INDUCTION THEOREMS
OF SURGERY OBSTRUCTION GROUPS

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Dedicated to Professor Anthony Bak for his sixtieth birthday

Abstract. Let $G$ be a finite group. It is well known that a Mackey functor $\{H \mapsto M(H)\}$ is a module over the Burnside ring functor $\{H \mapsto \Omega(H)\}$, where $H$ ranges over the set of all subgroups of $G$. For a fixed homomorphism $w : G \to \{-1, 1\}$, the Wall group functor $\{H \mapsto L^h_{w}(\mathbb{Z}[H], w|_H)\}$ is not a Mackey functor if $w$ is nontrivial. In this paper, we show that the Wall group functor is a module over the Burnside ring functor as well as over the Grothendieck-Witt ring functor $\{H \mapsto GW_0(\mathbb{Z}, H)\}$. In fact, we prove a more general result, that the functor assigning the equivariant surgery obstruction group on manifolds with middle-dimensional singular sets to each subgroup of $G$ is a module over the Burnside ring functor as well as over the special Grothendieck-Witt ring functor. As an application, we obtain a computable property of the functor described with an element in the Burnside ring.

1. Introduction

Dress’ induction theory ([10], [11], [12]) of Mackey functors has been useful for algebraic computation of Wall’s surgery obstruction groups ([27]) with trivial orientation homomorphisms and related groups (cf. [6], [13], [14]) as well as for applications in transformation groups (e.g. [16], [18], [25], [26]). In this paper, we develop induction theory for surgery obstruction groups appearing in [4], [5] and [19], which allows nontrivial orientation homomorphisms, and by using this generalization and [22, Theorem 1.1] we can construct various group actions on smooth manifolds (e.g. [4], [15], [16], [17], [20], [21], [24]).

Throughout this paper, let $G$ be a finite group, $S(G)$ the set of all subgroups of $G$, and $R$ a principal ideal domain (possibly a commutative field). Hence $R$ is a commutative ring and any finitely generated projective $R$-module is free over $R$. An $R$-module is always assumed to be finitely generated over $R$, unless otherwise stated.

Let $GW_0(R, G)$ denote the Grothendieck-Witt ring in A. Dress [11]. It is well known that the functor $H \mapsto GW_0(R, H)$, $H \in S(G)$, with canonical correspondence of morphisms is a Green functor, which is a special case of Theorem [11, 3] since $GW_0(R, G) = GW_0(R, G, \emptyset)$. Let $C(G)$ denote the set of all cyclic subgroups.

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of $G$. By [11] Theorem 1, the functor $H \mapsto GW_0(R, H)$ is $\mathcal{C}(G)$-hypercomputable in the sense of A. Bak [2]. Let $w : G \to \{-1, 1\}$ be a homomorphism and $n = 2k$ an even integer. If $w$ is nontrivial, the Wall group functor $H \mapsto L_n^h(R[H], w|_H)$ (27), $H \in S(G)$, is not a Mackey functor. Since $L_n^h(R[G], w) = WQ_0(A, \emptyset)$ with $A = (R, G, \emptyset, \emptyset, (−1)^k, w)$, Propositions [12.7] and [2.9] imply that the Wall group functor is a $w$-Mackey functor in the sense of Definition [2.2] and a module over the Burnside ring functor. Furthermore, the Wall group functor is a module over the functor $H \mapsto GW_0(R, H)$, which is a special case of Theorem [12.10]. Thus, we obtain the theorem:

**Theorem 1.1.** Let $w : G \to \{-1, 1\}$ be a homomorphism and $n$ an even integer. Then the Wall group functor $H \mapsto L_n^h(R[H], w|_H)$, $H \in S(G)$, is $\mathcal{C}(G)$-hypercomputable.

The main purpose of this paper is to study the induction–restriction theory of the equivariant surgery obstruction group $SWQ_0(R, G, Q, S, \Theta_G)$ obtained by Bak and Morimoto [5], which consists of equivalence classes of special $\lambda$-quadratic $R[G]$-modules. This surgery obstruction group is determined by a datum

$$D = (R, G, Q, S, \lambda, w, \Theta_G, \rho^{(2)}).$$

The ingredient $\lambda$ stands for a symmetry, namely either 1 or $−1$. Let $G(2)$ denote the subset of $G$ consisting of all elements of order 2. An element $g \in G(2)$ is called $\lambda$-symmetric or $\lambda$-quadratic if $g = \lambda w(g)g^{-1}$ or $g = −\lambda w(g)g^{-1}$, respectively. The ingredients $Q$ and $S$ are conjugation-invariant subsets of $G(2)$ consisting of $\lambda$-quadratic elements and $\lambda$-symmetric ones, respectively. Let $\mathfrak{P}(S)$ denote the set of all subsets of $S$. In a general case, $\Theta_G$ stands for a finite $G$-set and $\rho^{(2)}$ is a $G$-map $\Theta_G \mapsto \mathfrak{P}(S)$. In the case where $S$ and $\Theta_G$ are both empty and $\lambda = (−1)^k$, the group $SWQ_0(Z, G, Q, S, \Theta_G)$ coincides with the Bak group $W_{2k}(Z[G], \Gamma Q, w)$ (see [19]); if moreover $Q$ is also empty, then the group is nothing but the Wall group $L_{2k}^h(Z[G], w)$ (see [27]).

In the current section, since the case $\Theta_G = S$ has interesting applications (e.g. [4], [15], [16]), we let $\Theta_G$ and $\rho^{(2)}$ be the same as the set $S$ and the map $s \mapsto \{s\}$, $s \in S$, respectively.

We detail the pairing

$$SGW_0(Z, G, S, S) \times SWQ_0(Z, G, Q, S, S) \to SWQ_0(Z, G, Q, S, S)$$
in Sections [9] and [10] and show that $SWQ_0(Z, G, Q, S, S)$ is a module over the special Grothendieck-Witt ring $SGW_0(Z, G, S, S)$, which corrects the invalid description [15] page 513, lines 9–10] of the pairing.

The groups $GW_0(R, G)$ and $L_n^h(R[G], w)$ with $n = 2k$ have the hyperelementary computability. Dress proved this fact by studying the index of the subgroup $I(\mathcal{O}_G, GW_0(R, G))$ (11 Theorem 1), which we call the Dress index. The theorem looks technical but is fundamental. It is natural to regard the Burnside ring as a generalization of the ring of integers in the theory of transformation groups. Thus, one expects that some computability of the groups $SGW_0(Z, G, S, S)$ and $SWQ_0(Z, G, Q, S, S)$ can be described with an element in the Burnside ring instead of the Dress index. The following theorems are obtained in this respect.

Let $1_{\Omega(G)}$ denote the unit of the Burnside ring $\Omega(G)$.
Theorem 1.2. Let $S$ be a conjugation-invariant subset of $G$ consisting of elements of order 2, let $F$ be a conjugation-invariant set of subgroups of $G$ such that

$$S \times S \subset \bigcup_{H \in F} H \times H,$$

and let $\beta$ be an element of the Burnside ring $\Omega(G)$ such that

$$\text{Res}^G_H \beta = 1 \Omega(H)$$

for any $H \in F$. If $F$ contains all 2-hyperelementary (resp. cyclic) subgroups of $G$, then, for an arbitrary element $x \in \text{SGW}_0(R,G,S,S)$,

$$(1 - \beta)^2 x = 0$$

(resp. $(1 - \beta)^{2k+3} x = 0$, where $|G| = 2^k m$ with $m$ odd).

We say that $R$ is square identical if

$$r^2 \equiv r \mod 2R \quad \text{for all } r \in R.$$

Theorem 1.3. Let $S$, $\beta$ and $F$ be as in the theorem above. Suppose that $R$ is square identical, and each element of $S$ is $\lambda$-symmetric. Let $Q$ be a conjugation-invariant subset of $G$ consisting of $\lambda$-quadratic elements of order 2. If $F$ contains all 2-hyperelementary (resp. cyclic) subgroups of $G$, then for an arbitrary element $x$ of $\text{SWQ}_0(R,G,Q,S,S)$,

$$(1 - \beta)^2 x = 0$$

(resp. $(1 - \beta)^{2k+3} x = 0$, where $|G| = 2^k m$ with $m$ odd).

Note that the datum $D = (R, G, Q, S, \lambda, w, S, \rho(2))$, where $\rho(2) : S \rightarrow \mathfrak{Q}(S)$ is the “identity map” $s \mapsto \{s\}$, yields the datum $D = (R, H, Q \cap H, S \cap H, \lambda, w|_{H}, S \cap H, \rho(2)|_{S \cap H})$ and determines the group $\text{SWQ}_0(R, H, Q \cap H, S \cap H, S \cap H)$ for each subgroup $H$ of $G$.

Theorem 1.4. Let $G$ be a nonsolvable group and let $R$, $Q$ and $S$ be as in the previous theorem. Then

$$\text{SWQ}_0(R, G, Q, S, S) = \sum_H \text{Ind}^G_H \text{SWQ}_0(R, H, Q \cap H, S \cap H, S \cap H),$$

and the restriction homomorphism

$$\text{Res} : \text{SWQ}_0(R, G, Q, S, S) \longrightarrow \bigoplus_H \text{SWQ}_0(R, H, Q \cap H, S \cap H, S \cap H)$$

is injective, where $H$ ranges over the set of all solvable subgroups of $G$.

Each of Theorems 1.2–1.4 is slightly generalized in Section 13.

The organization of the paper is as follows. In Section 2 we define a $w$-Mackey functor, a Green functor, and a module over a Green functor. In Section 3 we observe basic properties of $\Theta$-positioned $R[G]$-modules, namely induction-restriction properties and the Mackey double coset formula. Section 4 is devoted to observing induction-restriction properties of $\Theta$-positioned Hermitian $R[G]$-modules as well as defining their Grothendieck-Witt rings. In Section 5 we introduce the $\nabla$-invariant of $\Theta$-positioned Hermitian $R[G]$-modules and define the special Grothendieck-Witt groups. Similarly to Wall’s surgery theory, $R[G]$-valued $\lambda$-Hermitian forms are indispensable objects to equivariant surgery theory on manifolds with middle-dimensional singular sets. Section 6 is devoted to observing induction-restriction
properties of $R[G]$-valued $\lambda$-Hermitian modules. Sections 7 and 8 are devoted to defining the Witt groups and the special Witt groups of $\Theta$-positioned quadratic $R[G]$-modules, respectively. The tensor product of a Hermitian $R[G]$-module and a quadratic $R[G]$-module is introduced in Section 9 and it is discussed with $\nabla$-invariants in Section 10. Section 11 is devoted to showing that the Grothendieck-Witt rings and special Grothendieck-Witt rings are Green functors (possibly without unit). In Section 12 we show that the bifunctor assigning the $H$-surgery obstruction group to a subgroup $H$ of $G$ is a module over the special Grothendieck-Witt ring functor. In Section 13 we present applications relevant to $G$-surgery.

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2. Bifunctors, $w$-Mackey functors and Green functors

Let $\mathcal{G}$ denote the category whose objects are subgroups of $G$ and whose morphisms are inclusions $j_{H,K} : H \to K$, where $H \subseteq K \subseteq G$, conjugations $c_{(H,g)} : H \to gHg^{-1}$, where $H \subseteq G$ and $g \in G$, and compositions of those maps. Let $\mathcal{A}$ stand for the category whose objects are abelian groups and whose morphisms are group homomorphisms. We denote by $\mathbb{Z}[S(G)]$ the free abelian group generated by all elements of $S(G)$; hence each element of $\mathbb{Z}[S(G)]$ has the form $\sum_H n_H H$ with $n_H \in \mathbb{Z}$. Let $\Omega(G)$ denote the Burnside ring of $G$ (cf. [7], [8], [9], [23]). In fact, $\Omega(G)$ is the free abelian group generated by all $G$-isomorphism classes $[G/H]$ of finite $G$-sets $G/H$ with $H \in S(G)$. Clearly, one has the canonical homomorphism from $\mathbb{Z}[S(G)]$ to $\Omega(G)$ such that $H \mapsto [G/H]$. In this paper, we mean by a bifunctor

$$L = (L^*, L_*) : \mathcal{G}(G) \to \mathcal{A}$$

a pair consisting of a contravariant functor $L^* : \mathcal{G}(G) \to \mathcal{A}$ and a covariant functor $L_* : \mathcal{G}(G) \to \mathcal{A}$ such that $L_*(H) = L^*(H)$, which is written as $L(H)$, for all $H \in S(G)$. If the context is clear, $f^*$ and $f_*$ stand for $L^*(f)$ and $L_*(f)$ respectively, and $\text{Res}^G_H$ and $\text{Ind}^G_H$ stand for $L^*(j_{H,K})$ and $L_*(j_{H,K})$ respectively. Each bifunctor $L = (L^*, L_*) : \mathcal{G} \to \mathcal{A}$ possesses the canonical pairing

$$\mathbb{Z}[S(G)] \times L(G) \to L(G); \quad \left( \sum_H n_H H, x \right) \mapsto \sum_H n_H \text{Ind}^G_H(\text{Res}^G_H x),$$

for $n_H \in \mathbb{Z}$ and $x \in L(G)$. It is interesting to look for a sufficient condition so that the pairing (2.1) factors through a pairing

$$\Omega(G) \times L(G) \to L(G).$$

If $L$ is a Mackey functor, then, as was seen in [7] Proposition 6.2.3, the pairing (2.1) factors through a pairing (2.2). In the case where the orientation homomorphism $w : G \to \{-1, 1\}$ is not trivial, the Wall group functor $H \mapsto L_H^b(\mathbb{Z}[H], w|_H)$, $H \in S(G)$, is not a Mackey functor; however, it will turn out that the associated pairing (2.1) factors through (2.2).

Let $L : \mathcal{G} \to \mathcal{A}$ be a bifunctor. Note that the kernel of the canonical map $\mathbb{Z}[S(G)] \to \Omega(G)$ is

$$\langle H - gHg^{-1} \mid H \in S(G), \ g \in G \rangle \mathbb{Z},$$

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If

\[(2.3)\quad L^*(j_{H,G})L^*(j_{H,G}) = L^*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G}) \quad (\forall H \in S(G), \forall g \in G),\]

then the pairing \[(2.1)\] factors through \[(2.2)\].

**Proposition 2.1.** Suppose \(L^*(c_{gHg^{-1},g^{-1}}) = L^*(c_{(H,g)})\) for all \(H \in S(G)\) and \(g \in G\). Then the equality \[(2.3)\] holds if and only if

\[(1)\quad L^*(c_{(G,g)})L^*(j_{H,G}) = L^*(j_{H,G})L^*(c_{(G,g)})\]

for all \(H \in S(G)\) and \(g \in G\).

**Proof.** By definition, the diagrams

\[
\begin{array}{ccc}
L(G) & \xrightarrow{L^*(j_{H,G})} & L(H) \\
\downarrow & & \downarrow \\
L^*(c_{(G,g^{-1})}) & \xrightarrow{L^*(c_{gHg^{-1},g^{-1}})} & L(gHg^{-1}) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
L(H) & \xrightarrow{L^*(j_{H,G})} & L(G) \\
\downarrow & & \downarrow \\
L^*(c_{(gHg^{-1},g^{-1})}) & \xrightarrow{L^*(c_{(G,g^{-1})})} & L(G) \\
\end{array}
\]

commute. By using the hypothesis above, we obtain the commutative diagram

\[
\begin{array}{ccc}
L(G) & \xrightarrow{L^*(j_{H,G})L^*(j_{H,G})} & L(G) \\
\downarrow & & \downarrow \\
L^*(c_{(G,g^{-1})}) & \xrightarrow{L^*(c_{(G,g^{-1})})} & L(G) \\
\end{array}
\]

Thus \[(2.3)\] holds if and only if

\[
L^*(c_{(gHg^{-1},G)})L^*(j_{gHg^{-1},G}) = L^*(j_{gHg^{-1},G})L^*(c_{(G,g^{-1})}),
\]

namely

\[
L^*(c_{(G,g^{-1})})L^*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G}) = L^*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G})L^*(c_{(G,g^{-1})}).
\]

This concludes the proposition. \(\square\)

Let \(w : G \to \{-1, 1\}\) be a homomorphism. We introduce a slight generalization of a Mackey functor (cf. [2], [7]).

**Definition 2.2.** A bifunctor \(M = (M^*, M_*\) from \(\mathcal{G}\) to \(\mathcal{A}\) is called a \(w\)-Mackey functor if the following conditions (1)–(3) are fulfilled:

1. \(M_*(c_{(H,g)}) = M^*(c_{gHg^{-1},g^{-1}})\) for all \(H \in S(G)\) and \(g \in G\),
2. \(M^*(c_{(H,h)}) = w(h)id_{M(H)}\) (hence \(M_*(c_{(H,h)}) = w(h)id_{M(H)}\)) for all \(H \in S(G)\) and \(h \in H\),
3. \(M^*(j_{K,G}) \circ M_*(j_{H,G})\) coincides with

\[
\bigoplus_{KgH \in K\setminus G/H} M_*(j_{KgHg^{-1},K}) \circ (w(g)M_*(c_{(H,g^{-1}Kg,g)})) \circ M^*(j_{Hg^{-1}Kg,H})
\]
for any $H, K \in S(G)$.

A $w$-Mackey functor for trivial $w$ is an ordinary Mackey functor. We will see that if $w$ is nontrivial, then the Wall group functor $H \mapsto L^n_{\mathbb{H}}(\mathbb{H}[H], w|_H)$ is not an ordinary Mackey functor but a $w$-Mackey functor (cf. Propositions 6.6 6.7 6.8 12.3 12.5 and 12.6). The next proposition is clear by definition.

**Proposition 2.3.** If $M = (M^*, M_*)$ is a $w$-Mackey functor, then $L = (L^*, L_*)$, given so that

\[
L \quad \text{ordinary Mackey functor but a} \quad w-Mackey \quad \text{functor.}
\]

where $H \in S(G)$. Such a pairing is called a Frobenius pairing if the following conditions (1)–(3) are satisfied for any morphism $f : H \to K$ in $\mathcal{S}$:

1. $N^*(f)(x \cdot y) = (L^*(f) x \cdot (M^*(f)y)$ for all $x \in L(K), y \in M(K)$,
2. $L_*(f)(x) \cdot y = N_*(f)((L^*(f)x) \cdot y)$ for all $x \in L(K), y \in M(H)$,
3. $L_*(f)(y) = N_*(f)(x \cdot M^*(f)y)$ for all $x \in L(H), y \in M(K)$.

Each of (2), (3) is referred to as the Frobenius reciprocity law.

Let us recall the definition of a Green functor.

