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COMPOSITION OPERATORS ACTING ON HOLOMORPHIC SOBOLEV SPACES

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Abstract. We study the action of composition operators on Sobolev spaces of analytic functions having fractional derivatives in some weighted Bergman space or Hardy space on the unit disk. Criteria for when such operators are bounded or compact are given. In particular, we find the precise range of orders of fractional derivatives for which all composition operators are bounded on such spaces. Sharp results about boundedness and compactness of a composition operator are also given when the inducing map is polygonal.

1. Introduction and Statement of Results

Let $\mathbf{D}$ be the unit disk in the complex plane. We shall write $H(\mathbf{D})$ for the class of all holomorphic functions on $\mathbf{D}$. Let $s \geq 0$ be a real number. Following [BB], we define the fractional derivative for $f \in H(\mathbf{D})$ of order s by

$$\mathcal{R}^s f(z) = \sum_{n=0}^{\infty} (1+n)^s a_n z^n, \qquad z \in \mathbf{D}$$

where $\sum_{n=0}^{\infty} a_n z^n$ is the Taylor series of f.

In this paper, we are going to investigate composition operators acting on holomorphic Sobolev spaces defined in terms of fractional derivatives. To introduce those holomorphic Sobolev spaces, let us first recall some well-known function spaces. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α is the space of all $f \in H(\mathbf{D})$ for which

$$||f||^p_{A^p_\alpha} = \int_{\mathbf{D}} |f(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty,$$

where dA is area measure on $\mathbf{D}$. Also, the Hardy space H^p is the space of all $g \in H(\mathbf{D})$ for which

$$||g||^p_{H^p} = \sup_{0<r<1} \int_0^{2\pi} |g(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

We will often use the following notation to allow unified statements:

$$A^p_{-1} = H^p.$$

This notation is justified by the weak-star convergence of $(\alpha+1)(1-|z|^2)^\alpha \, dA(z)/\pi$ to $d\theta/2\pi$ as $\alpha \to -1$.

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Now, for $p > 0$, $s \geq 0$ and $\alpha \geq -1$, the holomorphic Sobolev space $A^p_{\alpha,s}$ is defined to be the space of all $f \in H(\mathbf{D})$ for which $\mathcal{R}^s f \in A^p_\alpha$. We will often write $H^p_s = A^p_{-1,s}$. We define the norm of $f \in A^p_{\alpha,s}$ by

$$||f||_{A^p_{\alpha,s}} = ||\mathcal{R}^s f||_{A^p_\alpha}.$$

Of course, we are abusing the term "norm", since $||\cdot||_{A^p_{\alpha,s}}$ does not satisfy the triangle inequality for $0 < p < 1$, but in this case $(f,g) \mapsto ||f-g||^p_{A^p_{\alpha,s}}$ defines a translation-invariant metric on A^p_α which turns A^p_α into a complete topological vector space.

A function $\varphi \in H(\mathbf{D})$ that satisfies $\varphi(\mathbf{D}) \subset \mathbf{D}$ induces the composition operator C_φ, defined on $H(\mathbf{D})$ by

$$C_\varphi f = f \circ \varphi.$$

Throughout this paper the symbol φ will always represent a holomorphic self-map of $\mathbf{D}$. In this paper we study the action of composition operators on holomorphic Sobolev spaces. This setting allows a unified treatment of composition operators on Hardy spaces ($H^p = A^p_{-1,0}$), weighted Bergman spaces ($A^p_\alpha = A^p_{\alpha,0}$, $\alpha > -1$), and Dirichlet-type spaces ($A^p_{\alpha,1}$), where extensive research has already been done. The book [CM] is a good introduction to this work. The main results in this paper may be viewed as summarizing well-known boundedness and compactness results for composition operators on these spaces, and then extending them to the Sobolev setting.

It is a well-known consequence of Littlewood's Subordination Principle that every composition operator is bounded on A^p_α for every $p > 0$ and $\alpha \geq -1$; see [MS]. It is natural to ask how this extends to the spaces $A^p_{\alpha,s}$ when $s > 0$. For $p > 0$, $\alpha_j > -1$, $s_j \geq 0$ $(j = 1, 2)$ with $\alpha_1 - \alpha_2 = p(s_1 - s_2)$, we have the following equivalence (see Theorem 5.12 in [BB]):

$$A^p_{\alpha_1,s_1} \approx A^p_{\alpha_2,s_2}. \tag{1.1}$$

That is, these spaces are isomorphic and have equivalent norms. In particular, when $s < \frac{\alpha+1}{p}$ we have $A^p_{\alpha,s} \approx A^p_{\alpha-sp}$. Thus it follows that every composition operator is bounded on $A^p_{\alpha,s}$ when $s < \frac{\alpha+1}{p}$. The general situation is described in the following theorem. Just the statement of this and our other main results are given in this section. The proofs will come later.

Theorem 1.1. *Let $p > 0$, $s \geq 0$ and $\alpha \geq -1$.*

(a) *If $s < \frac{\alpha+1}{p}$, then every composition operator is bounded on $A^p_{\alpha,s}$.*
(b) *If $s = \frac{\alpha+1}{p}$ and*
  (i) *$p \geq 2$ or $\alpha = -1$, then every composition operator is bounded on $A^p_{\alpha,s}$.*
  (ii) *$p < 2$ and $\alpha > -1$, then some composition operators are not bounded on $A^p_{\alpha,s}$.*
(c) *If $s > \frac{\alpha+1}{p}$, then some composition operators are not bounded on $A^p_{\alpha,s}$.*

The case $\alpha = -1$ in part (b) corresponds to $s = 0$, and as previously mentioned every composition operator is bounded on $H^p = A^p_{-1,0}$. The case $\alpha = -1$ in part (c) shows that this does not extend to H^p_s for a range of positive s, as was the case for the Bergman-Sobolev spaces.

The bounds on s in Theorem 1.1 can be extended when the inducing map of the composition operator is univalent or, more generally, of bounded valence.

Theorem 1.2. *Let $p > 0$, $s \geq 0$ and $\alpha > -1$. Assume that φ is of bounded valence.*

(a) *If $p \geq 2$ and $s \leq \frac{\alpha+2}{p}$, then C_φ is bounded on $A^p_{\alpha,s}$.*
(b) *If $p < 2$ and $s < \frac{\alpha+1}{p} + \frac{1}{2}$, then C_φ is bounded on $A^p_{\alpha,s}$.*

The upper bound $s \leq \frac{\alpha+2}{p}$ in part (a), for $p \geq 2$, is sharp; an example will be given in §6. We do not know whether the upper bound $s < \frac{\alpha+1}{p} + \frac{1}{2}$ in part (b) is sharp, but another example will be given that shows that the upper bound cannot be extended to the bound $\frac{\alpha+2}{p}$ from part (a).

