $\langle G_n \rangle$ such that $\bar{\tau}(E_{n_i})$ and $\bar{\tau}(F_{n_i})$ are all finite for all natural numbers i and $\bar{\tau} \times \bar{\tau}(B \setminus \bigcup_{n=1}^\infty G_n) = 0$. Then for each natural number n_i , there are two sequences $\langle \lambda_{n_i, j} \rangle$ and $\langle \mu_{n_i, j} \rangle$ such that $\bar{\tau}(E_{n_i}) = \sum_{j=1}^\infty \bar{m}_{\lambda_{n_i, j}}(E_{n_i})$ and $\bar{\tau}(F_{n_i}) = \sum_{j=1}^\infty \bar{m}_{\mu_{n_i, j}}(F_{n_i})$.

Then

(3.9)

$$\begin{aligned} & \int_B f(x, y) d\bar{\tau} \times \bar{\tau}(x, y) \\ &= \int_{B \cap (\cup_{i=1}^\infty E_{n_i} \times F_{n_i})} f(x, y) d\bar{\tau} \times \bar{\tau}(x, y) \\ & \quad + \int_{B \setminus (\cup_{i=1}^\infty E_{n_i} \times F_{n_i})} f(x, y) d\bar{\tau} \times \bar{\tau}(x, y) \\ &\leq \sum_{i=1}^\infty \int_{E_{n_i} \times F_{n_i}} f(x, y) d\bar{\tau} \times \bar{\tau}(x, y) \\ &\leq \sum_{i=1}^\infty \sum_{j=1}^\infty \sum_{k=1}^\infty \int_{C_{\lambda_{n_i, j}} \times C_{\mu_{n_i, k}}} f(x, y) d\bar{m}_{\lambda_{n_i, j}} \times \bar{m}_{\mu_{n_i, k}}(x, y) \\ & \quad (\lambda_{n_i, j}, \mu_{n_i, k}) \notin T \\ &= 0. \end{aligned}$$

Hence, $\int_M f(x, y) d\bar{\tau} \times \bar{\tau}(x, y) = \sup_D \int_D f(x, y) d\bar{\tau} \times \bar{\tau}(x, y) = 0$, as desired where D ranges over all measurable subsets of M with $\bar{\tau} \times \bar{\tau}(D)$ finite. □

Example 3.11. In general, Theorem 3.10 does not hold. Let $(\mathbb{R}, \mathcal{M})$ be a Lebesgue measurable space. For a positive real number λ , let μ_λ be the Dirac measure on $(\mathbb{R}, \mathcal{M})$. For a positive real number λ with $\lambda \neq 1$, let ν_λ be the zero measure on $(\mathbb{R}, \mathcal{M})$, and let ν_1 be the usual Lebesgue measure on $(\mathbb{R}, \mathcal{M})$. For E in \mathcal{M} , let

$$(3.10) \quad \mu(E) = \sup_A \sum_{\lambda \in A} \mu_\lambda(E) \quad \text{and}$$

$$(3.11) \quad \nu(E) = \sup_A \sum_{\lambda \in A} \nu_\lambda(E)$$

where A ranges over all finite subsets of \mathbb{R} . Then μ is the counting measure on $(\mathbb{R}, \mathcal{M})$ and ν is the Lebesgue measure on $(\mathbb{R}, \mathcal{M})$. Let $\Delta = \{(x, x) \mid x \text{ is in } \mathbb{R}\}$. Then Δ is measurable in the product space $\mathbb{R} \times \mathbb{R}$, and

$$\int \int_{\mathbb{R} \times \mathbb{R}} \chi_\Delta(x, y) d\mu \times \nu(x, y) = +\infty,$$

but

$$\sup_A \sup_B \sum_{t \in A} \sum_{s \in B} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_\Delta(x, y) d\nu_s(y) d\mu_t(x) = 0$$

where A and B range over all finite subsets of \mathbb{R} .

4. TRANSLATION IN WIENER SPACE

It is a well-known fact that if X is an infinite-dimensional vector space, there is no nonzero quasi-invariant measure μ on (X, \mathcal{B}_R) where R is a linear subspace of the dual space of X and \mathcal{B}_R is the smallest σ -algebra on X in which every f in R is measurable; that is, there is no nonzero measure μ such that $\mu(N) = 0$ implies $\mu(N + y) = 0$ for all y in X [15, Corollary 1, p. 142]. Since the Wiener space $C_0[a, b]$ with the supremum norm is a real separable infinite-dimensional Banach

space, there is no nonzero quasi-invariant measure on Wiener space $C_0[a, b]$. It is natural to wonder how many y in $C_0[a, b]$ exist such that $\bar{\tau}(N) = 0$ but $\bar{\tau}(N+y) \neq 0$.