**Definition 2.4.** Let $L$, $M$ and $N$ be bifunctors from $\mathcal{G}$ to $\mathcal{A}$. A pairing $L \times M \to N$ is a family of biadditive maps

\[
L(H) \times M(H) \to N(H); \quad (x, y) \mapsto x \cdot y,
\]

where $H$ runs over $S(G)$. Such a pairing is called a Frobenius pairing if the following conditions (1)–(3) are satisfied for any morphism $f : H \to K$ in $\mathcal{G}$:

1. $N^*(f)(x \cdot y) = (L^*(f) x \cdot (M^*(f)y)$ for all $x \in L(K), y \in M(K)$,
2. $L_*(f)(x) \cdot y = N_*(f)((L^*(f)x) \cdot y)$ for all $x \in L(K), y \in M(H)$,
3. $L_*(f)(y) = N_*(f)(x \cdot M^*(f)y)$ for all $x \in L(H), y \in M(K)$.

Each of (2), (3) is referred to as the Frobenius reciprocity law.

Let us recall the definition of a Green functor.

**Definition 2.5.** A Mackey functor $M = (M_*, M^*) : \mathcal{G} \to \mathcal{A}$ is called a Green functor if each $M(H)$, $H \in S(G)$, is a ring with unit and the associated pairing $M \times M \to M$ is a Frobenius pairing. If the existence of the unit in $M(H)$ is not guaranteed, then $M$ is referred as a Green functor, possibly without unit.

The Burnside ring functor $H \mapsto \Omega(G)$ is a Green functor. Let $U : \mathcal{G} \to \mathcal{A}$ be a Green functor. We mean by a $U$-module $L$ (or a module $L$ over $U$) a bifunctor $L : \mathcal{G} \to \mathcal{A}$ equipped with a Frobenius pairing $U \times L \to L$.

**Proposition 2.6.** A $w$-Mackey functor $M$ is a module over the Burnside ring functor.

*Proof.* Let $L$ be the Mackey functor associated with $M$ in Proposition 2.3. By [7 Proposition 6.2.3], $L$ is a module over the Burnside ring functor. Hence, $L$ satisfies the equality (1) in Proposition 2.1. By using the relations between $M$ and $L$ in Proposition 2.3, we can check that $M$ satisfies the equality (1) in Proposition 2.1 and furthermore that $M$ is a module over the Burnside ring functor. □

**Proposition 2.7.** A module over a Green functor is a module over the Burnside ring functor.

*Proof.* Let $L = (L^*, L_*) : \mathcal{G} \to \mathcal{A}$ be a module over a Green functor $U = (U^*, U_*) : \mathcal{G} \to \mathcal{A}$. Then the associated pairing

\[
\Omega(H) \times L(H) \to L(H)
\]
can be defined so that \( a \cdot x = (a \cdot 1_{U(H)}) \cdot x \) for \( a \in \Omega(H) \) and \( x \in L(H) \), where \( 1_{U(H)} \) is the identity element of \( U(H) \). It is straightforward to check the Frobenius reciprocity laws of the pairing.

\[ \Box \]

3. \( \Theta \)-Positioned \( R[G] \)-modules

Let \( \Theta \) be a finite \( G \)-set. A pair \((M, \alpha)\) consisting of an \( R[G] \)-module \( M \) and a \( G \)-map \( \alpha : \Theta \to M \) is called a \( \Theta \)-positioned \( R[G] \)-module. Let \( H \) and \( K \) be finite groups and \( \varphi : H \to K \) a homomorphism. For a finite \( H \)-set \( X \), we define the \( K \)-set \( K \times_{H, \varphi} X \) as the quotient set of \( K \times X \) with respect to the equivalence relation \( \sim \) generated by \((k\varphi(h), x) \sim (khx), h \in H\). The set \( K \times_{H, \varphi} X \) is also denoted by \( K \times X \) or \( K \times_{H, \varphi} X \) if the context is clear. For an \( R[H] \)-module \( M \), the \( R[K] \)-module \( R[K] \otimes_{R[H], \varphi} M \) is defined as follows. Let \( F(R[K] \times M) \) denote the \( R \)-free module with basis \( R[K] \times M \) which may not be finitely generated over \( R \).

Let \( T \) denote the \( R \)-submodule generated by all elements of the form

\[
\begin{align*}
  &r(a, x) - (ra, x), \quad r(a, x) - (a, rx), \\
  &\quad (a + b, x) - (a, x) - (b, x), \quad (a, x + y) - (a, x) - (a, y), \quad \text{or} \\
  &\quad (a\varphi(h), x) - (a, hx),
\end{align*}
\]

where \( r \) ranges over \( R \), \( a \) and \( b \) over \( R[K] \), \( x \) and \( y \) over \( M \), and \( h \) over \( H \). Then \( R[K] \otimes_{R[H], \varphi} M \) is defined to be the quotient module \( F(R[K] \times M) / T \), which will also be denoted by \( R[K] \otimes_{R[H], \varphi} M \) or \( R[K] \otimes_{R[H]} M \). The element of the module represented by \((a, x) \in F(R[K] \times M)\) is denoted by \( a \otimes_{R[H], \varphi} x \), which will also be written as \( a \otimes_{\varphi} x \), \( a \otimes_{R[H]} x \) or \( a \otimes x \) if the context is clear. The \( K \)-action on \( R[K] \otimes_{R[H], \varphi} M \) is given by \((k, a) \otimes_{R[H], \varphi} x \mapsto (ka) \otimes_{R[H], \varphi} x \).

Let \( \Theta_H \) be a finite \( H \)-set, \( \Theta_K \) a finite \( K \)-set, and \( \psi : \Theta_H \to \Theta_K \) a \( \varphi \)-equivariant map, namely

\[
\psi(ht) = \varphi(h)\psi(t) \quad (h \in H, \ t \in \Theta_H).
\]

Let \( \varphi \) stand for the pair \((\varphi, \psi)\).

For a \( \Theta_K \)-positioned \( R[K] \)-module \( N = (N, \beta) \), we define the \( \Theta_H \)-positioned \( R[H] \)-module \( \varphi^#N = (\varphi^#N, \psi^#\beta) \) so that the underlying \( R \)-module of \( \varphi^#N \) is the same as \( N \) but the \( H \)-action on \( \varphi^#N \) is given by \((h, x) \mapsto \varphi(h)x\) for \( h \in H \), \( x \in \varphi^#N \), and \( \psi^#\beta : \Theta_H \to \varphi^#N \) is given by \( \psi^#\beta(t) = \beta(\psi(t)) \) for \( t \in \Theta_H \).

Proposition 3.1. Let \( \varphi : H \to K \) and \( \psi : \Theta_H \to \Theta_K \) be as above and let \( N_i = (N_i, \beta_i), \ i = 1, 2 \), be \( \Theta_K \)-positioned \( R[K] \)-modules. Then \( \varphi^#(N_1 \otimes_R N_2) = \varphi^#(N_1 \otimes_R \varphi^#N_2) \); namely, \((\varphi^#N_1 \otimes_R \varphi^#N_2, \psi^#(\beta_1 \otimes_R \beta_2))\) is canonically isomorphic to \((\varphi^#(N_1 \otimes_R N_2), \psi^#(\beta_1 \otimes_R \beta_2))\).

Proof. By definition, the underlying \( R \)-modules of \( \varphi^#N_1 \otimes_R \varphi^#N_2 \) and \( \varphi^#N_1 \otimes_R N_2 \) are \( N_1 \otimes_R N_2 \). One can check without difficulties that the \( K \)-actions of the two modules coincide. Moreover, we have

\[
(\psi^#(\beta_1 \otimes_R \psi^#(\beta_2))t) = \beta_1(\psi(t)) \otimes_R \beta_2(\psi(t)) = \psi^#(\beta_1 \otimes_R \beta_2)t
\]

for all \( t \in \Theta_H \). \( \Box \)

To the contrary, for a \( \Theta_H \)-positioned \( R[H] \)-module \( M = (M, \alpha) \), we define the \( \Theta_K \)-positioned \( R[K] \)-module \( \varphi^#M = (\varphi^#M, \psi^#\alpha) \) by \( \varphi^#M = R[K] \otimes_{R[H], \varphi} M \) and

\[
\psi^#\alpha(t) = \sum_{[k, t']} \{ k \otimes_{\varphi} \alpha(t') \mid [k, t'] \in K \times_{H, \varphi} \Theta_H \ \text{such that} \ k \psi(t') = t \} \quad \text{for} \ t \in \Theta_K.
\]
Let \( K \)-equivariance of the map \( \psi_\# \alpha \) holds because, for \( a \in K \) and \( t \in \Theta_K \),

\[
\psi_\# \alpha(at) = \sum_{[k,t'] \in K \times H, \varnothing H} \{ k \otimes \varphi \alpha(t') \mid k\psi(t') = at \} \\
= \sum_{[k,t'] \in K \times H, \varnothing H} \{ k \otimes \varphi \alpha(t') \mid a^{-1}k\psi(t') = t \} \\
= \sum_{[ak',t'] \in K \times H, \varnothing H} \{ ak' \otimes \varphi \alpha(t') \mid k'\psi(t') = t \} \\
= a \sum_{[ak',t'] \in K \times H, \varnothing H} \{ k' \otimes \varphi \alpha(t') \mid k'\psi(t') = t \} \\
= a \sum_{[k',t'] \in K \times H, \varnothing H} \{ k' \otimes \varphi \alpha(t') \mid k'\psi(t') = t \} \\
= a \psi_\# \alpha(t).
\]

**Proposition 3.2.** Let \( H \) be a subgroup of \( G \), \( M = (M, \alpha) \) a \( \Theta_H \)-positioned \( R[H] \)-module, \( g \) an element of \( G \), and \( \psi : \Theta_H \rightarrow \Theta_{gH^{-1}} \) a \( c_{H,g} \)-equivariant bijection. Then the diagram

\[
\begin{array}{ccc}
\Theta_{gH^{-1}} \quad \psi_\# \alpha \quad c_{(H,g) \#} M \\
\downarrow \psi^{-\#} \alpha \quad \downarrow f_0 \\
\psi_\# c_{gH^{-1},g^{-1}} M \\
\end{array}
\]

commutes, where \( f_0 : c_{(H,g) \#} M \rightarrow c_{(gH^{-1},g^{-1})} M \) is the \( R[gHg^{-1}] \)-isomorphism such that

\[
f_0(e \otimes_{H,c_{(H,g)}} x) = x \quad \text{for } x \in M.
\]

**Proof.** Let \( t \) be an element of \( \Theta_H \). Then by definition we have \( \psi_\# \alpha(\psi(t)) = e \otimes_{H,c_{(H,g)}} \alpha(t) \) and \( \psi^{-\#} \alpha(\psi(t)) = \alpha(t) \), which concludes the proposition. \( \Box \)

**Proposition 3.3.** Let \((H,\Theta_H), (K,\Theta_K)\), and \( \varphi = (\varphi, \psi) \) be as above. Then for a \( \Theta_H \)-positioned \( R[H] \)-module \( (M, \alpha) \) and a \( \Theta_K \)-positioned \( R[K] \)-module \( (N, \beta) \), the Frobenius reciprocity law holds; namely, the following diagram commutes:

\[
\Theta_K \xrightarrow{(\psi_\# \alpha) \otimes R \beta} (R[K] \otimes_{R[H],\varphi} M) \otimes_R N \\
\downarrow \psi_\#(\alpha \otimes_R \psi_\# \beta) \quad \downarrow f \\
R[K] \otimes_{R[H],\varphi} (M \otimes_R \varphi_\# N),
\]

where \( f \) is the canonical isomorphism such that \( f((k \otimes \varphi x) \otimes y) = k \otimes \varphi (x \otimes k^{-1}y) \) for \( k \in K, x \in M \) and \( y \in N \).

The commutability above is referred to as \( (\psi_\# \alpha) \otimes_R \beta = \psi_\#(\alpha \otimes_R \psi_\# \beta) \).
Proof. The proof runs as follows:

\[ ((\psi^\# \alpha) \otimes_R \beta)(t) = \sum_{[k, t'] \in K \times_H \Theta_H} \{ k \otimes_\varphi \alpha(t') \mid k \psi(t') = t \} \otimes \beta(t) \]

\[ = \sum_{[k, t'] \in K \times_H \Theta_H} \{ (k \otimes_\varphi \alpha(t')) \otimes \beta(t) \mid k \psi(t') = t \} \]

\[ = \sum_{[k, t'] \in K \times_H \Theta_H} \{ k \otimes_\varphi (\alpha(t') \otimes \beta(\psi(t'))) \mid k \psi(t') = t \} \]

\[ \leq \sum_{[k, t'] \in K \times_H \Theta_H} \{ k \otimes_\varphi (\alpha(t') \otimes (\psi^\# \beta)(t')) \mid k \psi(t') = t \} \]

\[ = \sum_{[k, t'] \in K \times_H \Theta_H} \{ k \otimes_\varphi (\alpha \otimes \psi^\# \beta)(t') \mid k \psi(t') = t \} \]

\[ = \sum_{[k, t'] \in K \times_H \Theta_H} \{ k \otimes_\varphi (\alpha \otimes \psi^\# \beta)(t') \mid k \psi(t') = t \} \]

\[ = \psi^\#_\ell (\alpha \otimes_R \psi^\#_t \beta)(t). \]

\[ \square \]

Let \( H \) be a subgroup of \( G \) and \( g \) an element of \( G \). Let \( c_{(H, g)} : H \to gHg^{-1} \) stand for the conjugation map by \( g \), i.e., \( c_{(H, g)}(h) = ghg^{-1} \) for \( h \in H \). Let \( Z \) be a finite \( G \)-set, \( \Theta_H \) an \( H \)-invariant subset of \( Z \), and \( \Theta_{gHg^{-1}} \) a \( gHg^{-1} \)-invariant subset of \( Z \) such that \( g\Theta_H = \Theta_{gHg^{-1}} \). Then the left translation by \( g \), namely the map \( \ell_{(H, g)} : \Theta_H \to \Theta_{gHg^{-1}} ; t \mapsto gt \), is a \( c_{(H, g)} \)-equivariant bijection. Let \( c_{(H, g)} \) denote the pair \((c_{(H, g)}), \ell_{(H, g)})\). If the context is clear, then we abuse \( c_{(H, g)} \) for \( \ell_{(H, g)} \). Let \( c_{(H, g)} \) denote the map \( c_{(H, g)} \) is a map from \( H \) to itself. Note that the map

\[ f_1 : c_{(H, g)} \# M \to M ; e \otimes_{c_{(H, g)}} x \mapsto gx \]

is an \( R[H] \)-isomorphism. In addition, the map

\[ f_2 : c_{(H, g)}^\# M \to M ; x \mapsto g^{-1}x \]

is an \( R[H] \)-isomorphism.

**Proposition 3.4.** Let \( H \) be a subgroup of \( G \) and \( \Theta_H \) a finite \( H \)-set. Then for any \( \Theta_H \)-positioned \( R[H] \)-module \((M, \alpha)\) and \( g \in H \), the following diagrams commute:
where $f_1$ and $f_2$ are the $R[H]$-isomorphisms given above.

These commutabilities are referred to as $\ell_{(H,g)} \circ \alpha = \alpha$ (or $c_{(H,g)} \circ \alpha = \alpha$) and $\ell_{(H,g)}^\# \circ \alpha = \alpha$ (or $c_{(H,g)}^\# \circ \alpha = \alpha$), respectively.

Proof. The commutabilities follow from the equalities

$$f_1(\ell_{(H,g)}^\# \circ \alpha(t)) = \sum_{[ghg^{-1},t'] \in H \times H \times (H,g)} \{ f_1(ghg^{-1} \otimes c_{(H,g)} \circ \alpha(t')) | ghg^{-1}(gt') = t \}$$

and

$$f_2(\ell_{(H,g)}^\# \circ \alpha(t)) = f_2(\alpha(\ell_{(H,g)}^\#(t))) = f_2(\alpha(gt)) = g^{-1} \alpha(gt) = \alpha(t),$$

for $t \in \Theta_H$. \qed

Let $Z$ be a finite $G$-set. Let $S(G)$ and $\Psi(Z)$ denote the set of all subgroups of $G$ and the set of all subsets of $Z$, respectively. We regard $S(G)$ as a $G$-set by conjugation, and $\Psi(Z)$ has the canonical $G$-action. Let $\Theta : S(G) \to \Psi(Z)$; $H \mapsto \Theta_H$, be a $G$-map. We say that $\Theta$ is intersection preserving if

$$\Theta_H \cap \Theta_K = \Theta_{H \cap K} \quad \text{for all } H, K \in S(G). \quad (3.1)$$

Let $H \subset K$ be subgroups of $G$. Then (3.1) implies $\Theta_H \subset \Theta_K$. Thus, the inclusion map $j_{H,K} : H \to K$ is automatically associated with the inclusion map $j_{\Theta_H, \Theta_K} : \Theta_H \to \Theta_K$, and hence yields the pair $j_{H,K} = (j_{H,K}, j_{\Theta_H, \Theta_K})$.

Usually, we use $\text{Ind}_H^K$ for $j_{H,K}$, $j_{\Theta_H, \Theta_K}$ and $j_{H,K}^\#$ and $\text{Res}_H^K$ for $j_{H,K}^\#$, $j_{\Theta_H, \Theta_K}$ and $j_{H,K}^\#$, if the context is clear.

Next, let $g$ be an element of $G$. Since $\Theta$ is a $G$-map, $\Theta_{gHg^{-1}} = g\Theta_H$ holds for any subgroup $H$ of $G$.