The equivalence (1.1) does not extend to the limiting case $\alpha_2 = -1$. However, for $\alpha_1 \geq -1$ with $\alpha_1 + 1 = p(s_1 - s_2)$, we have the following Littlewood-Paley-type inclusion relations:

$$p \leq 2 \implies A^p_{\alpha_1,s_1} \subset A^p_{-1,s_2}, \tag{1.2}$$

$$p \geq 2 \implies A^p_{-1,s_2} \subset A^p_{\alpha_1,s_1}. \tag{1.3}$$

Inclusion relations for different values of p are also known. For $0 < p_1 < p_2$, $\alpha_j \geq -1$, $s_j \geq 0$ $(j = 1, 2)$ with $\frac{\alpha_1+2}{p_1} - \frac{\alpha_2+2}{p_2} = s_1 - s_2$, we have

$$A^{p_1}_{\alpha_1,s_1} \subset A^{p_2}_{\alpha_2,s_2}. \tag{1.4}$$

Let $p > 0$, $\alpha \geq -1$, and $s \geq 0$. Note that we have $A^p_{\alpha,s} \subset A^{p(\alpha+2)/(\alpha+2-ps)}_\alpha$ for $ps < \alpha + 2$, as a special case of the above inclusion ($s_2 = 0$, $\alpha_1 = \alpha_2$). In case $ps \geq \alpha + 2$, inclusion relations with other types of function spaces are known as follows:

$$0 < ps - (\alpha + 2) < p \implies A^p_{\alpha,s} \subset \Lambda_{s-(\alpha+2)/p}, \tag{1.5}$$

$$ps = \alpha + 2 \implies A^p_{\alpha,s} \subset \mathrm{VMOA}. \tag{1.6}$$

Here, Λ_ε denotes the holomorphic Lipschitz space of order ε, $0 < \varepsilon < 1$, and VMOA denotes the space of holomorphic functions of vanishing mean oscillation. The definitions and more information on these spaces can be found in [CM] for Λ_ε and [G] for VMOA. For details of all the inclusions mentioned above, see Theorem 5.12, Theorem 5.13, and Theorem 5.14 in [BB].

The boundedness (compactness) of a composition operator on a smaller space often implies the boundedness (compactness) of the operator on larger spaces. This general philosophy and the inclusion relations mentioned above lead to natural conjectures. The methods developed below in §2 to address these conjectures require some restriction on the parameters. In particular, the case (1.2) is left open since our methods do not apply when the target space is a Hardy-Sobolev space.

Theorem 1.3. *Let $X \subset Y$ be any of the inclusion relations in* (1.3) – (1.5), *and assume for inclusion* (1.3) *that $s_2 < 1$, and for inclusion* (1.4) *that $\alpha_2 > -1$ and $s_2 < 1 + (1 + \alpha_2)/p_2$.*

(a) *If $C_\varphi : X \to X$ is bounded, then $C_\varphi : Y \to Y$ is bounded.*
(b) *If $C_\varphi : X \to X$ is compact, then $C_\varphi : Y \to Y$ is compact.*

Inclusion (1.6) was left out of the preceding theorem, but we have the following partial result in that case.

Theorem 1.4. *Let $p > 0$, $\alpha \geq -1$, $s \geq 0$ and assume $ps = \alpha + 2$.*

(a) *If $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is bounded, then $C_\varphi : \mathrm{VMOA} \to \mathrm{VMOA}$ is bounded.*

(b) *If φ is univalent and $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is compact, then $C_\varphi : \mathrm{VMOA} \to \mathrm{VMOA}$ is compact.*

We also mention the elementary inclusion relations that, for all $p > 0$, $s \geq 0$, $\alpha \geq -1$, and $\varepsilon \geq 0$,

$$A^p_{\alpha,s+\varepsilon} \subset A^p_{\alpha,s} \subset A^p_{\alpha+\varepsilon,s}.$$

In §3 we will give a result analogous to Theorem 1.3 for these inclusions, but with some restrictions on the parameters.

As a first application of Theorem 1.3, notice that it can be used to prove the case $\alpha > -1$, $p \geq 2$, and $s = \frac{\alpha+1}{p}$ in Theorem 1.1. Then $H^p \subset A^p_{\alpha,s}$ by (1.3), and so every composition operator is bounded on $A^p_{\alpha,s}$ by Theorem 1.3. In the other direction, once criteria for C_φ to be bounded or compact on the larger spaces are known, Theorem 1.3 can be used to provide necessary conditions for boundedness or compactness of C_φ on the smaller spaces. For example, by taking Λ_ε as the larger space, we have the following consequence, which has been known for $p \geq 2$ (Theorem 4.13 in [CM]), while it has been known to be false for $p = 1$ (p. 193 in [CM]). So, the gap $1 < p < 2$ is now filled in. A more general version is proved as Theorem 3.3 below.

Theorem 1.5. *Let $p > 1$ and suppose $C_\varphi : H^p_1 \to H^p_1$ is bounded. Then the angular derivative of φ exists at all points $\zeta \in \partial\mathbf{D}$ where the radial limit $\varphi(\zeta)$ of φ exists and satisfies $|\varphi(\zeta)| = 1$.*

A basic problem in the study of composition operators is to relate function-theoretic properties of φ to operator-theoretic properties of the restriction of C_φ to various spaces, as in Theorem 1.5. When $ps < \alpha + 1$, we have $A^p_{\alpha,s} \approx A^p_{\alpha-sp}$ by (1.1), and criteria for $C_\varphi : A^{p_1}_{\alpha_1} \to A^{p_2}_{\alpha_2}$ to be bounded or compact are known. The characterization is that a generalized Nevanlinna counting function for φ satisfies a growth condition if $p_2 \geq p_1$, or an integrability condition if $p_2 < p_1$; see [Sm1] and [SY]. The results in [Sm1] and [SY] do not apply when $ps \geq \alpha + 1$ in either the domain or the target space. In that case, criteria in the form of Carleson measure conditions for a measure defined using a modified counting function can be obtained as in Theorem 2.6 below, with some restrictions on the parameters α_j, p_j, and s_j. This Carleson-type criteria in Theorem 2.6 will be used to prove Theorems 1.2 and 1.3. We also mention that for the special case $p = 2$ other techniques are available, since the norm of a function in $A^2_{\alpha,s}$ may be given in terms of its power series coefficients. These spaces are examples of what are called weighted Hardy spaces in [CM], which is a good reference for composition operators acting on these spaces.

