

INDUCTION THEOREMS OF SURGERY OBSTRUCTION GROUPS

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Dedicated to Professor Anthony Bak for his sixtieth birthday

ABSTRACT. Let G be a finite group. It is well known that a Mackey functor $\{H \mapsto M(H)\}$ is a module over the Burnside ring functor $\{H \mapsto \Omega(H)\}$, where H ranges over the set of all subgroups of G . For a fixed homomorphism $w : G \rightarrow \{-1, 1\}$, the Wall group functor $\{H \mapsto L_n^h(\mathbb{Z}[H], w|_H)\}$ is not a Mackey functor if w is nontrivial. In this paper, we show that the Wall group functor is a module over the Burnside ring functor as well as over the Grothendieck-Witt ring functor $\{H \mapsto \text{GW}_0(\mathbb{Z}, H)\}$. In fact, we prove a more general result, that the functor assigning the equivariant surgery obstruction group on manifolds with middle-dimensional singular sets to each subgroup of G is a module over the Burnside ring functor as well as over the special Grothendieck-Witt ring functor. As an application, we obtain a computable property of the functor described with an element in the Burnside ring.

1. INTRODUCTION

Dress' induction theory ([10], [11], [12]) of Mackey functors has been useful for algebraic computation of Wall's surgery obstruction groups ([27]) with trivial orientation homomorphisms and related groups (cf. [6], [13], [14]) as well as for applications in transformation groups (e.g. [16], [18], [25], [26]). In this paper, we develop induction theory for surgery obstruction groups appearing in [4], [5] and [19], which allows nontrivial orientation homomorphisms, and by using this generalization and [22, Theorem 1.1] we can construct various group actions on smooth manifolds (e.g. [4], [15], [16], [17], [20], [21], [24]).

Throughout this paper, let G be a finite group, $\mathcal{S}(G)$ the set of all subgroups of G , and R a principal ideal domain (possibly a commutative field). Hence R is a commutative ring and any finitely generated projective R -module is free over R . An R -module is always assumed to be finitely generated over R , unless otherwise stated.

Let $\text{GW}_0(R, G)$ denote the Grothendieck-Witt ring in A. Dress [11]. It is well known that the functor $H \mapsto \text{GW}_0(R, H)$, $H \in \mathcal{S}(G)$, with canonical correspondence of morphisms is a Green functor, which is a special case of Theorem 11.3 since $\text{GW}_0(R, G) = \text{GW}_0(R, G, \emptyset)$. Let $\mathcal{C}(G)$ denote the set of all cyclic subgroups

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of G . By [11, Theorem 1], the functor $H \mapsto \mathrm{GW}_0(R, H)$ is $\mathcal{C}(G)$ -hypercomputable in the sense of A. Bak [2]. Let $w : G \rightarrow \{-1, 1\}$ be a homomorphism and $n = 2k$ an even integer. If w is nontrivial, the Wall group functor $H \mapsto L_n^h(R[H], w|_H)$ ([27]), $H \in \mathcal{S}(G)$, is not a Mackey functor. Since $L_n^h(R[G], w) = \mathrm{WQ}_0(\mathbf{A}, \emptyset)$ with $\mathbf{A} = (R, G, \emptyset, \emptyset, (-1)^k, w)$, Propositions 12.7 and 2.6 imply that the Wall group functor is a w -Mackey functor in the sense of Definition 2.2 and a module over the Burnside ring functor. Furthermore, the Wall group functor is a module over the functor $H \mapsto \mathrm{GW}_0(R, H)$, which is a special case of Theorem 12.10. Thus, we obtain the theorem:

Theorem 1.1. *Let $w : G \rightarrow \{-1, 1\}$ be a homomorphism and n an even integer. Then the Wall group functor $H \mapsto L_n^h(R[H], w|_H)$, $H \in \mathcal{S}(G)$, is $\mathcal{C}(G)$ -hypercomputable.*

The main purpose of this paper is to study the induction–restriction theory of the equivariant surgery obstruction group $\mathrm{SWQ}_0(R, G, Q, S, \Theta_G)$ obtained by Bak and Morimoto [5], which consists of equivalence classes of special λ -quadratic $R[G]$ -modules. This surgery obstruction group is determined by a datum

$$\mathcal{D} = (R, G, Q, S, \lambda, w, \Theta_G, \rho^{(2)}).$$

The ingredient λ stands for a symmetry, namely either 1 or -1 . Let $G(2)$ denote the subset of G consisting of all elements of order 2. An element $g \in G(2)$ is called λ -symmetric or λ -quadratic if $g = \lambda w(g)g^{-1}$ or $g = -\lambda w(g)g^{-1}$, respectively. The ingredients Q and S are conjugation-invariant subsets of $G(2)$ consisting of λ -quadratic elements and λ -symmetric ones, respectively. Let $\mathfrak{P}(S)$ denote the set of all subsets of S . In a general case, Θ_G stands for a finite G -set and $\rho^{(2)}$ is a G -map $\Theta_G \rightarrow \mathfrak{P}(S)$. In the case where S and Θ_G are both empty and $\lambda = (-1)^k$, the group $\mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, \Theta_G)$ coincides with the Bak group $W_{2k}(\mathbb{Z}[G], \Gamma Q, w)$ (see [19]); if moreover Q is also empty, then the group is nothing but the Wall group $L_{2k}^h(\mathbb{Z}[G], w)$ (see [27]).

In the current section, since the case $\Theta_G = S$ has interesting applications (e.g. [4], [15], [16]), we let Θ_G and $\rho^{(2)}$ be the same as the set S and the map $s \mapsto \{s\}$, $s \in S$, respectively.

We detail the pairing

$$\mathrm{SGW}_0(\mathbb{Z}, G, S, S) \times \mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, S) \longrightarrow \mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, S)$$

in Sections 9 and 10, and show that $\mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, S)$ is a module over the special Grothendieck-Witt ring $\mathrm{SGW}_0(\mathbb{Z}, G, S, S)$, which corrects the invalid description [15, page 513, lines 9–10] of the pairing.

The groups $\mathrm{GW}_0(R, G)$ and $L_n^h(R[G], w)$ with $n = 2k$ have the hyperelementary computability. Dress proved this fact by studying the index of the subgroup $I(\mathfrak{H}_\Sigma(G), \mathrm{GW}_0)$ of $\mathrm{GW}_0(R, G)$ ([11, Theorem 1]), which we call the *Dress index*. The theorem looks technical but is fundamental. It is natural to regard the Burnside ring as a generalization of the ring of integers in the theory of transformation groups. Thus, one expects that some computability of the groups $\mathrm{SGW}_0(\mathbb{Z}, G, S, S)$ and $\mathrm{SWQ}_0(\mathbb{Z}, G, Q, S, S)$ can be described with an element in the Burnside ring instead of the Dress index. The following theorems are obtained in this respect.

Let $1_{\Omega(G)}$ denote the unit of the Burnside ring $\Omega(G)$.

Theorem 1.2. *Let S be a conjugation-invariant subset of G consisting of elements of order 2, let \mathcal{F} be a conjugation-invariant set of subgroups of G such that*

$$S \times S \subset \bigcup_{H \in \mathcal{F}} H \times H,$$

and let β be an element of the Burnside ring $\Omega(G)$ such that

$$\text{Res}_H^G \beta = 1_{\Omega(H)} \quad \text{for any } H \in \mathcal{F}.$$

If \mathcal{F} contains all 2-hyerelementary (resp. cyclic) subgroups of G , then, for an arbitrary element $x \in \text{SGW}_0(R, G, S, S)$,

$$(1_{\Omega(G)} - \beta)^2 x = 0$$

(resp. $(1_{\Omega(G)} - \beta)^{2k+3} x = 0$, where $|G| = 2^k m$ with m odd).

We say that R is square identical if

$$(1.1) \quad r^2 \equiv r \pmod{2R} \quad \text{for all } r \in R.$$

Theorem 1.3. *Let S , β and \mathcal{F} be as in the theorem above. Suppose that R is square identical, and each element of S is λ -symmetric. Let Q be a conjugation-invariant subset of G consisting of λ -quadratic elements of order 2. If \mathcal{F} contains all 2-hyerelementary (resp. cyclic) subgroups of G , then for an arbitrary element x of $\text{SWQ}_0(R, G, Q, S, S)$,*

$$(1_{\Omega(G)} - \beta)^2 x = 0$$

(resp. $(1_{\Omega(G)} - \beta)^{2k+3} x = 0$, where $|G| = 2^k m$ with m odd).

Note that the datum $\mathcal{D} = (R, G, Q, S, \lambda, w, S, \rho^{(2)})$, where $\rho^{(2)} : S \rightarrow \mathfrak{P}(S)$ is the “identity map” $s \mapsto \{s\}$, yields the datum $\mathcal{D} = (R, H, Q \cap H, S \cap H, \lambda, w|_H, S \cap H, \rho^{(2)}|_{S \cap H})$ and determines the group $\text{SWQ}_0(R, H, Q \cap H, S \cap H, S \cap H)$ for each subgroup H of G .

Theorem 1.4. *Let G be a nonsolvable group and let R , Q and S be as in the previous theorem. Then*

$$\text{SWQ}_0(R, G, Q, S, S) = \sum_H \text{Ind}_H^G \text{SWQ}_0(R, H, Q \cap H, S \cap H, S \cap H),$$

and the restriction homomorphism

$$\text{Res} : \text{SWQ}_0(R, G, Q, S, S) \longrightarrow \bigoplus_H \text{SWQ}_0(R, H, Q \cap H, S \cap H, S \cap H)$$

is injective, where H ranges over the set of all solvable subgroups of G .

Each of Theorems 1.2–1.4 is slightly generalized in Section 13.

The organization of the paper is as follows. In Section 2, we define a w -Mackey functor, a Green functor, and a module over a Green functor. In Section 3, we observe basic properties of Θ -positioned $R[G]$ -modules, namely induction-restriction properties and the Mackey double coset formula. Section 4 is devoted to observing induction-restriction properties of Θ -positioned Hermitian $R[G]$ -modules as well as defining their Grothendieck-Witt rings. In Section 5, we introduce the ∇ -invariant of Θ -positioned Hermitian $R[G]$ -modules and define the special Grothendieck-Witt groups. Similarly to Wall’s surgery theory, $R[G]$ -valued λ -Hermitian forms are indispensable objects to equivariant surgery theory on manifolds with middle-dimensional singular sets. Section 6 is devoted to observing induction-restriction

properties of $R[G]$ -valued λ -Hermitian modules. Sections 7 and 8 are devoted to defining the Witt groups and the special Witt groups of Θ -positioned quadratic $R[G]$ -modules, respectively. The tensor product of a Hermitian $R[G]$ -module and a quadratic $R[G]$ -module is introduced in Section 9, and it is discussed with ∇ -invariants in Section 10. Section 11 is devoted to showing that the Grothendieck-Witt rings and special Grothendieck-Witt rings are Green functors (possibly without unit). In Section 12 we show that the bifunctor assigning the H -surgery obstruction group to a subgroup H of G is a module over the special Grothendieck-Witt ring functor. In Section 13, we present applications relevant to G -surgery.

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2. BIFUNCTORS, w -MACKEY FUNCTORS AND GREEN FUNCTORS

Let \mathcal{G} denote the category whose objects are subgroups of G and whose morphisms are inclusions $j_{H,K} : H \rightarrow K$, where $H \subset K \subset G$, conjugations $c_{(H,g)} : H \rightarrow gHg^{-1}$; $a \mapsto gag^{-1}$, where $H \subset G$ and $g \in G$, and compositions of those maps. Let \mathcal{A} stand for the category whose objects are abelian groups and whose morphisms are group homomorphisms. We denote by $\mathbb{Z}[\mathcal{S}(G)]$ the free abelian group generated by all elements of $\mathcal{S}(G)$; hence each element of $\mathbb{Z}[\mathcal{S}(G)]$ has the form $\sum_H n_H H$ with $n_H \in \mathbb{Z}$. Let $\Omega(G)$ denote the Burnside ring of G (cf. [7], [8], [9], [23]). In fact, $\Omega(G)$ is the free abelian group generated by all G -isomorphism classes $[G/H]$ of finite G -sets G/H with $H \in \mathcal{S}(G)$. Clearly, one has the canonical homomorphism from $\mathbb{Z}[\mathcal{S}(G)]$ to $\Omega(G)$ such that $H \mapsto [G/H]$. In this paper, we mean by a *bifunctor*

$$L = (L^*, L_*) : \mathcal{G}(G) \rightarrow \mathcal{A}$$

a pair consisting of a contravariant functor $L^* : \mathcal{G}(G) \rightarrow \mathcal{A}$ and a covariant functor $L_* : \mathcal{G}(G) \rightarrow \mathcal{A}$ such that $L_*(H) = L^*(H)$, which is written as $L(H)$, for all $H \in \mathcal{S}(G)$. If the context is clear, f^* and f_* stand for $L^*(f)$ and $L_*(f)$ respectively, and Res_H^K and Ind_H^K stand for $L^*(j_{H,K})$ and $L_*(j_{H,K})$ respectively. Each bifunctor $L = (L^*, L_*) : \mathcal{G} \rightarrow \mathcal{A}$ possesses the canonical pairing

$$(2.1) \quad \mathbb{Z}[\mathcal{S}(G)] \times L(G) \longrightarrow L(G); \left(\sum_H n_H H, x \right) \longmapsto \sum_H n_H \text{Ind}_H^G(\text{Res}_H^G x),$$

for $n_H \in \mathbb{Z}$ and $x \in L(G)$. It is interesting to look for a sufficient condition so that the pairing (2.1) factors through a pairing

$$(2.2) \quad \Omega(G) \times L(G) \longrightarrow L(G).$$

If L is a Mackey functor, then, as was seen in [7, Proposition 6.2.3], the pairing (2.1) factors through a pairing (2.2). In the case where the orientation homomorphism $w : G \rightarrow \{-1, 1\}$ is not trivial, the Wall group functor $H \mapsto L_n^h(\mathbb{Z}[H], w|_H)$, $H \in \mathcal{S}(G)$, is not a Mackey functor; however, it will turn out that the associated pairing (2.1) factors through (2.2).

Let $L : \mathcal{G} \rightarrow \mathcal{A}$ be a bifunctor. Note that the kernel of the canonical map $\mathbb{Z}[\mathcal{S}(G)] \rightarrow \Omega(G)$ is

$$\langle H - gHg^{-1} \mid H \in \mathcal{S}(G), g \in G \rangle_{\mathbb{Z}}.$$

If

$$(2.3) \quad L_*(j_{H,G})L^*(j_{H,G}) = L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G}) \quad (\forall H \in \mathcal{S}(G), \forall g \in G),$$

then the pairing (2.1) factors through (2.2).

Proposition 2.1. *Suppose $L_*(c_{(gHg^{-1},g^{-1})}) = L^*(c_{(H,g)})$ for all $H \in \mathcal{S}(G)$ and $g \in G$. Then the equality (2.3) holds if and only if*

$$(1) \quad L^*(c_{(G,g)})L_*(j_{H,G})L^*(j_{H,G}) = L_*(j_{H,G})L^*(j_{H,G})L^*(c_{(G,g)})$$

for all $H \in \mathcal{S}(G)$ and $g \in G$.

Proof. By definition, the diagrams

$$\begin{array}{ccc} L(G) & \xrightarrow{L^*(j_{H,G})} & L(H) \\ L^*(c_{(G,g^{-1})}) \downarrow & & \downarrow L^*(c_{(gHg^{-1},g^{-1})}) \\ L(G) & \xrightarrow{L^*(j_{gHg^{-1},G})} & L(gHg^{-1}) \end{array}$$

and

$$\begin{array}{ccc} L(H) & \xrightarrow{L_*(j_{H,G})} & L(G) \\ L_*(c_{(gHg^{-1},g^{-1})}) \uparrow & & \uparrow L_*(c_{(G,g^{-1})}) \\ L(gHg^{-1}) & \xrightarrow{L_*(j_{gHg^{-1},G})} & L(G) \end{array}$$

commute. By using the hypothesis above, we obtain the commutative diagram

$$\begin{array}{ccc} L(G) & \xrightarrow{L_*(j_{H,G})L^*(j_{H,G})} & L(G) \\ L^*(c_{(G,g^{-1})}) \downarrow & & \uparrow L_*(c_{(G,g^{-1})}) \\ L(G) & \xrightarrow{L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G})} & L(G). \end{array}$$

Thus (2.3) holds if and only if

$$L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G}) = L_*(c_{(G,g^{-1})})L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G})L^*(c_{(G,g^{-1})}),$$

namely

$$L^*(c_{(G,g^{-1})})L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G}) = L_*(j_{gHg^{-1},G})L^*(j_{gHg^{-1},G})L^*(c_{(G,g^{-1})}).$$

This concludes the proposition. \square

Let $w : G \rightarrow \{-1, 1\}$ be a homomorphism. We introduce a slight generalization of a Mackey functor (cf. [2], [7]).

Definition 2.2. A bifunctor $M = (M^*, M_*)$ from \mathcal{G} to \mathcal{A} is called a w -Mackey functor if the following conditions (1)–(3) are fulfilled:

- (1) $M_*(c_{(H,g)}) = M^*(c_{(gHg^{-1},g^{-1})})$ for all $H \in \mathcal{S}(G)$ and $g \in G$,
- (2) $M^*(c_{(H,h)}) = w(h)id_{M(H)}$ (hence $M_*(c_{(H,h)}) = w(h)id_{M(H)}$) for all $H \in \mathcal{S}(G)$ and $h \in H$,
- (3) $M^*(j_{K,G}) \circ M_*(j_{H,G})$ coincides with

$$\bigoplus_{KgH \in K \setminus G/H} M_*(j_{K \cap gHg^{-1},K}) \circ (w(g)M_*(c_{(H \cap g^{-1}Kg,g)})) \circ M^*(j_{H \cap g^{-1}Kg,H})$$

for any $H, K \in \mathcal{S}(G)$.

A w -Mackey functor for trivial w is an ordinary Mackey functor. We will see that if w is nontrivial, then the Wall group functor $H \mapsto L_n^h(\mathbb{Z}[H], w|_H)$ is not an ordinary Mackey functor but a w -Mackey functor (cf. Propositions 6.6, 6.7, 6.8, 12.4, 12.5 and 12.6). The next proposition is clear by definition.

Proposition 2.3. *If $M = (M^*, M_*)$ is a w -Mackey functor, then $L = (L^*, L_*)$, given so that $L(H) = M(H)$, $L^*(j_{H,K}) = M^*(j_{H,K})$, $L_*(j_{H,K}) = M_*(j_{H,K})$, $L^*(c(H, g)) = w(g)M^*(c(H, g))$ and $L_*(c(H, g)) = w(g)M_*(c(H, g))$ for all $H \subset K$ and $g \in G$, is a Mackey functor.*

In the case above, we say that L is the *Mackey functor associated with M* .

We use the term ‘‘Frobenius pairing’’ in a sense slightly more general than [7], where relevant bifunctors were assumed to be Mackey functors.

Definition 2.4. Let L, M and N be bifunctors from \mathcal{G} to \mathcal{A} . A *pairing* $L \times M \rightarrow N$ is a family of biadditive maps

$$L(H) \times M(H) \longrightarrow N(H); (x, y) \longmapsto x \cdot y,$$

where H runs over $\mathcal{S}(G)$. Such a pairing is called a *Frobenius pairing* if the following conditions (1)–(3) are satisfied for any morphism $f : H \rightarrow K$ in \mathcal{G} :

- (1) $N^*(f)(x \cdot y) = (L^*(f)x) \cdot (M^*(f)y)$ for all $x \in L(K)$, $y \in M(K)$,
 - (2) $x \cdot M_*(f)(y) = N_*(f)(L^*(f)(x) \cdot y)$ for all $x \in L(K)$, $y \in M(H)$,
 - (3) $L_*(f)(x) \cdot y = N_*(f)(x \cdot M^*(f)(y))$ for all $x \in L(H)$, $y \in M(K)$.
- Each of (2), (3) is referred to as the *Frobenius reciprocity law*.

Let us recall the definition of a Green functor.

Definition 2.5. A Mackey functor $M = (M_*, M^*) : \mathcal{G} \rightarrow \mathcal{A}$ is called a *Green functor* if each $M(H)$, $H \in \mathcal{S}(G)$, is a ring with unit and the associated pairing $M \times M \rightarrow M$ is a Frobenius pairing. If the existence of the unit in $M(H)$ is not guaranteed, then M is referred as a *Green functor, possibly without unit*.