**Proposition 3.5.** Let $\Theta : S(G) \to \Psi(Z)$ be an intersection-preserving $G$-map. Then for arbitrary subgroups $H$ and $K$ of $G$, each $\Theta_H$-positioned $R[H]$-module $M = (M, \alpha)$ satisfies the Mackey double coset formula. Namely,

$$\text{Res}_K^K(\text{Ind}_H^K M) = \bigoplus_{KgH \in K \setminus G/H} \text{Ind}_K^K(\text{Res}_{KgH} H \cap gHg^{-1}, Kg, \alpha) \text{Ind}_H^K H \cap gHg^{-1}, Kg, M).$$
More precisely, the following diagram commutes:

\[
\begin{array}{ccc}
\Theta_H & \xrightarrow{\gamma} & \bigoplus_{K,g,H \in K \setminus G/H} M(K, g, H) \\
\text{Res}_K^\omega \text{Ind}_H^G & \downarrow & \omega \\
\text{Res}_K^G(\text{Ind}_H^G M) & & \end{array}
\]

where

\[
M(K, g, H) = \text{Ind}_{K \cap gHg^{-1}, K,g,H}^K \text{Res}_{H \cap g^{-1}Kg}^H M(K, g, H)
\]

is the following diagram:

\[
\begin{array}{ccc}
\Theta_H & \xrightarrow{\gamma} & \bigoplus_{K,g,H \in K \setminus G/H} M(K, g, H) \\
\text{Res}_K^\omega \text{Ind}_H^G & \downarrow & \omega \\
\text{Res}_K^G(\text{Ind}_H^G M) & & \end{array}
\]

and \(\omega\) is the \(R[K]\)-isomorphism such that

\[
\omega(k \otimes (a \otimes c_{(H \cap g^{-1}Kg, a)} x)) = kg \otimes (g^{-1}ag)x \quad \text{for} \quad k \in K, \quad a \in K \cap gHg^{-1}, \quad x \in M.
\]

**Proof.** Let \(\alpha : \Theta_H \to M\) be an \(H\)-map, and let \(\{g_1, \ldots, g_\ell\}\) be a complete set of representatives of \(K \setminus G/H\). For \(t \in \Theta_K\), we have

\[
(\text{Res}_K^G \text{Ind}_H^G \alpha)(t)
\]

\[
= \sum_{j=1}^\ell \{g \otimes \alpha(t') \mid [g, t'] \in G \times_H \Theta_H, \ g t' = t\}
\]

\[
= \sum_{j=1}^\ell \sum_{j=1}^\ell \{gg_j \otimes \alpha(t') \mid [gg_j, t'] \in K g_j H \times_H \Theta_H, \ g \in K, \ gg_j t' = t\}
\]

\[
= \sum_{j=1}^\ell \sum_{j=1}^\ell \{gg_j \otimes \alpha(t') \mid [gg_j, t'] \in K g_j \times_{H \cap g_j^{-1}Kg_j} \Theta_H, \ g \in K, \ gg_j t' = t\}
\]

\[
= \sum_{j=1}^\ell \sum_{j=1}^\ell \{gg_j \otimes \alpha(t') \mid [gg_j, t'] \in K g_j \times_{H \cap g_j^{-1}Kg_j} \Theta_H \cap g_j^{-1}Kg_j, \ g \in K, \ gg_j t' = t\}
\]

in \(\text{Res}_K^G \text{Ind}_H^G M\)

and

\[
(\text{Ind}_{K \cap g_j H g_j^{-1}, K,g_j}^K \text{Res}_{H \cap g_j^{-1}Kg_j}^H \alpha)(t)
\]

\[
= \sum_{j=1}^\ell \{g \otimes \ell_{(H \cap g_j^{-1}Kg_j, g_j)} \text{Res}_{H \cap g_j^{-1}Kg_j}^H \alpha(t') \mid [g, t'] \in K \times_{K \cap g_j H g_j^{-1}} \Theta_K \cap g_j H g_j^{-1}, \ g t' = t\}
\]

\[
= \sum_{j=1}^\ell \{g \otimes (e \otimes \alpha(g_j^{-1} t')) \mid [g, t'] \in K \times_{K \cap g_j H g_j^{-1}} \Theta_K \cap g_j H g_j^{-1}, \ g t' = t\}
\]

\[
= \sum_{j=1}^\ell \{g \otimes (e \otimes \alpha(t'')) \mid [gg_j, t''] \in K g_j \times_{H \cap g_j^{-1}Kg_j} \Theta_H \cap g_j^{-1}Kg_j, \ gg_j t'' = t\}
\]

\[
= \sum_{j=1}^\ell \{gg_j \otimes \alpha(t'') \mid [gg_j, t''] \in K g_j \times_{H \cap g_j^{-1}Kg_j} \Theta_H \cap g_j^{-1}Kg_j, \ gg_j t'' = t\}.
\]

The proposition follows immediately from these equalities. \(\square\)
4. POSITIONED HERMITIAN $R[G]$-MODULES

In this section we introduce the Grothendieck-Witt rings of Θ-positioned Hermitian $R[G]$-modules.

**Definition 4.1.** Let $M$ be an $R[G]$-module. A map $B : M \times M \rightarrow R$ is called a Hermitian form on $M$ if the following conditions (1)–(3) are satisfied:

1. $B$ is $R$-bilinear,
2. $B$ is $G$-invariant, namely $B(gx, gy) = B(x, y)$,
3. $B$ is symmetric, namely $B(x, y) = B(y, x)$,

for all $x, y \in M$ and $g \in G$. A couple $(M, B)$ consisting of an $R[G]$-module $M$ and a Hermitian form $B$ on $M$ is called a Hermitian $R[G]$-module (or simply Hermitian module).

A Hermitian $R[G]$-module $(M, B)$ such that $M$ is a free $R$-module is said to be nonsingular if the associated map

$$M \rightarrow M^\# = \text{Hom}_R(M, R); \ x \mapsto B(x, -)$$

is bijective.

Let $H$ and $K$ be finite groups and $\varphi : H \rightarrow K$ a monomorphism. A Hermitian $R[K]$-module $(N, B)$ yields a Hermitian $R[H]$-module $(\varphi^# N, \varphi^# B)$ in a canonical way. By definition $\varphi^# N$ is nothing but $N$ as an $R$-module. The map $\varphi^# B : \varphi^# N \times \varphi^# N \rightarrow R$ is also the same as $B : N \times N \rightarrow R$. Clearly, $\varphi^# B$ is $R$-bilinear and symmetric. It is obvious that if $B$ is nonsingular, then so is $\varphi^# B$. Since

$$\varphi^# B(hx, hy) = B(\varphi(h)x, \varphi(h)y) = B(x, y)$$

for $h \in H$, $x, y \in \varphi^# N$, it follows that $\varphi^# B$ is $H$-invariant.

**Proposition 4.2.** Let $\varphi : H \rightarrow K$ be a monomorphism and let $(N_i, B_i)$, $i = 1, 2$, be Hermitian $R[K]$-modules. Then

$$(\varphi^# N_1 \otimes_R \varphi^# N_2, \varphi^# B_1 \otimes_R \varphi^# B_2) = (\varphi^# (N_1 \otimes_R N_2), \varphi^# (B_1 \otimes B_2)).$$

This proposition is obviously true.

Let $(M, B)$ be a Hermitian $R[H]$-module. Then, by definition,

$$\varphi^# M = R[K] \otimes_{R[H], \varphi} M.$$

We define the $R$-bilinear form

$$\varphi^# B : \varphi^# M \times \varphi^# M \rightarrow R,$$

so that

$$\varphi^# B(a \otimes \varphi x, b \otimes \varphi y) = \delta_{a \varphi(H), b \varphi(H)} B(x, \varphi^{-1}(a^{-1} b)y),$$

for $a, b \in K$ and $x, y \in M$, where $\delta_{a \varphi(H), b \varphi(H)} = 1$ if $a \varphi(H) = b \varphi(H)$, and $\delta_{a \varphi(H), b \varphi(H)} = 0$ otherwise. It is clear that $\varphi^# B$ is $K$-invariant and symmetric. If $B$ is nonsingular, then so is $\varphi^# B$. 

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Proposition 4.3. Let $H$ be a subgroup of $G$, $B$ a Hermitian form on an $R[H]$-module $M$, and $g$ an element of $G$. Then the diagram
\[
c_{(H,g)^{-1}}M \times c_{(H,g)^{-1}}M \quad \xrightarrow{f_0 \times f_0} \quad c_{(H,g)^{-1}}B \\
\xrightarrow{c_{(gHg^{-1},g^{-1})}M \times c_{(gHg^{-1},g^{-1})}M} \quad \xrightarrow{c_{(gHg^{-1},g^{-1})}B} \quad R
\]
commutes, where $f_0$ is the canonical $R[gHg^{-1}]$-isomorphism (cf. Proposition 3.2).

The proof is straightforward.

Proposition 4.4. Let $\varphi : H \to K$ be a monomorphism, and let $B$ and $B'$ be Hermitian forms on an $R[H]$-module $M$ and an $R[K]$-module $N$, respectively. Then the following diagram commutes:
\[
M_1 \times M_1 \quad \xrightarrow{f \times f} \quad M_2 \times M_2 \quad \xrightarrow{\varphi_0 B \otimes_R B'} \quad R,
\]
where $M_1 = (R[K] \otimes_R [H],\varphi M) \otimes_R N$, $M_2 = R[K] \otimes_R [H],\varphi (M \otimes_R \varphi^# N)$, and $f$ is the canonical isomorphism (cf. Proposition 3.3).

Proof: The commutativity follows from
\[
\varphi_0 B \otimes_R B'((a \otimes x) \otimes u, (b \otimes y) \otimes v) = \varphi_0 B(a \otimes x, (b \otimes y))B'(u, v)
\]
and
\[
\varphi_0 (B \otimes_R \varphi^# B')(a \otimes (x \otimes a^{-1}u), b \otimes (y \otimes b^{-1}v))
\]
\[
= \delta_{u\varphi(H),b\varphi(H)}(B(x, \varphi^{-1}(a^{-1}b)y)B'(a^{-1}u, \varphi^{-1}(a^{-1}b)v), \quad \text{for } a, b \in K, x, y \in M, \text{ and } u, v \in N.
\]

Proposition 4.5. Let $H$ be a subgroup of $G$ and $(M, B)$ a Hermitian $R[H]$-module. Then for any $g \in H$, the following diagrams commute:
\[
c_{(H,g)^{-1}}M \times c_{(H,g)^{-1}}M \quad \xrightarrow{f_1 \times f_1} \quad c_{(H,g)^{-1}}B \\
\xrightarrow{c_{(gHg^{-1},g^{-1})}M \times c_{(gHg^{-1},g^{-1})}M} \quad \xrightarrow{c_{(gHg^{-1},g^{-1})}B} \quad R
\]
where \( f_1 \) and \( f_2 \) are the canonical isomorphisms (cf. Proposition 3.4).

**Proof.** The commutability of the first diagram follows from
\[
c_{(H,g)} B(e \otimes x, e \otimes y) = B(x, y)
\]
and
\[
B(f_1(e \otimes x), f_1(e \otimes y)) = B(gx, gy) = B(x, y).
\]
The commutability of the second diagram follows from
\[
c_{(H,g)} B(x, y) = B(x, y)
\]
and
\[
B(f_2(x), f_2(y)) = B(g^{-1}x, g^{-1}y) = B(x, y).
\]

\[
\square
\]

**Proposition 4.6.** For any subgroups \( H \) and \( K \) of \( G \), each Hermitian \( R[H] \)-module \((M, B)\) satisfies the Mackey double coset formula. Namely,
\[
\text{Res}_K^G \text{Ind}_H^K B = \bigoplus_{KgH \in K \backslash G / H} \text{Ind}_K^K c_{(H,g^{-1}Kg,g)} \text{Res}_H^K B.
\]

More precisely, the following diagram commutes:
\[
\begin{array}{ccc}
\bigoplus_{KgH} M(K, g, H) & \times & \bigoplus_{KgH} M(K, g, H) \\
\omega \times \omega & \downarrow & \\
\text{Res}_K^G \text{Ind}_H^K M & \times & \text{Res}_K^G \text{Ind}_H^K M \\
\end{array}
\]
\[
\begin{array}{c}
\rightarrow R, \\
\omega \times \omega
\end{array}
\]
\[
\text{Res}_K^G \text{Ind}_H^K M \times \text{Res}_K^G \text{Ind}_H^K M \\
\text{Res}_K^G \text{Ind}_H^K B
\]
\[
\begin{array}{c}
\rightarrow R, \\
\omega \times \omega
\end{array}
\]
\[
\text{Res}_K^G \text{Ind}_H^K B
\]

where \( KgH \) runs over \( K \backslash G / H \),
\[
M(K, g, H) = \text{Ind}_K^K c_{(H,g^{-1}Kg,g)} \text{Res}_H^K B(KgHg^{-1}Kg,M)
\]
and \( \omega \) is the canonical isomorphism (cf. Proposition 3.4).

**Proof.** For \( u, v \in R[K] \otimes R[KgHg^{-1}Kg] \) \( c_{(H,g^{-1}Kg,g)} \text{Res}_H^K B(KgHg^{-1}Kg,M) \) with \( u = a \otimes (e \otimes x) \) and \( v = b \otimes (e \otimes x) \) respectively, where \( a, b \in K, x, y \in \text{Res}_H^K B(KgHg^{-1}Kg,M) \), we have
\[
\text{Ind}_K^K c_{(H,g^{-1}Kg,g)} \text{Res}_H^K B(u, v) = \delta_{a(KgHg^{-1}), b(KgHg^{-1})} c_{(H,g^{-1}Kg,g)} \text{Res}_H^K B(e \otimes x, a^{-1}b(e \otimes y)) = \delta_{a(KgHg^{-1}), b(KgHg^{-1})} B(x, g^{-1}a^{-1}bgy)
\]
We say that \( \text{Im}(\alpha) = \delta_{agH,bgH}B(x, (ag)^{-1}bgy) \) satisfy, where
\[
\text{Res}_K^G\text{Ind}_H^G B(ag \otimes x, bg \otimes y) = \delta_{agH,bgH}B(x, (ag)^{-1}bgy) \\
= \delta_{agH,bgH}B(x, g^{-1}a^{-1}bgy) \\
= \delta_{a(K \cap gHg^{-1}), b(K \cap gHg^{-1})}B(x, g^{-1}a^{-1}bgy).
\]
Thus we obtain the proposition. \(\square\)

**Definition 4.7.** Let \( \Theta \) be a finite \( G \)-set. A triple \((M, B, \alpha)\) consisting of a Hermitian \( R[G] \)-module \((M, B)\) and a \( G \)-map \( \alpha : \Theta \to M \) is called a \( \Theta \)-positioned Hermitian \( R[G] \)-module (or simply \( \Theta \)-positioned Hermitian module).

Let \( \mathcal{H}(R, G, \Theta) \) stand for the family of all \( \Theta \)-positioned Hermitian \( R[G] \)-modules \((M, B, \alpha)\) such that \( M \) is an \( R \)-free \( R[G] \)-module and \( B : M \times M \to R \) is nonsingular. We say that \( \alpha \) is totally isotropic (resp. trivial) if \( B(\text{Im}(\alpha), \text{Im}(\alpha)) = 0 \) (resp. \( \text{Im}(\alpha) = 0 \)). We set
\[
\mathcal{H}(R, G, \Theta)^{t-iso} = \{(M, B, \alpha) \in \mathcal{H}(R, G, \Theta) \mid \alpha \text{ is totally isotropic}\}, \\
\mathcal{H}(R, G, \Theta)^{triv} = \{(M, B, \alpha) \in \mathcal{H}(R, G, \Theta) \mid \alpha \text{ is trivial}\}.
\]
Let \( K \mathcal{H}_0(R, G, \Theta), \ K \mathcal{H}_0(R, G, \Theta)^{t-iso} \) and \( K \mathcal{H}_0(R, G) \) denote the Grothendieck groups of \( \mathcal{H}(R, G, \Theta), \mathcal{H}(R, G, \Theta)^{t-iso} \) and \( \mathcal{H}(R, G, \Theta)^{triv} \), respectively, under orthogonal sum.

Let \( M = (M, B, \alpha) \) be an object in \( \mathcal{H}(R, G, \Theta) \). An \( R \)-direct summand, \( R[G] \)-submodule \( U \) of \( M \) is called a Quillen submodule of \( M \) if \( U \subseteq U^\perp \) and \( \text{Im}(\alpha) \subseteq U \) both hold, where
\[
U^\perp = \{x \in M \mid B(x, y) = 0 \ (\forall y \in U)\}.
\]
In this case, \( (M, U) \) is called a Quillen pair. If \( M \in \mathcal{H}(R, G, \Theta) \) admits a Quillen submodule, then \( M \) belongs to \( \mathcal{H}(R, G, \Theta)^{t-iso} \) by definition. For a Quillen pair \( (M, U) \), we have the well-defined map
\[
B^+: U^\perp/U \times U^\perp/U \to R; \ B^+(x + U, y + U) = B(x, y) \ (x, y \in U^\perp).
\]

**Proposition 4.8.** Let \( (M, U) \), where \( M = (M, B, \alpha) \), be a Quillen pair. Then \( U^\perp/U \) is an \( R \)-free \( R[G] \)-module and \( B^+ \) is a nonsingular Hermitian form on \( U^\perp/U \).

**Proof.** Since \( U \) is an \( R \)-direct summand of \( M \), \( M \) factors to \( M = U \oplus N \) as \( R \)-modules. It follows that \( U \) and \( N \) both are \( R \)-free, and so are \( U^\# = \text{Hom}_R(U, R) \) and \( M/U \). Thus, the exact sequence
\[
0 \to U^\perp/U \to M/U \to U^\# \to 0
\]
spits via \( R \)-homomorphisms, and hence \( U^\perp/U \) is an \( R \)-direct summand of \( M/U \). In particular, \( U^\perp/U \) is \( R \)-free.

It is obvious that \( B^+ \) is \( R \)-bilinear, \( G \)-invariant and symmetric. So, it suffices to prove that \( B^+ \) is nonsingular. Since \( B \) is nonsingular, we can take an \( R \)-basis
\[
\{u_1, \ldots, u_m, y_1, \ldots, y_n, v_1, \ldots, v_m\}
\]
of \( M \) so that \( \{u_1, \ldots, u_m\} \) is an \( R \)-basis of \( U \), \( y_j \subseteq U^\perp \), and \( B(v_i, u_j) = \delta_{i,j} \) and \( B(v_i, y_j) = 0 \), where \( \delta_{i,j} = 1 \) if \( i = j \) and \( \delta_{i,j} = 0 \) otherwise. Let \( V \) denote the \( R \)-submodule of \( M \) generated by \( \{v_1, \ldots, v_m\} \). There exist elements \( z_1, \ldots, z_n \) of \( M \)
such that \( B(z_i, u_j) = 0, B(z_i, y_j) = \delta_{i,j} \) and \( B(z_i, v_j) = 0 \). Write \( z_i = y'_i + v'_i \) with \( y'_i \in U^\perp \) and \( v'_i \in V \). Then
\[
B(y'_i, y_j) = B(y'_i + v'_i, y_j) = B(z_i, y_j) = \delta_{i,j}.
\]
This shows that \( B^\perp : U^\perp/Y \times U^\perp/Y \to R \) is nonsingular. □

By the proposition, a Quillen pair \((M, U)\) induces an object \((U^\perp/Y, B^\perp/Y, \text{triv})\) of \(\mathcal{H}(R, G, \Theta)\), where \(\text{triv}: \Theta \to U^\perp/Y \) is the trivial map.