Characterizing when a composition operator is bounded on H^p_s, $s > 0$, seems much harder. The difficulty is that (1.1) does not provide an isomorphism with a space of functions defined with full derivatives, and the methods used to prove Theorem 2.6 do not apply. We have from Theorem 1.1 that, for any $p > 0$ and $s > 0$, there exists a function φ such that C_φ is not bounded on H^p_s. A positive result is that C_φ is compact on certain H^p_s whenever φ is of bounded valence and $\varphi(\mathbf{D})$ is contained in a polygonal region contained in $\mathbf{D}$. This is the special case $p_1 = p_2$ of the following result. For a polygon P inscribed in the unit circle, let $\theta(P)$ denote $1/\pi$ times the measure of the largest vertex angle of P.

Theorem 1.6. *Let $p_2 \geq p_1 > 0$ and assume $0 \leq s < \min\{\frac{1}{p_2}, \frac{1}{2}\}$. Let φ be a holomorphic function of bounded valence taking $\mathbf{D}$ into a polygon P inscribed in the unit circle. If $\theta(P) < \frac{p_1(1-sp_2)}{p_2(1-sp_1)}$, then $C_\varphi : H_s^{p_1} \to H_s^{p_2}$ is compact.*

When $s = 0$ and $p_1 = p_2$, this has long been known; see [ST]. When $s = 0$ and $p_2 \geq p_1$, this result is basically contained in [Sm1]. These results (when $s = 0$) do not require the hypothesis of bounded valence. We will prove a more general result in Theorem 5.5.

In the next section we develop the change of variable methods that we use to study composition operators, which we then use to give Carleson measure-type criteria for these operators to be bounded or compact. These criteria are then used in §3 to prove Theorem 1.3. Next, in §4, the proofs of Theorem 1.1 and Theorem 1.2 are given. Simple geometric criteria are then developed in §5 for boundedness and compactness of a composition operator between holomorphic Sobolev spaces when the inducing map is polygonal. The paper concludes, in §6, with several examples which demonstrate that our theorems are sharp.

2. Background: Carleson-Type Criteria

Our approach to studying composition operators on the spaces $A^p_{\alpha,s}$ involves a change of variable from z to $w = \varphi(z)$. The equivalence (1.1) allows us to assume that s is an integer, and then standard non-univalent change of variable methods can be applied. This gets quite complicated when s is an integer greater than 1. Thus, for simplicity and clarity of presentation, we confine our attention to the case $s = 1$. This enables us to cover parameters p, α and s with $\alpha + (1-s)p > -1$ by using the equivalence $A^p_{\alpha,s} \approx A^p_{\alpha+(1-s)p,1}$ from (1.1). The change of variable method for $s = 1$ is summarized as follows.

For a holomorphic map $\varphi : \mathbf{D} \to \mathbf{D}$ and $w \in \mathbf{D}$, define the modified counting function $N_{p,\alpha}(\varphi, w)$ corresponding to the measure $(1-|z|^2)^\alpha dA(z)$ by

$$N_{p,\alpha}(\varphi, w) = \sum |\varphi'(z)|^{p-2}(1-|z|^2)^\alpha$$

where the sum is over the set $\{z : \varphi(z) = w\}$. As usual, the zeros of $\varphi - w$ are repeated according to their multiplicity. The change of variable formula we need uses the measure

$$d\mu^\varphi_{p,\alpha}(w) = N_{p,\alpha}(\varphi, w)dA(w).$$

Then, by the area formula (see Theorem 2.32 in [CM]), we have the following change of variable formula.

Proposition 2.1. *Let $p > 0$ and $\alpha > -1$. Then, we have*

$$\int_{\mathbf{D}} |(f \circ \varphi)'(z)|^p (1-|z|^2)^\alpha dA(z) = \int_{\varphi(\mathbf{D})} |f'(w)|^p d\mu^\varphi_{p,\alpha}(w)$$

for functions $f \in H(\mathbf{D})$.

Note that Proposition 2.1 cannot be directly applied to the case $s = 1$, because $\mathcal{R}f(z) = f(z) + zf'(z)$ by our definition. This difficulty is overcome by the following proposition. We will often write $X \lesssim Y$ if $X \leq CY$ for some positive constant C dependent only on allowed parameters, and $X \approx Y$ if $X \lesssim Y \lesssim X$.

Proposition 2.2. *Let $p > 0$, $\alpha \geq -1$ and $a \in \mathbf{D}$. Then, for every positive integer n, we have*

$$||f||_{A^p_{\alpha,n}} \approx \sum_{k=0}^{n-1} |f^{(k)}(a)| + ||f^{(n)}||_{A^p_\alpha}$$

for $f \in H(\mathbf{D})$.

Proof. We prove the proposition for $a = 0$. The proof for general a is similar. The equivalence $||f||_{A^p_{\alpha,n}} \approx \sum_{k=0}^{n} ||f^{(k)}||_{A^p_\alpha}$ is proved in Theorem 5.3 of [BB]. Thus, $||f||_{A^p_{\alpha,n}} \gtrsim \sum_{k=0}^{n-1} |f^{(k)}(0)| + ||f^{(n)}||_{A^p_\alpha}$ is clear by subharmonicity.