Actually, Cameron showed in [2] that there is an \bar{m}_1 -measurable set E such that $E + y$ is not \bar{m}_1 -measurable for \bar{m}_1 -a.e. y , and Johnson and Skoug, in their 1979 paper [8], raised the following open problems.

- (1) Let E be in \mathcal{S} . Is it true that $E + y$ is in \mathcal{S} for s-a.e. y ?
- (2) Let N be in \mathcal{N} . Is $N + y$ in \mathcal{N} true for s-a.e. y ?

Theorem 4.1. *Let N be $\bar{\tau}$ -null. Then $N + y$ and $N - y$ are $\bar{\tau}$ -null for $\bar{\tau}$ -a.e. y .*

Proof. Let N be $\bar{\tau}$ -null. Consider a function $\Phi : C_0[a, b] \times C_0[a, b] \rightarrow C_0[a, b]$ such that $\Phi(x, y) = x - y$. Since N is $\bar{\tau}$ -null, there is a Borel set N^* in $C_0[a, b]$ such that $N \subset N^*$ and $\bar{\tau}(N^*) = 0$. Because Φ is continuous, $\Phi^{-1}(N^*)$ is also a Borel subset of $C_0[a, b] \times C_0[a, b]$. Hence, by Theorem 2.1 and Theorem 3.10,

$$\begin{aligned}
 & \int_{C_0[a,b] \times C_0[a,b]} \chi_{\Phi^{-1}(N^*)}(x, y) \, d\bar{\tau} \times \bar{\tau}(x, y) \\
 &= \sup_A \sup_B \sum_{\lambda \in A} \sum_{\mu \in B} \int_{C_0[a,b] \times C_0[a,b]} \chi_{\Phi^{-1}(N^*)}(x, y) \, d\bar{m}_\lambda \times \bar{m}_\mu(x, y) \\
 &= \sup_A \sup_B \sum_{\lambda \in A} \sum_{\mu \in B} \int_{C_0[a,b] \times C_0[a,b]} \chi_{N^*+y}(x) \, d\bar{m}_\lambda \times \bar{m}_\mu(x, y) \\
 &= \sup_A \sup_B \sum_{\lambda \in A} \sum_{\mu \in B} \int_{C_0[a,b]} \chi_{N^*}(z) \, d\bar{m}_{\sqrt{\lambda^2+\mu^2}}(z) \\
 (4.1) \quad &= 0
 \end{aligned}$$

where A and B range over all finite subsets of positive real numbers. Hence, $\bar{\tau} \times \bar{\tau}(\Phi^{-1}(N^*)) = 0$. By [12, Lemma 17, p. 306] for $\bar{\tau}$ -a.e. y ,

$$\bar{\tau}([\Phi^{-1}(N^*)]_y) = \bar{\tau}(N^* + y) = 0$$

where $[A]_y$ means the y cross section of a set A . Since $N \subset N^*$, we have that $\bar{\tau}(N + y) = 0$ for $\bar{\tau}$ -a.e. y . Similarly, it follows that $N - y$ is $\bar{\tau}$ -null for $\bar{\tau}$ -a.e. y , which concludes the proof of Theorem 4.1. □

Theorem 4.2. *If E is in \mathcal{T} , then $E + y$ and $E - y$ are in \mathcal{T} for $\bar{\tau}$ -a.e. y .*

Proof. Let E be in \mathcal{T} . Then there are two sets B and N such that B is in \mathcal{B} , N is $\bar{\tau}$ -null, $B \cap N = \emptyset$ and $E = B \cup N$. Trivially, $B + y$ and $B - y$ are in \mathcal{B} for all y in $C_0[a, b]$, and by Theorem 4.1, $N + y$ and $N - y$ are $\bar{\tau}$ -null for $\bar{\tau}$ -a.e. y . Hence $E + y = (B + y) \cup (N + y)$ and $E - y = (B - y) \cup (N - y)$ are in \mathcal{T} for $\bar{\tau}$ -a.e. y , as desired. □

Theorem 4.3. *If f is a \mathcal{T} -measurable function, then, for $\bar{\tau}$ -a.e. y , $f(x + y)$ and $f(x - y)$ are \mathcal{T} -measurable functions of x .*