The Burnside ring functor $H \mapsto \Omega(G)$ is a Green functor. Let $U : \mathcal{G} \rightarrow \mathcal{A}$ be a Green functor. We mean by a *U -module L* (or a module L over U) a bifunctor $L : \mathcal{G} \rightarrow \mathcal{A}$ equipped with a Frobenius pairing $U \times L \rightarrow L$.

Proposition 2.6. *A w -Mackey functor M is a module over the Burnside ring functor.*

Proof. Let L be the Mackey functor associated with M in Proposition 2.3. By [7, Proposition 6.2.3], L is a module over the Burnside ring functor. Hence, L satisfies the equality (1) in Proposition 2.1. By using the relations between M and L in Proposition 2.3, we can check that M satisfies the equality (1) in Proposition 2.1, and furthermore that M is a module over the Burnside ring functor. \square

Proposition 2.7. *A module over a Green functor is a module over the Burnside ring functor.*

Proof. Let $L = (L^*, L_*) : \mathcal{G} \rightarrow \mathcal{A}$ be a module over a Green functor $U = (U^*, U_*) : \mathcal{G} \rightarrow \mathcal{A}$. Then the associated pairing

$$\Omega(H) \times L(H) \longrightarrow L(H)$$

can be defined so that $a \cdot x = (a \cdot 1_{U(H)}) \cdot x$ for $a \in \Omega(H)$ and $x \in L(H)$, where $1_{U(H)}$ is the identity element of $U(H)$. It is straightforward to check the Frobenius reciprocity laws of the pairing. \square

3. Θ -POSITIONED $R[G]$ -MODULES

Let Θ be a finite G -set. A pair (M, α) consisting of an $R[G]$ -module M and a G -map $\alpha : \Theta \rightarrow M$ is called a Θ -positioned $R[G]$ -module. Let H and K be finite groups and $\varphi : H \rightarrow K$ a homomorphism. For a finite H -set X , we define the K -set $K \times_{H, \varphi} X$ as the quotient set of $K \times X$ with respect to the equivalence relation \sim generated by $(k\varphi(h), x) \sim (k, hx)$, $h \in H$. The set $K \times_{H, \varphi} X$ is also denoted by $K \times_{\varphi} X$ or $K \times_H X$ if the context is clear. For an $R[H]$ -module M , the $R[K]$ -module $R[K] \otimes_{R[H], \varphi} M$ is defined as follows. Let $F(R[K] \times M)$ denote the R -free module with basis $R[K] \times M$ which may not be finitely generated over R .

Let T denote the R -submodule generated by all elements of the form

$$\begin{aligned} & r(a, x) - (ra, x), \quad r(a, x) - (a, rx), \\ & (a + b, x) - (a, x) - (b, x), \quad (a, x + y) - (a, x) - (a, y), \quad \text{or} \\ & (a\varphi(h), x) - (a, hx), \end{aligned}$$

where r ranges over R , a and b over $R[K]$, x and y over M , and h over H . Then $R[K] \otimes_{R[H], \varphi} M$ is defined to be the quotient module $F(R[K] \times M)/T$, which will also be denoted by $R[K] \otimes_{\varphi} M$ or $R[K] \otimes_{R[H]} M$. The element of the module represented by $(a, x) \in F(R[K] \times M)$ is denoted by $a \otimes_{R[H], \varphi} x$, which will also be written as $a \otimes_{\varphi} x$, $a \otimes_{R[H]} x$ or $a \otimes x$ if the context is clear. The K -action on $R[K] \otimes_{R[H], \varphi} M$ is given by $(k, a \otimes_{R[H], \varphi} x) \mapsto (ka) \otimes_{R[H], \varphi} x$.

Let Θ_H be a finite H -set, Θ_K a finite K -set, and $\psi : \Theta_H \rightarrow \Theta_K$ a φ -equivariant map, namely

$$\psi(ht) = \varphi(h)\psi(t) \quad (h \in H, t \in \Theta_H).$$

Let φ stand for the pair (φ, ψ) .

For a Θ_K -positioned $R[K]$ -module $\mathbf{N} = (N, \beta)$, we define the Θ_H -positioned $R[H]$ -module $\varphi^{\#}\mathbf{N} = (\varphi^{\#}N, \psi^{\#}\beta)$ so that the underlying R -module of $\varphi^{\#}N$ is the same as N but the H -action on $\varphi^{\#}N$ is given by $(h, x) \mapsto \varphi(h)x$ for $h \in H$, $x \in \varphi^{\#}N$, and $\psi^{\#}\beta : \Theta_H \rightarrow \varphi^{\#}N$ is given by $\psi^{\#}\beta(t) = \beta(\psi(t))$ for $t \in \Theta_H$.

Proposition 3.1. *Let $\varphi : H \rightarrow K$ and $\psi : \Theta_H \rightarrow \Theta_K$ be as above and let $\mathbf{N}_i = (N_i, \beta_i)$, $i = 1, 2$, be Θ_K -positioned $R[K]$ -modules. Then $\varphi^{\#}\mathbf{N}_1 \otimes_R \varphi^{\#}\mathbf{N}_2 = \varphi^{\#}(\mathbf{N}_1 \otimes_R \mathbf{N}_2)$; namely, $(\varphi^{\#}N_1 \otimes_R \varphi^{\#}N_2, \psi^{\#}\beta_1 \otimes_R \psi^{\#}\beta_2)$ is canonically isomorphic to $(\varphi^{\#}(N_1 \otimes_R N_2), \psi^{\#}(\beta_1 \otimes_R \beta_2))$.*

Proof. By definition, the underlying R -modules of $\varphi^{\#}N_1 \otimes_R \varphi^{\#}N_2$ and $\varphi^{\#}(N_1 \otimes_R N_2)$ are $N_1 \otimes_R N_2$. One can check without difficulties that the K -actions of the two modules coincide. Moreover, we have

$$(\psi^{\#}\beta_1 \otimes_R \psi^{\#}\beta_2)(t) = \beta_1(\psi(t)) \otimes_R \beta_2(\psi(t)) = \psi^{\#}(\beta_1 \otimes_R \beta_2)(t)$$

for all $t \in \Theta_H$. \square

To the contrary, for a Θ_H -positioned $R[H]$ -module $\mathbf{M} = (M, \alpha)$, we define the Θ_K -positioned $R[K]$ -module $\varphi_{\#}\mathbf{M} = (\varphi_{\#}M, \psi_{\#}\alpha)$ by $\varphi_{\#}M = R[K] \otimes_{R[H], \varphi} M$ and

$$\psi_{\#}\alpha(t) = \sum_{[k, t']} \{k \otimes_{\varphi} \alpha(t') \mid [k, t'] \in K \times_{H, \varphi} \Theta_H \text{ such that } k\psi(t') = t\} \quad \text{for } t \in \Theta_K.$$

The K -equivariance of the map $\psi_{\#}\alpha$ holds because, for $a \in K$ and $t \in \Theta_K$,

$$\begin{aligned} \psi_{\#}\alpha(at) &= \sum_{[k,t'] \in K \times_{H,\varphi} \Theta_H} \{k \otimes_{\varphi} \alpha(t') \mid k\psi(t') = at\} \\ &= \sum_{[k,t'] \in K \times_{H,\varphi} \Theta_H} \{k \otimes_{\varphi} \alpha(t') \mid a^{-1}k\psi(t') = t\} \\ &= \sum_{[ak',t'] \in K \times_{H,\varphi} \Theta_H} \{ak' \otimes_{\varphi} \alpha(t') \mid k'\psi(t') = t\} \\ &= a \sum_{[ak',t'] \in K \times_{H,\varphi} \Theta_H} \{k' \otimes_{\varphi} \alpha(t') \mid k'\psi(t') = t\} \\ &= a \sum_{[k',t'] \in K \times_{H,\varphi} \Theta_H} \{k' \otimes_{\varphi} \alpha(t') \mid k'\psi(t') = t\} \\ &= a\psi_{\#}\alpha(t). \end{aligned}$$

Proposition 3.2. *Let H be a subgroup of G , $\mathbf{M} = (M, \alpha)$ a Θ_H -positioned $R[H]$ -module, g an element of G , and $\psi : \Theta_H \rightarrow \Theta_{gHg^{-1}}$ a $c_{H,g}$ -equivariant bijection. Then the diagram*

$$\begin{array}{ccc} \Theta_{gHg^{-1}} & \xrightarrow{\psi_{\#}\alpha} & c_{(H,g)\#}M \\ & \searrow \psi^{-1\#}\alpha & \downarrow f_0 \\ & & c_{(gHg^{-1},g^{-1})}^{\#}M \end{array}$$

commutes, where $f_0 : c_{(H,g)\#}M \rightarrow c_{(gHg^{-1},g^{-1})}^{\#}M$ is the $R[gHg^{-1}]$ -isomorphism such that

$$f_0(e \otimes_{H,c_{(H,g)}} x) = x \quad \text{for } x \in M.$$

Proof. Let t be an element of Θ_H . Then by definition we have $\psi_{\#}\alpha(\psi(t)) = e \otimes_{H,c_{(H,g)}} \alpha(t)$ and $\psi^{-1\#}\alpha(\psi(t)) = \alpha(t)$, which concludes the proposition. \square

Proposition 3.3. *Let (H, Θ_H) , (K, Θ_K) , and $\varphi = (\varphi, \psi)$ be as above. Then for a Θ_H -positioned $R[H]$ -module (M, α) and a Θ_K -positioned $R[K]$ -module (N, β) , the Frobenius reciprocity law holds; namely, the following diagram commutes:*

$$\begin{array}{ccc} \Theta_K & \xrightarrow{(\psi_{\#}\alpha) \otimes_R \beta} & (R[K] \otimes_{R[H],\varphi} M) \otimes_R N \\ & \searrow \psi_{\#}(\alpha \otimes_R \psi^{\#}\beta) & \downarrow f \\ & & R[K] \otimes_{R[H],\varphi} (M \otimes_R \varphi^{\#}N), \end{array}$$

where f is the canonical isomorphism such that $f((k \otimes_{\varphi} x) \otimes y) = k \otimes_{\varphi} (x \otimes k^{-1}y)$ for $k \in K$, $x \in M$ and $y \in N$.

The commutability above is referred to as $(\psi_{\#}\alpha) \otimes_R \beta = \psi_{\#}(\alpha \otimes_R \psi^{\#}\beta)$.

Proof. The proof runs as follows:

$$\begin{aligned}
 ((\psi_{\#}\alpha) \otimes_R \beta)(t) &= \sum_{[k,t'] \in K \times_H \Theta_H} \{k \otimes_{\varphi} \alpha(t') \mid k\psi(t') = t\} \otimes \beta(t) \\
 &= \sum_{[k,t'] \in K \times_H \Theta_H} \{(k \otimes_{\varphi} \alpha(t')) \otimes \beta(t) \mid k\psi(t') = t\} \\
 &= \sum_{[k,t'] \in K \times_H \Theta_H} \{(k \otimes_{\varphi} \alpha(t')) \otimes k\beta(\psi(t')) \mid k\psi(t') = t\} \\
 &\stackrel{f}{=} \sum_{[k,t'] \in K \times_H \Theta_H} \{k \otimes_{\varphi} (\alpha(t') \otimes \beta(\psi(t'))) \mid k\psi(t') = t\} \\
 &= \sum_{[k,t'] \in K \times_H \Theta_H} \{k \otimes_{\varphi} (\alpha(t') \otimes (\psi^{\#}\beta)(t')) \mid k\psi(t') = t\} \\
 &= \sum_{[k,t'] \in K \times_H \Theta_H} \{k \otimes_{\varphi} (\alpha \otimes \psi^{\#}\beta)(t') \mid k\psi(t') = t\} \\
 &= \psi_{\#}(\alpha \otimes_R \psi^{\#}\beta)(t).
 \end{aligned}$$

□

Let H be a subgroup of G and g an element of G . Let $c_{(H,g)} : H \rightarrow gHg^{-1}$ stand for the conjugation map by g , i.e., $c_{(H,g)}(h) = ghg^{-1}$ for $h \in H$. Let Z be a finite G -set, Θ_H an H -invariant subset of Z , and $\Theta_{gHg^{-1}}$ a gHg^{-1} -invariant subset of Z such that $g\Theta_H = \Theta_{gHg^{-1}}$. Then the left translation by g , namely the map $\ell_{(H,g)} : \Theta_H \rightarrow \Theta_{gHg^{-1}}; t \mapsto gt$, is a $c_{(H,g)}$ -equivariant bijection. Let $\mathbf{c}_{(H,g)}$ denote the pair $(c_{(H,g)}, \ell_{(H,g)})$. If the context is clear, then we abuse $c_{(H,g)\#}$ for $\ell_{(H,g)\#}$, and $c_{(H,g)\#}^{\#}$ for $\ell_{(H,g)\#}^{\#}$.

In the special case where $g \in H$, the conjugation map $c_{(H,g)}$ is a map from H to itself. Note that the map

$$f_1 : c_{(H,g)\#}M \longrightarrow M; e \otimes_{c_{(H,g)}} x \longmapsto gx$$

is an $R[H]$ -isomorphism. In addition, the map

$$f_2 : c_{(H,g)\#}^{\#}M \longrightarrow M; x \longmapsto g^{-1}x$$

is an $R[H]$ -isomorphism.

Proposition 3.4. *Let H be a subgroup of G and Θ_H a finite H -set. Then for any Θ_H -positioned $R[H]$ -module (M, α) and $g \in H$, the following diagrams commute:*

$$\begin{array}{ccc}
 \Theta_H & \xrightarrow{\ell_{(H,g)\#}\alpha} & c_{(H,g)\#}M \\
 & \searrow \alpha & \downarrow f_1 \\
 & & M, \\
 \\
 \Theta_H & \xrightarrow{\ell_{(H,g)\#}^{\#}\alpha} & c_{(H,g)\#}^{\#}M \\
 & \searrow \alpha & \downarrow f_2 \\
 & & M,
 \end{array}$$

where f_1 and f_2 are the $R[H]$ -isomorphisms given above.

These commutabilities are referred to as $\ell_{(H,g)\#}\alpha = \alpha$ (or $c_{(H,g)\#}\alpha = \alpha$) and $\ell_{(H,g)}^\#\alpha = \alpha$ (or $c_{(H,g)}^\#\alpha = \alpha$), respectively.

Proof. The commutabilities follow from the equalities

$$\begin{aligned} f_1(\ell_{(H,g)\#}\alpha(t)) &= \sum_{[ghg^{-1},t'] \in H \times_{H,c_{(H,g)}} \Theta_H} \{f_1(ghg^{-1} \otimes_{c_{(H,g)}} \alpha(t')) \mid ghg^{-1}(gt') = t\} \\ &= \sum_{[ghg^{-1},t'] \in H \times_{H,c_{(H,g)}} \Theta_H} \{f_1(e \otimes_{c_{(H,g)}} \alpha(ht')) \mid ght' = t\} \\ &= \sum_{[e,t''] \in H \times_{H,c_{(H,g)}} \Theta_H} \{f_1(e \otimes_{c_{(H,g)}} \alpha(t'')) \mid gt'' = t\} \\ &= \sum_{[e,t''] \in H \times_{H,c_{(H,g)}} \Theta_H} \{g\alpha(t'') \mid gt'' = t\} \\ &= \sum_{[e,t''] \in H \times_{H,c_{(H,g)}} \Theta_H} \{\alpha(t) \mid gt'' = t\} \\ &= \alpha(t), \end{aligned}$$

and

$$\begin{aligned} f_2(\ell_{(H,g)}^\#\alpha(t)) &= f_2(\alpha(\ell_{(H,g)}(t))) \\ &= f_2(\alpha(gt)) \\ &= g^{-1}\alpha(gt) \\ &= \alpha(t), \end{aligned}$$

for $t \in \Theta_H$. □

Let Z be a finite G -set. Let $\mathcal{S}(G)$ and $\mathfrak{P}(Z)$ denote the set of all subgroups of G and the set of all subsets of Z , respectively. We regard $\mathcal{S}(G)$ as a G -set by conjugation, and $\mathfrak{P}(Z)$ has the canonical G -action. Let $\Theta : \mathcal{S}(G) \rightarrow \mathfrak{P}(G)$; $H \mapsto \Theta_H$, be a G -map. We say that Θ is *intersection preserving* if

$$(3.1) \quad \Theta_H \cap \Theta_K = \Theta_{H \cap K} \quad \text{for all } H, K \in \mathcal{S}(G).$$

Let $H \subset K$ be subgroups of G . Then (3.1) implies $\Theta_H \subset \Theta_K$. Thus, the inclusion map $j_{H,K} : H \rightarrow K$ is automatically associated with the inclusion map $j_{\Theta_H, \Theta_K} : \Theta_H \rightarrow \Theta_K$, and hence yields the pair $\mathbf{j}_{H,K} = (j_{H,K}, j_{\Theta_H, \Theta_K})$.

Usually, we use Ind_H^K for $j_{H,K}\#$, $j_{\Theta_H, \Theta_K}\#$ and $\mathbf{j}_{H,K}\#$, and Res_H^K for $j_{H,K}^\#$, $j_{\Theta_H, \Theta_K}^\#$ and $\mathbf{j}_{H,K}^\#$, if the context is clear.

Next, let g be an element of G . Since Θ is a G -map, $\Theta_{gHg^{-1}} = g\Theta_H$ holds for any subgroup H of G .

Proposition 3.5. *Let $\Theta : \mathcal{S}(G) \rightarrow \mathfrak{P}(Z)$ be an intersection-preserving G -map. Then for arbitrary subgroups H and K of G , each Θ_H -positioned $R[H]$ -module $\mathbf{M} = (M, \alpha)$ satisfies the Mackey double coset formula. Namely,*

$$\text{Res}_K^G(\text{Ind}_H^G \mathbf{M}) = \bigoplus_{KgH \in K \backslash G / H} \text{Ind}_{K \cap gHg^{-1}c_{(H \cap g^{-1}Kg, g)}\#}^K \text{Res}_{H \cap g^{-1}Kg}^H \mathbf{M}.$$

More precisely, the following diagram commutes:

$$\begin{array}{ccc}
 \Theta_H & \xrightarrow{\gamma} & \bigoplus_{KgH \in K \backslash G/H} M(K, g, H) \\
 & \searrow \text{Res}_K^G \text{Ind}_H^G \alpha & \downarrow \omega \\
 & & \text{Res}_K^G (\text{Ind}_H^G M),
 \end{array}$$

where

$$\begin{aligned}
 M(K, g, H) &= \text{Ind}_{K \cap gHg^{-1}c(H \cap g^{-1}Kg, g)}^K \text{Res}_{H \cap g^{-1}Kg}^H M \\
 &= R[K] \otimes_{R[K \cap gHg^{-1}]} (R[K \cap gHg^{-1}] \otimes_{R[H \cap g^{-1}Kg], c(H \cap g^{-1}Kg, g)} \text{Res}_{H \cap g^{-1}Kg}^H M), \\
 \text{Res}_K^G (\text{Ind}_H^G M) &= \text{Res}_K^G (R[G] \otimes_{R[H]} M), \\
 \gamma &= \bigoplus_{KgH \in K \backslash G/H} \text{Ind}_{K \cap gHg^{-1}c(H \cap g^{-1}Kg, g)}^K (\text{Res}_{H \cap g^{-1}Kg}^H \alpha),
 \end{aligned}$$

and ω is the $R[K]$ -isomorphism such that

$$\omega(k \otimes (a \otimes_{c(H \cap g^{-1}Kg, g)} x)) = kg \otimes (g^{-1}ag)x \quad \text{for } k \in K, a \in K \cap gHg^{-1}, x \in M.$$

Proof. Let $\alpha : \Theta_H \rightarrow M$ be an H -map, and let $\{g_1, \dots, g_\ell\}$ be a complete set of representatives of $K \backslash G/H$. For $t \in \Theta_K$, we have

$$\begin{aligned}
 (\text{Res}_K^G \text{Ind}_H^G \alpha)(t) &= \sum \{g \otimes \alpha(t') \mid [g, t'] \in G \times_H \Theta_H, gt' = t\} \\
 &= \sum_{j=1}^{\ell} \sum \{gg_j \otimes \alpha(t') \mid [gg_j, t'] \in Kg_jH \times_H \Theta_H, g \in K, gg_j t' = t\} \\
 &= \sum_{j=1}^{\ell} \sum \{gg_j \otimes \alpha(t') \mid [gg_j, t'] \in Kg_j \times_{H \cap g_j^{-1}Kg_j} \Theta_H, g \in K, gg_j t' = t\} \\
 &= \sum_{j=1}^{\ell} \sum \{gg_j \otimes \alpha(t') \mid [gg_j, t'] \in Kg_j \times_{H \cap g_j^{-1}Kg_j} \Theta_{H \cap g_j^{-1}Kg_j}, \\
 &\quad g \in K, gg_j t' = t\} \quad \text{in } \text{Res}_K^G \text{Ind}_H^G M
 \end{aligned}$$

and

$$\begin{aligned}
 &(\text{Ind}_{K \cap g_jHg_j^{-1}c(H \cap g_j^{-1}Kg_j, g_j)}^K \text{Res}_{H \cap g_j^{-1}Kg_j}^H \alpha)(t) \\
 &= \sum \{g \otimes \ell_{(H \cap g_j^{-1}Kg_j, g_j)} \text{Res}_{H \cap g_j^{-1}Kg_j}^H \alpha(t') \mid \\
 &\quad [g, t'] \in K \times_{K \cap g_jHg_j^{-1}} \Theta_{K \cap g_jHg_j^{-1}}, gt' = t\} \\
 &= \sum \{g \otimes (e \otimes \alpha(g_j^{-1}t')) \mid [g, t'] \in K \times_{K \cap g_jHg_j^{-1}} \Theta_{K \cap g_jHg_j^{-1}}, gt' = t\} \\
 &= \sum \{g \otimes (e \otimes \alpha(t'')) \mid [gg_j, t''] \in Kg_j \times_{H \cap g_j^{-1}Kg_j} \Theta_{H \cap g_j^{-1}Kg_j}, gg_j t'' = t\} \\
 &\stackrel{\cong}{=} \sum \{gg_j \otimes \alpha(t'') \mid [gg_j, t''] \in Kg_j \times_{H \cap g_j^{-1}Kg_j} \Theta_{H \cap g_j^{-1}Kg_j}, gg_j t'' = t\}.
 \end{aligned}$$

The proposition follows immediately from these equalities. □

4. POSITIONED HERMITIAN $R[G]$ -MODULES

In this section we introduce the Grothendieck-Witt rings of Θ -positioned Hermitian $R[G]$ -modules.