Now, we prove the other direction of the inequalities. Since $H(\overline{\mathbf{D}})$ is dense in all holomorphic Sobolev spaces by Lemma 5.2 of [BB], it is sufficient to show that

$$||f||_{A^p_\alpha} \lesssim |f(0)| + ||f'||_{A^p_\alpha}, \qquad f \in H(\overline{\mathbf{D}}). \tag{2.1}$$

First, assume either $\alpha > -1$ or $0 < p \leq 1$. Let $f \in H(\overline{\mathbf{D}})$. For each $\beta > -1$, we have by Theorem 1.9 of [BB],

$$f(z) = \frac{1}{\pi} \int_{\mathbf{D}} \mathcal{R}f(w) G_\beta(z\bar{w})(1-|w|^2)^\beta dA(w)$$

where

$$G_\beta(z) = \frac{1}{z}\left\{\frac{1}{(1-z)^{1+\beta}} - 1\right\}.$$

Therefore, choosing $\beta > -1$ sufficiently large, we have by Lemma 4.1 of [BB] ($\alpha > -1$ or $0 < p \leq 1$ is used here),

$$||f||_{A^p_\alpha} \lesssim ||\mathcal{R}f||_{A^p_{\alpha+p}} \approx ||f||_{A^p_{\alpha+p}} + ||f'||_{A^p_{\alpha+p}}. \tag{2.2}$$

It is easy to see that, given $\varepsilon > 0$, there exist a constant $C > 0$ and a compact subset $K = \{z \in \mathbf{D} : |z| \leq r < 1\}$ of $\mathbf{D}$ such that

$$||f||_{A^p_{\alpha+p}} \leq \varepsilon ||f||_{A^p_\alpha} + C \sup_{z\in K} |f(z)|.$$

Taking $\varepsilon > 0$ sufficiently small, we have by (2.2),

$$\begin{aligned} ||f||_{A^p_\alpha} &\lesssim ||f'||_{A^p_{\alpha+p}} + \sup_{z\in K} |f(z)| \\ &\lesssim |f(0)| + ||f'||_{A^p_\alpha} + \sup_{z\in K} |f(z) - f(0)| \\ &\lesssim |f(0)| + ||f'||_{A^p_\alpha} + \sup_{z\in K} |f'(z)|. \end{aligned}$$

Since $\sup_{z\in K} |f'(z)| \lesssim ||f'||_{A^p_\alpha}$ by the subharmonicity of $|f'|^p$, we obtain (2.1) as desired.

Now, consider the case $\alpha = -1$ and $p > 1$. Note that

$$|f(e^{i\theta}) - f(0)| \leq \int_0^1 |f'(te^{i\theta})| dt.$$

Therefore, by Minkowski's inequality, we have

$$||f - f(0)||_{H^p} \leq \int_0^1 \left\{\frac{1}{2\pi}\int_0^{2\pi} |f'(te^{i\theta})|^p d\theta\right\}^{1/p} dt \leq ||f'||_{H^p},$$

which implies (2.1). The proof is complete. □

Having seen Proposition 2.1 and Proposition 2.2, it is now clear that the behavior of C_φ, when the target space is $A^p_{\alpha,1}$, depends on that of the measure $\mu^\varphi_{p,\alpha}$. For boundedness and compactness of C_φ, the criteria for $\mu^\varphi_{p,\alpha}$ turn out to be Carleson-type conditions in certain cases. To prove it, we need a couple of lemmas.

Lemma 2.3. *A bounded subset of any of the spaces $A^p_{\alpha,s}$, where $p > 0$, $s \geq 0$, and $\alpha \geq -1$, is a normal family.*

Proof. First assume $\alpha > -1$. Using (1.1), a bounded set X in $A^p_{\alpha,s}$ is also bounded in some $A^p_{\beta,n}$, where n is a nonnegative integer. Recall that there is a constant C such that

$$|g(w)| \leq C\|g\|_{A^p_\beta}(1-|w|)^{-(\beta+2)/p}$$

for all $g \in A^p_\beta$ (see, for example, Theorem 7.2.5 in [R1]). By Proposition 2.2, this shows that the functions in X are uniformly bounded on compact subsets of $\mathbf{D}$. Hence X is a normal family. The proof of the result for $\alpha = -1$ is similar, since $A^p_{-1,s} \subset A^{2p}_{0,s}$ by (1.4). The proof is complete. □

In the next lemma we will need the estimate that if $\alpha > -1$ and $\beta > 0$, then

$$\int_{\mathbf{D}} \frac{(1-|z|^2)^\alpha}{|1-\overline{a}z|^{2+\alpha+\beta}} dA(z) \approx \frac{1}{(1-|a|^2)^\beta} \quad (|a| \to 1-). \tag{2.3}$$

A reference is Theorem 1.7 of [HKZ].

Lemma 2.4. *Let $p > 0$, $\alpha \geq -1$ and $s \geq 0$. Let $N > \frac{\alpha+2}{p} - s$. Put $g_a(z) = (1-z\bar{a})^{-N}$ for $a, z \in \mathbf{D}$. Then, we have*

$$||g_a||_{A^p_{\alpha,s}} \approx (1-|a|)^{-N-s+\frac{\alpha+2}{p}}, \qquad a \in \mathbf{D}$$

where the constants in this estimate depend on N, s, α, and p, but are independent of a.

Proof. First, assume $\alpha > -1$. Let k be the smallest integer satisfying $k \geq s$. Then, we have $A^p_{\alpha,s} \approx A^p_{\alpha+(k-s)p,k}$ by (1.1). Thus, by Proposition 2.2, we have

$$||g_a||_{A^p_{\alpha,s}} \approx 1 + \sum_{j=1}^{k-1} c_{N,j}|a|^j + c_{N,k}|a|^k \left\{ \int_{\mathbf{D}} \frac{(1-|z|^2)^{\alpha+(k-s)p}}{|1-z\bar{a}|^{p(N+k)}} dA(z) \right\}^{1/p}$$

where $c_{N,j} = N(N+1)\dots(N+j-1)$. Thus, by (2.3), we have

$$||g_a||_{A^p_{\alpha,s}} \approx 1 + \sum_{j=1}^{k-1} c_{N,j}|a|^j + c_{N,k}C_1|a|^k(1-|a|)^{-N-s+\frac{\alpha+2}{p}},$$

where $C_1 = C_1(N,s,\alpha,p)$. The desired estimate follows. Next, assume $\alpha = -1$. Note that $A^{p/2}_{0,3/p+s} \subset A^p_{-1,s} \subset A^{2p}_{0,s}$ by (1.4) and thus

$$||g_a||_{A^{2p}_{0,s}} \lesssim ||g_a||_{A^p_{-1,s}} \lesssim ||g_a||_{A^{p/2}_{0,3/p+s}}.$$

On the other hand, we have

$$||g_a||_{A^{2p}_{0,s}} \approx (1-|a|)^{-N-s+\frac{1}{p}} \approx ||g_a||_{A^{p/2}_{0,3/p+s}}$$

by what we have just proved for the case $\alpha > -1$. This completes the proof. □

For any arc $I \subset \partial\mathbf{D}$ define the Carleson square over I to be

$$SI = \{re^{i\theta} : 1 - |I| \le r < 1,\ e^{i\theta} \in I\},$$

where $|I|$ is $1/(2\pi)$ times the Euclidean length of I. Also, let ∂ denote the complex differential operator, i.e., $\partial f = f'$ for $f \in H(\mathbf{D})$.