We define the Grothendieck-Witt groups
\[
GW_0(R, G, \Theta), \quad GW_0(R, G, \Theta)^{t\text{-iso}}, \quad GW_0(R, G)
\]
by
\[
GW_0(R, G, \Theta) = KH_0(R, G, \Theta)/[M] - [U^\perp/Y, B^\perp/Y, \text{triv}],
\]
\[
GW_0(R, G, \Theta)^{t\text{-iso}} = KH_0(R, G, \Theta)^{t\text{-iso}}/[M] - [U^\perp/Y, B^\perp/Y, \text{triv}],
\]
\[
GW_0(R, G) = KH_0(R, G)/[M] - [U^\perp/Y, B^\perp/Y, \text{triv}],
\]
where \((M, U)\) ranges over all Quillen pairs in \(\mathcal{H}(R, G, \Theta), \mathcal{H}(R, G, \Theta)^{t\text{-iso}}\) and \(\mathcal{H}(R, G, \Theta)^{\text{triv}}\), respectively. By definition, there are canonical homomorphisms
\[
GW_0(R, G) \to GW_0(R, G, \Theta)^{t\text{-iso}}
\]
and
\[
GW_0(R, G, \Theta)^{t\text{-iso}} \to GW_0(R, G, \Theta).
\]

**Proposition 4.9.** The homomorphisms
\[
GW_0(R, G) \to GW_0(R, G, \Theta)^{t\text{-iso}} \quad \text{and} \quad GW_0(R, G, \Theta)^{t\text{-iso}} \to GW_0(R, G, \Theta)
\]
are both injective. Moreover, the homomorphism \(GW_0(R, G) \to GW_0(R, G, \Theta)^{t\text{-iso}}\) is an isomorphism.

**Proof.** Consider the homomorphism
\[
GW_0(R, G, \Theta) \to GW_0(R, G)
\]
assigning \([M, B, \text{triv}]\) to \([M, B, \alpha]\). Since the composition
\[
GW_0(R, G) \to GW_0(R, G, \Theta)^{t\text{-iso}} \to GW_0(R, G, \Theta) \to GW_0(R, G)
\]
is the identity map, the homomorphisms
\[
GW_0(R, G) \to GW_0(R, G, \Theta)^{t\text{-iso}} \quad \text{and} \quad GW_0(R, G) \to GW_0(R, G, \Theta)
\]
are injective.

Let \(M = (M, B, \alpha)\) be a \(\Theta\)-positioned \(R[G]\)-Hermitian module such that \(\alpha\) is totally isotropic. Then, let \(L\) denote the \(R[G]\)-submodule of \(M\) generated by \(\alpha(\Theta)\), and set
\[
U = \{x \in M \mid rx \in L \text{ for some } r \in R \text{ with } r \neq 0\}.
\]
Then \(B(U, U) = 0\), and \(U\) is an \(R\)-direct summand, \(R[G]\)-submodule of \(M\). Thus, we have
\[
[M, B, \alpha] = [U^\perp/Y, B^\perp/Y, \text{triv}] \in GW_0(R, G, \Theta)^{t\text{-iso}}.
\]
This implies that the canonical homomorphism \(GW_0(R, G) \to GW_0(R, G, \Theta)^{t\text{-iso}}\) is surjective. □

For \(\Theta\)-positioned Hermitian \(R[G]\)-modules \(M_1 = (M_1, B_1, \alpha_1)\) and \(M_2 = (M_2, B_2, \alpha_2)\), we define the tensor product \(M_1 \otimes_R M_2\) over \(R\) as the \(\Theta\)-positioned Hermitian \(R[G]\)-module \((M_1 \otimes_R M_2, B_1 \otimes_R B_2, \alpha_1 \otimes_R \alpha_2)\).
Proposition 4.10. Let $\Theta$ be a finite $G$-set. Then $GW_0(R,G,\Theta)$ and $GW_0(R,G)$
($= GW_0(R,G,\Theta)^{\text{t-iso}}$) are commutative rings under the multiplication induced from
the tensor product over $R$. Moreover, the rings $GW_0(R,G,\Theta)$ and $GW_0(R,G)$ possess
units. Actually, the units of $GW_0(R,G,\Theta)$ and $GW_0(R,G)$ are the equivalence
classes of

$$(R,B : R \times R \to R, \alpha : \Theta \to R) \text{ and } (R,B : R \times R \to R, \text{triv} : \Theta \to R),$$
respectively, where $G$ acts trivially on $R$, $B$ is the map defined by $B(r_1,r_2) = r_1r_2$
for $r_1, r_2 \in R$, and $\alpha$ is the map defined by $\alpha(t) = 1$ for $t \in \Theta$.

5. The special Grothendieck-Witt rings

Let $S$ be a conjugation-invariant subset of
$$G(2) = \{g \in G | g^2 = e, \ g \neq e\}$$
and let $\Psi(S)$ denote the set of all subsets of $S$. Then the $G$-action on $S$ by conju-
gation yields a $G$-action on $\Psi(S)$. Let $\Theta$ be a finite $G$-set and $\rho^{(2)} : \Theta \to \Psi(S)$ a
$G$-map.

For a $G$-map $\alpha : \Theta \to M$, where $M$ is an $R[G]$-module, we define the map
$\Delta_{\alpha} : S \to M$ by

$$\Delta_{\alpha}(s) = \sum_{t} \{ \alpha(t) | t \in \Theta, \ \rho^{(2)}(t) \ni s \} \quad (s \in S).$$

Proposition 5.1. The map $\Delta_{\alpha}$ above is a $G$-map, namely $\Delta_{\alpha}(gsg^{-1}) = g\Delta_{\alpha}(s)$
for $g \in G$ and $s \in S$.

Proof. The proof runs as follows:

$$g\Delta_{\alpha}(s) = g \sum_{t} \{ \alpha(t) | t \in \Theta, \ \rho^{(2)}(t) \ni s \}$$

$$= \sum_{t} \{ \alpha(gt) | t \in \Theta, \ \rho^{(2)}(t) \ni s \}$$

$$= \sum_{t'} \{ \alpha(t') | g^{-1}t' \in \Theta, \ \rho^{(2)}(g^{-1}t') \ni s \}$$

$$= \sum_{t'} \{ \alpha(t') | t' \in \Theta, \ \rho^{(2)}(t') \ni gsg^{-1} \}$$

$$= \Delta_{\alpha}(gsg^{-1}).$$


Let $M = (M,B,\alpha)$ be an object in $\mathcal{H}(R,G,\Theta)$. We introduce a map

$$\nabla_{M} : M \to \text{Map}(S,R/2R),$$

which plays a key role in this paper. Define $\nabla_{M}(x)(s) \in R/2R$ for $x \in M$ and $s \in S$ by

$$\nabla_{M}(x)(s) = B(\Delta_{\alpha}(s) - x, sx).$$

Proposition 5.2. The map $\nabla_{M} : M \to \text{Map}(S,R/2R)$ is a $\mathbb{Z}[G]$-homomorphism.
Namely, the following hold:

1. $\nabla_{M}(x + y)(s) = \nabla_{M}(x)(s) + \nabla_{M}(y)(s)$ \quad $(x, y \in M, s \in S)$,
2. $\nabla_{M}(gx)(s) = g^{-1}\nabla_{M}(x)(gsg^{-1})$ \quad $(x \in M, s \in S)$.

If $R$ is square identical, $\nabla_{M} : M \to \text{Map}(S,R/2R)$ is an $R[G]$-homomorphism.
Proof. The formula (1) is obtained as follows:
\[
\nabla_M(x + y)(s) = B(\Delta_\alpha(s) - (x + y), s(x + y)) = \nabla_M(x)(s) + \nabla_M(y)(s) + B(-x, sy) + B(-y, sx) \\
= \nabla_M(x)(s) + \nabla_M(y)(s) - (B(x, sy) + B(y, sx)) = \nabla_M(x)(s) + \nabla_M(y)(s) \text{ in } R/2R.
\]

The formula (2) holds because
\[
\nabla_M(gx)(s) = B(\Delta_\alpha(s) - gx, sgx) = B(g^{-1}\Delta_\alpha(s) - x, g^{-1}sgx) = B(\Delta_\alpha(g^{-1}sg) - x, g^{-1}sgx) = \nabla_M(x)(g^{-1}sg) \text{ in } R/2R.
\]

The last assertion in the proposition is true since
\[
\nabla_M(rx)(s) = B(\Delta_\alpha(s) - rx, srx) = B(\Delta_\alpha(s), sx) - B(rx, sx) = rB(\Delta_\alpha(s), sx) - rB(x, sx) = rB(\Delta_\alpha(s) - x, sx) = r\nabla_M(x)(s) \text{ in } R/2R.
\]

We have established the proposition above. \Box

Let \(\mathcal{SH}(R, G, S, \Theta), \mathcal{SH}(R, G, S, \Theta)^{t\text{-iso}}\) and \(\mathcal{SH}(R, G, S, \Theta)^{triv}\) denote the family consisting of objects \(M\) with \(\nabla_M = 0\) of \(\mathcal{H}(R, G, \Theta), \mathcal{H}(R, G, \Theta)^{t\text{-iso}}\) and \(\mathcal{H}(R, G, \Theta)^{triv}\), respectively. We denote the Grothendieck groups of these under orthogonal sum by

\[
\text{KSH}_0(R, G, S, \Theta), \text{KSH}_0(R, G, S, \Theta)^{t\text{-iso}} \text{ and } \text{KSH}_0(R, G, S),
\]

respectively. Moreover, we define the special Grothendieck-Witt groups

\[
\text{SGW}_0(R, G, S, \Theta), \text{SGW}_0(R, G, S, \Theta)^{t\text{-iso}}, \text{SGW}_0(R, G, S)
\]

by

\[
\text{SGW}_0(R, G, S, \Theta) = \text{KSH}_0(R, G, S, \Theta)/[M] - [U^\perp/U, B^\perp, \text{triv}], \text{SGW}_0(R, G, S, \Theta)^{t\text{-iso}} = \text{KSH}_0(R, G, S, \Theta)^{t\text{-iso}}/[M] - [U^\perp/U, B^\perp, \text{triv}], \text{SGW}_0(R, G, S) = \text{KSH}_0(R, G, S)/[M] - [U^\perp/U, B^\perp, \text{triv}],
\]

where \((M, U)\) ranges over all Quillen pairs in \(\mathcal{SH}(R, G, S, \Theta), \mathcal{SH}(R, G, S, \Theta)^{t\text{-iso}}\) and \(\mathcal{SH}(R, G, S, \Theta)^{triv}\), respectively. Here we remark that if \(M \in \mathcal{SH}(R, G, S, \Theta)\) admits a Quillen submodule, then \(M\) belongs to \(\mathcal{SH}(R, G, S, \Theta)^{t\text{-iso}}\). By definition, there are canonical homomorphisms

\[
\text{SGW}_0(R, G, S) \rightarrow \text{SGW}_0(R, G, S, \Theta)^{t\text{-iso}}
\]

and

\[
\text{SGW}_0(R, G, S, \Theta)^{t\text{-iso}} \rightarrow \text{SGW}_0(R, G, S, \Theta).
\]
Proposition 5.3. The homomorphism $SGW_0(R, G, S) \to SGW_0(R, G, S, \Theta)^{t-iso}$ is surjective, and the homomorphism $SGW_0(R, G, S, \Theta)^{t-iso} \to SGW_0(R, G, S, \Theta)$ is injective.

Proof. The proof of the surjectivity of $SGW_0(R, G, S) \to SGW_0(R, G, S, \Theta)^{t-iso}$ is the same as that of $GW_0(R, G) \to GW_0(R, G, \Theta)^{t-iso}$ (see Proposition 4.9).

Let $M$ be an object of $SH(R, G, S, \Theta)^{t-iso}$ such that $[M] = 0$ in $SGW_0(R, G, S, \Theta)$.

Then there exist objects $M' = (M', B', \alpha')$, $M_1 = (M_1, B_1, \alpha_1)$ with a Quillen submodule $U_1$, and $M_2 = (M_2, B_2, \alpha_2)$ with a Quillen submodule $U_2$ of $SH(R, G, S, \Theta)$ such that

\[
M \oplus M' \oplus M_1 \oplus (U_2^+ \cap U_1, B_2^+, \text{triv}) \cong M' \oplus M_2 \oplus (U_1^+ \cap U_1, B_1^+, \text{triv}).
\]

By definition, both $M_1$ and $M_2$ belong to $SH(R, G, S, \Theta)^{t-iso}$. The object $M'$ above may be replaced by

\[
M'' = (M', B', \alpha') \oplus (M', -B', -\alpha').
\]

Then $M''$ has the Quillen submodule

\[
U'' = \{(x, x) \in M' \oplus M' \mid x \in M'\},
\]

and hence belongs to $SH(R, G, S, \Theta)^{t-iso}$, which lets us conclude that $[M] = 0$ in $SGW_0(R, G, S, \Theta)^{t-iso}$.

Proposition 5.4. If, for each $s \in S$, there is at most one element $t \in \Theta$ such that $\rho^{(2)}(t) \ni s$, then $SGW_0(R, G, S, \Theta)$, $SGW_0(R, G, S, \Theta)^{t-iso}$ and $SGW_0(R, G, S)$ are commutative rings, possibly without unit. If $R$ is square identical, and for each $s \in S$ there exists exactly one element $t \in \Theta$ such that $\rho^{(2)}(t) \ni s$, then $SGW_0(R, G, S, \Theta)$ is a commutative ring with unit.

Proof. Let $M_1 = (M_1, B_1, \alpha_1)$ and $M_2 = (M_2, B_2, \alpha_2)$ be objects of $H(R, G, \Theta)$ and $SH(R, G, S, \Theta)$, respectively. Then

\[
\nabla_{M_1 \otimes_R M_2}(x_1 \otimes x_2) = B_1 \otimes_R B_2(\Delta_{\alpha_1 \otimes R \alpha_2}(s) - x_1 \otimes x_2, s(x_1 \otimes x_2))
\]

\[
= B_1 \otimes_R B_2(\Delta_{\alpha_1}(s) \otimes \Delta_{\alpha_2}(s) - x_1 \otimes x_2, sx_1 \otimes sx_2)
\]

\[
= B_1(\Delta_{\alpha_1}(s) - x_1, sx_1)B_2(\Delta_{\alpha_2}(s), sx_2) - B_1(x_1, sx_1)B_2(x_2, sx_2)
\]

\[
= B_1(\Delta_{\alpha_1}(s) - x_1, sx_1)B_2(\Delta_{\alpha_2}(s), sx_2)
\]

\[
+ B_1(x_1, sx_1)B_2(\Delta_{\alpha_2}(s) - x_2, sx_2)
\]

\[
= \nabla_{M_1}(x_1)(s)B_2(\Delta_{\alpha_2}(s), sx_2) + B_1(x_1, sx_1)\nabla_{M_2}(x_2)(s)
\]

\[
= \nabla_{M_1}(x_1)(s)B_2(\Delta_{\alpha_2}(s), sx_2)
\]

in $R/2R$.

By using this and Proposition 5.2 (1), we can show that the product $M_1 \otimes_R M_2$ belongs to $SH(R, G, S, \Theta)$ if $M_1$ does. Therefore, the special Grothendieck-Witt groups are commutative rings.

Next we shall prove the last claim in the proposition. Let $(R, B, \alpha)$ denote the object in $H(R, G, \Theta)$ such that $G$ acts trivially on $R$, $B(r_1, r_2) = r_1r_2$ ($r_1, r_2 \in R$) and $\alpha(t) = 1$ ($t \in \Theta$). Then, the associated $\nabla : M \to \text{Map}(S, R/2R)$ is trivial, since

\[
\nabla(x)(s) = B(1 - r, sr) = r - r^2 = 0
\]

in $R/2R$.

Thus, $(R, B, \alpha)$ belongs to $SH(R, G, S, \Theta)$, and therefore we can now conclude that the ring $SGW_0(R, G, S, \Theta)$ possesses a unit.
Let \( \Delta \) to \( \text{SH} \) for any \( r \)
conditions (1)–(3) are satisfied:
\[ R \]
\begin{align*}
\text{Lemma 6.2.} & \quad \text{B} \text{ is an R[G]-module. A map } B : M \times M \rightarrow R[G] \text{ is called an R[G]-valued } \lambda \text{-Hermitian form (or } \lambda \text{-Hermitian form) on } M \text{ if the following conditions (1)–(3) are satisfied:
}\end{align*}
\begin{enumerate}
\item B is R-bilinear,
\item \( B(ax, by) = bB(y, x) \),
\item \( B(x, y) = \lambda B(y, x) \),
\end{enumerate}
for all \( x, y \in M \), \( a, b \in R[G] \).

Let \( B : M \times M \rightarrow R[G] \) be a \( \lambda \)-Hermitian form. For \( x, y \in M \), \( B(x, y) \) can be written as \( \sum_{g \in G} B(x, y)_g \) with \( B(x, y)_g \in R \). Define the R-homomorphism \( \varepsilon : R[G] \rightarrow R \) by
\[ \sum_{g \in G} \varepsilon (r_g) = \sum_{g \in G} w(g) r_g g^{-1} \quad (r_g \in R). \]

\begin{align*}
\text{Lemma 6.2.} & \quad B(x, y)_g = \varepsilon (B(x, y)_g) \text{ for all } x, y \in M \text{ and } g \in G, \text{ and consequently}
\quad B(x, y) = \sum_{g \in G} \varepsilon (B(x, y)_g) g.
\end{align*}

\begin{align*}
\text{Proof.} & \quad \text{By definition, we have } B(x, y)_e = \varepsilon (B(x, y)_g). \text{ By observing the coefficients of } g \text{ in } B(x, y) \text{ and}
\quad gB(x, y)_g = \sum_{h \in G} B(x, y)_h g h,
\end{align*}
we have \( B(x, y)_e = B(x, y)_g \). Thus, \( B(x, y)_g = \varepsilon (B(x, y)_g) \).

\begin{align*}
\text{Lemma 6.3.} & \quad \text{Let } M \text{ be as above. Then the composition } \varepsilon \circ B : M \times M \rightarrow R \text{ is a } \lambda \text{-symmetric, (G, w)-invariant, R-bilinear form on } M. \text{ Namely, the following hold:
}\end{align*}
\begin{enumerate}
\item \( \varepsilon (B(x + x', ry)) = r \varepsilon (B(x, y)) + r \varepsilon (B(x', y)) \),
\item \( \varepsilon (B(x, y)) = \lambda \varepsilon (B(y, x)) \),
\item \( \varepsilon (B(gx, gy)) = w(g) \varepsilon (B(x, y)) \),
\end{enumerate}
for any \( r \in R \), \( x, x', y \in M \) and \( g \in G \).
Proof. (1) The proof is straightforward.
(2) The equality follows from \( B(x, y) = \lambda B(y, x) \).
(3) By comparing the coefficients of \( e \) in \( B(x, gy) \) and \( w(g)B(g^{-1}x, y) \):

\[
B(x, gy) = \sum_{h \in G} B(x, gy)h, \\
w(g)B(g^{-1}x, y) = \sum_{h \in G} w(g)B(g^{-1}x, y)h,
\]

we have \( \varepsilon(B(x, gy)) = w(g)\varepsilon(B(g^{-1}x, y)) \), which is equivalent to the equality (3). \( \square \)

An \( R[G] \)-valued \( \lambda \)-Hermitian form \( B \) on an \( R[G] \)-projective module \( M \) is said to be nonsingular if the associated map

\[
M \to \text{Hom}_{R[G]}(M, R[G]); \quad x \mapsto B(x, -)
\]

is bijective.