Definition 4.1. Let M be an $R[G]$ -module. A map $B : M \times M \rightarrow R$ is called a *Hermitian form* on M if the following conditions (1)–(3) are satisfied:

- (1) B is R -bilinear,
- (2) B is G -invariant, namely $B(gx, gy) = B(x, y)$,
- (3) B is symmetric, namely $B(x, y) = B(y, x)$,

for all $x, y \in M$ and $g \in G$. A couple (M, B) consisting of an $R[G]$ -module M and a Hermitian form B on M is called a *Hermitian $R[G]$ -module* (or simply *Hermitian module*).

A Hermitian $R[G]$ -module (M, B) such that M is a free R -module is said to be *nonsingular* if the associated map

$$M \longrightarrow M^\# = \text{Hom}_R(M, R); \quad x \mapsto B(x, -)$$

is bijective.

Let H and K be finite groups and $\varphi : H \rightarrow K$ a monomorphism. A Hermitian $R[K]$ -module (N, B) yields a Hermitian $R[H]$ -module $(\varphi^\# N, \varphi^\# B)$ in a canonical way. By definition $\varphi^\# N$ is nothing but N as an R -module. The map $\varphi^\# B : \varphi^\# N \times \varphi^\# N \rightarrow R$ is also the same as $B : N \times N \rightarrow R$. Clearly, $\varphi^\# B$ is R -bilinear and symmetric. It is obvious that if B is nonsingular, then so is $\varphi^\# B$. Since

$$\varphi^\# B(hx, hy) = B(\varphi(h)x, \varphi(h)y) = B(x, y)$$

for $h \in H$, $x, y \in \varphi^\# N$, it follows that $\varphi^\# B$ is H -invariant.

Proposition 4.2. Let $\varphi : H \rightarrow K$ be a monomorphism and let (N_i, B_i) , $i = 1, 2$, be Hermitian $R[K]$ -modules. Then

$$(\varphi^\# N_1 \otimes_R \varphi^\# N_2, \varphi^\# B_1 \otimes_R \varphi^\# B_2) = (\varphi^\# (N_1 \otimes_R N_2), \varphi^\# (B_1 \otimes_R B_2)).$$

This proposition is obviously true.

Let (M, B) be a Hermitian $R[H]$ -module. Then, by definition,

$$\varphi_\# M = R[K] \otimes_{R[H], \varphi} M.$$

We define the R -bilinear form

$$\varphi_\# B : \varphi_\# M \times \varphi_\# M \rightarrow R,$$

so that

$$\varphi_\# B(a \otimes_\varphi x, b \otimes_\varphi y) = \delta_{a\varphi(H), b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y),$$

for $a, b \in K$ and $x, y \in M$, where $\delta_{a\varphi(H), b\varphi(H)} = 1$ if $a\varphi(H) = b\varphi(H)$, and $\delta_{a\varphi(H), b\varphi(H)} = 0$ otherwise. It is clear that $\varphi_\# B$ is K -invariant and symmetric. If B is nonsingular, then so is $\varphi_\# B$.

Proposition 4.3. *Let H be a subgroup of G , B a Hermitian form on an $R[H]$ -module M , and g an element of G . Then the diagram*

$$\begin{array}{ccc}
 c_{(H,g)\#} M \times c_{(H,g)\#} M & & \\
 \downarrow f_0 \times f_0 & \searrow c_{(H,g)\#} B & \\
 c_{(gHg^{-1},g^{-1})\#} M \times c_{(gHg^{-1},g^{-1})\#} M & \xrightarrow{c_{(gHg^{-1},g^{-1})\#} B} & R
 \end{array}$$

commutes, where f_0 is the canonical $R[gHg^{-1}]$ -isomorphism (cf. Proposition 3.2).

The proof is straightforward.

Proposition 4.4. *Let $\varphi : H \rightarrow K$ be a monomorphism, and let B and B' be Hermitian forms on an $R[H]$ -module M and an $R[K]$ -module N , respectively. Then the following diagram commutes:*

$$\begin{array}{ccc}
 M_1 \times M_1 & & \\
 \downarrow f \times f & \searrow \varphi\# B \otimes_R B' & \\
 M_2 \times M_2 & \xrightarrow{\varphi\#(B \otimes_R \varphi\# B')} & R,
 \end{array}$$

where $M_1 = (R[K] \otimes_{R[H],\varphi} M) \otimes_R N$, $M_2 = R[K] \otimes_{R[H],\varphi} (M \otimes_R \varphi\# N)$, and f is the canonical isomorphism (cf. Proposition 3.3).

Proof. The commutability follows from

$$\begin{aligned}
 \varphi\# B \otimes_R B'((a \otimes_\varphi x) \otimes u, (b \otimes_\varphi y) \otimes v) &= \varphi\# B(a \otimes_\varphi x, (b \otimes_\varphi y))B'(u, v) \\
 &= \delta_{a\varphi(H), b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y)B'(u, v)
 \end{aligned}$$

and

$$\begin{aligned}
 \varphi\#(B \otimes_R \varphi\# B')(a \otimes_\varphi (x \otimes a^{-1}u), b \otimes_\varphi (y \otimes b^{-1}v)) & \\
 = \delta_{a\varphi(H), b\varphi(H)} (B \otimes_R \varphi\# B')(x \otimes a^{-1}u, \varphi^{-1}(a^{-1}b)(y \otimes b^{-1}v)) & \\
 = \delta_{a\varphi(H), b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y)B'(a^{-1}u, \varphi(\varphi^{-1}(a^{-1}b))b^{-1}v) & \\
 = \delta_{a\varphi(H), b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y)B'(a^{-1}u, a^{-1}v) & \\
 = \delta_{a\varphi(H), b\varphi(H)} B(x, \varphi^{-1}(a^{-1}b)y)B'(u, v), &
 \end{aligned}$$

for $a, b \in K$, $x, y \in M$, and $u, v \in N$. □

Proposition 4.5. *Let H be a subgroup of G and (M, B) a Hermitian $R[H]$ -module. Then for any $g \in H$, the following diagrams commute:*

$$\begin{array}{ccc}
 c_{(H,g)\#} M \times c_{(H,g)\#} M & & \\
 \downarrow f_1 \times f_1 & \searrow c_{(H,g)\#} B & \\
 M \times M & \xrightarrow{B} & R,
 \end{array}$$

$$\begin{array}{ccc}
 c_{(H,g)}^\# M \times c_{(H,g)}^\# M & & \\
 \downarrow f_2 \times f_2 & \searrow c_{(H,g)}^\# B & \\
 M \times M & \xrightarrow{B} & R,
 \end{array}$$

where f_1 and f_2 are the canonical isomorphisms (cf. Proposition 3.4).

Proof. The commutability of the first diagram follows from

$$c_{(H,g)}^\# B(e \otimes x, e \otimes y) = B(x, y)$$

and

$$B(f_1(e \otimes x), f_1(e \otimes y)) = B(gx, gy) = B(x, y).$$

The commutability of the second diagram follows from

$$c_{(H,g)}^\# B(x, y) = B(x, y)$$

and

$$B(f_2(x), f_2(y)) = B(g^{-1}x, g^{-1}y) = B(x, y).$$

□

Proposition 4.6. For any subgroups H and K of G , each Hermitian $R[H]$ -module (M, B) satisfies the Mackey double coset formula. Namely,

$$\text{Res}_K^G \text{Ind}_H^K B = \bigoplus_{KgH \in K \backslash G/H} \text{Ind}_{K \cap gHg^{-1}}^K c_{(H \cap g^{-1}Kg, g)}^\# \text{Res}_{H \cap g^{-1}Kg}^H B.$$

More precisely, the following diagram commutes:

$$\begin{array}{ccc}
 \left(\bigoplus_{KgH} M(K, g, H) \right) \times \left(\bigoplus_{KgH} M(K, g, H) \right) & & \\
 \downarrow \omega \times \omega & \searrow \bigoplus \text{Ind}_{c_{(H \cap g^{-1}Kg, g)}^\#}^K \text{Res}_{H \cap g^{-1}Kg}^H B & \\
 \text{Res}_K^G \text{Ind}_H^K M \times \text{Res}_K^G \text{Ind}_H^K M & \xrightarrow{\text{Res}_K^G \text{Ind}_H^K B} & R,
 \end{array}$$

where KgH runs over $K \backslash G/H$,

$$M(K, g, H) = \text{Ind}_{K \cap gHg^{-1}}^K c_{(H \cap g^{-1}Kg, g)}^\# \text{Res}_{H \cap g^{-1}Kg}^H M,$$

and ω is the canonical isomorphism (cf. Proposition 3.5).

Proof. For $u, v \in R[K] \otimes_{R[K \cap gHg^{-1}]} c_{(H \cap g^{-1}Kg, g)}^\# \text{Res}_{H \cap g^{-1}Kg}^H M$ with $u = a \otimes (e \otimes x)$ and $v = b \otimes (e \otimes x)$ respectively, where $a, b \in K, x, y \in \text{Res}_{H \cap g^{-1}Kg}^H M$, we have

$$\begin{aligned}
 & \text{Ind}_{K \cap gHg^{-1}}^K c_{(H \cap g^{-1}Kg, g)}^\# \text{Res}_{H \cap g^{-1}Kg}^H B(u, v) \\
 &= \delta_{a(K \cap gHg^{-1}), b(K \cap gHg^{-1})} c_{(H \cap g^{-1}Kg, g)}^\# \text{Res}_{H \cap g^{-1}Kg}^H B(e \otimes x, a^{-1}b(e \otimes y)) \\
 &= \delta_{a(K \cap gHg^{-1}), b(K \cap gHg^{-1})} B(x, g^{-1}a^{-1}bgy)
 \end{aligned}$$

and

$$\begin{aligned} \text{Res}_K^G \text{Ind}_H^G B(ag \otimes x, bg \otimes y) &= \delta_{agH, bgH} B(x, (ag)^{-1} bgy) \\ &= \delta_{agH, bgH} B(x, g^{-1} a^{-1} bgy) \\ &= \delta_{a(K \cap gHg^{-1}), b(K \cap gHg^{-1})} B(x, g^{-1} a^{-1} bgy). \end{aligned}$$

Thus we obtain the proposition. □

Definition 4.7. Let Θ be a finite G -set. A triple (M, B, α) consisting of a Hermitian $R[G]$ -module (M, B) and a G -map $\alpha : \Theta \rightarrow M$ is called a Θ -positioned Hermitian $R[G]$ -module (or simply Θ -positioned Hermitian module).

Let $\mathcal{H}(R, G, \Theta)$ stand for the family of all Θ -positioned Hermitian $R[G]$ -modules (M, B, α) such that M is an R -free $R[G]$ -module and $B : M \times M \rightarrow R$ is nonsingular. We say that α is *totally isotropic* (resp. *trivial*) if $B(\text{Im}(\alpha), \text{Im}(\alpha)) = 0$ (resp. $\text{Im}(\alpha) = 0$). We set

$$\begin{aligned} \mathcal{H}(R, G, \Theta)^{\text{t-iso}} &= \{(M, B, \alpha) \in \mathcal{H}(R, G, \Theta) \mid \alpha \text{ is totally isotropic}\}, \\ \mathcal{H}(R, G, \Theta)^{\text{triv}} &= \{(M, B, \alpha) \in \mathcal{H}(R, G, \Theta) \mid \alpha \text{ is trivial}\}. \end{aligned}$$

Let

$$\text{KH}_0(R, G, \Theta), \text{KH}_0(R, G, \Theta)^{\text{t-iso}} \text{ and } \text{KH}_0(R, G, \Theta)^{\text{triv}}$$

denote the Grothendieck groups of $\mathcal{H}(R, G, \Theta)$, $\mathcal{H}(R, G, \Theta)^{\text{t-iso}}$ and $\mathcal{H}(R, G, \Theta)^{\text{triv}}$, respectively, under orthogonal sum.

Let $\mathbf{M} = (M, B, \alpha)$ be an object in $\mathcal{H}(R, G, \Theta)$. An R -direct summand, $R[G]$ -submodule U of M is called a *Quillen submodule* of M if $U \subset U^\perp$ and $\text{Im}(\alpha) \subset U$ both hold, where

$$U^\perp = \{x \in M \mid B(x, y) = 0 \ (\forall y \in U)\}.$$

In this case, (\mathbf{M}, U) is called a *Quillen pair*. If $\mathbf{M} \in \mathcal{H}(R, G, \Theta)$ admits a Quillen submodule, then \mathbf{M} belongs to $\mathcal{H}(R, G, \Theta)^{\text{t-iso}}$ by definition. For a Quillen pair (\mathbf{M}, U) , we have the well-defined map

$$B^\perp : U^\perp/U \times U^\perp/U \rightarrow R; \quad B^\perp(x + U, y + U) = B(x, y) \quad (x, y \in U^\perp).$$

Proposition 4.8. *Let (\mathbf{M}, U) , where $\mathbf{M} = (M, B, \alpha)$, be a Quillen pair. Then U^\perp/U is an R -free $R[G]$ -module and B^\perp is a nonsingular Hermitian form on U^\perp/U .*

Proof. Since U is an R -direct summand of M , M factors to $M = U \oplus N$ as R -modules. It follows that U and N both are R -free, and so are $U^\# = \text{Hom}_R(U, R)$ and M/U . Thus, the exact sequence

$$0 \longrightarrow U^\perp/U \longrightarrow M/U \longrightarrow U^\# \longrightarrow 0$$

splits via R -homomorphisms, and hence U^\perp/U is an R -direct summand of M/U . In particular, U^\perp/U is R -free.

It is obvious that B^\perp is R -bilinear, G -invariant and symmetric. So, it suffices to prove that B^\perp is nonsingular. Since B is nonsingular, we can take an R -basis

$$\{u_1, \dots, u_m, y_1, \dots, y_n, v_1, \dots, v_m\}$$

of M so that $\{u_1, \dots, u_m\}$ is an R -basis of U , $y_j \in U^\perp$, and $B(v_i, u_j) = \delta_{i,j}$ and $B(v_i, y_j) = 0$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. Let V denote the R -submodule of M generated by $\{v_1, \dots, v_m\}$. There exist elements z_1, \dots, z_n of M

such that $B(z_i, u_j) = 0$, $B(z_i, y_j) = \delta_{i,j}$ and $B(z_i, v_j) = 0$. Write z_i as $z_i = y'_i + v'_i$ with $y'_i \in U^\perp$ and $v'_i \in V$. Then

$$B(y'_i, y_j) = B(y'_i + v'_i, y_j) = B(z_i, y_j) = \delta_{i,j}.$$

This shows that $B^\perp : U^\perp/U \times U^\perp/U \rightarrow R$ is nonsingular. □

By the proposition, a Quillen pair (\mathbf{M}, U) induces an object $(U^\perp/U, B^\perp, \text{triv})$ of $\mathcal{H}(R, G, \Theta)$, where $\text{triv} : \Theta \rightarrow U^\perp/U$ is the trivial map.

We define the Grothendieck-Witt groups

$$\text{GW}_0(R, G, \Theta), \quad \text{GW}_0(R, G, \Theta)^{\text{t-iso}}, \quad \text{GW}_0(R, G)$$

by

$$\begin{aligned} \text{GW}_0(R, G, \Theta) &= \text{KH}_0(R, G, \Theta) / \langle [\mathbf{M}] - [U^\perp/U, B^\perp, \text{triv}] \rangle, \\ \text{GW}_0(R, G, \Theta)^{\text{t-iso}} &= \text{KH}_0(R, G, \Theta)^{\text{t-iso}} / \langle [\mathbf{M}] - [U^\perp/U, B^\perp, \text{triv}] \rangle, \\ \text{GW}_0(R, G) &= \text{KH}_0(R, G) / \langle [\mathbf{M}] - [U^\perp/U, B^\perp, \text{triv}] \rangle, \end{aligned}$$

where (\mathbf{M}, U) ranges over all Quillen pairs in $\mathcal{H}(R, G, \Theta)$, $\mathcal{H}(R, G, \Theta)^{\text{t-iso}}$ and $\mathcal{H}(R, G, \Theta)^{\text{triv}}$, respectively. By definition, there are canonical homomorphisms

$$\text{GW}_0(R, G) \rightarrow \text{GW}_0(R, G, \Theta)^{\text{t-iso}}$$

and

$$\text{GW}_0(R, G, \Theta)^{\text{t-iso}} \rightarrow \text{GW}_0(R, G, \Theta).$$

Proposition 4.9. *The homomorphisms*

$\text{GW}_0(R, G) \rightarrow \text{GW}_0(R, G, \Theta)^{\text{t-iso}}$ and $\text{GW}_0(R, G, \Theta)^{\text{t-iso}} \rightarrow \text{GW}_0(R, G, \Theta)$ are both injective. Moreover, the homomorphism $\text{GW}_0(R, G) \rightarrow \text{GW}_0(R, G, \Theta)^{\text{t-iso}}$ is an isomorphism.

Proof. Consider the homomorphism

$$\text{GW}_0(R, G, \Theta) \rightarrow \text{GW}_0(R, G)$$

assigning $[M, B, \text{triv}]$ to $[M, B, \alpha]$. Since the composition

$$\text{GW}_0(R, G) \rightarrow \text{GW}_0(R, G, \Theta)^{\text{t-iso}} \rightarrow \text{GW}_0(R, G, \Theta) \rightarrow \text{GW}_0(R, G)$$

is the identity map, the homomorphisms

$$\text{GW}_0(R, G) \rightarrow \text{GW}_0(R, G, \Theta)^{\text{t-iso}} \quad \text{and} \quad \text{GW}_0(R, G) \rightarrow \text{GW}_0(R, G, \Theta)$$

are injective.