Proof. Let f be \mathcal{T} -measurable. Then there is a sequence $\langle \varphi_n \rangle$ of \mathcal{T} -simple functions with $\bar{\tau}(K_1^c) = 0$ where $K_1 \equiv \{x \text{ in } C_0[a, b] \mid \text{the limit } \lim_{n \rightarrow \infty} \varphi_n(x) \text{ exists and equals } f(x)\}$. Let $\Phi : C_0[a, b] \times C_0[a, b] \rightarrow C_0[a, b]$ be a function with $\Phi(x, y) = x - y$. By a method similar to that used in the proof of Theorem 4.1, we can show that $\Phi^{-1}(K_1^c)$ is $\bar{\tau} \times \bar{\tau}$ -null, and so for $\bar{\tau}$ -a.e. y , $[\Phi^{-1}(K_1^c)]_y$ is $\bar{\tau}$ -null. Since if φ_n is \mathcal{T} -simple then there is a $\bar{\tau}$ -measurable subset K_2 of $C_0[a, b]$ such that $\bar{\tau}(K_2) = 0$ and $\varphi_n(x + y)$ is

also \mathcal{T} -simple for y in K_2 and $[\Phi^{-1}(K_1^c)]_y = \{x \text{ in } C_0[a, b] \mid \text{the limit } \lim_{n \rightarrow \infty} \varphi_n(x+y) \text{ exists and equals } f(x+y)\}^c$, it follows that $f(x+y)$ is a \mathcal{T} -measurable function of x for y in $K_1 \cap K_2^c$, and $\bar{\tau}(K_1^c \cup K_2) = 0$. By a method similar to that used in the proof of this theorem, it is not hard to prove that for $\bar{\tau}$ -a.e. y , $f(x-y)$ is a $\bar{\tau}$ -measurable function of x , which concludes the proof of Theorem 4.3. \square

Theorem 4.4. *Let $f : C_0[a, b] \rightarrow \mathbb{R}$ be nonnegative and $\bar{\tau}$ -measurable. Then $\int_{C_0[a,b] \times C_0[a,b]} f(x+y) d\bar{\tau} \times \bar{\tau}(x, y)$ is either zero or infinite.*

Proof. From Theorem 2.1 and Theorem 3.10, we have

$$\begin{aligned}
 & \int_{C_0[a,b] \times C_0[a,b]} f(x+y) d\bar{\tau} \times \bar{\tau}(x, y) \\
 &= \sup_A \sup_B \sum_{\lambda \in A} \sum_{\mu \in B} \int_{C_0[a,b] \times C_0[a,b]} f(x+y) d\bar{m}_\lambda \times \bar{m}_\mu(x, y) \\
 (4.2) \quad &= \sup_A \sup_B \sum_{\lambda \in A} \sum_{\mu \in B} \int_{C_0[a,b]} f(z) d\bar{m}_{\sqrt{\lambda^2 + \mu^2}}(z)
 \end{aligned}$$

where A and B range over all finite subsets of positive real numbers.

If $\int_{C_0[a,b]} f(z) d\bar{m}_{\sqrt{\lambda_0^2 + \mu_0^2}}(z)$ is positive for some positive real numbers λ_0 and μ_0 , then for all (λ, μ) in $K = \{(\lambda, \mu) \mid \sqrt{\lambda^2 + \mu^2} = \sqrt{\lambda_0^2 + \mu_0^2}, \lambda > 0 \text{ and } \mu > 0\}$ the integral $\int_{C_0[a,b]} f(z) d\bar{m}_{\sqrt{\lambda^2 + \mu^2}}(z)$ is positive and K is uncountable; so

$$\int_{C_0[a,b] \times C_0[a,b]} f(x+y) d\bar{\tau} \times \bar{\tau}(x, y)$$

is infinite. \square

5. THE CONVERSE MEASURABILITY THEOREM AND THE WIENER INTEGRATION FORMULA

In this section, we will treat the converse measurability theorem and the Wiener integration formula in the light of $\bar{\tau}$ -measurability.

Throughout this section, we adopt the following notation:

(1) Let $\vec{t} = (t_1, t_2, \dots, t_n)$ be a vector in \mathbb{R}^n with $a = t_0 < t_1 < t_2 < \dots < t_n \leq b$.

(2) Let $P_{\vec{t}} : C_0[a, b] \rightarrow \mathbb{R}^n$ be a function with $P_{\vec{t}}(x) = (x(t_1), x(t_2), \dots, x(t_n))$.

(3) For a positive real number λ , let

$$W_n(\lambda, \vec{t}, \vec{u}) = \left[(2\pi\lambda)^n \prod_{i=1}^n (t_i - t_{i-1}) \right]^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2\lambda} \sum_{i=1}^n \frac{(u_i - u_{i-1})^2}{t_i - t_{i-1}} \right\}.$$

Theorem 5.1 (The converse measurability theorem). *Let E be a subset of \mathbb{R}^n . The following statements are equivalent:*

(1) E is Lebesgue measurable.

(2) $P_{\vec{t}}^{-1}(E)$ is $\bar{\tau}$ -measurable.

(3) For any positive real number λ , $P_{\vec{t}}^{-1}(E)$ is \bar{m}_λ -measurable, that is, $P_{\vec{t}}^{-1}(E)$ is in \mathcal{S} .