**Lemma 6.4.** Let \( B \) be an \( R[G] \)-valued \( \lambda \)-Hermitian form on an \( R[G] \)-projective module \( M \). Then \( B \) is nonsingular if and only if the induced \( R \)-bilinear form \( \varepsilon \circ B : M \times M \to R \) is nonsingular.

Let \( H \) and \( K \) be finite groups with homomorphisms \( w_H : H \to \{-1,1\} \) and \( w_K : K \to \{-1,1\} \), respectively. Let \( \varphi : H \to K \) be a monomorphism such that \( w_K \circ \varphi = w_H \). Let \( N \) be an \( R[K] \)-module and \( B : N \times N \to R[K] \) a \( \lambda \)-Hermitian form. We define the map \( \varphi^\#B : \varphi^\#N \times \varphi^\#N \to R[H] \) by

\[
\varphi^\#B(x, y) = \sum_{h \in H} \varepsilon(B(x, \varphi(h)^{-1}y))h \quad (x, y \in \varphi^\#N).
\]

It immediately follows that \( \varphi^\#B \) is an \( R[H] \)-valued \( \lambda \)-Hermitian form on \( \varphi^\#N \). If \( B \) is nonsingular, then so is \( \varphi^\#B \). Next let \( M \) be a stably free \( R[H] \)-module. Then

\[
\varphi^\#M = R[K] \otimes_{R[H], \varphi} M
\]

is clearly a stably \( R[K] \)-free module. Let \( B : M \times M \to R[H] \) be a \( \lambda \)-Hermitian form. We define the \( R \)-bilinear map \( \varphi^\#B : \varphi^\#M \times \varphi^\#M \to R[K] \) so that

\[
\varphi^\#B(a \otimes \varphi x, b \otimes \varphi y) = \sum_{k \in K} w_K(a)\delta_{\varphi(H),k^{-1}b \varphi(H)}\varepsilon(B(x, \varphi^{-1}(a^{-1}k^{-1}b)y))k,
\]

for \( a, b \in K, x, y \in M \).

**Lemma 6.5.** Let \( \varphi^\#B \) be as above. Then

\[
\varphi^\#B(a \otimes \varphi x, b \otimes \varphi y) = b\varphi'(B(x, y))\overline{a},
\]

for \( a, b \in K, x, y \in M \); and \( \varphi^\#B \) is an \( R[K] \)-valued \( \lambda \)-Hermitian form on \( \varphi^\#M \), where \( \varphi' : R[H] \to R[K] \) is the ring homomorphism canonically induced by \( \varphi : H \to K \). If \( B \) is nonsingular, then so is \( \varphi^\#B \).
Proof. The formula in the lemma is true because
\[
\phi(B(a \otimes x, b \otimes y)) = \sum_{k \in K} w_k(B(a, \phi^{-1}(a^{-1}k^{-1}b)y))k
\]
\[
= b(\sum_{k \in K} \delta_{\phi(H), a^{-1}k^{-1}b} \varepsilon(B(x, \phi^{-1}(a^{-1}k^{-1}b)y))b^{-1}ka)\overline{\alpha}
\]
\[
= b(\sum_{k' \in K} \delta_{\phi(H), k'^{-1}} \varepsilon(B(x, \phi^{-1}(k'^{-1}y))k')\overline{\alpha}
\]
\[
= b\phi'(\sum_{k' \in K} \delta_{\phi(H), k'^{-1}} \varepsilon(B(x, \phi^{-1}(k'^{-1}y))\phi^{-1}(k'))\overline{\alpha}
\]
\[
= b\phi'(B(x, y))\overline{\alpha}.
\]
One can check the latter claim in the lemma by using this formula. \qed

Proposition 6.6. Let \( H \) be a subgroup of \( G \), \( B \) an \( R[H] \)-valued \( \lambda \)-Hermitian form on an \( R[H] \)-module \( M \), and \( g \) an element of \( G \). Provided \( w_H = w_{gHg^{-1}} \circ c_{(H,g)} \), the diagram

\[
\begin{array}{ccc}
c_{(H,g)} \# M \times c_{(H,g)} \# M & \xrightarrow{f_0 \times f_0} & c_{(gHg^{-1},g^{-1})} \# M \times c_{(gHg^{-1},g^{-1})} \# M \\
& \xrightarrow{c_{(H,g)} \# B} & R[gHg^{-1}]
\end{array}
\]

commutes, where \( f_0 \) is the canonical \( R[gHg^{-1}] \)-isomorphism (cf. Proposition 3.2).

The proof of the proposition is straightforward.

Given a datum \( D = (R, G, w, \lambda) \) as above, we obtain the datum
\[
D_H = (R, H, w|_H, \lambda)
\]
for each subgroup \( H \) of \( G \).

Proposition 6.7. Let \( H \) be a subgroup of \( G \) and \( B : M \times M \to R[H] \) a \( \lambda \)-Hermitian form on an \( R[H] \)-module \( M \). Then for each \( g \in H \), the following diagrams commute:

\[
\begin{array}{ccc}
c_{(H,g)} \# M \times c_{(H,g)} \# M & \xrightarrow{f_1 \times f_2} & R[H], \\
\end{array}
\]

\[
\begin{array}{ccc}
c_{(H,g)} \# M \times c_{(H,g)} \# M & \xrightarrow{f_2 \times f_2} & R[H], \\
\end{array}
\]

where \( f_1 \) and \( f_2 \) are the canonical isomorphisms (cf. Proposition 3.4).
Proof. The commutability of the first diagram follows from
\[(c_{(H,g)}) B)(e \otimes x, e \otimes y) = \sum_{h \in H} \varepsilon(B(x, g^{-1} h^{-1} gy)) h\]
and
\[B(f_1(e \otimes x), f_1(e \otimes y)) = B(gx, gy) = \sum_{h \in H} \varepsilon(B(gx, h^{-1} gy)) h = \omega(g) \sum_{h \in H} \varepsilon(B(x, g^{-1} h^{-1} gy)) h.\]

The commutability of the second diagram follows from
\[(c_{(H,g)}) B)(x, y) = \sum_{h \in H} \varepsilon(B(x, g h^{-1} g^{-1} y) h)
and
\[B(f_2(x), f_2(y)) = B(g^{-1} x, g^{-1} y) = \sum_{h \in H} \varepsilon(B(g^{-1} x, h^{-1} g^{-1} y)) h = \omega(g) \sum_{h \in H} \varepsilon(B(x, g^{-1} h^{-1} gy)) h.\]

\[\square\]

Proposition 6.8. For any subgroups \(H\) and \(K\) of \(G\), each \(R[H]\)-valued \(\lambda\)-Hermitian form \(B : M \times M \to R[H]\) on an \(R[H]\)-module \(M\) satisfies the \(w\)-Mackey double coset formula. Namely,
\[(\text{Res}_G^K \text{Ind}_H^G B) \circ (\omega \times \omega) = \sum_{KgH \in K \backslash G / H} \omega(g) \text{Ind}_K^K H \text{Res}_H^K M \text{Res}_{K \cap gH}^H B(a \otimes (e \otimes x), b \otimes (e \otimes y)),\]
where \(\omega\) is the canonical isomorphism (cf. Proposition 3.5). Particularly, in the case \(w(G) = \{1\}\), \(B\) satisfies the Mackey double coset formula.

Proof. It suffices to prove that
\[(\text{Res}_K^K \text{Ind}_H^G B)(ag \otimes x, bg \otimes y) = \omega(g)(\text{Ind}_K^K H \text{Res}_H^K M \text{Res}_{K \cap gH}^H B)(a \otimes (e \otimes x), b \otimes (e \otimes y))\]
for any \(g \in G, a, b \in K, x, y \in \text{Res}_{K \cap gH}^H M\). This equality holds because
\[(\text{Res}_K^K \text{Ind}_H^G B)(ag \otimes x, bg \otimes y) = \sum_{k \in K} \omega(a) \delta_{agH,k^{-1}bgH} \varepsilon(B(x, (ag)^{-1} k^{-1} bg y) k = \omega(g) \sum_{k \in K} \omega(a) \delta_{agH,k^{-1}bgH} \varepsilon(B(x, g^{-1}(a^{-1} k^{-1} b) gy) k)\]
and
\[
(\text{Ind}^K_G \text{Res}^H_{G \cap g^{-1} K H} \varphi) (a \otimes (e \otimes x), b \otimes (e \otimes y)) = \sum_{k \in K} w(a) \delta_{a(K \cap g^{-1} H, k^{-1} b(K \cap g^{-1} H))}.
\]
\[
\cdot (c(H \cap g^{-1} K H) \text{Res}^H_{G \cap g^{-1} K H} B)(e \otimes x, a^{-1} k^{-1} b(e \otimes y)) k
\]
\[
= \sum_{k \in K} w(a) \delta_{a(K \cap g^{-1} H, k^{-1} b(K \cap g^{-1} H))} B(x, g^{-1}(a^{-1} k^{-1} b)g y) k.
\]


In this paper $\lambda$ stands for either 1 or $-1$. Let $w : G \to \{1, -1\}$ be a group homomorphism. Set
\[
G^\lambda(2) = \{g \in G(2) \mid w(g) = \lambda\},
\]
\[
G^{-\lambda}(2) = \{g \in G(2) \mid w(g) = -\lambda\}.
\]
Clearly we have $g = \lambda g$ for $g \in G^\lambda(2)$ and $g = -\lambda g$ for $g \in G^{-\lambda}(2)$. Let $S$ and $Q$ be conjugation-invariant subsets of $G^\lambda(2)$ and $G^{-\lambda}(2)$, respectively. We shall define the Witt group of $\Theta$-positioned quadratic $R[G]$-modules, which is the Wall group (cf. [27]) in the case where $Q$, $S$ and $\Theta$ are the empty set, and the Bak group (cf. [1], [19]) in the case where $S$ and $\Theta$ are the empty set. The datum
\[
A = (R, G, Q, S, \lambda, w)
\]
is relevant to the group. Define $R$-submodules $A_s = A_s(G, S; R)$, $A_q = A_q(G, S; R)$ and $\Lambda = \Lambda(G, Q; R)$ of $A := R[G]$ as follows:
\[
A_s = R[S] = \{s \mid s \in S\} R,
\]
\[
A_q = R[G \setminus S] = \{g \mid g \in G \setminus S\} R,
\]
\[
\Lambda = \langle x - \lambda x \mid x \in A \rangle_R + \langle g \mid g \in Q \rangle_R.
\]
This module $\Lambda$ is called the form parameter generated by $Q$.

Definition 7.1. A map $q : M \to A_q/\Lambda$ is called an $A$-quadratic form (or quadratic form) on $M$ with respect to $B$ if the following conditions (1)–(3) are fulfilled:

1. $q(gx) = gq(x)g^{-1}$ and $q(rx) = r^2 q(x)$ in $A_q/\Lambda = A/(\Lambda + A_s)$,
2. $q(x + y) - q(x) - q(y) = B(x, y)$ in $A_q/\Lambda = A/(\Lambda + A_s)$,
3. $\overline{q(x) + \lambda q(x)} = B(x, x)$ in $A_q = A/A_s$,

for all $x, y \in M$, $g \in G$, $r \in R$, where $q(x) \in A_q$ is a lifting of $q(x)$.


Let $\Theta$ be a finite $G$-set. A quadruple $(M, B, q, \alpha)$ consisting of an $A$-quadratic $R[G]$-module $(M, B, q)$ and a $G$-map $\alpha : \Theta \to M$ is called a $\Theta$-positioned $A$-quadratic $R[G]$-module (or $\Theta$-positioned $\lambda$-quadratic $R[G]$-module).

Let $Q(A, \Theta)$ (or $Q(R, G, Q, S, \Theta)$) denote the family of all $\Theta$-positioned $A$-quadratic $R[G]$-modules $(M, B, q, \alpha)$ such that $M$ is a stably free $R[G]$-module and $B$ is nonsingular.
Let \( M = (M, B, q, \alpha) \in \mathcal{Q}(A, \Theta) \). The map \( \alpha \) is said to be \textit{totally isotropic} (resp. \textit{trivial}) if \( B(\operatorname{Im}(\alpha), \operatorname{Im}(\alpha)) = 0 \) and \( q(\operatorname{Im}(\alpha)) = 0 \) (resp. \( \operatorname{Im}(\alpha) = 0 \)). Set
\[
\mathcal{Q}(A, \Theta)^{\text{t-iso}} = \left\{ (M, B, q, \alpha) \in \mathcal{Q}(A, \Theta) \mid \alpha \text{ is totally isotropic} \right\},
\]
\[
\mathcal{Q}(A, \Theta)^{\text{triv}} = \left\{ (M, B, q, \alpha) \in \mathcal{Q}(A, \Theta) \mid \alpha \text{ is trivial} \right\}.
\]

Let \( \mathcal{K}_0(A, \Theta) \), \( \mathcal{K}_0(A, \Theta)^{\text{t-iso}} \) and \( \mathcal{K}_0(A) \) denote the Grothendieck groups of \( \mathcal{Q}(A, \Theta) \), \( \mathcal{Q}(A, \Theta)^{\text{t-iso}} \) and \( \mathcal{Q}(A, \Theta)^{\text{triv}} \), respectively, under orthogonal sum.

A stably \( R[G]\)-free, \( R[G]\)-direct summand \( L \) of \( M \) is called a \textit{Lagrangian submodule} of \( M \) if \( B(L, L) = 0 \), \( q(L) = 0 \), \( L^\perp = L \) and \( \operatorname{Im}(\alpha) \subseteq L \), where
\[
L^\perp = \{ x \in M \mid B(x, y) = 0 \text{ (}\forall y \in L \text{)} \}.
\]

If \( M \) has a Lagrangian submodule, then \( M \) is called a \textit{null module}. The groups defined by
\[
\mathcal{W}_0(A, \Theta) = \mathcal{K}_0(A, \Theta)/\langle \text{null modules in } \mathcal{Q}(A, \Theta) \rangle,
\]
\[
\mathcal{W}_0(A, \Theta)^{\text{t-iso}} = \mathcal{K}_0(A, \Theta)^{\text{t-iso}}/\langle \text{null modules in } \mathcal{Q}(A, \Theta)^{\text{t-iso}} \rangle,
\]
\[
\mathcal{W}_0(A) = \mathcal{K}_0(A)/\langle \text{null modules in } \mathcal{Q}(A, \Theta)^{\text{triv}} \rangle
\]
are called the \textit{Witt groups} of \( \Theta \)-positioned \( A \)-quadratic \( R[G]\)-modules. If the context is clear, those Witt groups are also denoted by
\[
\mathcal{W}_0(R, G, Q, S, \Theta), \quad \mathcal{W}_0(R, G, Q, S)^{\text{t-iso}}, \quad \mathcal{W}_0(R, G, Q, S),
\]
respectively.

8. \textsc{The special Witt groups}

Let \( A = (R, G, Q, S, \lambda, w) \) be as in the previous section, \( \Theta \) a finite \( G \)-set and \( \rho^{(2)} : \Theta \rightarrow \mathfrak{S}(S) \) a \( G \)-map (cf. Section 5). Let \( M = (M, B, q, \alpha) \) be a \( \Theta \)-positioned \( A \)-quadratic \( R[G]\)-module, where \( \alpha : \Theta \rightarrow M \). The associated map \( \nabla_M : M \rightarrow \operatorname{Map}(S, R/2R) \) is defined by
\[
\nabla_M(x)(s) = \varepsilon(B(\Delta_{\alpha}(s) - x, sx)),
\]
for \( x \in M \) and \( s \in S \), where \( \Delta_{\alpha} : S \rightarrow M \) is the map defined by \( \langle \Delta_{\alpha} \rangle \).

If \( M \in \mathcal{Q}(A, \Theta) \) satisfies \( \nabla_M = 0 \), then we call \( M \) a \textit{special} \( \Theta \)-positioned \( A \)-quadratic \( R[G]\)-module (or a \textit{special} \( \Theta \)-positioned \( \lambda \)-quadratic \( R[G]\)-module). Set
\[
\mathcal{S}\mathcal{Q}(A, \Theta) = \{ M \in \mathcal{Q}(A, \Theta) \mid \nabla_M = 0 \},
\]
\[
\mathcal{S}\mathcal{Q}(A, \Theta)^{\text{t-iso}} = \{ M \in \mathcal{Q}(A, \Theta)^{\text{t-iso}} \mid \nabla_M = 0 \},
\]
\[
\mathcal{S}\mathcal{Q}(A, \Theta)^{\text{triv}} = \{ M \in \mathcal{Q}(A, \Theta)^{\text{triv}} \mid \nabla_M = 0 \}.
\]

The corresponding Grothendieck groups are denoted by
\[
\mathcal{K}\mathcal{S}_0(A, \Theta), \quad \mathcal{K}\mathcal{S}_0(A, \Theta)^{\text{t-iso}}, \quad \mathcal{K}\mathcal{S}_0(A),
\]
respectively, or by
\[
\mathcal{K}\mathcal{S}_0(R, G, Q, S, \Theta), \quad \mathcal{K}\mathcal{S}_0(R, G, Q, S)^{\text{t-iso}}, \quad \mathcal{K}\mathcal{S}_0(R, G, Q, S),
\]
respectively. Further, define the \textit{special Witt groups}
\[
\mathcal{S}\mathcal{W}_0(A, \Theta) \quad (= \mathcal{S}\mathcal{W}_0(R, G, Q, S, \Theta)),
\]
\[
\mathcal{S}\mathcal{W}_0(A, \Theta)^{\text{t-iso}} \quad (= \mathcal{S}\mathcal{W}_0(R, G, Q, S)^{\text{t-iso}}),
\]
\[
\mathcal{S}\mathcal{W}_0(A) \quad (= \mathcal{S}\mathcal{W}_0(R, G, Q, S))
\]
by
\[ SWQ_0(A, \Theta) = KSQ_0(A, \Theta)/\langle \text{null modules in } SQ(A, \Theta) \rangle, \]
\[ SWQ_0(A, \Theta)^{t-iso} = KSQ_0(A, \Theta)^{t-iso}/\langle \text{null modules in } SQ(A, \Theta)^{t-iso} \rangle, \]
\[ SWQ_0(A) = KSQ_0(A)/\langle \text{null modules in } SQ(A, \Theta)^{t-triv} \rangle, \]
respectively.