Let $\mathbf{M} = (M, B, \alpha)$ be a Θ -positioned $R[G]$ -Hermitian module such that α is totally isotropic. Then, let L denote the $R[G]$ -submodule of M generated by $\alpha(\Theta)$, and set

$$U = \{x \in M \mid rx \in L \text{ for some } r \in R \text{ with } r \neq 0\}.$$

Then $B(U, U) = 0$, and U is an R -direct summand, $R[G]$ -submodule of M . Thus, we have

$$[M, B, \alpha] = [U^\perp/U, B^\perp, \text{triv}] \text{ in } \text{GW}_0(R, G, \Theta)^{\text{t-iso}}.$$

This implies that the canonical homomorphism $\text{GW}_0(R, G) \rightarrow \text{GW}_0(R, G, \Theta)^{\text{t-iso}}$ is surjective. □

For Θ -positioned Hermitian $R[G]$ -modules $\mathbf{M}_1 = (M_1, B_1, \alpha_1)$ and $\mathbf{M}_2 = (M_2, B_2, \alpha_2)$, we define the tensor product $\mathbf{M}_1 \otimes_R \mathbf{M}_2$ over R as the Θ -positioned Hermitian $R[G]$ -module $(M_1 \otimes_R M_2, B_1 \otimes_R B_2, \alpha_1 \otimes_R \alpha_2)$.

Proposition 4.10. *Let Θ be a finite G -set. Then $\text{GW}_0(R, G, \Theta)$ and $\text{GW}_0(R, G)$ ($= \text{GW}_0(R, G, \Theta)^{\text{t-iso}}$) are commutative rings under the multiplication induced from the tensor product over R . Moreover, the rings $\text{GW}_0(R, G, \Theta)$ and $\text{GW}_0(R, G)$ possess units. Actually, the units of $\text{GW}_0(R, G, \Theta)$ and $\text{GW}_0(R, G)$ are the equivalence classes of*

$$(R, B : R \times R \rightarrow R, \alpha : \Theta \rightarrow R) \quad \text{and} \quad (R, B : R \times R \rightarrow R, \text{triv} : \Theta \rightarrow R),$$

respectively, where G acts trivially on R , B is the map defined by $B(r_1, r_2) = r_1 r_2$ for $r_1, r_2 \in R$, and α is the map defined by $\alpha(t) = 1$ for $t \in \Theta$.

5. THE SPECIAL GROTHENDIECK-WITT RINGS

Let S be a conjugation-invariant subset of

$$G(2) = \{g \in G \mid g^2 = e, g \neq e\}$$

and let $\mathfrak{P}(S)$ denote the set of all subsets of S . Then the G -action on S by conjugation yields a G -action on $\mathfrak{P}(S)$. Let Θ be a finite G -set and $\rho^{(2)} : \Theta \rightarrow \mathfrak{P}(S)$ a G -map.

For a G -map $\alpha : \Theta \rightarrow M$, where M is an $R[G]$ -module, we define the map $\Delta_\alpha : S \rightarrow M$ by

$$(5.1) \quad \Delta_\alpha(s) = \sum_t \{\alpha(t) \mid t \in \Theta, \rho^{(2)}(t) \ni s\} \quad (s \in S).$$

Proposition 5.1. *The map Δ_α above is a G -map, namely $\Delta_\alpha(gsg^{-1}) = g\Delta_\alpha(s)$ for $g \in G$ and $s \in S$.*

Proof. The proof runs as follows:

$$\begin{aligned} g\Delta_\alpha(s) &= g \sum_t \{\alpha(t) \mid t \in \Theta, \rho^{(2)}(t) \ni s\} \\ &= \sum_t \{\alpha(gt) \mid t \in \Theta, \rho^{(2)}(t) \ni s\} \\ &= \sum_{t'} \{\alpha(t') \mid g^{-1}t' \in \Theta, \rho^{(2)}(g^{-1}t') \ni s\} \\ &= \sum_{t'} \{\alpha(t') \mid t' \in \Theta, \rho^{(2)}(t') \ni gsg^{-1}\} \\ &= \Delta_\alpha(gsg^{-1}). \end{aligned}$$

□

Let $\mathbf{M} = (M, B, \alpha)$ be an object in $\mathcal{H}(R, G, \Theta)$. We introduce a map

$$\nabla_{\mathbf{M}} : M \rightarrow \text{Map}(S, R/2R),$$

which plays a key role in this paper. Define $\nabla_{\mathbf{M}}(x)(s) \in R/2R$ for $x \in M$ and $s \in S$ by

$$(5.2) \quad \nabla_{\mathbf{M}}(x)(s) = B(\Delta_\alpha(s) - x, sx).$$

Proposition 5.2. *The map $\nabla_{\mathbf{M}} : M \rightarrow \text{Map}(S, R/2R)$ is a $\mathbb{Z}[G]$ -homomorphism. Namely, the following hold:*

- (1) $\nabla_{\mathbf{M}}(x + y)(s) = \nabla_{\mathbf{M}}(x)(s) + \nabla_{\mathbf{M}}(y)(s) \quad (x, y \in M, s \in S),$
- (2) $\nabla_{\mathbf{M}}(gx)(s) = \nabla_{\mathbf{M}}(x)(g^{-1}sg) \quad (x \in M, s \in S).$

If R is square identical, $\nabla_{\mathbf{M}} : M \rightarrow \text{Map}(S, R/2R)$ is an $R[G]$ -homomorphism.

Proof. The formula (1) is obtained as follows:

$$\begin{aligned}\nabla_{\mathbf{M}}(x+y)(s) &= B(\Delta_\alpha(s) - (x+y), s(x+y)) \\ &= \nabla_{\mathbf{M}}(x)(s) + \nabla_{\mathbf{M}}(y)(s) + B(-x, sy) + B(-y, sx) \\ &= \nabla_{\mathbf{M}}(x)(s) + \nabla_{\mathbf{M}}(y)(s) - (B(x, sy) + B(y, sx)) \\ &= \nabla_{\mathbf{M}}(x)(s) + \nabla_{\mathbf{M}}(y)(s) \quad \text{in } R/2R.\end{aligned}$$

The formula (2) holds because

$$\begin{aligned}\nabla_{\mathbf{M}}(gx)(s) &= B(\Delta_\alpha(s) - gx, sgx) \\ &= B(g^{-1}\Delta_\alpha(s) - x, g^{-1}sgx) \\ &= B(\Delta_\alpha(g^{-1}sg) - x, g^{-1}sgx) \\ &= \nabla_{\mathbf{M}}(x)(g^{-1}sg) \quad \text{in } R/2R.\end{aligned}$$

The last assertion in the proposition is true since

$$\begin{aligned}\nabla_{\mathbf{M}}(rx)(s) &= B(\Delta_\alpha(s) - rx, srx) \\ &= B(\Delta_\alpha(s), srx) - B(rx, srx) \\ &= rB(\Delta_\alpha(s), sx) - r^2B(x, sx) \\ &= rB(\Delta_\alpha(s), sx) - rB(x, sx) \\ &= rB(\Delta_\alpha(s) - x, sx) \\ &= r\nabla_{\mathbf{M}}(x)(s) \quad \text{in } R/2R.\end{aligned}$$

We have established the proposition above. \square

Let $\mathcal{SH}(R, G, S, \Theta)$, $\mathcal{SH}(R, G, S, \Theta)^{\text{t-iso}}$ and $\mathcal{SH}(R, G, S, \Theta)^{\text{triv}}$ denote the family consisting of objects \mathbf{M} with $\nabla_{\mathbf{M}} = 0$ of $\mathcal{H}(R, G, \Theta)$, $\mathcal{H}(R, G, \Theta)^{\text{t-iso}}$ and $\mathcal{H}(R, G, \Theta)^{\text{triv}}$, respectively. We denote the Grothendieck groups of these under orthogonal sum by

$$\text{KSH}_0(R, G, S, \Theta), \quad \text{KSH}_0(R, G, S, \Theta)^{\text{t-iso}} \quad \text{and} \quad \text{KSH}_0(R, G, S),$$

respectively. Moreover, we define the *special Grothendieck-Witt groups*

$$\text{SGW}_0(R, G, S, \Theta), \quad \text{SGW}_0(R, G, S, \Theta)^{\text{t-iso}}, \quad \text{SGW}_0(R, G, S)$$

by

$$\begin{aligned}\text{SGW}_0(R, G, S, \Theta) &= \text{KSH}_0(R, G, S, \Theta) / \langle [\mathbf{M}] - [U^\perp/U, B^\perp, \text{triv}] \rangle, \\ \text{SGW}_0(R, G, S, \Theta)^{\text{t-iso}} &= \text{KSH}_0(R, G, S, \Theta)^{\text{t-iso}} / \langle [\mathbf{M}] - [U^\perp/U, B^\perp, \text{triv}] \rangle, \\ \text{SGW}_0(R, G, S) &= \text{KSH}_0(R, G, S) / \langle [\mathbf{M}] - [U^\perp/U, B^\perp, \text{triv}] \rangle,\end{aligned}$$

where (\mathbf{M}, U) ranges over all Quillen pairs in $\mathcal{SH}(R, G, S, \Theta)$, $\mathcal{SH}(R, G, S, \Theta)^{\text{t-iso}}$ and $\mathcal{SH}(R, G, S, \Theta)^{\text{triv}}$, respectively. Here we remark that if $\mathbf{M} \in \mathcal{SH}(R, G, S, \Theta)$ admits a Quillen submodule, then \mathbf{M} belongs to $\mathcal{SH}(R, G, S, \Theta)^{\text{t-iso}}$. By definition, there are canonical homomorphisms

$$\text{SGW}_0(R, G, S) \rightarrow \text{SGW}_0(R, G, S, \Theta)^{\text{t-iso}}$$

and

$$\text{SGW}_0(R, G, S, \Theta)^{\text{t-iso}} \rightarrow \text{SGW}_0(R, G, S, \Theta).$$

Proposition 5.3. *The homomorphism $SGW_0(R, G, S) \rightarrow SGW_0(R, G, S, \Theta)^{t\text{-iso}}$ is surjective, and the homomorphism $SGW_0(R, G, S, \Theta)^{t\text{-iso}} \rightarrow SGW_0(R, G, S, \Theta)$ is injective.*

Proof. The proof of the surjectivity of $SGW_0(R, G, S) \rightarrow SGW_0(R, G, S, \Theta)^{t\text{-iso}}$ is the same as that of $GW_0(R, G) \rightarrow GW_0(R, G, \Theta)^{t\text{-iso}}$ (see Proposition 4.9).

Let \mathbf{M} be an object of $\mathcal{SH}(R, G, S, \Theta)^{t\text{-iso}}$ such that $[\mathbf{M}] = 0$ in $SGW_0(R, G, S, \Theta)$. Then there exist objects $\mathbf{M}' = (M', B', \alpha')$, $\mathbf{M}_1 = (M_1, B_1, \alpha_1)$ with a Quillen submodule U_1 , and $\mathbf{M}_2 = (M_2, B_2, \alpha_2)$ with a Quillen submodule U_2 of $\mathcal{SH}(R, G, S, \Theta)$ such that

$$\mathbf{M} \oplus \mathbf{M}' \oplus \mathbf{M}_1 \oplus (U_2^\perp/U_2, B_2^\perp, \text{triv}) \cong \mathbf{M}' \oplus \mathbf{M}_2 \oplus (U_1^\perp/U_1, B_1^\perp, \text{triv}).$$

By definition, both \mathbf{M}_1 and \mathbf{M}_2 belong to $\mathcal{SH}(R, G, S, \Theta)^{t\text{-iso}}$. The object \mathbf{M}' above may be replaced by

$$\mathbf{M}'' = (M', B', \alpha') \oplus (M', -B', -\alpha').$$

Then \mathbf{M}'' has the Quillen submodule

$$U'' = \{(x, x) \in M' \oplus M' \mid x \in M'\},$$

and hence belongs to $\mathcal{SH}(R, G, S, \Theta)^{t\text{-iso}}$, which lets us conclude that

$$[\mathbf{M}] = 0 \text{ in } SGW_0(R, G, S, \Theta)^{t\text{-iso}}.$$

□

Proposition 5.4. *If, for each $s \in S$, there is at most one element $t \in \Theta$ such that $\rho^{(2)}(t) \ni s$, then $SGW_0(R, G, S, \Theta)$, $SGW_0(R, G, S, \Theta)^{t\text{-iso}}$ and $SGW_0(R, G, S)$ are commutative rings, possibly without unit. If R is square identical, and for each $s \in S$ there exists exactly one element $t \in \Theta$ such that $\rho^{(2)}(t) \ni s$, then $SGW_0(R, G, S, \Theta)$ is a commutative ring with unit.*

Proof. Let $\mathbf{M}_1 = (M_1, B_1, \alpha_1)$ and $\mathbf{M}_2 = (M_2, B_2, \alpha_2)$ be objects of $\mathcal{H}(R, G, \Theta)$ and $\mathcal{SH}(R, G, S, \Theta)$, respectively. Then

$$\begin{aligned} \nabla_{\mathbf{M}_1 \otimes_R \mathbf{M}_2}(x_1 \otimes x_2)(s) &= B_1 \otimes_R B_2(\Delta_{\alpha_1 \otimes_R \alpha_2}(s) - x_1 \otimes x_2, s(x_1 \otimes x_2)) \\ &= B_1 \otimes_R B_2(\Delta_{\alpha_1}(s) \otimes \Delta_{\alpha_2}(s) - x_1 \otimes x_2, sx_1 \otimes sx_2) \\ &= B_1(\Delta_{\alpha_1}(s), sx_1)B_2(\Delta_{\alpha_2}(s), sx_2) - B_1(x_1, sx_1)B_2(x_2, sx_2) \\ &= B_1(\Delta_{\alpha_1}(s) - x_1, sx_1)B_2(\Delta_{\alpha_2}(s), sx_2) \\ &\quad + B_1(x_1, sx_1)B_2(\Delta_{\alpha_2}(s) - x_2, sx_2) \\ &= \nabla_{\mathbf{M}_1}(x_1)(s)B_2(\Delta_{\alpha_2}(s), sx_2) + B_1(x_1, sx_1)\nabla_{\mathbf{M}_2}(x_2)(s) \\ &= \nabla_{\mathbf{M}_1}(x_1)(s)B_2(\Delta_{\alpha_2}(s), sx_2) \text{ in } R/2R. \end{aligned}$$

By using this and Proposition 5.2 (1), we can show that the product $\mathbf{M}_1 \otimes_R \mathbf{M}_2$ belongs to $\mathcal{SH}(R, G, S, \Theta)$ if \mathbf{M}_1 does. Therefore, the special Grothendieck-Witt groups are commutative rings.

Next we shall prove the last claim in the proposition. Let (R, B, α) denote the object in $\mathcal{H}(R, G, \Theta)$ such that G acts trivially on R , $B(r_1, r_2) = r_1 r_2$ ($r_1, r_2 \in R$) and $\alpha(t) = 1$ ($t \in \Theta$). Then, the associated $\nabla : M \rightarrow \text{Map}(S, R/2R)$ is trivial, since

$$\nabla(x)(s) = B(1 - r, sr) = r - r^2 = 0 \text{ in } R/2R.$$

Thus, (R, B, α) belongs to $\mathcal{SH}(R, G, S, \Theta)$, and therefore we can now conclude that the ring $SGW_0(R, G, S, \Theta)$ possesses a unit. □

Proposition 5.5. *The group $SGW_0(R, G, S)$ is a module over the ring $GW_0(R, G)$.*

Proof. Let $\mathbf{M}_1 = (M_1, B_1, \text{triv})$ and $\mathbf{M}_2 = (M_2, B_2, \text{triv})$ be arbitrary objects of $\mathcal{H}(R, G, \Theta)^{\text{triv}}$ and $\mathcal{SH}(R, G, S, \Theta)^{\text{triv}}$, respectively. Then, as in the proof of Proposition 5.4, we have

$$\nabla_{\mathbf{M}_1 \otimes \mathbf{M}_2}(x_1 \otimes x_2)(s) = \nabla_{\mathbf{M}_1}(x_1)(s)B_2(\Delta_{\alpha_2}(s), sx_2) + B_1(x_1, sx_1)\nabla_{\mathbf{M}_2}(x_2)(s).$$

Since $\Delta_{\alpha_2}(s) = 0$ and $\nabla_{\mathbf{M}_2}(x_2)(s) = 0$, $\nabla_{\mathbf{M}_1 \otimes \mathbf{M}_2}$ vanishes. Thus $\mathbf{M}_1 \otimes \mathbf{M}_2$ belongs to $\mathcal{SH}(R, G, S, \Theta)^{\text{triv}}$. \square

6. $R[G]$ -VALUED λ -HERMITIAN FORMS

Let λ stand for 1 or -1 and let $w : G \rightarrow \{-1, 1\}$ be a homomorphism. The group ring $A = R[G]$ is equipped with the anti-involution $-$ defined by

$$\overline{\sum_{g \in G} r_g g} = \sum_{g \in G} w(g)r_g g^{-1} \quad (r_g \in R).$$

Definition 6.1. Let M be an $R[G]$ -module. A map $B : M \times M \rightarrow R[G]$ is called an $R[G]$ -valued λ -Hermitian form (or λ -Hermitian form) on M if the following conditions (1)–(3) are satisfied:

- (1) B is R -bilinear,
- (2) $B(ax, by) = bB(y, x)\bar{a}$,
- (3) $B(x, y) = \lambda\overline{B(y, x)}$,

for all $x, y \in M, a, b \in R[G]$.

Let $B : M \times M \rightarrow R[G]$ be a λ -Hermitian form. For $x, y \in M, B(x, y)$ can be written as $\sum_{g \in G} B(x, y)_g g$ with $B(x, y)_g \in R$. Define the R -homomorphism $\varepsilon : R[G] \rightarrow R$ by

$$(6.1) \quad \varepsilon \left(\sum_{g \in G} r_g g \right) = r_e \quad (r_g \in R).$$

Lemma 6.2. $B(x, y)_g = \varepsilon(B(x, g^{-1}y))$ for all $x, y \in M$ and $g \in G$, and consequently

$$B(x, y) = \sum_{g \in G} \varepsilon(B(x, g^{-1}y))g.$$

Proof. By definition, we have $B(x, y)_e = \varepsilon(B(x, y))$. By observing the coefficients of g in $B(x, y)$ and

$$gB(x, g^{-1}y) = \sum_{h \in G} B(x, g^{-1}y)_h gh,$$

we have $B(x, g^{-1}y)_e = B(x, y)_g$. Thus, $B(x, y)_g = \varepsilon(B(x, g^{-1}y))$. \square

Lemma 6.3. *Let \mathbf{M} be as above. Then the composition $\varepsilon \circ B : M \times M \rightarrow R$ is a λ -symmetric, (G, w) -invariant, R -bilinear form on M . Namely, the following hold:*

- (1) $\varepsilon(B(x + x', ry)) = r\varepsilon(B(x, y)) + r\varepsilon(B(x', y))$,
- (2) $\varepsilon(B(x, y)) = \lambda\varepsilon(B(y, x))$,
- (3) $\varepsilon(B(gx, gy)) = w(g)\varepsilon(B(x, y))$,

for any $r \in R, x, x', y \in M$ and $g \in G$.

Proof. (1) The proof is straightforward.

(2) The equality follows from $B(x, y) = \lambda \overline{B(y, x)}$.

(3) By comparing the coefficients of e in $B(x, gy)$ and $w(g)B(g^{-1}x, y)$:

$$B(x, gy) = \sum_{h \in G} B(x, gy)_h h,$$

$$w(g)B(g^{-1}x, y) = \sum_{h \in G} w(g)B(g^{-1}x, y)_h h,$$

we have $\varepsilon(B(x, gy)) = w(g)\varepsilon(B(g^{-1}x, y))$, which is equivalent to the equality (3). □

An $R[G]$ -valued λ -Hermitian form B on an $R[G]$ -projective module M is said to be *nonsingular* if the associated map

$$M \rightarrow \text{Hom}_{R[G]}(M, R[G]); \quad x \mapsto B(x, -)$$

is bijective.