The next lemma asserts that certain operators are compact. We review the definition, since when $p < 1$ the spaces involved are not Banach spaces. Suppose X and Y are complete topological vectors spaces whose topologies are induced by metrics. A continuous linear operator $T : X \to Y$ is said to be compact if the image of every bounded set in X is relatively compact in Y. Due to the metric topology of Y, T will be compact if and only if the image of every bounded sequence in X has a subsequence that converges in Y. Also, linearity of T allows us to only consider sequences in the unit ball of X.

In the following lemma, part (a) is well known; see Theorems 2.2 and 3.1 in [L1]. Part (b) is certainly known to experts. For example, the case $k = 0$, $p = q$, and $\alpha > -1$ occurs as Theorem 4.3 in [MS]. A proof is included here since we do not know a reference. In our application, we will take $k \le 1$.

Lemma 2.5. *Assume that one of the following three conditions holds:*

$$(i)\ \alpha > -1, 0 < p \le q; \quad (ii)\ \alpha = -1, p = q \ge 2; \quad (iii)\ \alpha = -1, 0 < p < q.$$

Let k be a nonnegative integer and μ be a positive finite Borel measure on $\mathbf{D}$.

(a) *$\partial^k : A^p_\alpha \to L^q(d\mu)$ is bounded if and only if*

$$\mu(SI) = O\left(|I|^{kq+q(\alpha+2)/p}\right), \qquad I \subset \partial\mathbf{D}.$$

(b) *$\partial^k : A^p_\alpha \to L^q(d\mu)$ is compact if and only if*

$$\mu(SI) = o\left(|I|^{kq+q(\alpha+2)/p}\right), \qquad |I| \to 0. \tag{2.4}$$

Moreover, the norm of the map in (a) *satisfies the inequality $\|\partial^k\|^q \le C\|\mu\|$, where $\|\mu\|$ is the supremum of the quantity $\mu(SI)/|I|^{kq+q(\alpha+2)/p}$ over $I \subset \partial\mathbf{D}$.*

Proof. We provide a proof of (b). We first prove the sufficiency. So, assume (2.4) and let $\{f_n\}$ be a bounded sequence in A^p_α, say of norm at most $1/2$. We must show that $\{f_n\}$ contains a subsequence whose k-th derivatives converge in $L^q(d\mu)$. Recall that we have observed that a bounded set in A^p_α is a normal family, and so by subtracting the limit function and re-indexing an appropriate subsequence, we may assume that $\|f_n\|_{A^p_\alpha} \le 1$ and that $\{f_n\}$ and hence $\{f_n^{(k)}\}$ converges to 0 uniformly on compact subsets of $\mathbf{D}$. We need to show that $\{f_n^{(k)}\}$ converges to 0 in $L^q(d\mu)$. Let $\varepsilon > 0$ and write

$$\|f_n^{(k)}\|^q_{L^q(d\mu)} = \int_{r\mathbf{D}} |f_n^{(k)}|^q d\mu + \int_{\mathbf{D}\setminus r\mathbf{D}} |f_n^{(k)}|^q d\mu, \qquad 0 < r < 1.$$

The first term is easily handled. For any fixed $r \in (0,1)$, the uniform convergence of $\{f_n^{(k)}\}$ to 0 on $r\mathbf{D}$ allows us to find $N(r)$ such that

$$\int_{r\mathbf{D}} |f_n^{(k)}|^q d\mu < \varepsilon, \quad n \ge N(r).$$

Turning to the second term, by hypothesis we can choose $r = r_\varepsilon \in (0,1)$ so that the measure $d\nu(w) = \chi_{\mathbf{D}\setminus r\mathbf{D}}(w)d\mu(w)$ satisfies $\nu(SI) \le \varepsilon|I|^\beta$, whenever $|I| \le 1 - r$,

where $\beta = kq + q(2+\alpha)/p$. For $|I| > 1-r$, we subdivide I into m arcs of length at most $1-r$, where $m \le |I|/(1-r)+1 \le 2|I|/(1-r)$, and observe that $SI \cap (\mathbf{D} \setminus r\mathbf{D})$ is contained in the Carleson squares associated with the smaller arcs. Thus, the previous estimate shows that

$$\nu(SI) \le \frac{2|I|}{1-r}\varepsilon(1-r)^\beta \le 2\varepsilon|I|^\beta$$

in this case as well. Note that we used $\beta \ge 1$, which is a consequence of the hypotheses, for the last inequality. Thus, we see from (a) that there is a constant C_1 such that

$$\sup_n \int_{\mathbf{D}} |f_n^{(k)}|^q d\nu \le C_1\varepsilon.$$

Combined with the previous estimate, this shows that $\|f_n^{(k)}\|_{L^q(d\mu)} \to 0$ as required.

Now, we prove the necessity. Suppose (2.4) is false. Then there exist a constant $C_2 > 0$ and a sequence of arcs $I_n \subset \partial\mathbf{D}$ such that $|I_n| \to 0$ and

$$\mu(SI_n) \ge C_2|I_n|^{kq+q(\alpha+2)/p}. \tag{2.5}$$

Let $\delta_n = |I_n|$ and $\zeta_n \in \partial\mathbf{D}$ be the center of I_n for each n. Fix a large integer $N > (\alpha+2)/p$. Let $g_n(z) = (1-(1-\delta_n)z\bar{\zeta}_n)^{-N}$ and put $f_n = g_n||g_n||_{A^p_\alpha}^{-1}$. Note that $||g_n||^p_{A^p_\alpha} \approx \delta_n^{-Np+\alpha+2}$ by Lemma 2.4. Thus, $\{f_n\}$ converges uniformly to 0 on compact subsets of $\mathbf{D}$. Now, using the compactness of $\partial^k : A^p_\alpha \to L^q(d\mu)$, pick a subsequence of $\{f_n\}$ whose k-th derivatives converge to 0 in $L^q(d\mu)$ and use the same notation for that subsequence. Note that $|1-(1-\delta_n)z\bar{\zeta}_n| \approx \delta_n$ for $z \in SI_n$ and n large. Thus, by (2.5), we have

$$\begin{aligned} ||f_n^{(k)}||^q_{L^q(d\mu)} &\ge \int_{SI_n} |f_n^{(k)}|^q d\mu \\ &\gtrsim \delta_n^{Nq-q(\alpha+2)/p} \int_{SI_n} |1-(1-\delta_n)z\bar{\zeta}_n|^{-Nq-kq} d\mu(z) \\ &\gtrsim C_2 \end{aligned}$$

for all large n. This is a contradiction, because $||f_n^{(k)}||^q_{L^q(d\mu)} \to 0$. The proof is complete. □

Now, a change of variables and standard arguments give us the following Carleson measure characterizations of boundedness and compactness. As discussed in the first paragraph of this section, we restrict our consideration of the orders of differentiation to certain ranges; analysis of the general case seems too complicated for this paper. We also mention again that when $sp < \alpha+1$ or $p = 2$, other methods are available and much more is known; see the discussion following Theorem 1.5 in the Introduction.