Proof. Let E be a Lebesgue measurable set in \mathbb{R}^n . Then we can write $E = B \cup N$ where B is a Borel measurable set in \mathbb{R}^n , N is a Lebesgue null set and $B \cap N = \emptyset$.

Since $P_{\vec{t}}$ is continuous, $P_{\vec{t}}^{-1}(B)$ is Borel measurable. Also, there is a Borel null subset N^* of \mathbb{R}^n such that $N \subset N^*$. Then, for any positive real number λ ,

$$\begin{aligned} m_\lambda(P_{\vec{t}}^{-1}(N^*)) &= m_1(\lambda^{-1}P_{\vec{t}}^{-1}(N^*)) = m_1(P_{\vec{t}}^{-1}(\lambda^{-1}N^*)) \\ &= \int_{\mathbb{R}^n} \chi_{\lambda^{-1}N^*}(u_1, u_2, \dots, u_n) W_n(1, \vec{t}, \vec{u}) dm_L(\vec{u}) \\ (5.1) \qquad &= 0, \end{aligned}$$

which implies that $P_{\vec{t}}^{-1}(N^*)$ is τ -null. Since $P_{\vec{t}}^{-1}(N) \subset P_{\vec{t}}^{-1}(N^*)$ and $P_{\vec{t}}^{-1}(E) = P_{\vec{t}}^{-1}(B) \cup P_{\vec{t}}^{-1}(N)$, $P_{\vec{t}}^{-1}(E)$ is $\bar{\tau}$ -measurable. Hence (1) implies (2). It is clear that (2) implies (3). By the classical converse measurability theorem [5], we know that (3) implies (1). \square

Theorem 5.2 (The Wiener integration formula). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. Then the following statements are equivalent:*

- (1) f is Lebesgue measurable.
- (2) $f \circ P_{\vec{t}}$ is $\bar{\tau}$ -measurable.
- (3) $f \circ P_{\vec{t}}$ is \mathcal{S} -measurable.

Furthermore, if any one of the conditions (1), (2), or (3) holds, then for any positive real number λ ,

$$\begin{aligned} \int_{C_0[a,b]} f \circ P_{\vec{t}}(x) d\bar{m}_1(x) &= \int_{C_\lambda} f \circ P_{\vec{t}}(\lambda x) d\bar{m}_\lambda(x) \\ &= \int_{\mathbb{R}^n} f(u) W_n(1, \vec{t}, \vec{u}) dm_L(\vec{u}) \\ (5.2) \qquad &= \int_{\mathbb{R}^n} f(\lambda^{-\frac{1}{2}}\vec{u}) W_n(\lambda, \vec{t}, \vec{u}) dm_L(\vec{u}) \end{aligned}$$

where if any one of the four integrals in equation (5.2) is defined, whether finite or infinite, then each of the other three integrals is also defined and equality holds.

Proof. Let E be a Lebesgue measurable set in \mathbb{R}^n and let $f(\vec{u}) = \chi_E(\vec{u})$. Then we can write $E = B \cup N$ where B is a Borel subset of \mathbb{R}^n , N is a Lebesgue null set and $B \cap N = \emptyset$. By the proof of Theorem 5.1, $P_{\vec{t}}^{-1}(N)$ is $\bar{\tau}$ -null. Since $P_{\vec{t}}^{-1}(B)$ is Borel, trivially, $f \circ P_{\vec{t}}(x) = \chi_{P_{\vec{t}}^{-1}(B)}(x) + \chi_{P_{\vec{t}}^{-1}(N)}(x)$ is $\bar{\tau}$ -measurable. We assume that f is simple, say $f = \sum_{i=1}^n a_i \chi_{E_i}$ where the a_i ($i = 1, 2, \dots, n$) are real numbers and the E_i ($i = 1, 2, \dots, n$) are pairwise disjoint Lebesgue measurable sets in \mathbb{R}^n . Then clearly $f \circ P_{\vec{t}}(x) = \sum_{i=1}^n a_i \chi_{P_{\vec{t}}^{-1}(E_i)}(x)$ is $\bar{\tau}$ -measurable. Now, let f be an arbitrary measurable function on \mathbb{R}^n . Then there is a sequence $\langle \varphi_n \rangle$ of simple functions such that $\langle \varphi_n \rangle$ converges to f , m_L -a.e. Let $K \equiv \{ \vec{u} \text{ in } \mathbb{R}^n \mid \text{the limit } \lim_{n \rightarrow \infty} \varphi_n(\vec{u}) \text{ exists and } \lim_{n \rightarrow \infty} \varphi_n(\vec{u}) = f(\vec{u}) \}$. Then K^c is Lebesgue null and $\{ x \text{ in } C_0[a, b] \mid \text{the limit } \lim_{n \rightarrow \infty} \varphi_n \circ P_{\vec{t}}(x) \text{ exists and } \lim_{n \rightarrow \infty} \varphi_n \circ P_{\vec{t}}(x) = f \circ P_{\vec{t}}(x) \} = P_{\vec{t}}^{-1}(K^c)$. By the proof of Theorem 5.1, $P_{\vec{t}}^{-1}(K^c)$ is $\bar{\tau}$ -null. So for all natural numbers n , $\varphi_n \circ P_{\vec{t}}$ is $\bar{\tau}$ -measurable and $P_{\vec{t}}^{-1}(K)$ is $\bar{\tau}$ -null. Hence $f \circ P_{\vec{t}}$ is $\bar{\tau}$ -measurable. So the proof of “(1) implies (2)” is finished. It is trivial that (2) implies (3) by Theorem 3.8. To prove that (3) implies (1), we assume that $f \circ P_{\vec{t}}$ is \mathcal{S} -measurable. Then, for any