Lemma 6.4. *Let B be an $R[G]$ -valued λ -Hermitian form on an $R[G]$ -projective module M . Then B is nonsingular if and only if the induced R -bilinear form $\varepsilon \circ B : M \times M \rightarrow R$ is nonsingular.*

Let H and K be finite groups with homomorphisms $w_H : H \rightarrow \{-1, 1\}$ and $w_K : K \rightarrow \{-1, 1\}$, respectively. Let $\varphi : H \rightarrow K$ be a monomorphism such that $w_K \circ \varphi = w_H$. Let N be an $R[K]$ -module and $B : N \times N \rightarrow R[K]$ a λ -Hermitian form. We define the map $\varphi^\# B : \varphi^\# N \times \varphi^\# N \rightarrow R[H]$ by

$$(6.2) \quad \varphi^\# B(x, y) = \sum_{h \in H} \varepsilon(B(x, \varphi(h)^{-1}y))h \quad (x, y \in \varphi^\# N).$$

It immediately follows that $\varphi^\# B$ is an $R[H]$ -valued λ -Hermitian form on $\varphi^\# N$. If B is nonsingular, then so is $\varphi^\# B$. Next let M be a stably free $R[H]$ -module. Then

$$\varphi_\# M = R[K] \otimes_{R[H], \varphi} M$$

is clearly a stably $R[K]$ -free module. Let $B : M \times M \rightarrow R[H]$ be a λ -Hermitian form. We define the R -bilinear map $\varphi_\# B : \varphi_\# M \times \varphi_\# M \rightarrow R[K]$ so that

$$(6.3) \quad \varphi_\# B(a \otimes_\varphi x, b \otimes_\varphi y) = \sum_{k \in K} w_K(a) \delta_{a\varphi(H), k^{-1}b\varphi(H)} \varepsilon(B(x, \varphi^{-1}(a^{-1}k^{-1}b)y))k,$$

for $a, b \in K, x, y \in M$.

Lemma 6.5. *Let $\varphi_\# B$ be as above. Then*

$$\varphi_\# B(a \otimes_\varphi x, b \otimes_\varphi y) = b\varphi'(B(x, y))\bar{a},$$

for $a, b \in K, x, y \in M$; and $\varphi_\# B$ is an $R[K]$ -valued λ -Hermitian form on $\varphi_\# M$, where $\varphi' : R[H] \rightarrow R[K]$ is the ring homomorphism canonically induced by $\varphi : H \rightarrow K$. If B is nonsingular, then so is $\varphi_\# B$.

Proof. The formula in the lemma is true because

$$\begin{aligned}
 \varphi_{\#}B(a \otimes_{\varphi} x, b \otimes_{\varphi} y) &= \sum_{k \in K} w_K(a) \delta_{a\varphi(H), k^{-1}b\varphi(H)} \varepsilon(B(x, \varphi^{-1}(a^{-1}k^{-1}b)y))k \\
 &= b \left(\sum_{k \in K} \delta_{\varphi(H), a^{-1}k^{-1}b\varphi(H)} \varepsilon(B(x, \varphi^{-1}(a^{-1}k^{-1}b)y)) \right) b^{-1}ka \bar{a} \\
 &= b \left(\sum_{k' \in K} \delta_{\varphi(H), k'^{-1}\varphi(H)} \varepsilon(B(x, \varphi^{-1}(k'^{-1})y)) \right) k' \bar{a} \\
 &= b\varphi' \left(\sum_{k' \in K} \delta_{\varphi(H), k'^{-1}\varphi(H)} \varepsilon(B(x, \varphi^{-1}(k'^{-1})y)) \varphi^{-1}(k') \right) \bar{a} \\
 &= b\varphi'(B(x, y))\bar{a}.
 \end{aligned}$$

One can check the latter claim in the lemma by using this formula. □

Proposition 6.6. *Let H be a subgroup of G , B an $R[H]$ -valued λ -Hermitian form on an $R[H]$ -module M , and g an element of G . Provided $w_H = w_{gHg^{-1}} \circ c_{(H,g)}$, the diagram*

$$\begin{array}{ccc}
 c_{(H,g)\#}M \times c_{(H,g)\#}M & & \\
 \downarrow f_0 \times f_0 & \searrow c_{(H,g)\#}B & \\
 c_{(gHg^{-1},g^{-1})\#}M \times c_{(gHg^{-1},g^{-1})\#}M & \xrightarrow{c_{(gHg^{-1},g^{-1})\#}B} & R[gHg^{-1}]
 \end{array}$$

commutes, where f_0 is the canonical $R[gHg^{-1}]$ -isomorphism (cf. Proposition 3.2).

The proof of the proposition is straightforward.
 Given a datum $\mathcal{D} = (R, G, w, \lambda)$ as above, we obtain the datum

$$\mathcal{D}_H = (R, H, w|_H, \lambda)$$

for each subgroup H of G .

Proposition 6.7. *Let H be a subgroup of G and $B : M \times M \rightarrow R[H]$ a λ -Hermitian form on an $R[H]$ -module M . Then for each $g \in H$, the following diagrams commute:*

$$\begin{array}{ccc}
 c_{(H,g)\#}M \times c_{(H,g)\#}M & & \\
 \downarrow f_1 \times f_1 & \searrow c_{(H,g)\#}B & \\
 M \times M & \xrightarrow{w(g)B} & R[H], \\
 \\
 c_{(H,g)\#}^{\#}M \times c_{(H,g)\#}^{\#}M & & \\
 \downarrow f_2 \times f_2 & \searrow c_{(H,g)\#}^{\#}B & \\
 M \times M & \xrightarrow{w(g)B} & R[H],
 \end{array}$$

where f_1 and f_2 are the canonical isomorphisms (cf. Proposition 3.4).

Proof. The commutability of the first diagram follows from

$$(c_{(H,g)\#}B)(e \otimes x, e \otimes y) = \sum_{h \in H} \varepsilon(B(x, g^{-1}h^{-1}gy))h$$

and

$$\begin{aligned} B(f_1(e \otimes x), f_1(e \otimes y)) &= B(gx, gy) \\ &= \sum_{h \in H} \varepsilon(B(gx, h^{-1}gy))h \\ &= w(g) \sum_{h \in H} \varepsilon(B(x, g^{-1}h^{-1}gy))h. \end{aligned}$$

The commutability of the second diagram follows from

$$(c_{(H,g)\#}B)(x, y) = \sum_{h \in H} \varepsilon(B(x, gh^{-1}g^{-1}y))h$$

and

$$\begin{aligned} B(f_2(x), f_2(y)) &= B(g^{-1}x, g^{-1}y) \\ &= \sum_{h \in H} \varepsilon(B(g^{-1}x, h^{-1}g^{-1}y))h \\ &= w(g) \sum_{h \in H} \varepsilon(B(x, g^{-1}h^{-1}gy))h. \end{aligned}$$

□

Proposition 6.8. *For any subgroups H and K of G , each $R[H]$ -valued λ -Hermitian form $B : M \times M \rightarrow R[H]$ on an $R[H]$ -module M satisfies the w -Mackey double coset formula. Namely,*

$$\begin{aligned} (\text{Res}_K^G \text{Ind}_H^G B) \circ (\omega \times \omega) \\ = \sum_{KgH \in K \backslash G / H} w(g) (\text{Ind}_{K \cap gHg^{-1}}^K c_{(H \cap g^{-1}Kg, g)\#} \text{Res}_{H \cap g^{-1}Kg}^H B), \end{aligned}$$

where ω is the canonical isomorphism (cf. Proposition 3.5). Particularly, in the case $w(G) = \{1\}$, B satisfies the Mackey double coset formula.

Proof. It suffices to prove that

$$\begin{aligned} (\text{Res}_K^G \text{Ind}_H^G B)(ag \otimes x, bg \otimes y) \\ = w(g) (\text{Ind}_{K \cap gHg^{-1}}^K c_{(H \cap g^{-1}Kg, g)\#} \text{Res}_{H \cap g^{-1}Kg}^H B)(a \otimes (e \otimes x), b \otimes (e \otimes y)) \end{aligned}$$

for any $g \in G$, $a, b \in K$, $x, y \in \text{Res}_{H \cap g^{-1}Kg}^H M$. This equality holds because

$$\begin{aligned} (\text{Res}_K^G \text{Ind}_H^G B)(ag \otimes x, bg \otimes y) \\ = \sum_{k \in K} w(ag) \delta_{agH, k^{-1}bgH} \varepsilon(B(x, (ag)^{-1}k^{-1}bgy))k \\ = w(g) \sum_{k \in K} w(a) \delta_{agH, k^{-1}bgH} \varepsilon(B(x, g^{-1}(a^{-1}k^{-1}b)gy))k \end{aligned}$$

and

$$\begin{aligned} & (\text{Ind}_{K \cap gHg^{-1}c(H \cap g^{-1}Kg, g)}^K \text{Res}_{H \cap g^{-1}Kg}^H B)(a \otimes (e \otimes x), b \otimes (e \otimes y)) \\ &= \sum_{k \in K} w(a) \delta_{a(K \cap gHg^{-1}), k^{-1}b(K \cap gHg^{-1})} \\ & \quad \cdot (c_{(H \cap g^{-1}Kg, g)} \text{Res}_{H \cap g^{-1}Kg}^H B)(e \otimes x, a^{-1}k^{-1}b(e \otimes y))k \\ &= \sum_{k \in K} w(a) \delta_{a(K \cap gHg^{-1}), k^{-1}b(K \cap gHg^{-1})} B(x, g^{-1}(a^{-1}k^{-1}b)gy)k. \end{aligned}$$

□

7. POSITIONED QUADRATIC $R[G]$ -MODULES

In this paper λ stands for either 1 or -1 . Let $w : G \rightarrow \{-1, 1\}$ be a group homomorphism. Set

$$\begin{aligned} G^\lambda(2) &= \{g \in G(2) \mid w(g) = \lambda\}, \\ G^{-\lambda}(2) &= \{g \in G(2) \mid w(g) = -\lambda\}. \end{aligned}$$

Clearly we have $g = \lambda \bar{g}$ for $g \in G^\lambda(2)$ and $g = -\lambda \bar{g}$ for $g \in G^{-\lambda}(2)$. Let S and Q be conjugation-invariant subsets of $G^\lambda(2)$ and $G^{-\lambda}(2)$, respectively. We shall define the Witt group of Θ -positioned quadratic $R[G]$ -modules, which is the Wall group (cf. [27]) in the case where Q, S and Θ are the empty set, and the Bak group (cf. [1], [19]) in the case where S and Θ are the empty set. The datum

$$\mathbf{A} = (R, G, Q, S, \lambda, w)$$

is relevant to the group. Define R -submodules $A_s = A_s(G, S; R)$, $A_q = A_q(G, S; R)$ and $\Lambda = \Lambda(G, Q; R)$ of $A := R[G]$ as follows:

$$\begin{aligned} A_s &= R[S] (= \langle s \mid s \in S \rangle_R), \\ A_q &= R[G \setminus S] (= \langle g \mid g \in G \setminus S \rangle_R), \\ \Lambda &= \langle x - \lambda \bar{x} \mid x \in A \rangle_R + \langle g \mid g \in Q \rangle_R. \end{aligned}$$

This module Λ is called the *form parameter* generated by Q .

Definition 7.1. A map $q : M \rightarrow A_q/\Lambda$ is called an **A**-quadratic form (or quadratic form) on M with respect to B if the following conditions (1)–(3) are fulfilled:

- (1) $q(gx) = gq(x)\bar{g}$ and $q(rx) = r^2q(x)$ in $A_q/\Lambda = A/(\Lambda + A_s)$,
- (2) $q(x + y) - \overline{q(x)} - q(y) = B(x, y)$ in $A_q/\Lambda = A/(\Lambda + A_s)$,
- (3) $q(\widetilde{x}) + \lambda q(\overline{x}) = B(x, x)$ in $A_q = \widetilde{A/A_s}$,

for all $x, y \in M, g \in G, r \in R$, where $q(x) \in A_q$ is a lifting of $q(x)$.

A triple (M, B, q) consisting of an $R[G]$ -module M , an $R[G]$ -valued λ -Hermitian form B on M and an **A**-quadratic form q on M with respect to B , is called an **A**-quadratic $R[G]$ -module (or λ -quadratic $R[G]$ -module).

Let Θ be a finite G -set. A quadruple (M, B, q, α) consisting of an **A**-quadratic $R[G]$ -module (M, B, q) and a G -map $\alpha : \Theta \rightarrow M$ is called a Θ -positioned **A**-quadratic $R[G]$ -module (or Θ -positioned λ -quadratic $R[G]$ -module).

Let $\mathcal{Q}(\mathbf{A}, \Theta)$ (or $\mathcal{Q}(R, G, Q, S, \Theta)$) denote the family of all Θ -positioned **A**-quadratic $R[G]$ -modules (M, B, q, α) such that M is a stably free $R[G]$ -module and B is nonsingular.

Let $\mathbf{M} = (M, B, q, \alpha) \in \mathcal{Q}(\mathbf{A}, \Theta)$. The map α is said to be *totally isotropic* (resp. *trivial*) if $B(\text{Im}(\alpha), \text{Im}(\alpha)) = 0$ and $q(\text{Im}(\alpha)) = 0$ (resp. $\text{Im}(\alpha) = 0$). Set

$$\begin{aligned} \mathcal{Q}(\mathbf{A}, \Theta)^{\text{t-iso}} &= \{(M, B, q, \alpha) \in \mathcal{Q}(\mathbf{A}, \Theta) \mid \alpha \text{ is totally isotropic}\}, \\ \mathcal{Q}(\mathbf{A}, \Theta)^{\text{triv}} &= \{(M, B, q, \alpha) \in \mathcal{Q}(\mathbf{A}, \Theta) \mid \alpha \text{ is trivial}\}. \end{aligned}$$

Let $\text{KQ}_0(\mathbf{A}, \Theta)$, $\text{KQ}_0(\mathbf{A}, \Theta)^{\text{t-iso}}$ and $\text{KQ}_0(\mathbf{A})$ denote the Grothendieck groups of $\mathcal{Q}(\mathbf{A}, \Theta)$, $\mathcal{Q}(\mathbf{A}, \Theta)^{\text{t-iso}}$ and $\mathcal{Q}(\mathbf{A}, \Theta)^{\text{triv}}$, respectively, under orthogonal sum.

A stably $R[G]$ -free, $R[G]$ -direct summand L of M is called a *Lagrangian submodule* of \mathbf{M} if $B(L, L) = 0$, $q(L) = 0$, $L^\perp = L$ and $\text{Im}(\alpha) \subset L$, where

$$L^\perp = \{x \in M \mid B(x, y) = 0 \ (\forall y \in L)\}.$$

If \mathbf{M} has a Lagrangian submodule, then \mathbf{M} is called a *null module*. The groups defined by

$$\begin{aligned} \text{WQ}_0(\mathbf{A}, \Theta) &= \text{KQ}_0(\mathbf{A}, \Theta) / \langle \text{null modules in } \mathcal{Q}(\mathbf{A}, \Theta) \rangle, \\ \text{WQ}_0(\mathbf{A}, \Theta)^{\text{t-iso}} &= \text{KQ}_0(\mathbf{A}, \Theta)^{\text{t-iso}} / \langle \text{null modules in } \mathcal{Q}(\mathbf{A}, \Theta)^{\text{t-iso}} \rangle, \\ \text{WQ}_0(\mathbf{A}) &= \text{KQ}_0(\mathbf{A}) / \langle \text{null modules in } \mathcal{Q}(\mathbf{A}, \Theta)^{\text{triv}} \rangle \end{aligned}$$

are called the *Witt groups* of Θ -positioned \mathbf{A} -quadratic $R[G]$ -modules. If the context is clear, those Witt groups are also denoted by

$$\text{WQ}_0(R, G, Q, S, \Theta), \quad \text{WQ}_0(R, G, Q, S, \Theta)^{\text{t-iso}}, \quad \text{WQ}_0(R, G, Q, S),$$

respectively.

8. THE SPECIAL WITT GROUPS

Let $\mathbf{A} = (R, G, Q, S, \lambda, w)$ be as in the previous section, Θ a finite G -set and $\rho^{(2)} : \Theta \rightarrow \mathfrak{P}(S)$ a G -map (cf. Section 5). Let $\mathbf{M} = (M, B, q, \alpha)$ be a Θ -positioned \mathbf{A} -quadratic $R[G]$ -module, where $\alpha : \Theta \rightarrow M$. The associated map $\nabla_{\mathbf{M}} : M \rightarrow \text{Map}(S, R/2R)$ is defined by

$$(8.1) \quad \nabla_{\mathbf{M}}(x)(s) = \varepsilon(B(\Delta_\alpha(s) - x, sx)),$$

for $x \in M$ and $s \in S$, where $\Delta_\alpha : S \rightarrow M$ is the map defined by (5.1).

If $\mathbf{M} \in \mathcal{Q}(\mathbf{A}, \Theta)$ satisfies $\nabla_{\mathbf{M}} = 0$, then we call \mathbf{M} a *special Θ -positioned \mathbf{A} -quadratic $R[G]$ -module* (or a *special Θ -positioned λ -quadratic $R[G]$ -module*). Set

$$\begin{aligned} \mathcal{SQ}(\mathbf{A}, \Theta) &= \{\mathbf{M} \in \mathcal{Q}(\mathbf{A}, \Theta) \mid \nabla_{\mathbf{M}} = 0\}, \\ \mathcal{SQ}(\mathbf{A}, \Theta)^{\text{t-iso}} &= \{\mathbf{M} \in \mathcal{Q}(\mathbf{A}, \Theta)^{\text{t-iso}} \mid \nabla_{\mathbf{M}} = 0\}, \\ \mathcal{SQ}(\mathbf{A}, \Theta)^{\text{triv}} &= \{\mathbf{M} \in \mathcal{Q}(\mathbf{A}, \Theta)^{\text{triv}} \mid \nabla_{\mathbf{M}} = 0\}. \end{aligned}$$

The corresponding Grothendieck groups are denoted by

$$\text{KSQ}_0(\mathbf{A}, \Theta), \quad \text{KSQ}_0(\mathbf{A}, \Theta)^{\text{t-iso}}, \quad \text{KSQ}_0(\mathbf{A})$$

respectively, or by

$$\text{KSQ}_0(R, G, Q, S, \Theta), \quad \text{KSQ}_0(R, G, Q, S, \Theta)^{\text{t-iso}}, \quad \text{KSQ}_0(R, G, Q, S)$$

respectively. Further, define the *special Witt groups*

$$\begin{aligned} \text{SWQ}_0(\mathbf{A}, \Theta) &= \text{SWQ}_0(R, G, Q, S, \Theta), \\ \text{SWQ}_0(\mathbf{A}, \Theta)^{\text{t-iso}} &= \text{SWQ}_0(R, G, Q, S, \Theta)^{\text{t-iso}}, \\ \text{SWQ}_0(\mathbf{A}) &= \text{SWQ}_0(R, G, Q, S) \end{aligned}$$

by

$$\begin{aligned} \text{SWQ}_0(\mathbf{A}, \Theta) &= \text{KSQ}_0(\mathbf{A}, \Theta) / \langle \text{null modules in } \mathcal{SQ}(\mathbf{A}, \Theta) \rangle, \\ \text{SWQ}_0(\mathbf{A}, \Theta)^{\text{t-iso}} &= \text{KSQ}_0(\mathbf{A}, \Theta)^{\text{t-iso}} / \langle \text{null modules in } \mathcal{SQ}(\mathbf{A}, \Theta)^{\text{t-iso}} \rangle, \\ \text{SWQ}_0(\mathbf{A}) &= \text{KSQ}_0(\mathbf{A}) / \langle \text{null modules in } \mathcal{SQ}(\mathbf{A}, \Theta)^{\text{triv}} \rangle, \end{aligned}$$

respectively.

9. TENSOR PRODUCTS OF HERMITIAN MODULES AND QUADRATIC MODULES

Let $\mathbf{A} = (R, G, Q, S, \lambda, w)$ be as in Section 7, and Θ a finite G -set. Let $\mathbf{M} = (M, B, q)$ be an \mathbf{A} -quadratic $R[G]$ -module. By definition, B is a map $M \times M \rightarrow R[G]$ and q is a map $M \rightarrow A_q/\Lambda$. We write G as a disjoint union of the form

$$G = \{e\} \amalg G(2) \amalg C \amalg C^{-1},$$

where C is a subset of G consisting of elements of order ≥ 3 and $C^{-1} = \{g^{-1} \mid g \in C\}$. Set

$$\mathcal{Q}(G) = \{e\} \cup (G^\lambda(2) \setminus S) \cup (G^{-\lambda}(2) \setminus Q) \cup C.$$

Let R_g stand for the R -module defined by

$$R_g = \begin{cases} R/(1-\lambda)R & (g = e), \\ R & (g \in G^\lambda(2)), \\ R/2R & (g \in G^{-\lambda}(2)), \\ R & (\text{otherwise}), \end{cases}$$

for each $g \in G$. Then $q(x)$, $x \in M$, can be regarded as the formal sum

$$\sum_{g \in \mathcal{Q}(G)} q(x)_g g$$

with $q(x)_g \in R_g$; namely, $q : M \rightarrow A_q/\Lambda$ can be regarded as the map

$$M \rightarrow \bigoplus_{g \in \mathcal{Q}(G)} R_g; \quad x \mapsto (q(x)_g).$$

We set $q(x)_g = \lambda w(g)q(x)_{g^{-1}}$ for $g \in G$ with $g^{-1} \in \mathcal{Q}(G)$. This definition is compatible with the ambiguity of choice of $\mathcal{Q}(G)$, because

$$\widetilde{q(x)_g} = \lambda w(g)\widetilde{q(x)_{g^{-1}}} \pmod{\Lambda}.$$

Let $\mathbf{M}_1 = (M_1, B_1, \alpha_1)$ and $\mathbf{M}_2 = (M_2, B_2, q_2, \alpha_2)$ be objects in $\mathcal{H}(R, G, S, \Theta)$ and $\mathcal{Q}(\mathbf{A}, \Theta)$, respectively. We define an object $\mathbf{M}_1 \cdot \mathbf{M}_2$ in $\mathcal{Q}(\mathbf{A}, \Theta)$ as the product of \mathbf{M}_1 and \mathbf{M}_2 as follows. For the sake of convenience, $\mathbf{M} = (M, B, q, \alpha)$ stands for $\mathbf{M}_1 \cdot \mathbf{M}_2$ for a while.