9. Tensor Products of Hermitian Modules and Quadratic Modules

Let \( A = (R, G, Q, S, \lambda, w) \) be as in Section 8 and \( \Theta \) a finite \( G \)-set. Let \( M = (M, B, q) \) be an \( A \)-quadratic \( R[G] \)-module. By definition, \( B \) is a map \( M \times M \to R[G] \) and \( q \) is a map \( M \to A_q/\Lambda \). We write \( G \) as a disjoint union of the form
\[ G = \{ e \} \coprod G(2) \coprod C \coprod C^{-1}, \]
where \( C \) is a subset of \( G \) consisting of elements of order \( \geq 3 \) and \( C^{-1} = \{ g^{-1} | g \in C \} \).

Set
\[ Q(G) = \{ e \} \cup (G^\lambda(2) \setminus S) \cup (G^{-\lambda}(2) \setminus Q) \cup C. \]

Let \( R_g \) stand for the \( R \)-module defined by
\[ R_g = \begin{cases} R/(1-\lambda)R & (g = e), \\ R & (g \in G^\lambda(2)), \\ R/2R & (g \in G^{-\lambda}(2)), \\ R & (\text{otherwise}), \end{cases} \]
for each \( g \in G \). Then \( q(x), x \in M \), can be regarded as the formal sum
\[ \sum_{g \in Q(G)} q(x)_g g \]
with \( q(x)_g \in R_g \); namely, \( q : M \to A_q/\Lambda \) can be regarded as the map
\[ M \to \bigoplus_{g \in Q(G)} R_g; \quad x \mapsto (q(x)_g). \]

We set \( q(x)_g = \lambda w(g)q(x)_g^{-1} \) for \( g \in G \) with \( g^{-1} \in Q(G) \). This definition is compatible with the ambiguity of choice of \( Q(G) \), because
\[ \widetilde{q(x)}_g g = \lambda w(g)\widetilde{q(x)}_g g^{-1} \mod \Lambda. \]

Let \( M_1 = (M_1, B_1, \alpha_1) \) and \( M_2 = (M_2, B_2, q_2, \alpha_2) \) be objects in \( \mathcal{H}(R, G, S, \Theta) \) and \( Q(A, \Theta) \), respectively. We define an object \( M_1 \cdot M_2 \) in \( Q(A, \Theta) \) as the product of \( M_1 \) and \( M_2 \) as follows. For the sake of convenience, \( M = (M, B, q, \alpha) \) stands for \( M_1 \cdot M_2 \) for a while.

First, \( M \) is defined as the \( R \)-module \( M_1 \otimes_R M_2 \) with the \( G \)-action: \( (g, x \otimes y) \mapsto (gx) \otimes (gy) \), where \( g \in G, x \in M_1 \) and \( y \in M_2 \). Since \( M_1 \) is \( R \)-free and \( M_2 \) is stably \( R[G] \)-free, \( M \) is stably \( R[G] \)-free.

Second, \( B : M \times M \to R[G] \) is defined as the \( R \)-bilinear form such that
\[ B(x \otimes y, x' \otimes y') = \sum_{g \in G} B_1(x, g^{-1}x')\epsilon(B_2(y, g^{-1}y'))g. \]
The equality \( B(u, v) = \lambda B(v, u) \) \((u, v \in M)\) holds since
\[
B(x \otimes y, x' \otimes y') = \sum_{g \in G} B_1(x, g^{-1}x')\varepsilon(B_2(y, g^{-1}y'))g
\]
\[= \sum_{g \in G} \lambda B_1(g^{-1}x', x)\varepsilon(B_2(g^{-1}y', y))g\]
\[= \lambda \sum_{g \in G} w(g)B_1(x', gx)\varepsilon(B_2(y', gy))g\]
\[= \lambda \sum_{g \in G} B_1(x', gx)\varepsilon(B_2(y', gy))g^{-1}\]
\[= \lambda \sum_{g \in G} B_1(x', gx)\varepsilon(B_2(y', gy))g^{-1}\]
\[= \lambda B(x' \otimes y', x \otimes y).
\]
The equality \( B(au, bv) = bB(u, v)\) holds because
\[
B(a(x \otimes y), b(x' \otimes y')) = \sum_{g \in G} B_1(ax, g^{-1}bx')\varepsilon(B_2(ay, g^{-1}by'))g
\]
\[= b \sum_{h \in G} B_1(ax, h^{-1}x')\varepsilon(B_2(ay, h^{-1}y'))h\]
\[= b \sum_{h \in G} w(a)B_1(x, a^{-1}h^{-1}x')\varepsilon(B_2(y, a^{-1}h^{-1}y'))h\]
\[= b \sum_{h \in G} w(a)B_1(x, (ha)^{-1}x')\varepsilon(B_2(y, (ha)^{-1}y'))h\]
\[= b \sum_{k \in G} w(a)B_1(x, k^{-1}x')\varepsilon(B_2(y, k^{-1}y'))ka^{-1}\]
\[= b B(x \otimes y, x' \otimes y') \bar{\pi}.
\]
Thus, \( B \) is an \( R[G] \)-valued \( \lambda \)-Hermitian form on \( M \). Note that \( B_1 \) and \( \varepsilon \circ B_2 \) are both nonsingular. So, \( B_1 \otimes (\varepsilon \circ B_2) \) is nonsingular, which implies that \( B \) is nonsingular.

Third, we describe the definition of \( q : M \to A_q/\Lambda \). Let \( F(M_1 \times M_2) \) denote the \( R \)-free module with basis \( \{ (x, y) \mid x \in M_1, y \in M_2 \} \) (although it may not be finitely generated), \( T \) the subset of \( F(M_1 \times M_2) \) consisting of all elements of the form
\[
r(x, y) - (rx, y), \quad r(x, y) - (x, ry),
\]
\[
(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y'),
\]
where \( r \) ranges over \( R \), \( x \) and \( x' \) over \( M_1 \), \( y \) and \( y' \) over \( M_2 \); and let \( \lfloor \ \rfloor : F(M_1 \times M_2) \to M_1 \otimes M_2 \) denote the canonical map.

**Lemma 9.1.** Let \( f \) be a map from \( F(M_1 \times M_2) \) to \( A_q/\Lambda = A/(A_q + \Lambda) \). If the following conditions \((1)-(3)\) are fulfilled for all \( r \in R, u, v \in F(M_1 \times M_2) \) and \( t \in T \):
\[
(1) \quad f(ru) = r^2 f(u),
\]
\[
(2) \quad f(u + v) = f(u) + f(v) + B([u, v]),
\]
\[
(3) \quad f(t) = 0,
\]
then \( f \) factors through \( M_1 \otimes M_2 \to A_q/\Lambda \).
Moreover, in \( A \), we have

\[
f(q) = \sum_{i<j} (r_i + r_j)q^2(x_i, x_j, y_i, y_j) + \left( \sum_{i} r_i B(x_i \otimes y_i, x_j \otimes y_j) \right) g.
\]

By definition, we have \( f(r u) = r^2 f(u) \) for all \( r \in R \) and \( u \in F(M_1 \times M_2) \).

Note that for \( u = \sum r_i(x_i, y_i) \) and \( v = \sum r'_i(x_i, y_i) \), we have

\[
f(u + v) = \sum \sum (r_i + r'_i)^2 B(x_i, g^{-1} x_i)q^2(y_i) g + \sum (r_i + r'_i)(r_j + r'_j) B(x_i \otimes y_i, x_j \otimes y_j).
\]

Thus, we have

\[
f(u + v) - f(u) - f(v) = \sum \sum 2r_i r'_i B(x_i, g^{-1} x_i)q^2(y_i) g + \sum (r_i r'_j + r'_i r_j) B(x_i \otimes y_i, x_j \otimes y_j).
\]

On the other hand, in \( A_q / \Lambda \) we have

\[
B(\sum r_i x_i \otimes y_i, \sum r'_i x_i \otimes y_i) = \sum r_i r'_i B(x_i \otimes y_i, x_i \otimes y_i)
\]

\[
+ \sum_{i<j} (r_i r'_j) B(x_i \otimes y_i, x_j \otimes y_j) + r_j r'_i B(x_j \otimes y_j, x_i \otimes y_i)
\]

\[
= \sum r_i r'_i B(x_i \otimes y_i, x_i \otimes y_i)
\]

\[
+ \sum_{i<j} (r_i r'_j) B(x_i \otimes y_i, x_j \otimes y_j) + r'_i r_j B(x_i \otimes y_i, x_j \otimes y_j).
\]

Moreover, in \( A/(A + \Lambda) \) we have

\[
B(x_i \otimes y_i, x_i \otimes y_i) = \sum_{g \in G} B_1(x_i, g^{-1} x_i) \varepsilon(B_2(y_i, g^{-1} y_i)) g
\]

\[
= \sum_{g \in G} B_1(x_i, g^{-1} x_i) 2q^2(y_i) g.
\]

Thus we obtain \( f(u + v) - f(u) - f(v) = B([u], [v]) \) in \( A_q / \Lambda \).

It is clear that \( f(t) = 0 \) for all \( t \in T \).

Since the conditions (1)–(3) in Lemma 9.1 are satisfied, we obtain the map \( q : M \to A_q / \Lambda \) by \( q([u]) = f(u) \) for \( u \in F(M_1 \times M_2) \). Immediately we have \( q(r[u]) = r^2 q([u]) \) and \( q([u + v]) - q([u]) - q([v]) = B([u], [v]) \) for \( r \in R \) and \( u \).
Let \( v \in F(M_1 \times M_2) \). For \( g \in G \) and \( u = (x, y) \), we have
\[
q(gu) = f(gx, gy)
\]
\[
= \sum_{h \in \mathbb{Q}(G)} B_1(gx, h^{-1}gx)q_2(gy)h
\]
\[
= \sum_{h \in \mathbb{Q}(G)} w(g)B_1(x, g^{-1}h^{-1}gx)q_2(y)h^{-1}hg
\]
\[
= \sum_{h \in \mathbb{Q}(G)} w(g)B_1(x, k^{-1}x)q_2(y)kg^{-1}
\]
\[
= g \sum_{h \in \mathbb{Q}(G)} B_1(x, k^{-1}x)q_2(y)k\gamma
\]
\[
= g \frac{f(x \otimes y)\gamma}{gq([u]\gamma)},
\]
where \( k = g^{-1}hg \). Thus, \( q(gz) = gq(z)\gamma \) for all \( g \in G \) and \( z \in M \).

Next, we check the property (3) in Definition 7.1. For \( u = (x, y) \) we have
\[
\hat{q}([u]) + \lambda q([u]) = \sum_{g \in \mathbb{Q}(G)} B_1(x, g^{-1}x)(\hat{q_2(y)}g + \lambda \hat{q_2(y)}g)\gamma
\]
\[
= \sum_{g \in G} B_1(x, g^{-1}x)B_2(y, y)g\gamma
\]
\[
= B([u], [u]) \text{ in } A_q = A/A_z,
\]
which shows that \( \hat{q}(z) + \lambda q(z) = B(z, z) \) for all \( z \in M \).

Putting all together, we see that the current triple \((M, B, q)\) is an \( A \)-quadratic \( R[G] \)-module.

Defining \( \alpha : \Theta \to M \) by \( \alpha(t) = \alpha_1(t) \otimes \alpha_2(t) \) for \( t \in \Theta \), we establish \( M_1 \cdot M_2 \)
\((= M = (M, B, q, \alpha))\) from \( M_1 = (M_1, B_1, \alpha_1) \) and \( M_2 = (M_2, B_2, q_2, \alpha_2) \).

**Theorem 9.2.** Let \( A = (R, G, Q, S, \lambda, w) \) and \( \Theta \) be as above. Then
\[
WQ_0(A, \Theta), \ WQ_0(A, \Theta)^{t\text{-iso}} \text{ and } WQ_0(A)
\]
are modules over \( GW_0(R, G, S, \Theta) \), and \( WQ_0(A) \) is one over \( GW_0(R, G, S) \) by the pairing
\[
(M_1, M_2) \mapsto M_1 \cdot M_2.
\]

10. Tensor products and \( \nabla \)-invariants

In this section we invoke that \( R \) is square identical. Let \( Q, S, w, \lambda \) and \( \Theta \)
be as in Section 7 and let \( \rho^{(2)} : \Theta \to \mathfrak{P}(S) \) be a \( G \)-map such that for every \( s \in S \), there exists exactly one \( t \in \Theta \) with \( \rho^{(2)}(t) = s \). Hence, by Proposition 5.4
\( SGW_0(R, G, S, \Theta) \) is a commutative ring with unit.

**Proposition 10.1.** Let \( M_1 = (M_1, B_1, \alpha_1) \) and \( M_2 = (M_2, B_2, q_2, \alpha_2) \) be objects in \( SH(R, G, S, \Theta) \) and \( SQ(A, \Theta) \), respectively. Then \( M = M_1 \cdot M_2 = (M, B, q, \alpha) \)
defined in the previous section lies in \( SQ(A, \Theta) \).
Proof. It was already shown that $M = M_1 \cdot M_2$ belongs to $\mathcal{Q}(\mathcal{A}, \Theta)$. Therefore, it suffices to show that $\nabla_M = 0$. By definition, we have

$$\nabla_M(x \otimes y)(s) = \varepsilon(B(\Delta_\alpha(s) - x \otimes y, s(x \otimes y)))$$

$$= \varepsilon(B(\Delta_\alpha(s) \otimes \Delta_\alpha(s) - x \otimes y, sx \otimes sy))$$

$$= \varepsilon(B(\Delta_\alpha(s) - x \otimes y, sx \otimes sy)) - \varepsilon(B(x \otimes y, sx \otimes sy))$$

$$= B_1(\Delta_\alpha(s), sx)\varepsilon(B_2(\Delta_\alpha(s), sy)) - B_1(x, sx)\varepsilon(B_2(y, sy))$$

$$= B_1(\Delta_\alpha(s) - y, sy) + B_1(x, sx)\varepsilon((B_2(\Delta_\alpha(s), sy)) + B_1(x, sx)\nabla_{M_2}(y)(s)$$

$$= 0 \quad \text{in } R/2R$$

for $x \in M_1$, $y \in M_2$, and $s \in S$. By using Proposition 5.2, we have $\nabla_M = 0$. □

The next theorem follows.

**Theorem 10.2.** Let $\mathcal{A} = (R, G, Q, S, \lambda, w)$ and $\Theta$ be as above. Then $SWQ_0(\mathcal{A}, \Theta)$, $SWQ_0(\mathcal{A}, \Theta)^{\text{t-iso}}$ and $SWQ_0(\mathcal{A})$ are modules over $SGW_0(R, G, S, \Theta)$.

11. The Mackey and Green Structures of GW and SGW

Let $S$ be a conjugation-invariant subset of $G(2)$, and set

$$S_H = H \cap S$$

for each $H \in S(G)$. Let $Z^{(0)}$ be a finite $G$-set and let $\mathfrak{P}(Z^{(0)})$ stand for the set of all subsets of $Z^{(0)}$. Let $S(G) \rightarrow \mathfrak{P}(Z^{(0)}); H \mapsto Z^{(0)}_H$, be an intersection-preserving $G$-map (see (5.1)), where $S(G)$ is the set of all subgroups of $G$ on which $G$ acts by conjugation.

Define $\Theta_H$ by

$$\Theta_H = S_H \cup Z^{(0)}_H.$$ 

It immediately follows that the map $H \mapsto \Theta_H$ is intersection preserving. Define $\rho^{(2)}_H : \Theta_H \rightarrow \mathfrak{P}(S_H)$ by

$$\rho^{(2)}_H(t) = \begin{cases} \{t\} & (t \in S_H), \\ \emptyset & (t \in Z^{(0)}_H). \end{cases}$$

Then, obviously, for each $s \in S_H$, there exists exactly one $t \in \Theta_H$ with $s \in \rho^{(2)}_H(t)$. In this case, $GW_0(R, H, \Theta_H)$ is a commutative ring with unit for each subgroup $H$ of $G$, and so is $SGW_0(R, H, S_H, \Theta_H)$ if $R$ is square identical.

Now let $\varphi : H \rightarrow K$ be a morphism in $\mathcal{G}$, namely one of an inclusion map, a conjugation map, or a composition of such maps. Then we have the associated $\varphi$-equivariant map $\psi : \Theta_H \rightarrow \Theta_K$. Actually, if $\varphi$ is the inclusion map $j_{H,K} : H \rightarrow K$, then $S_H \subset S_K$ and $Z^{(0)}_H \subset Z^{(0)}_K$, and therefore the associated $\psi : \Theta_H \rightarrow \Theta_K$ is the inclusion map; if $\varphi$ is the conjugation map $c_{(H,g)} : H \rightarrow gHg^{-1}$, then the associated $\psi : \Theta_H \rightarrow \Theta_{gHg^{-1}} = g\Theta_K$ is the left translation $\ell_{(\Theta_H,g)}$ by $g$. Since the $G$-action on $S$ is given by conjugation, $\ell_{(\Theta_H,g)}|_{S_H}$ is the conjugation $c_{(H,g)}|_{S_H}$ by $g$. Thus, there are canonical correspondences

$$GW_0(R, H, \Theta_H) \rightarrow GW_0(R, K, \Theta_K); [M, B, \alpha] \mapsto [\varphi_\# M, \varphi_\# B, \psi_\# \alpha]$$
and

\[ GW_0(R, K, \Theta_K) \to GW_0(R, H, \Theta_H); \ [N, B, \alpha] \mapsto [\varphi^#N, \varphi^#B, \psi^#\alpha]. \]

**Lemma 11.1.** \( \nabla_{\varphi, M} = 0 \) for any morphism \( \varphi : H \to K \) in \( \mathcal{G} \) and any object \( M = (M, B, \alpha) \) in \( SH(R, H, \Theta_H) \).