Borel subset B of \mathbb{R} , $(f \circ P_{\vec{t}})^{-1}(B) = P_{\vec{t}}^{-1}(f^{-1}(B))$ is \overline{m}_1 -measurable. Hence, by the converse measurability theorem [5], $f^{-1}(B)$ is Lebesgue measurable, as desired. By the classical Wiener integration formula [7] and the change of variables, we can show that the equalities in (5.2) hold. \square

Remark 5.3. Let $\mathcal{M}(\vec{\tau})$ be the set of all $\vec{\tau}$ -measurable functions F of the form $F = g \circ P_{\vec{t}}$ on $C_0[a, b]$ where g is a complex-valued function on \mathbb{R}^n , and let $\mathcal{M}(m_L)$ be the set of all m_L -measurable functions on \mathbb{R}^n . For F in $\mathcal{M}(\vec{\tau})$ let

$$[F]_{\vec{\tau}} = \{G \text{ in } \mathcal{M}(\vec{\tau}) \mid G \overset{\vec{\tau}}{\approx} F\},$$

and for f in $\mathcal{M}(m_L)$ let

$$[f]_{m_L} = \{g \text{ in } \mathcal{M}(m_L) \mid g = f \text{ } m_L\text{-a.e.}\}.$$

Let $T : \mathcal{M}(m_L) \rightarrow \mathcal{M}(\vec{\tau})$ be a function with $T(f) = f \circ P_{\vec{t}}$. By the elementary theory of measure, one can check easily that the following statements hold.

- (1) T is bijective.
- (2) If $f = g$ m_L -a.e. in $\mathcal{M}(m_L)$, then $T(f) \overset{\vec{\tau}}{\approx} T(g)$; that is, if $[f]_{m_L} = [g]_{m_L}$, then $[T(f)]_{\vec{\tau}} = [T(g)]_{\vec{\tau}}$.
- (3) If $T(f) \overset{\vec{\tau}}{\approx} T(g)$, then $f = g$ m_L -a.e.; that is, if $[T(f)]_{\vec{\tau}} = [T(g)]_{\vec{\tau}}$, then $[f]_{m_L} = [g]_{m_L}$.

Remark 5.4. Let p be a given real number with $1 \leq p < +\infty$ and let λ be a positive real number. Let $\mathcal{L}_p(\vec{\tau}, \lambda)$ be the space of functions F of the form $F = g \circ P_{\vec{t}}$ on $C_0[a, b]$ where g is a complex-valued function on \mathbb{R}^n such that F is in $\mathcal{M}(\vec{\tau})$ and $[\int_{C_0[a,b]} |F|^p dm_\lambda]^{\frac{1}{p}} = \|F\|_{\mathcal{L}_p(\vec{\tau}, \lambda)}$ is finite. Let μ_λ be a measure on \mathbb{R}^n such that the Radon-Nikodým derivative $\frac{d\mu_\lambda}{dm_L}$ exists and $\frac{d\mu_\lambda}{dm_L}(\vec{u}) = W_n(\lambda, \vec{t}, \vec{u})$. Let $\mathcal{L}_p(\mu_\lambda)$ be the space of Lebesgue measurable functions f on \mathbb{R}^n such that $\|f\|_{\mathcal{L}_p(\mu_\lambda)} = [\int_{\mathbb{R}^n} |f(u_1, \dots, u_n)|^p d\mu_\lambda(u_1, \dots, u_n)]^{\frac{1}{p}}$ is finite. Let $M_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function with $M_\lambda(\vec{u}) = \lambda \vec{u}$ (that is, $M_\lambda(u_1, u_2, \dots, u_n) = (\lambda u_1, \lambda u_2, \dots, \lambda u_n)$), and let $\tilde{M}_\lambda : C_0[a, b] \rightarrow C_0[a, b]$ be a function with $\tilde{M}_\lambda(x) = \lambda x$. Let G_λ be an operator from $\mathcal{L}_p(\mu_1)$ into $\mathcal{L}_p(\mu_{\lambda^{-2}})$ given by $G_\lambda(f) = f \circ M_\lambda$, let H_λ be an operator from $\mathcal{L}_p(\vec{\tau}, 1)$ into $\mathcal{L}_p(\vec{\tau}, \lambda^{-1})$ given by $H_\lambda(F) = F \circ \tilde{M}_\lambda$, let T be an operator from $\mathcal{L}_p(\mu_1)$ into $\mathcal{L}_p(\vec{\tau}, 1)$ with $T(f) = f \circ P_{\vec{t}}$, and let \tilde{T} be an operator from $\mathcal{L}_p(\mu_{\lambda^{-2}})$ into $\mathcal{L}_p(\vec{\tau}, \lambda^{-1})$ with $\tilde{T}(f) = f \circ P_{\vec{t}}$. Then, by the elementary theory of measure, we can find that the following statements hold.