First, M is defined as the R -module $M_1 \otimes_R M_2$ with the G -action: $(g, x \otimes y) \mapsto (gx) \otimes (gy)$, where $g \in G$, $x \in M_1$ and $y \in M_2$. Since M_1 is R -free and M_2 is stably $R[G]$ -free, M is stably $R[G]$ -free.

Second, $B : M \times M \rightarrow R[G]$ is defined as the R -bilinear form such that

$$B(x \otimes y, x' \otimes y') = \sum_{g \in G} B_1(x, g^{-1}x')\varepsilon(B_2(y, g^{-1}y'))g.$$

The equality $B(u, v) = \overline{\lambda B(v, u)}$ ($u, v \in M$) holds since

$$\begin{aligned} B(x \otimes y, x' \otimes y') &= \sum_{g \in G} B_1(x, g^{-1}x')\varepsilon(B_2(y, g^{-1}y'))g \\ &= \sum_{g \in G} \lambda B_1(g^{-1}x', x)\varepsilon(B_2(g^{-1}y', y))g \\ &= \lambda \sum_{g \in G} w(g)B_1(x', gx)\varepsilon(B_2(y', gy))g \\ &= \lambda \sum_{g \in G} B_1(x', gx)\varepsilon(B_2(y', gy))\overline{g^{-1}} \\ &= \lambda \sum_{g \in G} B_1(x', gx)\varepsilon(B_2(y', gy))g^{-1} \\ &= \overline{\lambda B(x' \otimes y', x \otimes y)}. \end{aligned}$$

The equality $B(au, bv) = bB(u, v)\overline{a}$ ($a, b \in G, u, v \in M$) holds because

$$\begin{aligned} B(a(x \otimes y), b(x' \otimes y')) &= \sum_{g \in G} B_1(ax, g^{-1}bx')\varepsilon(B_2(ay, g^{-1}by'))g \\ &= b \sum_{h \in G} B_1(ax, h^{-1}x')\varepsilon(B_2(ay, h^{-1}y'))h \\ &= b \sum_{h \in G} w(a)B_1(x, a^{-1}h^{-1}x')\varepsilon(B_2(y, a^{-1}h^{-1}y'))h \\ &= b \sum_{h \in G} w(a)B_1(x, (ha)^{-1}x')\varepsilon(B_2(y, (ha)^{-1}y'))h \\ &= b \sum_{k \in G} w(a)B_1(x, k^{-1}x')\varepsilon(B_2(y, k^{-1}y'))ka^{-1} \\ &= b B(x \otimes y, x' \otimes y') \overline{a}. \end{aligned}$$

Thus, B is an $R[G]$ -valued λ -Hermitian form on M . Note that B_1 and $\varepsilon \circ B_2$ are both nonsingular. So, $B_1 \otimes (\varepsilon \circ B_2)$ is nonsingular, which implies that B is nonsingular.

Third, we describe the definition of $q : M \rightarrow A_q/\Lambda$. Let $F(M_1 \times M_2)$ denote the R -free module with basis $\{(x, y) \mid x \in M_1, y \in M_2\}$ (although it may not be finitely generated), T the subset of $F(M_1 \times M_2)$ consisting of all elements of the form

$$\begin{aligned} &r(x, y) - (rx, y), \quad r(x, y) - (x, ry), \\ &(x + x', y) - (x, y) - (x', y), \quad \text{or} \quad (x, y + y') - (x, y) - (x, y'), \end{aligned}$$

where r ranges over R , x and x' over M_1 , y and y' over M_2 ; and let $[\] : F(M_1 \times M_2) \rightarrow M_1 \otimes M_2$ denote the canonical map.

Lemma 9.1. *Let f be a map from $F(M_1 \times M_2)$ to $A_q/\Lambda = A/(A_s + \Lambda)$. If the following conditions (1)–(3) are fulfilled for all $r \in R, u, v \in F(M_1 \times M_2)$ and $t \in T$:*

- (1) $f(ru) = r^2 f(u)$,
- (2) $f(u + v) = f(u) + f(v) + B([u], [v])$,
- (3) $f(t) = 0$,

then f factors through $M_1 \otimes M_2 \rightarrow A_q/\Lambda$.

The proof is elementary, and we omit it.

Define a map $f : F(M_1 \times M_2) \rightarrow A_q/\Lambda = A/(A_s + \Lambda)$ by

$$f\left(\sum_i r_i(x_i, y_i)\right) = \sum_i \sum_{g \in \mathcal{Q}(G)} r_i^2 B_1(x_i, g^{-1}x_i) q_2(y_i)_g g + \sum_{i < j} r_i r_j B(x_i \otimes y_i, x_j \otimes y_j),$$

for finitely many distinct (x_i, y_i) with $x_i \in M_1$, $y_i \in M_2$, where $r_i \in R$.

By definition, we have $f(ru) = r^2 f(u)$ for all $r \in R$ and $u \in F(M_1 \times M_2)$.

Note that for $u = \sum_i r_i(x_i, y_i)$ and $v = \sum_i r'_i(x_i, y_i)$, we have

$$\begin{aligned} f(u+v) &= \sum_i \sum_{g \in \mathcal{Q}(G)} (r_i + r'_i)^2 B_1(x_i, g^{-1}x_i) q_2(y_i)_g g \\ &\quad + \sum_{i < j} (r_i + r'_i)(r_j + r'_j) B(x_i \otimes y_i, x_j \otimes y_j). \end{aligned}$$

Thus, we have

$$\begin{aligned} f(u+v) - f(u) - f(v) &= \sum_i \sum_{g \in \mathcal{Q}(G)} 2r_i r'_i B_1(x_i, g^{-1}x_i) q_2(y_i)_g g + \sum_{i < j} (r_i r'_j + r'_i r_j) B(x_i \otimes y_i, x_j \otimes y_j). \end{aligned}$$

On the other hand, in A_q/Λ we have

$$\begin{aligned} B\left(\sum_i r_i x_i \otimes y_i, \sum_i r'_i x_i \otimes y_i\right) &= \sum_i r_i r'_i B(x_i \otimes y_i, x_i \otimes y_i) \\ &\quad + \sum_{i < j} (r_i r'_j B(x_i \otimes y_i, x_j \otimes y_j) + r_j r'_i B(x_j \otimes y_j, x_i \otimes y_i)) \\ &= \sum_i r_i r'_i B(x_i \otimes y_i, x_i \otimes y_i) \\ &\quad + \sum_{i < j} (r_i r'_j B(x_i \otimes y_i, x_j \otimes y_j) + r'_i r_j B(x_i \otimes y_i, x_j \otimes y_j)). \end{aligned}$$

Moreover, in $A/(A_s + \Lambda)$ we have

$$\begin{aligned} B(x_i \otimes y_i, x_i \otimes y_i) &= \sum_{g \in G} B_1(x_i, g^{-1}x_i) \varepsilon(B_2(y_i, g^{-1}y_i)) g \\ &= \sum_{g \in \mathcal{Q}(G)} B_1(x_i, g^{-1}x_i) 2q_2(y_i)_g g. \end{aligned}$$

Thus we obtain $f(u+v) - f(u) - f(v) = B([u], [v])$ in A_s/Λ .

It is clear that $f(t) = 0$ for all $t \in T$.

Since the conditions (1)–(3) in Lemma 9.1 are satisfied, we obtain the map $q : M \rightarrow A_q/\Lambda$ by $q([u]) = f(u)$ for $u \in F(M_1 \times M_2)$. Immediately we have $q(r[u]) = r^2 q([u])$ and $q([u+v]) - q([u]) - q([v]) = B([u], [v])$ for $r \in R$ and $u,$

$v \in F(M_1 \times M_2)$. For $g \in G$ and $u = (x, y)$, we have

$$\begin{aligned} q(g[u]) &= f(gx, gy) \\ &= \sum_{h \in Q(G)} B_1(gx, h^{-1}gx)q_2(gy)_h h \\ &= \sum_{h \in Q(G)} w(g)B_1(x, g^{-1}h^{-1}gx)q_2(y)_{g^{-1}hg} h \\ &= \sum_{h \in Q(G)} w(g)B_1(x, k^{-1}x)q_2(y)_k g k g^{-1} \\ &= g \sum_{h \in Q(G)} B_1(x, k^{-1}x)q_2(y)_k k \bar{g} \\ &= gf(x \otimes y)\bar{g} \\ &= gq([u])\bar{g}, \end{aligned}$$

where $k = g^{-1}hg$. Thus, $q(gz) = gq(z)\bar{g}$ for all $g \in G$ and $z \in M$.

Next we check the property (3) in Definition 7.1. For $u = (x, y)$ we have

$$\begin{aligned} \widetilde{q([u])} + \lambda \widetilde{q([u])} &= \sum_{g \in Q(G)} B_1(x, g^{-1}x)(\widetilde{q_2(y)}_g g + \lambda \widetilde{q_2(y)}_g \bar{g}) \\ &= \sum_{g \in G} B_1(x, g^{-1}x)B_2(y, y)_g g \\ &= B([u], [u]) \quad \text{in } A_q = A/A_s, \end{aligned}$$

which shows that $\widetilde{q(z)} + \lambda \widetilde{q(z)} = B(z, z)$ for all $z \in M$.

Putting all together, we see that the current triple (M, B, q) is an \mathbf{A} -quadratic $R[G]$ -module.

Defining $\alpha : \Theta \rightarrow M$ by $\alpha(t) = \alpha_1(t) \otimes \alpha_2(t)$ for $t \in \Theta$, we establish $\mathbf{M}_1 \cdot \mathbf{M}_2$ ($= \mathbf{M} = (M, B, q, \alpha)$) from $\mathbf{M}_1 = (M_1, B_1, \alpha_1)$ and $\mathbf{M}_2 = (M_2, B_2, q_2, \alpha_2)$.

Theorem 9.2. *Let $\mathbf{A} = (R, G, Q, S, \lambda, w)$ and Θ be as above. Then*

$$\text{WQ}_0(\mathbf{A}, \Theta), \quad \text{WQ}_0(\mathbf{A}, \Theta)^{\text{t-iso}} \quad \text{and} \quad \text{WQ}_0(\mathbf{A})$$

are modules over $\text{GW}_0(R, G, S, \Theta)$, and $\text{WQ}_0(\mathbf{A})$ is one over $\text{GW}_0(R, G, S)$ by the pairing

$$(\mathbf{M}_1, \mathbf{M}_2) \longmapsto \mathbf{M}_1 \cdot \mathbf{M}_2.$$

10. TENSOR PRODUCTS AND ∇ -INVARIANTS

In this section we invoke that R is square identical. Let Q, S, w, λ and Θ be as in Section 7, and let $\rho^{(2)} : \Theta \rightarrow \mathfrak{P}(S)$ be a G -map such that for every $s \in S$, there exists exactly one $t \in \Theta$ with $\rho^{(2)}(t) = s$. Hence, by Proposition 5.4, $\text{SGW}_0(R, G, S, \Theta)$ is a commutative ring with unit.

Proposition 10.1. *Let $\mathbf{M}_1 = (M_1, B_1, \alpha_1)$ and $\mathbf{M}_2 = (M_2, B_2, q_2, \alpha_2)$ be objects in $\mathcal{SH}(R, G, S, \Theta)$ and $\mathcal{SQ}(\mathbf{A}, \Theta)$, respectively. Then $\mathbf{M} = \mathbf{M}_1 \cdot \mathbf{M}_2 = (M, B, q, \alpha)$ defined in the previous section lies in $\mathcal{SQ}(\mathbf{A}, \Theta)$.*

Proof. It was already shown that $\mathbf{M} = \mathbf{M}_1 \cdot \mathbf{M}_2$ belongs to $\mathcal{Q}(\mathbf{A}, \Theta)$. Therefore, it suffices to show that $\nabla_{\mathbf{M}} = 0$. By definition, we have

$$\begin{aligned} \nabla_{\mathbf{M}}(x \otimes y)(s) &= \varepsilon(B(\Delta_{\alpha}(s) - x \otimes y, s(x \otimes y))) \\ &= \varepsilon(B(\Delta_{\alpha_1}(s) \otimes \Delta_{\alpha_2}(s) - x \otimes y, sx \otimes sy)) \\ &= \varepsilon(B(\Delta_{\alpha_1}(s) \otimes \Delta_{\alpha_2}(s), sx \otimes sy)) - \varepsilon(B(x \otimes y, sx \otimes sy)) \\ &= B_1(\Delta_{\alpha_1}(s), sx)\varepsilon(B_2(\Delta_{\alpha_2}(s), sy)) - B_1(x, sx)\varepsilon(B_2(y, sy)) \\ &= B_1(\Delta_{\alpha_2}(s) - x, sx)\varepsilon(B_2(\Delta_{\alpha_2}(s), sy)) \\ &\quad + B_1(x, sx)\varepsilon(B_2(\Delta_{\alpha_2}(s) - y, sy)) \\ &= \nabla_{\mathbf{M}_1}(x)(s)\varepsilon(B_2(\Delta_{\alpha_2}(s), sy)) + B_1(x, sx)\nabla_{\mathbf{M}_2}(y)(s) \\ &= 0 \quad \text{in } R/2R \end{aligned}$$

for $x \in M_1, y \in M_2$, and $s \in S$. By using Proposition 5.2 (1), we have $\nabla_{\mathbf{M}} = 0$. \square

The next theorem follows.

Theorem 10.2. *Let $\mathbf{A} = (R, G, Q, S, \lambda, w)$ and Θ be as above. Then*

$$\text{SWQ}_0(\mathbf{A}, \Theta), \quad \text{SWQ}_0(\mathbf{A}, \Theta)^{\text{t-iso}} \quad \text{and} \quad \text{SWQ}_0(\mathbf{A})$$

are modules over $\text{SGW}_0(R, G, S, \Theta)$.

11. THE MACKEY AND GREEN STRUCTURES OF GW AND SGW

Let S be a conjugation-invariant subset of $G(2)$, and set

$$S_H = H \cap S$$

for each $H \in \mathcal{S}(G)$. Let $Z^{(0)}$ be a finite G -set and let $\mathfrak{P}(Z^{(0)})$ stand for the set of all subsets of $Z^{(0)}$. Let $\mathcal{S}(G) \rightarrow \mathfrak{P}(Z^{(0)}); H \mapsto Z_H^{(0)}$, be an intersection-preserving G -map (see (3.1)), where $\mathcal{S}(G)$ is the set of all subgroups of G on which G acts by conjugation.

Define Θ_H by

$$\Theta_H = S_H \amalg Z_H^{(0)}.$$

It immediately follows that the map $H \mapsto \Theta_H$ is intersection preserving. Define $\rho_H^{(2)} : \Theta_H \rightarrow \mathfrak{P}(S_H)$ by

$$\rho_H^{(2)}(t) = \begin{cases} \{t\} & (t \in S_H), \\ \emptyset & (t \in Z_H^{(0)}). \end{cases}$$

Then, obviously, for each $s \in S_H$, there exists exactly one $t \in \Theta_H$ with $s \in \rho_H^{(2)}(t)$. In this case, $\text{GW}_0(R, H, \Theta_H)$ is a commutative ring with unit for each subgroup H of G , and so is $\text{SGW}_0(R, H, S_H, \Theta_H)$ if R is square identical.

Now let $\varphi : H \rightarrow K$ be a morphism in \mathcal{G} , namely one of an inclusion map, a conjugation map, or a composition of such maps. Then we have the associated φ -equivariant map $\psi : \Theta_H \rightarrow \Theta_K$. Actually, if φ is the inclusion map $j_{H,K} : H \rightarrow K$, then $S_H \subset S_K$ and $Z_H^{(0)} \subset Z_K^{(0)}$, and therefore the associated $\psi : \Theta_H \rightarrow \Theta_K$ is the inclusion map; if φ is the conjugation map $c_{(H,g)} : H \rightarrow gHg^{-1}$, then the associated $\psi : \Theta_H \rightarrow \Theta_{gHg^{-1}} = g\Theta_H$ is the left translation $\ell_{(\Theta_H, g)}$ by g . Since the G -action on S is given by conjugation, $\ell_{(\Theta_H, g)}|_{S_H}$ is the conjugation $c_{(H,g)}|_{S_H}$ by g . Thus, there are canonical correspondences

$$\text{GW}_0(R, H, \Theta_H) \rightarrow \text{GW}_0(R, K, \Theta_K); [M, B, \alpha] \mapsto [\varphi_{\#}M, \varphi_{\#}B, \psi_{\#}\alpha]$$

and

$$\mathrm{GW}_0(R, K, \Theta_K) \rightarrow \mathrm{GW}_0(R, H, \Theta_H); [N, B, \alpha] \mapsto [\varphi^\# N, \varphi^\# B, \psi^\# \alpha].$$

Lemma 11.1. $\nabla_{\varphi^\#} \mathbf{M} = 0$ for any morphism $\varphi : H \rightarrow K$ in \mathcal{G} and any object $\mathbf{M} = (M, B, \alpha)$ in $\mathcal{SH}(R, H, \Theta_H)$.

Proof. For the proof, we may suppose that $\varphi = j_{H,K}$ or $c_{(H,g)}$. For any $z = k \otimes_\varphi x \in \varphi^\# M$ with $k \in K$, $x \in M$ and $s \in S_K$, we have

$$\begin{aligned} \nabla_{\varphi^\#} \mathbf{M}(k \otimes_\varphi x)(s) &= \varphi^\# B(\Delta_{\psi^\#} \alpha(s) - k \otimes_\varphi x, s(k \otimes_\varphi x)) \\ &= \varphi^\# B(\Delta_{\psi^\#} \alpha(s), s(k \otimes_\varphi x)) - \varphi^\# B(k \otimes_\varphi x, s(k \otimes_\varphi x)) \quad \text{in } R/2R. \end{aligned}$$

By definition, we have

$$\begin{aligned} \varphi^\# B(\Delta_{\psi^\#} \alpha(s), s(k \otimes_\varphi x)) &= \varphi^\# B(\Delta_{\psi^\#} \alpha(s), k \otimes_\varphi x) \\ &= \varphi^\# B(\psi^\# \alpha(s), k \otimes_\varphi x) \\ &= \sum_{[a, s'] \in K \times_{H, \varphi} \Theta_H} \{ \varphi^\# B(a \otimes_\varphi \alpha(s'), k \otimes_\varphi x) \mid s' \in S_H, a\varphi(s')a^{-1} = s \} \\ &= \sum_{[a, s'] \in K \times_{H, \varphi} \Theta_H} \{ \delta_{a\varphi(H), k\varphi(H)} B(\alpha(s'), \varphi^{-1}(a^{-1}k)x) \mid s' \in S_H, \varphi(s') = a^{-1}sa \} \\ &= \sum_{[a, s'] \in K \times_{H, \varphi} \Theta_H} \{ \delta_{\varphi(H), a^{-1}k\varphi(H)} B(\alpha(s'), \varphi^{-1}(a^{-1}k)x) \mid s' \in S_H, \varphi(s') = a^{-1}sa \} \\ &= \sum_{[k, s''] \in K \times_{H, \varphi} \Theta_H} \{ B(\alpha(s''), x) \mid s'' \in S_H, \varphi(s'') = k^{-1}sk \} \\ &= \sum_{[k, s''] \in K \times_{H, \varphi} \Theta_H} \{ B(\alpha(s''), s''x) \mid s'' \in S_H, \varphi(s'') = k^{-1}sk \} \\ &= \sum_{[k, s''] \in K \times_{H, \varphi} \Theta_H} \{ B(x, s''x) \mid s'' \in S_H, \varphi(s'') = k^{-1}sk \} \\ &= \begin{cases} B(x, \varphi^{-1}(k^{-1}sk)x) & (\text{if } k^{-1}sk \in \varphi(H)), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} \varphi^\# B((k \otimes_\varphi x), s(k \otimes_\varphi x)) &= \varphi^\# B(k \otimes_\varphi x, sk \otimes_\varphi x) \\ &= \delta_{k\varphi(H), sk\varphi(H)} B(x, \varphi^{-1}(k^{-1}sk)x) \\ &= \begin{cases} B(x, \varphi^{-1}(k^{-1}sk)x) & (\text{if } k^{-1}sk \in \varphi(H)), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

This gives us $\nabla_{\varphi^\#} \mathbf{M}(z)(s) = 0$ for all $z \in \varphi^\# M$ and $s \in \Theta_K$. □

Proposition 11.2. Let S_H, Z_H and Θ_H be as above. Then, the Grothendieck-Witt ring functor $H \mapsto \mathrm{GW}_0(R, H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a Mackey functor, and so is the special Grothendieck-Witt ring functor $\mathrm{SGW}_0(R, H, S_H, \Theta_H)$, $H \in \mathcal{S}(G)$.