**Proof.** For the proof, we may suppose that \( \varphi = j_{H,K} \) or \( c_{(H, H)} \). For any \( z = k \otimes_{\varphi} x \in \varphi_M \) with \( k \in K \), \( x \in M \) and \( s \in S_K \), we have

\[
\nabla_{\varphi, M}(k \otimes_{\varphi} x)(s) = \varphi#B(\Delta_{\varphi, \alpha}(s) - k \otimes_{\varphi} x, s(k \otimes_{\varphi} x)) \]

\[
= \varphi#B(\Delta_{\varphi, \alpha}(s), s(k \otimes_{\varphi} x)) - \varphi#B(k \otimes_{\varphi} x, s(k \otimes_{\varphi} x)) \in R/2R.
\]

By definition, we have

\[
\varphi#B(\Delta_{\varphi, \alpha}(s), s(k \otimes_{\varphi} x)) = \varphi#B(\Delta_{\varphi, \alpha}(s), k \otimes_{\varphi} x)
\]

\[
= \sum_{[a, s'] \in K \times H, \varphi(H)} \{ \varphi#B(a \otimes_{\varphi} \alpha(s'), k \otimes_{\varphi} x) \mid s' \in S_H, a\varphi(s')a^{-1} = s \}
\]

\[
= \sum_{[a, s'] \in K \times H, \varphi(H)} \{ \delta_{a\varphi(H), k\varphi(H)}(H, s', \varphi^{-1}(a^{-1}k)x) \mid s' \in S_H, \varphi(s') = a^{-1}sa \}
\]

\[
= \sum_{[a, s'] \in K \times H, \varphi(H)} \{ \delta_{\varphi(H), a^{-1}k\varphi(H)}(H, s', \varphi^{-1}(a^{-1}k)x) \mid s' \in S_H, \varphi(s') = a^{-1}sa \}
\]

\[
= \sum_{[a, s'] \in K \times H, \varphi(H)} \{ B(a(s''), x) \mid s'' \in S_H, \varphi(s'') = k^{-1}sk \}
\]

\[
= \sum_{[a, s''] \in K \times H, \varphi(H)} \{ B(a(s''), s''x) \mid s'' \in S_H, \varphi(s'') = k^{-1}sk \}
\]

\[
= \sum_{[a, s''] \in K \times H, \varphi(H)} \{ B(x, s''x) \mid s'' \in S_H, \varphi(s'') = k^{-1}sk \}
\]

\[
= \begin{cases} 
B(x, \varphi^{-1}(k^{-1}sk)x) & \text{if } k^{-1}sk \in \varphi(H), \\
0 & \text{(otherwise)}.
\end{cases}
\]

On the other hand,

\[
\varphi#B((k \otimes_{\varphi} x), s(k \otimes_{\varphi} x)) = \varphi#B((k \otimes_{\varphi} x), s(k \otimes_{\varphi} x))
\]

\[
= \delta_{k\varphi(H), s(k\varphi(H))}(H, x, \varphi^{-1}(k^{-1}sk)x)
\]

\[
= \begin{cases} 
B(x, \varphi^{-1}(k^{-1}sk)x) & \text{if } k^{-1}sk \in \varphi(H), \\
0 & \text{(otherwise)}.
\end{cases}
\]

This gives us \( \nabla_{\varphi, M}(z)(s) = 0 \) for all \( z \in \varphi_M \) and \( s \in \Theta_K \). \( \square \)

**Proposition 11.2.** Let \( S_H, Z_H \) and \( \Theta_H \) be as above. Then, the Grothendieck-Witt ring functor \( H \mapsto GW_0(R, H, \Theta_H) \), \( H \in S(G) \), is a Mackey functor, and so is the special Grothendieck-Witt ring functor \( SGW_0(R, H, S_H, \Theta_H) \), \( H \in S(G) \).

**Proof.** This follows from Propositions 3.2, 3.4, 3.5, 4.3, 4.5, 4.6 and Lemma 11.1. \( \square \)

**Theorem 11.3.** Let \( S_H, Z_H \) and \( \Theta_H \) be as above. Then, the Grothendieck-Witt ring functor \( H \mapsto GW_0(R, H, \Theta_H) \), \( H \in S(G) \), is a Green functor, and the special
Theorem 11.4. The special Grothendieck-Witt group functor

\[ H \mapsto \text{SGW}_0(R, H, S_H) \]

is a module over the Grothendieck-Witt ring functor \( H \mapsto \text{GW}_0(R, H, S_H) \).

Proof. By Proposition 5.5, \( \text{SGW}_0(R, H, S_H) \) is a module over \( \text{GW}_0(R, H) \). The required properties for a Frobenius pairing follow from Propositions 3.1, 3.3, 4.2, 4.4, 4.10 and 5.4. \( \square \)

12. The Pairing \( \text{SGW}_0 \times \text{SW}_0 \rightarrow \text{SW}_0 \)

Let \( S \subseteq G(2), S_H, Z^{(2)}_H, \Theta_H, \rho^{(2)}_H \) be as in Section 11, where \( H \in S(G) \). Let \( w : G \rightarrow \{-1, 1\} \) be a homomorphism and let \( \lambda \) stand for either 1 or \(-1\). In the current section we invoke

\[ S \subseteq G^\lambda(2) \]}

Let \( Q \) be a conjugation-invariant subset of \( G^{-\lambda}(2) \). We set \( Q_H = H \cap Q, A_H = R[H], \) and \( A_H = (R, H, Q_H, S_H, \lambda, w|_H) \) for \( H \in S(G) \).

Let \( \varphi : H \rightarrow K \), where \( H, K \in S(G) \), be a monomorphism such that \( w|_K \circ \varphi = w|_H \), \( \varphi(Q_H) \subseteq Q_K \), and \( \varphi(S_H) \subseteq S_K \).

Let \( N = (N, B, q) \) be an \( A_K \)-quadratic \( R[K] \)-module. We can write \( q(x) \) as \( \sum_{g \in Q(K)} q(x)_g g \), where \( Q(K) = K \cap Q(G) \) and \( q(x)_g \in R_g \). We define \( \varphi^# q : \varphi^# M \rightarrow (A_H)_q/\Lambda_H = R[H]/(R[S_H] + \Lambda_H) \) by

\[ \varphi^# q(x) = \sum_{h \in Q(H)} q(x)_{\varphi(h) h} \]

for \( x \in \varphi^# M \), where \( \Lambda_H \) is the smallest form parameter of \( R[H] \) including \( Q_H \).

Lemma 12.1. The \( \varphi^# q \) above is an \( A_H \)-quadratic form on \( \varphi^# N \) with respect to \( \varphi^# B \).

Proof. The proof is straightforward, as follows: For \( g \in H \) and \( x \in \varphi^# N \), we have

\[ \varphi^# q(gx) = \sum_{h \in Q(H)} q(gx)_{\varphi(h) h} = \sum_{h \in Q(H)} q(\varphi(g)x)_{\varphi(h) h} = \sum_{h \in Q(H)} w(\varphi(g))q(x)_{\varphi(g) -1 \varphi(h) \varphi(g)} h \]

\[ = g \left( \sum_{h \in Q(H)} q(x)_{\varphi(g -1 h) g -1 h g} \right) \]

\[ = g \varphi^# q(x). \]
For $x, y \in \varphi^#N$, we have
\[
\varphi^#q(x + y) - \varphi^#q(x) - \varphi^#q(x) = \sum_{h \in Q(H)} (q(x + y)_{\varphi(h)} - q(x)_{\varphi(h)} - q(y)_{\varphi(h)})h
\]
\[
= \sum_{h \in H} B(x, y)_{\varphi(h)}h
\]
\[
= \sum_{h \in H} \varepsilon(B(x, \varphi(h)^{-1}y))h
\]
\[
= \varphi^#B(x, y)
\]
in $A_H/(\Lambda_H + (A_H)_s)$.

For $x \in \varphi^#N$, we have
\[
\varphi^#q(x) + \lambda\varphi^#q(x) = \sum_{h \in Q(H)} (\varphi(x)_{\varphi(h)}h + \lambda q(x)_{\varphi(h)}h)
\]
\[
= \sum_{h \in Q(H)} (\varphi(x)_{\varphi(h)}h + \lambda w(h)q(x)_{\varphi(h)}h^{-1})
\]
\[
= \sum_{h \in Q(H)} (\varphi(x)_{\varphi(h)}h + q(x)_{\varphi(h)}^{-1}h^{-1})
\]
\[
= \varphi^#B(x, x) \text{ in } A_H/(A_H)_s.
\]

\[\square\]

**Proposition 12.2.** Let $\varphi : H \to K$, $A_H$ and $A_K$ be as above, and let $M_1 = (M_1, B_1)$ and $M_2 = (M_2, B_2, q_2)$ be a Hermitian $R[K]$-module and an $A_K$-quadratic module, respectively. Then $(\varphi^#M_1) \cdot (\varphi^#M_2) = \varphi^#(M_1 \cdot M_2)$.

**Proof.** Let $x, x' \in M_1$ and $y, y' \in M_2$. Then
\[
B_{\varphi^#M_1, \varphi^#M_2}(x \otimes y, x' \otimes y') = \sum_{h \in H} B_1(x, \varphi(h)^{-1}x')\varepsilon(B_2(y, \varphi(h)^{-1}y'))h
\]
\[
= B_{\varphi^#(M_1, M_2)}(x \otimes y, x' \otimes y').
\]

In addition,
\[
q_{\varphi^#M_1, \varphi^#M_2}(x \otimes y) = \sum_{h \in Q(H)} B_1(x, \varphi(h)^{-1}x)q_2(y)_{\varphi(h)}h
\]
\[
= q_{\varphi^#(M_1, M_2)}(x \otimes y).
\]
We have established the proposition. \[\square\]

Now let $M = (M, B, q)$ be an $A_H$-quadratic $R[H]$-module such that $M$ is stably $R[H]$-free and $B$ is nonsingular. Let $\{g_1, \ldots, g_{t}\}$ be a complete set of representatives of $K/\varphi(H)$, where $g_i$ are elements in $K$. We define $\varphi^#q : \varphi^#M \to (A_K)_q/\Lambda = A_K/(\Lambda_K + (A_K)_s)$ by
\[
\varphi^#q\left(\sum_{i=1}^t g_i \otimes_{\varphi} x_i\right) = \sum_{i=1}^t g_i \varphi(q(x_i))g_i + \sum_{1 \leq i < j \leq t} g_i g_j \varphi(B(x_i, x_j))g_i.
\]
Lemma 12.3. The \( \varphi_\# q \) above is a quadratic form on \( \varphi_\# M \) with respect to \( \varphi_\# B \). Namely, the following hold:

1. \( \varphi_\# q(ru) = r^2 \varphi_\# q(u) \).
2. \( \varphi_\# q(u + v) - \varphi_\# q(u) - \varphi_\# q(v) = \varphi_\# B(u, v) \).
3. \( \varphi_\# q(u) + \lambda \varphi_\# q(u) = \varphi_\# B(u, u) \) in \( A_K/(A_K) \),
4. \( \varphi_\# q(ku) = k \varphi_\# q(u) k \),

for all \( r \in R, u, v \in \varphi_\# M, k \in K \).

Proof. The equality (1) holds clearly.

The proof of (2) runs as follows:

\[
\varphi_\# q \left( \sum g_i \otimes \varphi x_i + \sum g_i \otimes \varphi y_i \right) - \varphi_\# q \left( \sum g_i \otimes \varphi x_i \right) - \varphi_\# q \left( \sum g_i \otimes \varphi y_i \right)
\]

\[
= \sum \limits_{i=1}^{\ell} g_i \left( \varphi(q(x_i + y_i)) - \varphi(q(x_i)) - \varphi(q(y_i)) \right)q_i
\]

\[
+ \sum \limits_{1 \leq i < j \leq \ell} g_j \varphi \left( B(x_i + y_i, y_j - B(x_i, y_j) - B(y_i, y_j) \right) q_i
\]

\[
= \sum \limits_{i=1}^{\ell} g_i \varphi \left( B(x_i, y_i) \right)q_i + \sum \limits_{1 \leq i < j \leq \ell} g_j \varphi \left( B(x_i, y_j) \right) q_i
\]

\[
= \varphi_\# B \left( \sum \limits_{i=1}^{\ell} g_i \otimes \varphi x_i, \sum \limits_{j=1}^{\ell} g_j \otimes \varphi y_j \right).
\]

The equality (3) holds because

\[
\varphi_\# q(g_i \otimes \varphi x) + \lambda \varphi_\# q(g_i \otimes \varphi x)
\]

\[
= g_i \varphi(q(x))q_i + \lambda g_i \varphi(q(x))q_i
\]

\[
= g_i \varphi(B(x, x))q_i
\]

\[
= \varphi_\# B(g_i \otimes \varphi x, g_i \otimes \varphi x).
\]

For \( k \in K \), we can write \( kg_i \) in the form \( g_{\sigma(i)} \varphi(h_i) \) with \( h_i \in H \). Then

\[
\varphi_\# q(k(g_i \otimes \varphi x)) = \varphi_\# q(g_{\sigma(i)} \otimes \varphi h_i x)
\]

\[
= g_{\sigma(i)} \varphi(q(h_i x))g_{\sigma(i)}
\]

\[
= g_{\sigma(i)} \varphi(h_i) \varphi(q(x))g_{\sigma(i)} \varphi(h_i)
\]

\[
= kg_i \varphi(q(x))k
\]

\[
= \varphi_\# q(g_i \otimes \varphi x) k.
\]

The equation (4) follows from this and (2) above. \( \square \)
Proposition 12.4. Let $H$ be a subgroup of $G$, $q : M \to (A_H)_q/\Lambda_H$ an $A_H$-quadratic form on $M$, and $g$ an element of $G$. Then the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{c_{(H,g)}} & (A_H)_q/\Lambda_H \\
\downarrow{f_0} & & \downarrow{w(g)q} \\
M & \xrightarrow{c_{(gH^{-1},g^{-1})}} & (A_{gH^{-1}})_q/\Lambda_{gH^{-1}}
\end{array}
\]

commutes, where $f_0$ is the canonical $R[gHg^{-1}]$-isomorphism (cf. Proposition 3.2).

The proof of the proposition is straightforward.

Proposition 12.5. Let $H$ be a subgroup of $G$ and $q : M \to (A_H)_q/\Lambda_H$ an $A_H$-quadratic form on $M$. Then for each $g \in H$, the following diagrams commute:

\[
\begin{array}{ccc}
M & \xrightarrow{c_{(H,g)}} & (A_H)_q/\Lambda_H \\
\downarrow{f_1} & & \downarrow{w(g)q} \\
M & \xrightarrow{c_{(gH^{-1},g^{-1})}} & (A_{gH^{-1}})_q/\Lambda_{gH^{-1}}
\end{array}
\]

and

\[
\begin{array}{ccc}
M & \xrightarrow{c_{(H,g)}} & (A_H)_q/\Lambda_H \\
\downarrow{f_2} & & \downarrow{w(g)q} \\
M & \xrightarrow{c_{(gH^{-1},g^{-1})}} & (A_{gH^{-1}})_q/\Lambda_{gH^{-1}}
\end{array}
\]

where $f_1$ and $f_2$ are the canonical isomorphisms (cf. Proposition 3.4).

The proposition follows straightforwardly from the definition.

Proposition 12.6. For any subgroups $H$ and $K$ of $G$, each $A_H$-quadratic form $q : M \to (A_H)_q/\Lambda_H$ satisfies the $w$-Mackey double coset formula. Namely,

\[
(\text{Res}_K^G \text{Ind}_H^G q) \circ \omega = \sum_{KgH \subseteq K \setminus G/H} w(g) \text{Ind}_K^K c_{(H \cap g^{-1} K, g, g)} \# (\text{Res}_{K^gHg^{-1}K}^H q),
\]

where $\omega$ is the canonical isomorphism (cf. Proposition 3.3). Particularly, in the case $w(G) = \{1\}$, $q$ satisfies the Mackey double coset formula.

Proof. It suffices to prove that

\[
(\text{Res}_K^G \text{Ind}_H^G q)(ag \otimes x) = w(g)(\text{Ind}_K^K c_{(H \cap g^{-1} K, g, g)} \# (\text{Res}_{K^gHg^{-1}K}^H q))(a \otimes (e \otimes x))
\]
for any \( g \in G, a \in K, x \in \text{Res}^H_{H \cap g^{-1}Kg}M \). This is valid because

\[
(\text{Res}^G_K \text{Ind}^G_H q)(ag \otimes x) = \sum_{k \in \mathbb{Q}(K)} (\text{Ind}^G_H q)(ag \otimes x)_k k
\]

and

\[
(\text{Ind}^K_K \cap g:H^{-1}\cap c(H \cap g^{-1}Kg, g)^{-1}\text{Res}^H_{H \cap g^{-1}Kg} q)(a \otimes (e \otimes x))
\]

\[
= \sum_{k \in \mathbb{Q}(K)} (a \text{c}(H \cap g^{-1}Kg, g)^{-1}\text{Res}^H_{H \cap g^{-1}Kg} q)(e \otimes x) k_k
\]

\[
= \sum_{k \in \mathbb{Q}(K)} (ag(\text{Res}^H_{H \cap g^{-1}Kg} q)(x) g^{-1} \pi) k_k
\]

\[
= w(g) \sum_{k \in \mathbb{Q}(K)} (ag(x) \pi g) k.
\]

\[
\square
\]

**Proposition 12.7.** Let \( A_H \) and \( \Theta_H \) be as above for each \( H \in S(G) \). Then the Witt group functor \( H \mapsto WQ_0(A_H, \Theta_H) \), \( H \in S(G) \), and the special Witt group functor \( H \mapsto SWQ_0(A_H, \Theta_H) \), \( H \in S(G) \), are both \( w \)-Mackey functors, and hence modules over the Burnside ring functor \( H \mapsto \Omega(G) \), \( H \in S(G) \).

**Proof.** The claim for the Witt group functor follows from Propositions 8.2, 8.4, 8.5, 6.4, 6.8, 12.3, and 12.6.

Let \( M = (M, B, q, \alpha) \) be a \( \Theta_H \)-positioned \( A_H \)-quadratic \( R[H] \)-module. By Lemma 5.1, \( \varepsilon \circ B : M \times M \to R \) is a \( \lambda \)-symmetric, \( (H, w[H]) \)-invariant, \( R \)-bilinear form. For a morphism \( \varphi : H \to K \) in \( \mathcal{G} \), the same argument as the proof of Lemma 11.1 shows that if \( \nabla^M = 0 \) (see (8.1)), then \( \nabla^\varphi_M = 0 \). (In fact, consider the case where \( R \) is replaced by \( R/2R \).) Thus, the claim for the special Witt group functor also follows. \[ \square \]

In the remainder of this section, let \( \varphi : H \to K \) be a morphism in \( \mathcal{G} \).