- (1) The operators G_λ, H_λ, T and \tilde{T} are all well defined, and their semi-operator norms are equal to one.
- (2) $f = g$ μ_1 -a.e. if and only if $f = g$ $\mu_{\lambda^{-2}}$ -a.e.
- (3) $f = g$ μ_1 -a.e. if and only if $T(f) \overset{\vec{\tau}}{\approx} T(g)$.
- (4) $f = g$ $\mu_{\lambda^{-2}}$ -a.e. if and only if $\tilde{T}(f) \overset{\vec{\tau}}{\approx} \tilde{T}(g)$.

(5) $H_\lambda \circ T = \tilde{T} \circ G_\lambda$; that is, the diagram

$$\begin{CD} \mathcal{L}_p(\mu_1) @>T>> \mathcal{L}_p(\bar{\tau}, 1) \\ @V G_\lambda VV @VV H_\lambda V \\ \mathcal{L}_p(\mu_{\lambda^{-2}}) @>\tilde{T}>> \mathcal{L}_p(\bar{\tau}, \lambda^{-1}) \end{CD}$$

commutes.

6. FOURIER-FEYNMAN TRANSFORM AND $\bar{\tau}$ -MEASURABILITY

Brue, in his Ph.D. thesis [1], written under the direction of Cameron, introduced the concept of a Fourier-Feynman transform. Since then, Fourier-Feynman transforms have been studied by Cameron, Storvick, Johnson, Skoug, Park, and many others; see [7, p. 636] for additional references. Because of measurability problems, all of the functions F on $C_0[a, b]$ and all of the functions f on \mathbb{R}^n considered in [4], [9] were assumed to be Borel measurable; also see [8, pp. 170-171] for further discussion. In this section we will extend various results in [4], [9], in particular, Theorem 0.1, Theorem 1.1, and Theorem 2.1 of [9], to the case where the functions F on $C_0[a, b]$ are $\bar{\tau}$ -measurable and the functions f on \mathbb{R}^n are Lebesgue measurable.

Definition 6.1. Let p be a given number with $1 < p \leq 2$. Let H_n ($n = 1, 2, \dots$) and H be $\bar{\tau}$ -measurable functions on $C_0[a, b]$. We say that H is the $\bar{\tau}$ -limit in the mean of order p' of H_n over $C_0[a, b]$ provided that for each $\rho > 0$,

$$(6.1) \quad \lim_{n \rightarrow \infty} \int_{C_0[a, b]} |H_n(\rho y) - H(\rho y)|^{p'} d\bar{m}_1(y) = 0.$$

Then we write

$$(6.2) \quad \lim_{n \rightarrow \infty} (W_\tau^{p'}) H_n \overset{\bar{\tau}}{\approx} H$$

and we call H the $\bar{\tau}$ -limit in the mean of order p' of H_n over $C_0[a, b]$. A similar definition is understood when n is replaced by a continuously varying parameter. Let q be a nonzero real number. We define the L_p -analytic Fourier-Feynman transform of F , denoted by $\hat{T}_q^{(p)}(F)$, by the formula

$$(6.3) \quad \hat{T}_q^{(p)}(F)(y) \equiv \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} (W_\tau^{p'}) T_\lambda(F)(y)$$

for $1 < p \leq 2$ whenever this limit exists. We define the L_1 -analytic Fourier-Feynman transform of F , which we denote by $\hat{T}_q^{(1)}(F)$, as the functional (if it exists) on $C_0[a, b]$ such that

$$(6.4) \quad \hat{T}_q^{(1)}(F)(y) \equiv \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} T_\lambda(F)(y)$$

for $\bar{\tau}$ -a.e. y .