Proof. This follows from Propositions 3.2, 3.4, 3.5, 4.3, 4.5 and 4.6, and Lemma 11.1. □

Theorem 11.3. Let S_H, Z_H and Θ_H be as above. Then, the Grothendieck-Witt ring functor $H \mapsto \mathrm{GW}_0(R, H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a Green functor, and the special

Grothendieck-Witt ring functor $H \mapsto \text{SGW}_0(R, H, S_H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a Green functor, possibly without unit. If R is square identical, then the functor $H \mapsto \text{SGW}_0(R, H, S_H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a Green functor.

Proof. The theorem follows from Propositions 3.1, 3.3, 4.2, 4.4, 4.10 and 5.4. \square

Theorem 11.4. *The special Grothendieck-Witt group functor*

$$H \mapsto \text{SGW}_0(R, H, S_H)$$

is a module over the Grothendieck-Witt ring functor $H \rightarrow \text{GW}_0(R, H, S_H)$.

Proof. By Proposition 5.5, $\text{SGW}_0(R, H, S_H)$ is a module over $\text{GW}_0(R, H)$. The required properties for a Frobenius pairing follow from Propositions 3.1, 3.3, 4.2 and 4.4. \square

12. THE PAIRING $\text{SGW}_0 \times \text{SWQ}_0 \rightarrow \text{SWQ}_0$

Let $S \subset G(2)$, S_H , $Z_H^{(0)}$, Θ_H , $\rho_H^{(2)}$ be as in Section 11, where $H \in \mathcal{S}(G)$. Let $w : G \rightarrow \{-1, 1\}$ be a homomorphism and let λ stand for either 1 or -1 . In the current section we invoke

$$S \subset G^\lambda(2).$$

Let Q be a conjugation-invariant subset of $G^{-\lambda}(2)$. We set $Q_H = H \cap Q$, $A_H = R[H]$, and $\mathbf{A}_H = (R, H, Q_H, S_H, \lambda, w|_H)$ for $H \in \mathcal{S}(G)$.

Let $\varphi : H \rightarrow K$, where $H, K \in \mathcal{S}(G)$, be a monomorphism such that $w|_K \circ \varphi = w|_H$, $\varphi(Q_H) \subset Q_K$, and $\varphi(S_H) \subset S_K$.

Let $\mathbf{N} = (N, B, q)$ be an \mathbf{A}_K -quadratic $R[K]$ -module. We can write $q(x)$ as $\sum_{g \in \mathcal{Q}(K)} q(x)_g g$, where $\mathcal{Q}(K) = K \cap \mathcal{Q}(G)$ and $q(x)_g \in R_g$. We define $\varphi^\# q : \varphi^\# M \rightarrow (A_H)_q / \Lambda_H = R[H] / (R[S_H] + \Lambda_H)$ by

$$\varphi^\# q(x) = \sum_{h \in \mathcal{Q}(H)} q(x)_{\varphi(h)} h$$

for $x \in \varphi^\# M$, where Λ_H is the smallest form parameter of $R[H]$ including Q_H .

Lemma 12.1. *The $\varphi^\# q$ above is an \mathbf{A}_H -quadratic form on $\varphi^\# N$ with respect to $\varphi^\# B$.*

Proof. The proof is straightforward, as follows: For $g \in H$ and $x \in \varphi^\# N$, we have

$$\begin{aligned} \varphi^\# q(gx) &= \sum_{h \in \mathcal{Q}(H)} q(gx)_{\varphi(h)} h \\ &= \sum_{h \in \mathcal{Q}(H)} q(\varphi(g)x)_{\varphi(h)} h \\ &= \sum_{h \in \mathcal{Q}(H)} w(\varphi(g)) q(x)_{\varphi(g)^{-1} \varphi(h) \varphi(g)} h \\ &= g \left(\sum_{h \in \mathcal{Q}(H)} q(x)_{\varphi(g^{-1}hg)} g^{-1}hg \right) \bar{g} \\ &= g \varphi^\# q(x) \bar{g}. \end{aligned}$$

For $x, y \in \varphi^\#N$, we have

$$\begin{aligned} \varphi^\#q(x+y) - \varphi^\#q(x) - \varphi^\#q(y) &= \sum_{h \in \mathcal{Q}(H)} (q(x+y)_{\varphi(h)} - q(x)_{\varphi(h)} - q(y)_{\varphi(h)})h \\ &= \sum_{h \in H} B(x, y)_{\varphi(h)}h \\ &= \sum_{h \in H} \varepsilon(B(x, \varphi(h)^{-1}y))h \\ &= \varphi^\#B(x, y) \end{aligned}$$

in $A_H/(\Lambda_H + (A_H)_s)$.

For $x \in \varphi^\#N$, we have

$$\begin{aligned} \widetilde{\varphi^\#q(x)} + \overline{\lambda\varphi^\#q(x)} &= \sum_{h \in \mathcal{Q}(H)} (\widetilde{q(x)_{\varphi(h)}}h + \lambda\overline{q(x)_{\varphi(h)}\bar{h}}) \\ &= \sum_{h \in \mathcal{Q}(H)} (\widetilde{q(x)_{\varphi(h)}}h + \lambda w(h)\overline{q(x)_{\varphi(h)}h^{-1}}) \\ &= \sum_{h \in \mathcal{Q}(H)} (\widetilde{q(x)_{\varphi(h)}}h + \overline{q(x)_{\varphi(h)^{-1}}h^{-1}}) \\ &= \varphi^\#B(x, x) \quad \text{in } A_H/(A_H)_s. \end{aligned}$$

□

Proposition 12.2. *Let $\varphi : H \rightarrow K$, \mathbf{A}_H and \mathbf{A}_K be as above, and let $\mathbf{M}_1 = (M_1, B_1)$ and $\mathbf{M}_2 = (M_2, B_2, q_2)$ be a Hermitian $R[K]$ -module and an \mathbf{A}_K -quadratic module, respectively. Then $(\varphi^\#\mathbf{M}_1) \cdot (\varphi^\#\mathbf{M}_2) = \varphi^\#(\mathbf{M}_1 \cdot \mathbf{M}_2)$.*

Proof. Let $x, x' \in M_1$ and $y, y' \in M_2$. Then

$$\begin{aligned} B_{\varphi^\#\mathbf{M}_1, \varphi^\#\mathbf{M}_2}(x \otimes y, x' \otimes y') &= \sum_{h \in H} B_1(x, \varphi(h)^{-1}x')\varepsilon(B_2(y, \varphi(h)^{-1}y'))h \\ &= B_{\varphi^\#(\mathbf{M}_1 \cdot \mathbf{M}_2)}(x \otimes y, x' \otimes y'). \end{aligned}$$

In addition,

$$\begin{aligned} q_{\varphi^\#\mathbf{M}_1, \varphi^\#\mathbf{M}_2}(x \otimes y) &= \sum_{h \in \mathcal{Q}(H)} B_1(x, \varphi(h)^{-1}x)q_2(y)_{\varphi(h)}h \\ &= q_{\varphi^\#(\mathbf{M}_1 \cdot \mathbf{M}_2)}(x \otimes y). \end{aligned}$$

We have established the proposition. □

Now let $\mathbf{M} = (M, B, q)$ be an \mathbf{A}_H -quadratic $R[H]$ -module such that M is stably $R[H]$ -free and B is nonsingular. Let $\{g_1, \dots, g_\ell\}$ be a complete set of representatives of $K/\varphi(H)$, where g_i are elements in K . We define $\varphi_\#q : \varphi_\#M \rightarrow (A_K)_q/\Lambda = A_K/(\Lambda_K + (A_K)_s)$ by

$$\varphi_\#q \left(\sum_{i=1}^{\ell} g_i \otimes_\varphi x_i \right) = \sum_{i=1}^{\ell} g_i \varphi(q(x_i))\bar{g}_i + \sum_{1 \leq i < j \leq \ell} g_j \varphi(B(x_i, x_j))\bar{g}_i.$$

Lemma 12.3. *The $\varphi_{\#}q$ above is a quadratic form on $\varphi_{\#}M$ with respect to $\varphi_{\#}B$. Namely, the following hold:*

- (1) $\varphi_{\#}q(ru) = r^2\varphi_{\#}q(u)$,
- (2) $\varphi_{\#}q(u+v) - \overline{\varphi_{\#}q(u)} - \varphi_{\#}q(v) = \varphi_{\#}B(u, v)$,
- (3) $\overline{\varphi_{\#}q(u)} + \lambda\overline{\varphi_{\#}q(u)} = \varphi_{\#}B(u, u)$ in $A_K/(A_K)_s$,
- (4) $\varphi_{\#}q(ku) = k\varphi_{\#}q(u)\overline{k}$,

for all $r \in R$, $u, v \in \varphi_{\#}M$, $k \in K$.

Proof. The equality (1) holds clearly.

The proof of (2) runs as follows:

$$\begin{aligned} \varphi_{\#}q\left(\sum_i g_i \otimes_{\varphi} x_i + \sum_i g_i \otimes_{\varphi} y_i\right) - \varphi_{\#}q\left(\sum_i g_i \otimes_{\varphi} x_i\right) - \varphi_{\#}q\left(\sum_i g_i \otimes_{\varphi} y_i\right) \\ = \sum_{i=1}^{\ell} g_i(\varphi(q(x_i + y_i)) - \varphi(q(x_i)) - \varphi(q(y_i)))\overline{g_i} \\ + \sum_{1 \leq i < j \leq \ell} g_j \varphi(B(x_i + y_i, x_j + y_j) - B(x_i, x_j) - B(y_i, y_j))\overline{g_i} \\ = \sum_{i=1}^{\ell} g_i \varphi(B(x_i, y_i))\overline{g_i} + \sum_{1 \leq i \neq j \leq \ell} g_j \varphi(B(x_i, y_j))\overline{g_i} \\ = \varphi_{\#}B\left(\sum_{i=1}^{\ell} g_i \otimes_{\varphi} x_i, \sum_{j=1}^{\ell} g_j \otimes_{\varphi} y_j\right). \end{aligned}$$

The equality (3) holds because

$$\begin{aligned} \overline{\varphi_{\#}q(g_i \otimes_{\varphi} x)} + \lambda\overline{\varphi_{\#}q(g_i \otimes_{\varphi} x)} \\ = g_i \varphi(q(x))\overline{g_i} + \lambda g_i \varphi(q(x))\overline{g_i} \\ = g_i \varphi(B(x, x))\overline{g_i} \\ = \varphi_{\#}B(g_i \otimes_{\varphi} x, g_i \otimes_{\varphi} x). \end{aligned}$$

For $k \in K$, we can write kg_i in the form $g_{\sigma(i)}\varphi(h_i)$ with $h_i \in H$. Then

$$\begin{aligned} \varphi_{\#}q(k(g_i \otimes_{\varphi} x)) &= \varphi_{\#}q(g_{\sigma(i)} \otimes_{\varphi} h_i x) \\ &= g_{\sigma(i)} \varphi(q(h_i x))\overline{g_{\sigma(i)}} \\ &= g_{\sigma(i)} \varphi(h_i) \varphi(q(x))\overline{g_{\sigma(i)} \varphi(h_i)} \\ &= kg_i \varphi(q(x))\overline{g_i k} \\ &= k\varphi_{\#}q(g_i \otimes_{\varphi} x)\overline{k}. \end{aligned}$$

The equation (4) follows from this and (2) above. \square

Proposition 12.4. *Let H be a subgroup of G , $q : M \rightarrow (A_H)_q/\Lambda_H$ an A_H -quadratic form on M , and g an element of G . Then the diagram*

$$\begin{array}{ccc}
 c_{(H,g)\#} M & & \\
 \downarrow f_0 & \searrow c_{(H,g)\#} q & \\
 c_{(gHg^{-1},g^{-1})\#} M & \xrightarrow{c_{(gHg^{-1},g^{-1})\#} q} & (A_{gHg^{-1}})_q/\Lambda_{gHg^{-1}}
 \end{array}$$

commutes, where f_0 is the canonical $R[gHg^{-1}]$ -isomorphism (cf. Proposition 3.2).

The proof of the proposition is straightforward.

Proposition 12.5. *Let H be a subgroup of G and $q : M \rightarrow (A_H)_q/\Lambda_H$ an A_H -quadratic form on M . Then for each $g \in H$, the following diagrams commute:*

$$\begin{array}{ccc}
 c_{(H,g)\#} M & & \\
 \downarrow f_1 & \searrow c_{(H,g)\#} q & \\
 M & \xrightarrow{w(g)q} & (A_H)_q/\Lambda_H,
 \end{array}$$

$$\begin{array}{ccc}
 c_{(H,g)\#}^{\#} M & & \\
 \downarrow f_2 & \searrow c_{(H,g)\#}^{\#} q & \\
 M & \xrightarrow{w(g)q} & (A_H)_q/\Lambda_H,
 \end{array}$$

where f_1 and f_2 are the canonical isomorphisms (cf. Proposition 3.4).

The proposition follows straightforwardly from the definition.

Proposition 12.6. *For any subgroups H and K of G , each A_H -quadratic form $q : M \rightarrow (A_H)_q/\Lambda_H$ satisfies the w -Mackey double coset formula. Namely,*

$$(\text{Res}_K^G \text{Ind}_H^G q) \circ \omega = \sum_{KgH \in K \backslash G/H} w(g) \text{Ind}_{K \cap gHg^{-1} c_{(H \cap g^{-1}Kg, g)\#}^K}^K \text{Res}_{H \cap g^{-1}Kg}^H q,$$

where ω is the canonical isomorphism (cf. Proposition 3.5). Particularly, in the case $w(G) = \{1\}$, q satisfies the Mackey double coset formula.

Proof. It suffices to prove that

$$(\text{Res}_K^G \text{Ind}_H^G q)(ag \otimes x) = w(g) (\text{Ind}_{K \cap gHg^{-1} c_{(H \cap g^{-1}Kg, g)\#}^K}^K \text{Res}_{H \cap g^{-1}Kg}^H q)(a \otimes (e \otimes x))$$

for any $g \in G$, $a \in K$, $x \in \text{Res}_{H \cap g^{-1}Kg}^H M$. This is valid because

$$\begin{aligned} (\text{Res}_K^G \text{Ind}_H^G q)(ag \otimes x) &= \sum_{k \in \mathcal{Q}(K)} (\text{Ind}_H^G q)(ag \otimes x)_k k \\ &= \sum_{k \in \mathcal{Q}(K)} (agq(x)\overline{ag})_k k \end{aligned}$$

and

$$\begin{aligned} &(\text{Ind}_{K \cap gHg^{-1}c(H \cap g^{-1}Kg, g)}^K \text{Res}_{H \cap g^{-1}Kg}^H q)(a \otimes (e \otimes x)) \\ &= \sum_{k \in \mathcal{Q}(K)} (a(c_{(H \cap g^{-1}Kg, g)} \text{Res}_{H \cap g^{-1}Kg}^H q)(e \otimes x)\overline{a})_k k \\ &= \sum_{k \in \mathcal{Q}(K)} (ag(\text{Res}_{H \cap g^{-1}Kg}^H q)(x)g^{-1}\overline{a})_k k \\ &= w(g) \sum_{k \in \mathcal{Q}(K)} (agq(x)\overline{ag})_k k. \end{aligned}$$

□

Proposition 12.7. *Let \mathbf{A}_H and Θ_H be as above for each $H \in \mathcal{S}(G)$. Then the Witt group functor $H \mapsto \text{WQ}_0(\mathbf{A}_H, \Theta_H)$, $H \in \mathcal{S}(G)$, and the special Witt group functor $H \mapsto \text{SWQ}_0(\mathbf{A}_H, \Theta_H)$, $H \in \mathcal{S}(G)$, are both w -Mackey functors, and hence modules over the Burnside ring functor $H \mapsto \Omega(G)$, $H \in \mathcal{S}(G)$.*

Proof. The claim for the Witt group functor follows from Propositions 3.2, 3.4, 3.5, 6.6, 6.7, 6.8, 12.4, 12.5, and 12.6.

Let $\mathbf{M} = (M, B, q, \alpha)$ be a Θ_H -positioned \mathbf{A}_H -quadratic $R[H]$ -module. By Lemma 6.3, $\varepsilon \circ B : M \times M \rightarrow R$ is a λ -symmetric, $(H, w|_H)$ -invariant, R -bilinear form. For a morphism $\varphi : H \rightarrow K$ in \mathcal{G} , the same argument as the proof of Lemma 11.1 shows that if $\nabla_{\mathbf{M}} = 0$ (see (8.1)), then $\nabla_{\varphi\#\mathbf{M}} = 0$. (In fact, consider the case where R is replaced by $R/2R$.) Thus, the claim for the special Witt group functor also follows. □

In the remainder of this section, let $\varphi : H \rightarrow K$ be a morphism in \mathcal{G} .

Proposition 12.8. *Let $\mathbf{M}_1 = (M_1, B_1, \alpha_1)$ and $\mathbf{M}_2 = (M_2, B_2, q_2, \alpha_2)$ be objects in $\mathcal{H}(R, K, \Theta_K)$ and $\mathcal{Q}(\mathbf{A}_H, \Theta_H)$, respectively. Let*

$$f : M_1 \otimes_R \varphi\#M_2 \rightarrow \varphi\#(\varphi\#M_1 \otimes_R M_2)$$

denote the canonical isomorphism, namely $f(x \otimes (k \otimes_\varphi y)) = k \otimes_\varphi (k^{-1}x \otimes y)$ for $k \in K$, $x \in M_1$ and $y \in M_2$. Then the diagram

$$\begin{array}{ccc} M_3 \times M_3 & & \\ \downarrow f \times f & \searrow B_1 \otimes_R \varphi\#B_2 & \\ M_4 \times M_4 & \xrightarrow{\varphi\#(\varphi\#B_1 \otimes_R B_2)} & A_K \end{array}$$

where $M_3 = (M_1 \otimes_R (R[K] \otimes_{R[H],\varphi} M_2))$ and $M_4 = R[K] \otimes_{R[H],\varphi} (\varphi^\# M_1 \otimes_R M_2)$, and the diagram

$$\begin{array}{ccc}
 M_1 \otimes_R (R[K] \otimes_{R[H],\varphi} M_2) & & \\
 \downarrow f & \searrow^{B_1 \otimes_R \varphi \# q_2} & \\
 R[K] \otimes_{R[H],\varphi} (\varphi^\# M_1 \otimes_R M_2) & \xrightarrow{\varphi_\#(\varphi^\# B_1 \otimes_R q_2)} & (A_K)_q / \Lambda_K
 \end{array}$$

commute.

Proof. Let $k, k' \in K$, $x, x' \in M_1$, and $y, y' \in M_2$.