**Proposition 12.8.** Let \( M_1 = (M_1, B_1, \alpha_1) \) and \( M_2 = (M_2, B_2, q_2, \alpha_2) \) be objects in \( H(R, K, \Theta_K) \) and \( Q(A_H, \Theta_H) \), respectively. Let

\[
f : M_1 \otimes_R \varphi_# M_2 \to \varphi_# (\varphi_# M_1 \otimes_R M_2)
\]

denote the canonical isomorphism, namely \( f(x \otimes (k \otimes \varphi y)) = k \otimes \varphi (k^{-1} x \otimes y) \) for \( k \in K, x \in M_1 \) and \( y \in M_2 \). Then the diagram

\[
\begin{array}{ccc}
M_3 \times M_3 & \xrightarrow{f \times f} & M_4 \times M_4 \\
\downarrow & & \downarrow \\
M_2 \otimes_R \varphi_# B_1 & & A_K \\
\end{array}
\]

\[
B_1 \otimes_R \varphi_# B_2
\]
where \( M_3 = (M_1 \otimes_R (R[K] \otimes_{R[H],\varphi} M_2)) \) and \( M_4 = R[K] \otimes_{R[H],\varphi} (\varphi^# M_1 \otimes_R M_2) \), and the diagram

\[
\begin{array}{ccc}
M_1 \otimes_R (R[K] \otimes_{R[H],\varphi} M_2) & \overset{f}{\longrightarrow} & R[K] \otimes_{R[H],\varphi} (\varphi^# M_1 \otimes_R M_2) \\
& \searrow^{B_1 \otimes_R \varphi^# q_2} & \downarrow_{\varphi^# (\varphi^# B_1 \otimes_R q_2)} \\
& & (A_K)_q / \Lambda_K
\end{array}
\]

commute.

**Proof.** Let \( k, k' \in K, x, x' \in M_1, \) and \( y, y' \in M_2. \)

The commutability \( B_1 \otimes (\varphi^# B_2) = \varphi^# ((\varphi^# B_1) \otimes B_2) \) via \( f \) holds because

\[
B_1 \otimes (\varphi^# B_2)(x \otimes (k \otimes_\varphi y), x' \otimes (k' \otimes_\varphi y')) = \sum_{g \in K} B_1(x, g^{-1}x') \varepsilon(\varphi^# B_2(k \otimes_\varphi y, g^{-1}(k' \otimes_\varphi y'))) g
\]

\[
= \sum_{g \in K} w(k) \delta_{k \varphi(H), g^{-1}k' \varphi(H)} B_1(x, g^{-1}x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1}g^{-1}k')y')) g
\]

and

\[
\varphi^# ((\varphi^# B_1) \otimes B_2)(k \otimes_\varphi (k^{-1}x \otimes y), k' \otimes_\varphi (k'^{-1}x' \otimes y')) = \sum_{g \in K} w(k) \delta_{k \varphi(H), g^{-1}k' \varphi(H)} \cdot \varepsilon \left( ((\varphi^# B_1 \otimes B_2)((k^{-1}x \otimes y), \varphi^{-1}(k^{-1}g^{-1}k')(k'^{-1}x' \otimes y'))) \right) g
\]

\[
= \sum_{g \in K} w(k) \delta_{k \varphi(H), g^{-1}k' \varphi(H)} \cdot B_1(k^{-1}x, (k^{-1}g^{-1}k')k'^{-1}x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1}g^{-1}k')y')) g
\]

\[
= \sum_{g \in K} w(k) \delta_{k \varphi(H), g^{-1}k' \varphi(H)} B_1(k^{-1}x, k^{-1}g^{-1}x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1}g^{-1}k')y')) g
\]

\[
= \sum_{g \in K} w(k) \delta_{k \varphi(H), g^{-1}k' \varphi(H)} B_1(x, g^{-1}x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1}g^{-1}k')y')) g.
\]
Proposition 12.9. Let $M_1 = (M_1, B_1, \alpha_1)$ and $M_2 = (M_2, B_2, q_2, \alpha_2)$ be objects in $\mathcal{H}(R, H, \Theta_H)$ and $\mathcal{Q}({A_K, \Theta_K})$, respectively. Let
\[
f' : (\varphi(B_1) \otimes_R M_2) \to \varphi(M_1 \otimes_R \varphi(B_2))
\]
denote the canonical isomorphism, namely $f'((k \otimes_R x) \otimes y) = k \otimes_R (x \otimes k^{-1}y)$ for $k \in K$, $x \in M_1$ and $y \in M_2$. Then the diagram

\[
\begin{array}{ccc}
M_3 \times M_3 & \overset{\varphi(B_1 \otimes_R B_2)}{\longrightarrow} & A_K \\
\downarrow f' \times f' & & \\
M_4 \times M_4 & \overset{\varphi(B_1 \otimes_R B_2)}{\longrightarrow} & A_K
\end{array}
\]
where $M_3 = (R[K] \otimes_{R[H], \varphi} M_1) \otimes_R M_2$ and $M_4 = R[K] \otimes_{R[H], \varphi} (M_1 \otimes_R \varphi^# M_2)$, and the diagram

\[
\begin{array}{ccc}
(R[K] \otimes_{R[H], \varphi} M_1) \otimes_R M_2 & \xrightarrow{f'} & (A_K)_{q}/\Lambda_K \\
& \varphi_{B_1 \otimes_R q_2} \downarrow & \\
R[K] \otimes_{R[H], \varphi} (M_1 \otimes_R \varphi^# M_2) & \xrightarrow{\varphi_{(B_1 \otimes_R \varphi^# q_2)}} & (A_K)_{q}/\Lambda_K
\end{array}
\]

commute.

Proof. Let $k, k' \in K$, $x, x' \in M_1$, and $y, y' \in M_2$.

The commutability $(\varphi_B \otimes_{B_1} B_2) = \varphi_{B_1 \otimes_R (\varphi_B \otimes_{B_2})}$ via $f'$ holds because

\[
(\varphi_B \otimes_{B_1} B_2)((k \otimes_{B_1} (x \otimes y), (k' \otimes_{B_1} x')) \otimes y') = \sum_{g \in K} \varphi_B((k \otimes_{B_1} x, g^{-1}(k' \otimes_{B_1} x')) \varepsilon(B_2(y, g^{-1}y')))
\]

and

\[
\varphi_B((k \otimes_{B_1} (x \otimes y), (k' \otimes_{B_1} x')) \otimes y') = \sum_{g \in K} \delta_{k, \varphi(H), g^{-1}, k', \varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k')x') \varepsilon(B_2(y, g^{-1}y')))
\]

The commutability $(\varphi_B \otimes q_2) = \varphi_B((k \otimes_{B_1} (x \otimes y)) \otimes y) = \sum_{g \in \Omega(K)} (\varphi_B B_1)(k \otimes_{B_1} x, g^{-1}(k \otimes_{B_1} x))q_2(y)g
\]

\[
= \sum_{g \in \Omega(K)} \delta_{k, \varphi(H), g^{-1}, k', \varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k)xq_2(y)g
\]

\[
= \sum_{g \in \Omega(K)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k)x)q_2(y)g
\]

\[
= k \sum_{h \in \Omega(H)} B_1(x, h^{-1}x)q_2(y)q_{k, (h^{-1})}(h^{-1}) \varphi(h)
\]

\[
= k \sum_{h \in \Omega(H)} w(k)B_1(x, h^{-1}x)q_2(k^{-1}y)q_{(h), \varphi(h)}(h^{-1})
\]

\[
= k \varphi \left( \sum_{h \in \Omega(H)} B_1(x, h^{-1}x)q_2(k^{-1}y)q_{(h), \varphi(h)}(h^{-1}) \right)
\]
and
\[ \varphi_#(B_1 \otimes (\varphi_# q_2))(k \otimes (x \otimes k^{-1} y)) = k\varphi(B_1 \otimes (\varphi_# q_2)(x \otimes k^{-1} y)) \]
\[ = k\varphi \left( \sum_{h \in \mathcal{Q}(H)} B_1(x, h^{-1} x)q_2((k^{-1} y)\varphi(h)h) \right) \mathcal{K}. \]

Let \( \psi : \Theta_H \to \Theta_K \) denote the map associated with \( \varphi \).

**Theorem 12.10.** Let \( A_H \) and \( \Theta_H \) be as above for each \( H \in S(G) \). Then the \( w \)-Mackey functor \( H \mapsto \mathcal{W}Q_0(A_H, \Theta_H), H \in S(G) \), is a module over the Green functor \( H \mapsto \mathcal{G}W_0(R, H, \Theta_H), H \in S(G) \). If \( R \) is square identical, then the \( w \)-Mackey functor \( H \mapsto \mathcal{S}WQ_0(A_H, \Theta_H), H \in S(G) \), is a module over the Green functor \( H \mapsto \mathcal{S}GW_0(R, H, S_H, \Theta_H), H \in S(G) \).

**Proof.** By Proposition 3.3 we have \( \alpha_1 \otimes (\psi_# \alpha_2) = \psi_#((\psi_# \alpha_1) \otimes \alpha_2) \) via \( f \) in Proposition 12.5. By Proposition 5.3 we have \( (\psi_# \alpha_1) \otimes \alpha_2 = \psi_#(\alpha_1 \otimes \psi_# \alpha_2) \) via \( f' \) in Proposition 12.9. The theorem follows from Propositions 12.2, 12.8 and 12.9. \( \square \)

13. Applications of induction and restriction

Let \( Z^0 \) be a finite \( G \)-set, and let \( S(G) \to \mathcal{P}(Z^0) \); \( H \mapsto Z^0_H \) be an intersection-preserving \( G \)-map. Let \( S \) be a conjugation-invariant subset of \( G(2) \). We set \( S_H = H \cap S \) and \( \Theta_H = S_H \cap Z^0_H \). Define \( \rho^{(2)}_H : \Theta_H \to \mathcal{P}(S_H) \) by
\[ \rho^{(2)}_H(t) = \begin{cases} \{t\} & \text{if } t \in S_H, \\ \emptyset & \text{if } t \in Z^0_H. \end{cases} \]

Further, let \( \mathcal{F} \) be a conjugation-invariant subset of \( S(G) \) such that
\[ (13.1) \quad \Theta_G \times \Theta_G = \bigcup_{H \in \mathcal{F}} \Theta_H \times \Theta_H, \]
and let \( \beta \) be an element in the Burnside ring \( \Omega(G) \) such that
\[ \text{Res}^G_H \beta = 1_{\Omega(H)} \quad \text{for any } H \in \mathcal{F}. \]

**Theorem 13.1.** Let \( x \) be an arbitrary element in \( \mathcal{S}GW_0(R, G, S, \Theta_G) \). If \( \mathcal{F} \) contains all 2-hyperelementary (resp. cyclic) subgroups of \( G \), then \((1_{\Omega(G)} - \beta)^2 x = 0 \)
(resp. \((1_{\Omega(G)} - \beta)^{2k+3} x = 0 \), where \( k \) is the integer such that \(|G| = 2^k m \) with an odd integer \( m \)).

For the proof, we recall two lemmas.

**Lemma 13.2** (A. Dress [11] Theorems 1 and 3). For a set \( \mathcal{H} \) of subgroups of \( G \), the restriction homomorphism
\[ \text{Res} : \mathcal{G}W_0(\mathbb{Z}, G) \to \bigoplus_{H \in \mathcal{H}} \mathcal{G}W_0(\mathbb{Z}, H) \]
has the following properties.

1. If \( \mathcal{H} \) contains all 2-hyperelementary subgroups of \( G \), then \( \text{Res} \) is injective.
2. If \( \mathcal{H} \) contains all cyclic subgroups of \( G \), then the kernel of \( \text{Res} \) is annihilated by 4.
For a subgroup $H$ of $G$, we denote by $\chi_H$ the homomorphism $\Omega(G) \to \mathbb{Z}$ such that $\chi_H([X]) = |X^H|$ for every finite $G$-set $X$.

**Lemma 13.3** ([13], Proposition 6.3). Let $x$ be an element of $\Omega(G)$ such that $\chi_H(x) \equiv 0 \mod 2$ for all $H \in S(G)$. Then $x^{k+1}$ lies in $2\Omega(G)$, where $k$ is the integer such that $|G| = 2^k m$ with an odd integer $m$.

**Proof of Theorem 13.1** Let $H$ be a 2-hyperelementary subgroup of $G$.

First consider the case where $F$ contains all 2-hyperelementary subgroups of $G$. Then, it is obvious that $\text{Res}^G_H((1_{\Omega(G)} - \beta)GW_0(\mathbb{Z}, G)) = 0$. For a subgroup $H$ of $G$, we have

$$\chi_K((1_{\Omega(G)} - \beta) = 0 \mod 2$$

for any subgroup $K$ of $H$, and hence $\text{Res}^G_H((1_{\Omega(G)} - \beta)^{2k+2} \text{ lies in } 4\Omega(H)$). So, we can write $\text{Res}^G_H((1_{\Omega(G)} - \beta)^{2k+2} = 4\gamma$ for some $\gamma \in \Omega(H)$. Clearly, $\text{Res}^G_H(\gamma) = 0$ for all cyclic subgroups of $H$. Thus by (2) of Dress’ Lemma, $\gamma GW_0(Z, H)$ is annihilated by 4, and hence $\text{Res}^G_H((1_{\Omega(G)} - \beta)^{2k+2} GW_0(Z, G)) = 0$. By (1) of Dress’ Lemma, we obtain

$$(1 - \beta)GW_0(Z, G) = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+2} GW_0(Z, G) = 0.$$ 

Since the canonical map $GW_0(Z, G) \to GW_0(R, G)$ is an $\Omega(G)$-homomorphism of a ring with unit, it follows that

$$(1 - \beta)GW_0(R, G) = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+2} GW_0(R, G) = 0.$$ 

Noting that the Mackey functor $SGW_0(R, - , S_\infty)$ is a module over the Green functor $GW_0(R, -)$, we obtain

$$(1 - \beta)SGW_0(R, G, S) = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+2} SGW_0(R, G, S) = 0.$$ 

Recall Proposition 5.28, namely the fact that the canonical homomorphism

$$SGW_0(R, G, S) \to SGW_0(R, G, S, \Theta_G)^{t-iso}$$

is surjective. In addition, the homomorphism is an $\Omega(G)$-homomorphism. Hence, we conclude that

$$(1 - \beta)SGW_0(R, G, S, \Theta_G)^{t-iso} = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+2} SGW_0(R, G, S, \Theta_G)^{t-iso} = 0.$$ 

On the other hand, it is easy to check that $(1 - \beta)SGW_0(R, G, S, \Theta_G)$ is contained in (the image by the canonical map from) $SGW_0(R, G, S, \Theta_G)^{t-iso}$.

Putting all together, we establish that

$$(1 - \beta)^2SGW_0(R, G, S, \Theta_G) = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+3} SGW_0(R, G, S, \Theta_G) = 0.$$ 

□

**Proof of Theorem 13.2** Here $Z^{(0)}$ is the empty set. Since $H \mapsto SGW(R, H, S_H, S_H)$ is a Mackey functor, it is a module over the Burnside ring functor $H \mapsto \Omega(H)$ by [13], Proposition 6.2.3]. For each subgroup $H$ of $G$ we have

$$\Theta_H \times \Theta_H = (S \cap H) \times (S \cap H) = (S \times S) \cap (H \times H).$$

Thus (13.1) is fulfilled, and Theorem 13.2 follows from Theorem 13.1 □
Now let \( w : G \to \{-1, 1\} \) be a homomorphism, \( \lambda = 1 \) or \(-1\), and let \( Q \) be a conjugation-invariant subset of \( G^{-\lambda}(2) \). Suppose \( S \subset G^{\lambda}(2) \). For each \( H \in S(G) \), we set \( A_H = R[H], Q_H = H \cap Q \), and \( A_H = (R, H, Q_H, S_H, \lambda, w|_H). \)

**Theorem 13.4.** Suppose \( R \) is square identical. Let \( x \) be an arbitrary element of the special Witt group \( \text{SWQ}_0(A_G, \Theta_G) \). If \( F \) contains all 2-hyperelementary (resp. cyclic) subgroups of \( G \), then \( (1\Omega(G) - \beta)^2 x = 0 \) (resp. \( (1\Omega(G) - \beta)^{2k+3} x = 0 \), where \( |G| = 2^k m \) with \( m \) odd).

**Proof.** The theorem follows from Proposition 12.10 and Theorem 13.1. \( \square \)

**Proof of Theorem 1.3.** Theorem 1.3 follows from Theorem 13.4. \( \square \)

**Theorem 13.5.** Suppose that \( R \) is square identical, \( F \) contains any cyclic subgroup of \( G \), and \( \beta \) has the form
\[
\beta = \sum_{H \in \tilde{F}} n_H[G/H],
\]
with \( n_H \in \mathbb{Z} \) for some lower closed subset \( \tilde{F} \) of \( S(G) \); namely, any subgroup \( H \) of \( G \) lies in \( \tilde{F} \) whenever \( K \in \tilde{F} \) and \( H \subset K \). Then
\[
\text{SWQ}_0(R, G, Q, S, \Theta_G) = \sum_{H \in \tilde{F}} \text{Ind}_H^G \text{SWQ}_0(R, H, Q_H, S_H, \Theta_H),
\]
and the restriction homomorphism
\[
\text{Res} : \text{SWQ}_0(R, G, Q, S, \Theta_G) \to \bigoplus_{H \in \tilde{F}} \text{SWQ}_0(R, H, Q_H, S_H, \Theta_H)
\]
is injective.

**Proof.** By hypothesis, we can write
\[
(1\Omega(G) - \beta)^{2[G]+3} = [G/G] - \sum_{H \in \tilde{F}} m_H[G/H]
\]
with \( m_H \in \mathbb{Z} \). For an arbitrary element \( x \in \text{SWQ}_0(R, G, Q, S, \Theta_G) \), Theorem 13.4 implies that
\[
x = \sum_{H \in \tilde{F}} m_H[G/H] \cdot x = \sum_{H \in \tilde{F}} m_H \text{Ind}_H^G(\text{Res}_H^G x).
\]
Moreover, if \( \text{Res}_H^G x = 0 \) for every \( H \in \tilde{F} \), then we conclude that \( x = 0 \). \( \square \)

**Proof of Theorem 1.4.** Since \( G \) is a nonsolvable group, there exists an idempotent \( \beta \in \Omega(G) \) such that \( \chi_K(\beta) = 0 \) for any nonsolvable subgroup \( K \) of \( G \) and \( \chi_H(\beta) = 1 \) for any solvable subgroup \( H \) of \( G \). This element \( \beta \) has the form \( \beta = \sum_{H \in \tilde{F}} n_H[G/H] \) with \( n_H \in \mathbb{Z} \), where \( H \) runs over the set of all solvable subgroups of \( G \). Thus, Theorem 1.4 follows from Theorem 13.5. \( \square \)

**References**


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