Theorem 2.6. *Assume that one of the following three conditions holds:*
(i) $\alpha_1 > -1, 0 < p_1 \le p_2$;
(ii) $\alpha_1 = -1, p_1 = p_2 \ge 2$;
(iii) $\alpha_1 = -1, 0 < p_1 < p_2$.
Also, assume $\alpha_2 > -1$ *and*

$$0 \le s_1 < 1 + \frac{\alpha_1+2}{p_1} - \frac{1}{p_2}, \qquad 0 \le s_2 < 1 + \frac{\alpha_2+1}{p_2}. \tag{2.6}$$

(a) $C_\varphi : A^{p_1}_{\alpha_1,s_1} \to A^{p_2}_{\alpha_2,s_2}$ *is bounded if and only if*

$$\mu^{\varphi}_{p_2,\alpha_2+(1-s_2)p_2}(SI) = O\left(|I|^{(2+\alpha_1)p_2/p_1+(1-s_1)p_2}\right), \qquad I \subset \partial\mathbf{D}. \tag{2.7}$$

(b) $C_\varphi : A^{p_1}_{\alpha_1,s_1} \to A^{p_2}_{\alpha_2,s_2}$ *is compact if and only if*

$$\mu^{\varphi}_{p_2,\alpha_2+(1-s_2)p_2}(SI) = o\left(|I|^{(2+\alpha_1)p_2/p_1+(1-s_1)p_2}\right), \qquad |I| \to 0. \tag{2.8}$$

Proof. Here, for brevity, we prove the sufficiency for boundedness and the necessity for compactness. The other implications can be seen by easy modifications. Also, let $\mu = \mu^{\varphi}_{p_2,\alpha_2+(1-s_2)p_2}$ for simplicity.

First, we prove the sufficiency for boundedness. One may easily modify the proof for compactness. So, suppose that μ satisfies (2.7).

Note that

$$[p_2 - 2 + (2+\alpha_1)p_2/p_1] - s_1p_2 > -1$$

by the first part of (2.6). Thus, by Lemma 2.5 (a) ($k = 0$), we have

$$\int_{\mathbf{D}} |g(z)|^{p_2} d\mu(z) \lesssim \int_{\mathbf{D}} |g(w)|^{p_2}(1-|w|)^{[p_2-2+(2+\alpha_1)p_2/p_1]-s_1p_2} dA(w) \tag{2.9}$$

for functions g holomorphic on $\mathbf{D}$.

Also, note that $\alpha_2 + (1-s_2)p_2 > -1$ by the second part of (2.6). It follows from (1.1), Proposition 2.2, Proposition 2.1 and (2.9) that

$$\begin{aligned}
\|f\circ\varphi\|_{A^{p_2}_{\alpha_2,s_2}} &\approx \|f\circ\varphi\|_{A^{p_2}_{\alpha_2+(1-s_1)p_2,1}} \\
&\approx |f(\varphi(0))| + ||f'||_{L^{p_2}(\mu)} \\
&\lesssim |f(\varphi(0))| + ||f'||_{A^{p_2}_{[p_2-2+(2+\alpha_1)p_2/p_1]-s_1p_2}}.
\end{aligned}$$

Now, by Proposition 2.2 and (1.1) again, we see that the sum in the last line above is equivalent to

$$\begin{aligned}
||f||_{A^{p_2}_{[p_2-2+(2+\alpha_1)p_2/p_1]-s_1p_2,1}} &\approx ||f||_{A^{p_2}_{p_2-2+(2+\alpha_1)p_2/p_1,1+s_1}} \\
&= ||\mathcal{R}^{s_1}f||_{A^{p_2}_{p_2-2+(2+\alpha_1)p_2/p_1,1}} \\
&\approx |\mathcal{R}^{s_1}f(0)| + \|\partial\mathcal{R}^{s_1}f\|_{A^{p_2}_{p_2-2+(2+\alpha_1)p_2/p_1}}.
\end{aligned}$$

Next, it is clear that $|\mathcal{R}^{s_1}f(0)| \lesssim \|f\|_{A^{p_1}_{\alpha_1,s_1}}$. Also, it is easy to verify using Lemma 2.5 (a) ($k = 1$) that

$$\|\partial\mathcal{R}^{s_1}f\|_{A^{p_2}_{p_2-2+(2+\alpha_1)p_2/p_1}} \lesssim \|\mathcal{R}^{s_1}f\|_{A^{p_1}_{\alpha_1}} = \|f\|_{A^{p_1}_{\alpha_1,s_1}}.$$

Putting these estimates together, we conclude the boundedness of $C_\varphi : A^{p_1}_{\alpha_1,s_1} \to A^{p_2}_{\alpha_2,s_2}$.

Next, we prove the necessity for compactness. So, suppose that $C_\varphi : A^{p_1}_{\alpha_1,s_1} \to A^{p_2}_{\alpha_2,s_2}$ is compact. Suppose that (2.8) does not hold. Then there exist a constant $C > 0$ and a sequence of arcs $I_n \subset \partial\mathbf{D}$ such that $|I_n| \to 0$ and

$$\mu(SI_n) \geq C|I_n|^{(2+\alpha_1)p_2/p_1+(1-s_1)p_2}.$$

Let $\delta_n = |I_n|$ and $\zeta_n \in \partial\mathbf{D}$ be the center of I_n for each n. Fix a large integer $N > (\alpha_1+2)/p - s_1$. Let $g_n(z) = (1-(1-\delta_n)z\bar{\zeta}_n)^{-N}$ and put $f_n = g_n||g_n||^{-1}_{A^{p_1}_{\alpha_1,s_1}}$. Note $||g_n||_{A^{p_1}_{\alpha_1,s_1}} \approx \delta_n^{-N-s_1+(2+\alpha_1)/p_1}$ by Lemma 2.4. Thus, $\{f_n\}$ converges uniformly to 0 on compact subsets of $\mathbf{D}$. Therefore, using the compactness of $C_\varphi : A^{p_1}_{\alpha_1,s_1} \to A^{p_2}_{\alpha_2,s_2}$, we may pick a subsequence of $\{f_n\circ\varphi\}$ that converges to 0 in $A^{p_2}_{\alpha_2,s_2}$ and