The commutability $B_1 \otimes (\varphi_\# B_2) = \varphi_\#((\varphi^\# B_1) \otimes B_2)$ via f holds because

$$\begin{aligned}
 & B_1 \otimes (\varphi_\# B_2)(x \otimes (k \otimes_\varphi y), x' \otimes (k' \otimes_\varphi y')) \\
 &= \sum_{g \in K} B_1(x, g^{-1} x') \varepsilon(\varphi_\# B_2(k \otimes_\varphi y, g^{-1}(k' \otimes_\varphi y'))) g \\
 &= \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1} k' \varphi(H)} B_1(x, g^{-1} x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1} g^{-1} k') y')) g
 \end{aligned}$$

and

$$\begin{aligned}
 & \varphi_\#((\varphi^\# B_1) \otimes B_2)(k \otimes_\varphi (k^{-1} x \otimes y), k' \otimes_\varphi (k'^{-1} x' \otimes y')) \\
 &= \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1} k' \varphi(H)} \\
 & \quad \cdot \varepsilon\left((\varphi^\# B_1 \otimes B_2)((k^{-1} x \otimes y), \varphi^{-1}(k^{-1} g^{-1} k')(k'^{-1} x' \otimes y'))\right) g \\
 &= \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1} k' \varphi(H)} \\
 & \quad \cdot B_1(k^{-1} x, (k^{-1} g^{-1} k') k'^{-1} x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1} g^{-1} k') y')) g \\
 &= \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1} k' \varphi(H)} B_1(k^{-1} x, k^{-1} g^{-1} x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1} g^{-1} k') y')) g \\
 &= \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1} k' \varphi(H)} B_1(x, g^{-1} x') \varepsilon(B_2(y, \varphi^{-1}(k^{-1} g^{-1} k') y')) g.
 \end{aligned}$$

The commutability $B_1 \otimes (\varphi\#q_2) = \varphi\#((\varphi\#B_1) \otimes q_2)$ via f follows from

$$\begin{aligned} B_1 \otimes (\varphi\#q_2)(x \otimes (k \otimes_\varphi y)) &= \sum_{g \in \mathcal{Q}(K)} B_1(x, g^{-1}x) \varphi\#q_2(k \otimes_\varphi y) g g \\ &= \sum_{g \in \mathcal{Q}(K)} B_1(x, g^{-1}x) (k\varphi(q_2(y))\bar{k})_g g \\ &= \sum_{g \in \mathcal{Q}(K)} B_1(x, g^{-1}x) \varphi(q_2(y))_{k^{-1}gk} w(k) g \\ &= \sum_{g \in \mathcal{Q}(K) \cap k\varphi(H)k^{-1}} B_1(x, g^{-1}x) q_2(y)_{\varphi^{-1}(k^{-1}gk)} w(k) g \\ &= k \left(\sum_{a \in k^{-1}\mathcal{Q}(K)k \cap \varphi(H)} B_1(x, ka^{-1}k^{-1}x) q_2(y)_{\varphi^{-1}(a)} a \right) \bar{k} \\ &= k \left(\sum_{b \in \mathcal{Q}(H)} B_1(x, k\varphi(b)^{-1}k^{-1}x) q_2(y)_b \varphi(b) \right) \bar{k} \end{aligned}$$

and

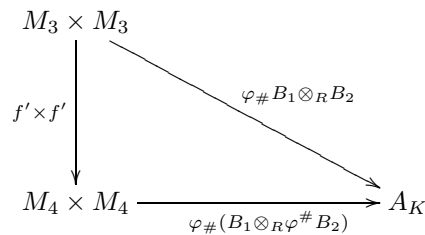
$$\begin{aligned} \varphi\#((\varphi\#B_1) \otimes q_2)(k \otimes_\varphi (k^{-1}x \otimes y)) &= k\varphi((\varphi\#B_1) \otimes q_2(k^{-1}x \otimes y))\bar{k} \\ &= k\varphi \left(\sum_{h \in \mathcal{Q}(H)} \varphi\#B_1(k^{-1}x, h^{-1}k^{-1}x) q_2(y)_h h \right) \bar{k} \\ &= k\varphi \left(\sum_{h \in \mathcal{Q}(H)} B_1(k^{-1}x, \varphi(h)^{-1}k^{-1}x) q_2(y)_h h \right) \bar{k} \\ &= k\varphi \left(\sum_{h \in \mathcal{Q}(H)} B_1(x, k\varphi(h)^{-1}k^{-1}x) q_2(y)_h h \right) \bar{k}. \end{aligned}$$

□

Proposition 12.9. Let $\mathbf{M}_1 = (M_1, B_1, \alpha_1)$ and $\mathbf{M}_2 = (M_2, B_2, q_2, \alpha_2)$ be objects in $\mathcal{H}(R, H, \Theta_H)$ and $\mathcal{Q}(\mathbf{A}_K, \Theta_K)$, respectively. Let

$$f' : (\varphi\#M_1) \otimes_R M_2 \rightarrow \varphi\#(M_1 \otimes_R \varphi\#M_2)$$

denote the canonical isomorphism, namely $f'((k \otimes_\varphi x) \otimes y) = k \otimes_\varphi (x \otimes k^{-1}y)$ for $k \in K$, $x \in M_1$ and $y \in M_2$. Then the diagram



where $M_3 = (R[K] \otimes_{R[H], \varphi} M_1) \otimes_R M_2$ and $M_4 = R[K] \otimes_{R[H], \varphi} (M_1 \otimes_R \varphi^\# M_2)$, and the diagram

$$\begin{array}{ccc}
 (R[K] \otimes_{R[H], \varphi} M_1) \otimes_R M_2 & & \\
 \downarrow f' & \searrow \varphi_\# B_1 \otimes_R q_2 & \\
 R[K] \otimes_{R[H], \varphi} (M_1 \otimes_R \varphi^\# M_2) & \xrightarrow{\varphi_\# (B_1 \otimes_R \varphi^\# q_2)} & (A_K)_q / \Lambda_K
 \end{array}$$

commute.

Proof. Let $k, k' \in K, x, x' \in M_1$, and $y, y' \in M_2$.

The commutability $(\varphi_\# B_1) \otimes B_2 = \varphi_\# (B_1 \otimes (\varphi^\# B_2))$ via f' holds because

$$\begin{aligned}
 & (\varphi_\# B_1) \otimes B_2((k \otimes_\varphi x) \otimes y, (k' \otimes_\varphi x') \otimes y') \\
 &= \sum_{g \in K} (\varphi_\# B_1)((k \otimes_\varphi x), g^{-1}(k' \otimes_\varphi x')) \varepsilon(B_2(y, g^{-1}y'))g \\
 &= \sum_{g \in K} \delta_{k\varphi(H), g^{-1}k'\varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k')x') \varepsilon(B_2(y, g^{-1}y'))g
 \end{aligned}$$

and

$$\begin{aligned}
 & \varphi_\# (B_1 \otimes (\varphi^\# B_2))(k \otimes_\varphi (x \otimes k^{-1}y), k' \otimes_\varphi (x' \otimes k'^{-1}y')) \\
 &= \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1}k'\varphi(H)} \\
 & \quad \cdot \varepsilon \left(B_1 \otimes (\varphi^\# B_2)(x \otimes k^{-1}y, \varphi^{-1}(k^{-1}g^{-1}k')(x' \otimes k'^{-1}y')) \right) g \\
 &= \sum_{g \in K} w(k) \delta_{k\varphi(H), g^{-1}k'\varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k')x') \varepsilon(B_2(k^{-1}y, k^{-1}g^{-1}y'))g \\
 &= \sum_{g \in K} \delta_{k\varphi(H), g^{-1}k'\varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k')x') \varepsilon(B_2(y, g^{-1}y'))g.
 \end{aligned}$$

The commutability $(\varphi_\# B_1) \otimes q_2 = \varphi_\# (B_1 \otimes (\varphi^\# q_2))$ via f' follows from

$$\begin{aligned}
 & (\varphi_\# B_1) \otimes q_2((k \otimes_\varphi x) \otimes y) = \sum_{g \in \mathcal{Q}(K)} (\varphi_\# B_1)(k \otimes_\varphi x, g^{-1}(k \otimes_\varphi x)) q_2(y)g \\
 &= \sum_{g \in \mathcal{Q}(K)} \delta_{k\varphi(H), g^{-1}k\varphi(H)} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k)x) q_2(y)g \\
 &= \sum_{g \in \mathcal{Q}(k\varphi(H)k^{-1})} B_1(x, \varphi^{-1}(k^{-1}g^{-1}k)x) q_2(y)g \\
 &= k \sum_{h \in \mathcal{Q}(H)} B_1(x, h^{-1}x) q_2(y)_{k\varphi(h)k^{-1}} \varphi(h)k^{-1} \\
 &= k \sum_{h \in \mathcal{Q}(H)} w(k) B_1(x, h^{-1}x) q_2(k^{-1}y)_{\varphi(h)} \varphi(h)k^{-1} \\
 &= k\varphi \left(\sum_{h \in \mathcal{Q}(H)} B_1(x, h^{-1}x) q_2(k^{-1}y)_{\varphi(h)} h \right) \bar{k}
 \end{aligned}$$

and

$$\begin{aligned} \varphi_{\#}(B_1 \otimes (\varphi^{\#}q_2))(k \otimes_{\varphi}(x \otimes k^{-1}y)) &= k\varphi(B_1 \otimes (\varphi^{\#}q_2)(x \otimes k^{-1}y))\bar{k} \\ &= k\varphi\left(\sum_{h \in \mathcal{Q}(H)} B_1(x, h^{-1}x)q_2(k^{-1}y)_{\varphi(h)}h\right)\bar{k}. \end{aligned}$$

□

Let $\psi : \Theta_H \rightarrow \Theta_K$ denote the map associated with φ .

Theorem 12.10. *Let \mathbf{A}_H and Θ_H be as above for each $H \in \mathcal{S}(G)$. Then the w -Mackey functor $H \mapsto \text{WQ}_0(\mathbf{A}_H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a module over the Green functor $H \mapsto \text{GW}_0(R, H, \Theta_H)$, $H \in \mathcal{S}(G)$. If R is square identical, then the w -Mackey functor $H \mapsto \text{SWQ}_0(\mathbf{A}_H, \Theta_H)$, $H \in \mathcal{S}(G)$, is a module over the Green functor $H \mapsto \text{SGW}_0(R, H, S_H, \Theta_H)$, $H \in \mathcal{S}(G)$.*

Proof. By Proposition 3.3 we have $\alpha_1 \otimes (\psi_{\#}\alpha_2) = \psi_{\#}((\psi^{\#}\alpha_1) \otimes \alpha_2)$ via f in Proposition 12.8. By Proposition 3.3 we have $(\psi_{\#}\alpha_1) \otimes \alpha_2 = \psi_{\#}(\alpha_1 \otimes \psi^{\#}\alpha_2)$ via f' in Proposition 12.9. The theorem follows from Propositions 12.2, 12.8 and 12.9. □

13. APPLICATIONS OF INDUCTION AND RESTRICTION

Let Z^0 be a finite G -set, and let $\mathcal{S}(G) \rightarrow \mathfrak{P}(Z^0)$; $H \mapsto Z_H^{(0)}$ be an intersection-preserving G -map. Let S be a conjugation-invariant subset of $G(2)$. We set $S_H = H \cap S$ and $\Theta_H = S_H \amalg Z_H^{(0)}$. Define $\rho_H^{(2)} : \Theta_H \rightarrow \mathfrak{P}(S_H)$ by

$$\rho_H^{(2)}(t) = \begin{cases} \{t\} & (t \in S_H), \\ \emptyset & (t \in Z_H^{(0)}). \end{cases}$$

Further, let \mathcal{F} be a conjugation-invariant subset of $\mathcal{S}(G)$ such that

$$(13.1) \quad \Theta_G \times \Theta_G = \bigcup_{H \in \mathcal{F}} \Theta_H \times \Theta_H,$$

and let β be an element in the Burnside ring $\Omega(G)$ such that

$$\text{Res}_H^G \beta = 1_{\Omega(H)} \quad \text{for any } H \in \mathcal{F}.$$

Theorem 13.1. *Let x be an arbitrary element in $\text{SGW}_0(R, G, S, \Theta_G)$. If \mathcal{F} contains all 2-hyerelementary (resp. cyclic) subgroups of G , then $(1_{\Omega(G)} - \beta)^2 x = 0$ (resp. $(1_{\Omega(G)} - \beta)^{2k+3} x = 0$, where k is the integer such that $|G| = 2^k m$ with an odd integer m).*

For the proof, we recall two lemmas.

Lemma 13.2 (A. Dress [11, Theorems 1 and 3]). *For a set \mathcal{H} of subgroups of G , the restriction homomorphism*

$$\text{Res} : \text{GW}_0(\mathbb{Z}, G) \rightarrow \bigoplus_{H \in \mathcal{H}} \text{GW}_0(\mathbb{Z}, H)$$

has the following properties.

- (1) *If \mathcal{H} contains all 2-hyerelementary subgroups of G , then Res is injective.*
- (2) *If \mathcal{H} contains all cyclic subgroups of G , then the kernel of Res is annihilated by 4.*

For a subgroup H of G , we denote by χ_H the homomorphism $\Omega(G) \rightarrow \mathbb{Z}$ such that $\chi_H([X]) = |X^H|$ for every finite G -set X .

Lemma 13.3 ([15, Proposition 6.3]). *Let x be an element of $\Omega(G)$ such that $\chi_H(x) \equiv 0 \pmod{2}$ for all $H \in \mathcal{S}(G)$. Then x^{k+1} lies in $2\Omega(G)$, where k is the integer such that $|G| = 2^k m$ with an odd integer m .*

Proof of Theorem 13.1. Let H be a 2-hyerelementary subgroup of G .

First consider the case where \mathcal{F} contains all 2-hyerelementary subgroups of G . Then, it is obvious that $\text{Res}_H^G(1_{\Omega(G)} - \beta) = 0$. Since the Green functor $\text{GW}_0(\mathbb{Z}, -)$ is a module over the Green functor $\Omega(-)$, $\text{Res}_H^G((1_{\Omega(G)} - \beta)\text{GW}_0(\mathbb{Z}, G)) = 0$.

Next, consider the case where \mathcal{F} contains all cyclic subgroups of G . Then

$$\chi_K(1_{\Omega(G)} - \beta) \equiv 0 \pmod{2}$$

for any subgroup K of H , and hence $\text{Res}_H^G(1_{\Omega(G)} - \beta)^{2k+2}$ lies in $4\Omega(H)$. So, we can write $\text{Res}_H^G(1_{\Omega(G)} - \beta)^{2k+2} = 4\gamma$ for some $\gamma \in \Omega(H)$. Clearly, $\text{Res}_C^H \gamma = 0$ for all cyclic subgroups of H . Thus by (2) of Dress' Lemma, $\gamma\text{GW}_0(\mathbb{Z}, H)$ is annihilated by 4, and hence $\text{Res}_H^G((1_{\Omega(G)} - \beta)^{2k+2}\text{GW}_0(\mathbb{Z}, G)) = 0$. By (1) of Dress' Lemma, we obtain

$$(1 - \beta)\text{GW}_0(\mathbb{Z}, G) = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+2}\text{GW}_0(\mathbb{Z}, G) = 0.$$

Since the canonical map $\text{GW}_0(\mathbb{Z}, G) \rightarrow \text{GW}_0(R, G)$ is an $\Omega(G)$ -homomorphism of a ring with unit, it follows that

$$(1 - \beta)\text{GW}_0(R, G) = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+2}\text{GW}_0(R, G) = 0.$$

Noting that the Mackey functor $\text{SGW}_0(R, -, S_-)$ is a module over the Green functor $\text{GW}_0(R, -)$, we obtain

$$(1 - \beta)\text{SGW}_0(R, G, S) = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+2}\text{SGW}_0(R, G, S) = 0.$$

Recall Proposition 5.3, namely the fact that the canonical homomorphism

$$\text{SGW}_0(R, G, S) \rightarrow \text{SGW}_0(R, G, S, \Theta_G)^{\text{t-iso}}$$

is surjective. In addition, the homomorphism is an $\Omega(G)$ -homomorphism. Hence, we conclude that

$$(1 - \beta)\text{SGW}_0(R, G, S, \Theta_G)^{\text{t-iso}} = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+2}\text{SGW}_0(R, G, S, \Theta_G)^{\text{t-iso}} = 0.$$

On the other hand, it is easy to check that $(1 - \beta)\text{SGW}_0(R, G, S, \Theta_G)$ is contained in (the image by the canonical map from) $\text{SGW}_0(R, G, S, \Theta_G)^{\text{t-iso}}$.

Putting all together, we establish that

$$(1 - \beta)^2\text{SGW}_0(R, G, S, \Theta_G) = 0 \text{ or } (1_{\Omega(G)} - \beta)^{2k+3}\text{SGW}_0(R, G, S, \Theta_G) = 0.$$

□

Proof of Theorem 1.2. Here $Z^{(0)}$ is the empty set. Since $H \mapsto \text{SGW}(R, H, S_H, S_H)$ is a Mackey functor, it is a module over the Burnside ring functor $H \mapsto \Omega(H)$ by [7, Proposition 6.2.3]. For each subgroup H of G we have

$$\Theta_H \times \Theta_H = (S \cap H) \times (S \cap H) = (S \times S) \cap (H \times H).$$

Thus (13.1) is fulfilled, and Theorem 1.2 follows from Theorem 13.1. □

Now let $w : G \rightarrow \{-1, 1\}$ be a homomorphism, $\lambda = 1$ or -1 , and let Q be a conjugation-invariant subset of $G^{-\lambda}(2)$. Suppose $S \subset G^\lambda(2)$. For each $H \in \mathcal{S}(G)$, we set $A_H = R[H]$, $Q_H = H \cap Q$, and $\mathbf{A}_H = (R, H, Q_H, S_H, \lambda, w|_H)$.

Theorem 13.4. *Suppose R is square identical. Let x be an arbitrary element of the special Witt group $\text{SWQ}_0(\mathbf{A}_G, \Theta_G)$. If \mathcal{F} contains all 2-hyperelementary (resp. cyclic) subgroups of G , then $(1_{\Omega(G)} - \beta)^2x = 0$ (resp. $(1_{\Omega(G)} - \beta)^{2k+3}x = 0$, where $|G| = 2^k m$ with m odd).*

Proof. The theorem follows from Proposition 12.10 and Theorem 13.1. \square

Proof of Theorem 1.3. Theorem 1.3 follows from Theorem 13.4. \square

Theorem 13.5. *Suppose that R is square identical, \mathcal{F} contains any cyclic subgroup of G , and β has the form*

$$\beta = \sum_{H \in \tilde{\mathcal{F}}} n_H [G/H],$$

with $n_H \in \mathbb{Z}$ for some lower closed subset $\tilde{\mathcal{F}}$ of $\mathcal{S}(G)$; namely, any subgroup H of G lies in $\tilde{\mathcal{F}}$ whenever $K \in \tilde{\mathcal{F}}$ and $H \subset K$. Then

$$\text{SWQ}_0(R, G, Q, S, \Theta_G) = \sum_{H \in \tilde{\mathcal{F}}} \text{Ind}_H^G \text{SWQ}_0(R, H, Q_H, S_H, \Theta_H),$$

and the restriction homomorphism

$$\text{Res} : \text{SWQ}_0(R, G, Q, S, \Theta_G) \rightarrow \bigoplus_{H \in \tilde{\mathcal{F}}} \text{SWQ}_0(R, H, Q_H, S_H, \Theta_H)$$

is injective.

Proof. By hypothesis, we can write

$$(1_{\Omega(G)} - \beta)^{2|G|+3} = [G/G] - \sum_{H \in \tilde{\mathcal{F}}} m_H [G/H]$$

with $m_H \in \mathbb{Z}$. For an arbitrary element $x \in \text{SWQ}_0(R, G, Q, S, \Theta_G)$, Theorem 13.4 implies that

$$x = \sum_{H \in \tilde{\mathcal{F}}} m_H [G/H] \cdot x = \sum_{H \in \tilde{\mathcal{F}}} m_H \text{Ind}_H^G (\text{Res}_H^G x).$$

Moreover, if $\text{Res}_H^G x = 0$ for every $H \in \tilde{\mathcal{F}}$, then we conclude that $x = 0$. \square

Proof of Theorem 1.4. Since G is a nonsolvable group, there exists an idempotent $\beta \in \Omega(G)$ such that $\chi_K(\beta) = 0$ for any nonsolvable subgroup K of G and $\chi_H(\beta) = 1$ for any solvable subgroup H of G . This element β has the form $\beta = \sum_H n_H [G/H]$ with $n_H \in \mathbb{Z}$, where H runs over the set of all solvable subgroups of G . Thus, Theorem 1.4 follows from Theorem 13.5. \square

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