Definition 6.2. Let n be a natural number and let $a = t_0 < t_1 < t_2 < \dots < t_n \leq b$. For $1 \leq p \leq 2$, let $\bar{\mathcal{A}}_n^{(p)}$ be the space of functions F of the form

$$(6.5) \quad F(x) = f(x(t_1), x(t_2), \dots, x(t_n))$$

$\bar{\tau}$ -a.e. where f is an element of $L_p(\mathbb{R}^n)$.

Let p be a real number with $1 \leq p \leq 2$, and let q be a nonzero real number. Let $\mathcal{M}(\bar{\tau})$ be the space of all $\bar{\tau}$ -measurable functionals on $C_0[a, b]$ and let $\mathcal{MF}(\bar{\tau})$ be the space of all $\bar{\tau}$ -measurable functionals F on $C_0[a, b]$ such that there exists a Fourier-Feynman transform $\hat{T}_q^{(p)}(F)$ of F . Then, by the Minkowski inequality, linearity of T_λ and the properties of limit, we can easily check that $\hat{T}_q^{(p)}$ is a linear transformation from $\mathcal{MF}(\bar{\tau})$ into $\mathcal{M}(\bar{\tau})$.

From Definitions 6.1 and 6.2, we can prove the next theorem, which is the counterpart to Theorem 0.1 in [9].

Theorem 6.3. *Let q be a nonzero real number and let $1 \leq p \leq 2$ be given. Let F_1 and F_2 be $\bar{\tau}$ -measurable on $C_0[a, b]$ such that $F_1 \overset{\bar{\tau}}{\approx} F_2$. Then the following statements hold.*

- (1) $F_1(\cdot + y) \overset{\bar{\tau}}{\approx} F_2(\cdot + y)$ for $\bar{\tau}$ -a.e. y .
- (2) If, for $\bar{\tau}$ -a.e. y , $T_\lambda(F_1)(y)$ exists for λ in \mathbb{C}_+ , then, for $\bar{\tau}$ -a.e. y , $T_\lambda(F_2)(y)$ exists for λ in \mathbb{C}_+ and $T_\lambda(F_1) \overset{\bar{\tau}}{\approx} T_\lambda(F_2)$.
- (3) If $\hat{T}_q^{(p)}(F_1)$ exists, then $\hat{T}_q^{(p)}(F_2)$ exists and $\hat{T}_q^{(p)}(F_1) \overset{\bar{\tau}}{\approx} \hat{T}_q^{(p)}(F_2)$.

Proof. Let $F = F_1 - F_2$ and $N = \{x \text{ in } C_0[a, b] \mid F(x) \neq 0\}$. Then N is $\bar{\tau}$ -null. By Theorem 4.1, for $\bar{\tau}$ -a.e. y , $N - y = \{x - y \text{ in } C_0[a, b] \mid F(x) \neq 0\} = \{z \text{ in } C_0[a, b] \mid F(z + y) \neq 0\}$ is $\bar{\tau}$ -null; hence we have proved statement (1). By statement (1) in this theorem, for $\bar{\tau}$ -a.e. y and for all positive real number λ ,

$$(6.6) \quad \int_{C_0[a, b]} F_1(\lambda^{-\frac{1}{2}}x + y) \, d\bar{m}_1(x) = \int_{C_0[a, b]} F_2(\lambda^{-\frac{1}{2}}x + y) \, d\bar{m}_1(x).$$

Defining $N_1 = \{y \text{ in } C_0[a, b] \mid T_\lambda(F_1)(y) \text{ does not exist for } \lambda \text{ in } \mathbb{C}_+\}$ and $N_2 = \{y \text{ in } C_0[a, b] \mid T_\lambda(F_1)(y) \text{ exists but the equality (6.6) fails}\}$, we see that $N_1 \cup N_2$ is $\bar{\tau}$ -null and for y in $(N_1 \cup N_2)^c$ the equality (6.6) holds, which implies that for $\bar{\tau}$ -a.e. y , $T_\lambda(F_2)(y)$ exists for λ in \mathbb{C}_+ and $T_\lambda(F_1) \overset{\bar{\tau}}{\approx} T_\lambda(F_2)$, as desired.

To prove statement (3), we will need to consider two cases: (a) $p = 1$ and (b) $1 < p \leq 2$.

(a) Let $N_3 = \{y \text{ in } C_0[a, b] \mid T_\lambda(F_1)(y) \neq T_\lambda(F_2)(y) \text{ for } \lambda \text{ in } \mathbb{C}_+\}$ and $N_4 = \{y \text{ in } C_0[a, b] \mid \hat{T}_q^{(1)}(F_1)(y) \text{ does not exist}\}$. Then $N_3 \cup N_4$ is $\bar{\tau}$ -null and, for y in $(N_3 \cup N_4)^c$,

$$(6.7) \quad \hat{T}_q^{(1)}(F_1)(y) = \lim_{\lambda \rightarrow -iq} T_\lambda(F_1)(y) = \lim_{\lambda \rightarrow -iq} T_\lambda(F_2)(y) = \hat{T}_q^{(1)}(F_2)(y),$$

which implies that the L_1 -Fourier-Feynman transform $\hat{T}_q^{(1)}(F_2)(y)$ of F_2 exists $\bar{\tau}$ -a.e. y and $\hat{T}_q^{(1)}(F_1) \overset{\bar{\tau}}{\approx} \hat{T}_q^{(1)}(F_2)$.