use the same notation for that subsequence. Now, first using Proposition 2.1 and then proceeding as in the proof of Lemma 2.5, we have

$$\begin{aligned}||f_n \circ \varphi||_{A^{p_2}_{\alpha_2,s_2}} &\approx ||f_n \circ \varphi||_{A^{p_2}_{\alpha_2+(1-s_2)p_2,1}} \\ &\gtrsim \delta_n^{N+s_1-(2+\alpha_1)/p_1} \left\{ \int_{SI_n} |1-(1-\delta_n)z\bar{\zeta}_n|^{-(N+1)p_2} d\mu(z) \right\}^{1/p_2} \\ &\gtrsim \delta_n^{s_1-1-(2+\alpha_1)/p_1} \mu(SI_n)^{1/p_2} \\ &\geq C\end{aligned}$$

for all large n. This is a contradiction, because $||f_n \circ \varphi||_{A^{p_2}_{\alpha_2,s_2}} \to 0$. The proof is complete. □

3. From Small Spaces to Larger Spaces

We now turn to the proof of Theorem 1.3. For convenience we divide the theorem into more easily managed pieces, considering each implication separately as well as boundedness and compactness.

Theorem 3.1. *Let p_j, s_j and α_j $(j = 1, 2)$ be as in the hypotheses of Theorem 2.6. In addition, assume that $\frac{\alpha_1+2}{p_1} - \frac{\alpha_2+2}{p_2} = s_1 - s_2$.*

(a) *$C_\varphi : A^{p_1}_{\alpha_1,s_1} \to A^{p_2}_{\alpha_2,s_2}$ is bounded (compact, resp.) if and only if C_φ is bounded (compact, resp.) on $A^{p_2}_{\alpha_2,s_2}$.*

(b) *If C_φ is bounded (compact, resp.) on $A^{p_1}_{\alpha_1,s_1}$, then so is C_φ on $A^{p_2}_{\alpha_2,s_2}$.*

Proof. Note that $(2+\alpha_1)p_2/p_1 + (1-s_1)p_2 = \alpha_2 + 2 + (1-s_2)p_2$. Thus, (a) follows from Theorem 2.6. Also, note that $A^{p_1}_{\alpha_1,s_1} \subset A^{p_2}_{\alpha_2,s_2}$ by (1.4). Thus, (b) follows from (a). □

It is straightforward to check that when $\alpha_1 = -1$ and $p_1 = p_2 \geq 2$, the hypotheses (2.6) in Theorem 2.6 are equivalent to $s_2 < 1$ in (1.3). Thus, Theorem 1.3 with inclusion (1.3) is an immediate consequence of Theorem 3.1. Similarly, when $\alpha_1 \geq -1$ and $0 < p_1 < p_2$, the hypotheses (2.6) in Theorem 2.6 are equivalent to $\alpha_2 > -1$ and $s_2 < 1 + (1+\alpha_2)/p_2$ in (1.4), and so Theorem 1.3 with inclusion (1.4) follows.

The proof of the next theorem uses some properties of the pseudo-hyperbolic distance ρ on $\mathbf{D}$. Recall that the pseudo-hyperbolic distance between points a and b in $\mathbf{D}$ is given by

$$\rho(a,b) = \left| \frac{a-b}{1-\overline{a}b} \right|.$$

We use $D(a,r)$ to denote the pseudo-hyperbolic disk of radius r and center a. Recall also the well-known and useful identity

$$1 - \left| \frac{a-b}{1-\overline{a}b} \right|^2 = \frac{(1-|a|^2)(1-|b|^2)}{|1-\overline{a}b|^2}, \qquad a, b \in \mathbf{D}.$$

In particular, it is a consequence of this that

$$|1-\overline{a}b| \approx 1-|a|^2 \approx 1-|b|^2 \tag{3.1}$$

whenever $b \in D(a, 1/2)$.

The next result covers the inclusion (1.5) in Theorem 1.3, and so completes its proof.

Theorem 3.2. *Let $p > 0$, $\alpha \geq -1$ and $\frac{\alpha+2}{p} < s < 1 + \frac{\alpha+2}{p}$. If C_φ is bounded (compact, resp.) on $A^p_{\alpha,s}$, then so is C_φ on $\Lambda_{s-(\alpha+2)/p}$.*

Proof. We first prove the assertion on boundedness with the additional assumptions that $\alpha > -1$, $p > 1$ and $\frac{\alpha+2}{p} < s < 1 + \frac{\alpha+1}{p}$. Note that $\alpha + (1-s)p > -1$ and therefore $A^p_{\alpha,s} \approx A^p_{\alpha+(1-s)p,1}$ by (1.1). Choose $a \in \mathbf{D}$ such that $|\varphi(a)| \geq 1/2$, and consider the test function $f_a(z) = \log(1 - \overline{\varphi(a)}z)$. Then, by Proposition 2.2, we have

$$\begin{aligned} ||f_a \circ \varphi||^p_{A^p_{\alpha+(1-s)p,1}} &\gtrsim \int_{\mathbf{D}} \frac{|\varphi(a)|^p}{|1 - \overline{\varphi(a)}\varphi(z)|^p} |\varphi'(z)|^p (1-|z|^2)^{\alpha+(1-s)p} dA(z) \\ &\geq \int_{D(a,1/2)} \frac{|\varphi(a)|^p}{|1 - \overline{\varphi(a)}\varphi(z)|^p} |\varphi'(z)|^p (1-|z|^2)^{\alpha+(1-s)p} dA(z). \end{aligned}$$

For $z \in D(a, 1/2)$, we have $1 - |z|^2 \approx 1 - |a|^2$, by (3.1). Also, the Schwarz-Pick Lemma tells us that $\varphi(z) \in D(\varphi(a), 1/2)$, and so $|1 - \overline{\varphi(a)}\varphi(z)| \approx 1 - |\varphi(a)|^2$ from (3.1). Using these estimates in the last term in the display above shows that

$$\begin{aligned} ||f_a \circ \varphi||^p_{A^p_{\alpha+(1-s)p,1}} &\gtrsim \int_{D(a,1/2)} \frac{|\varphi'(z)|^p}{(1-|\varphi(a)|^2)^p} (1-|a|^2)^{\alpha+(1-s)p} dA(z) \\ &\gtrsim \frac{|\varphi'(a)|^p (1-|a|^2)^{\alpha+2+(1-s)p}}{(1-|\varphi(a)|^2)^p}. \end{aligned} \tag{3.2}$$

For the last inequality we used that $D(a, 1/2)$ contains a Euclidean disk with center a and radius comparable to $1 - |a|^2$, and that $|\varphi'|^p$ is subharmonic.