(b) Let $N_5 = \{y \text{ in } C_0[a, b] \mid \hat{T}_q^{(p)}(F_1)(y) \text{ is not well-defined}\}$. Then $N_3 \cup N_5$ is $\bar{\tau}$ -null. Letting $\hat{T}_q^{(p)}(F_2)(y) = \hat{T}_q^{(p)}(F_1)(y)$ on $(N_3 \cup N_5)^c$ and $\hat{T}_q^{(p)}(F_2)(y) = 0$ on

$N_3 \cup N_5$, we get $\hat{T}_q^{(p)}(F_1) \overset{\bar{\tau}}{\approx} \hat{T}_q^{(p)}(F_2)$ and, for all positive real numbers ρ ,

$$\begin{aligned}
 & \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[a,b]} | (T_\lambda(F_2) - \hat{T}_q^{(p)}(F_2))(\rho y) |^{p'} d\bar{m}_1(y) \\
 &= \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[a,b] \setminus (N_3 \cup N_5)} | T_\lambda(F_2)(\rho y) - \hat{T}_q^{(p)}(F_2)(\rho y) |^{p'} d\bar{m}_1(y) \\
 &= \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[a,b] \setminus (N_3 \cup N_5)} | T_\lambda(F_1)(\rho y) - \hat{T}_q^{(p)}(F_2)(\rho y) |^{p'} d\bar{m}_1(y) \\
 &= \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[a,b]} | T_\lambda(F_1)(\rho y) - \hat{T}_q^{(p)}(F_2)(\rho y) |^{p'} d\bar{m}_1(y) \\
 (6.8) \quad &= 0.
 \end{aligned}$$

Hence, we have

$$(6.9) \quad \hat{T}_q^{(p)}(F_2) \overset{\bar{\tau}}{\approx} \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} (W_\tau^{p'}) T_\lambda(F_2),$$

as desired. □

Using Theorem 6.3, we can improve Theorems 1.1 and 1.2 in [9] as follows.

Theorem 6.4. *Let $1 \leq p \leq 2$ and let q be a non-zero real number. Let F in $\bar{\mathcal{A}}_n^{(p)}$ be given by (6.5). Then:*

(1) *The L_p -analytic Fourier-Feynman transform of F , namely $\hat{T}_q^{(p)}(F)$, exists, is in $\bar{\mathcal{A}}_n^{(p)}$ and is given by the formula*

$$(6.10) \quad \hat{T}_q^{(p)}(F) \overset{\bar{\tau}}{\approx} (K_{-iq} f).$$

(2) *For each $\rho > 0$,*

$$(6.11) \quad \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} \int_{C_0[a,b]} | T_{\bar{\lambda}} T_\lambda(F)(\rho y) - F(\rho y) |^p d\bar{m}_1(y) = 0.$$

(3) *$T_{\bar{\lambda}} T_\lambda(F) \rightarrow F$ $\bar{\tau}$ -a.e. as $\lambda \rightarrow -iq$ through \mathbb{C}_+ .*

Proof. Using the above Theorem 5.2 and Lemmas 1.1–1.3 in [9], we can prove statements (1) and (2) by the same method as in the proofs of Lemmas 1.1–1.3 in [9] if we replace Borel measurability by Lebesgue measurability and the concept of s-a.e. by the concept of $\bar{\tau}$ -a.e. By statement (1) in this theorem, we know that $T_\lambda(F)$ and $T_{\bar{\lambda}} T_\lambda(F)$ are in $\bar{\mathcal{A}}_n^{(p)}$ for λ in \mathbb{C}_+ , and by Theorem 5.2, $T_\lambda(F)$ and $T_{\bar{\lambda}} T_\lambda(F)$ are all $\bar{\tau}$ -measurable. Hence, putting $N = \{y \text{ in } C_0[a,b] \mid \text{the limit } \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} T_{\bar{\lambda}} T_\lambda(F)(y) \text{ does not converge to } F(y)\}$, we see that N is $\bar{\tau}$ -measurable. By

statement (3) in Lemma 1.3 in [9], we know that ρN is \bar{m}_1 -null for all positive real numbers ρ . Therefore, N is $\bar{\tau}$ -null, as desired. □

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