Meanwhile, since $f(0) = 0$, we have

$$\begin{aligned} ||f_a||^p_{A^p_{\alpha+(1-s)p,1}} &\approx \int_{\mathbf{D}} \frac{|\varphi(a)|^p}{|1 - \overline{\varphi(a)}z|^p} (1-|z|^2)^{\alpha+(1-s)p} dA(z) \\ &\approx (1-|\varphi(a)|^2)^{\alpha+2-sp}, \end{aligned} \tag{3.3}$$

where the last equivalence holds by (2.3), because $sp > \alpha + 2$. Putting these estimates together with the assumption that $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is bounded, we get

$$\sup_{a \in \mathbf{D}} |\varphi'(a)| \left\{ \frac{(1-|\varphi(a)|^2)}{(1-|a|^2)} \right\}^{-1+s-(\alpha+2)/p} < \infty.$$

This is equivalent to the boundedness of C_φ on $\Lambda_{s-(\alpha+2)/p}$; see [Ma] or Theorem 4.9 in [CM].

Now, consider the general case $\alpha \geq -1$ and $\frac{\alpha+2}{p} < s < 1 + \frac{\alpha+2}{p}$. Choose $q > p$ so large that $q > 1$ and $s < 1 + \frac{\alpha+2}{p} - \frac{1}{q}$. Put $\beta = \frac{(\alpha+2)q}{p} - 2$. Then, $\beta > \alpha \geq -1$ and $\frac{\beta+2}{q} = \frac{\alpha+2}{p}$. Now, by (1.4) and (1.5), we have

$$A^p_{\alpha,s} \subset A^q_{\beta,s} \subset \Lambda_{s-(\beta+2)/q} = \Lambda_{s-(\alpha+2)/p}.$$

Also, note that $\frac{\beta+2}{q} < s < 1 + \frac{\beta+1}{q}$. Now, suppose that $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is bounded. Then $C_\varphi : A^q_{\beta,s} \to A^q_{\beta,s}$ is bounded by Theorem 3.1 and thus so is $C_\varphi : \Lambda_{s-(\alpha+2)/p} \to \Lambda_{s-(\alpha+2)/p}$ by the result for the special case we proved first. This proves the assertion on boundedness.

We now prove the assertion on compactness. Note that $\Lambda_{s-(\alpha+2)/p}$ and $A^p_{\alpha,s}$ are Möbius invariant, in the sense that every composition operator induced by a conformal automorphism of the unit disk maps each space into itself, and contained

in the disk algebra of holomorphic functions on the unit disk that extend to be continuous on the closed disk. Thus a general theorem of J. H. Shapiro [Sh] asserts that compactness of C_φ on each of these spaces implies that $\varphi(\mathbf{D})$ is a relatively compact subset of $\mathbf{D}$. We recall also that

$$|f(0)| + \sup\{(1-|z|^2)^{1-\beta}|f'(z)| : z \in \mathbf{D}\}$$

is an equivalent norm on Λ_β; see Theorem 4.1 in [CM]. Now, let $\{f_n\}$ be a bounded sequence in $\Lambda_{s-(\alpha+2)/p}$. We must show that some subsequence of $\{f_n \circ \varphi\}$ converges in $\Lambda_{s-(\alpha+2)/p}$. We know that $\{f_n\}$ is a normal family, and thus a subsequence (which we still call $\{f_n\}$) converges to some $f \in H(\mathbf{D})$ uniformly on compact subsets of $\mathbf{D}$. Also, if C_φ is compact on $A^p_{\alpha,s}$, then it is bounded and so $\varphi = C_\varphi z \in A^p_{\alpha,s} \subset \Lambda_{s-(\alpha+2)/p}$. Hence $(1-|z|^2)^{1-s+(\alpha+2)/p}|\varphi'(z)|$ is uniformly bounded on $\mathbf{D}$, and it follows that

$$|f_n \circ \varphi(0) - f \circ \varphi(0)| + (1-|z|^2)^{1-s+(\alpha+2)/p}|f_n' \circ \varphi(z) - f' \circ \varphi(z)||\varphi'(z)| \to 0$$

uniformly on $\mathbf{D}$ as $n \to \infty$, since $\varphi(\mathbf{D})$ is contained in a compact subset of $\mathbf{D}$. This means that $\{f_n \circ \varphi\}$ converges to the function $g = f \circ \varphi$ in $\Lambda_{s-(\alpha+2)/p}$, and so $C_\varphi : \Lambda_{s-(\alpha+2)/p} \to \Lambda_{s-(\alpha+2)/p}$ is compact. The proof is complete. □

Criteria for C_φ to be bounded or compact on Λ_ε are known. So Theorem 3.2 can be used to provide necessary conditions for boundedness or compactness of C_φ on the smaller spaces. In particular, we recall that the boundedness on Λ_ε implies the existence of the angular derivative of φ at all points of the unit circle where φ has a radial limit of modulus 1; see Corollary 4.10 in [CM]. This proves the following theorem.

Theorem 3.3. *Let $p > 0$, $\alpha \geq -1$ and $\frac{\alpha+2}{p} < s < 1 + \frac{\alpha+2}{p}$. If $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is bounded, then the angular derivative of φ exists at all points $\zeta \in \partial\mathbf{D}$ where the radial limit $\varphi(\zeta)$ of φ exists and satisfies $|\varphi(\zeta)| = 1$.*

As mentioned in the introduction, the conclusion of Theorem 3.3 is false for $\alpha = -1, s = 1$ and $p = 1$. Thus, for $\alpha = -1$, the lower bound $1/p$ for s cannot be decreased in general. We also give an example which shows that the lower bound $s > \frac{\alpha+2}{p}$ in Theorem 3.3 is sharp in case $\alpha > -1$. See Example 6.3 below.

The proof of the next theorem is based on Theorem 2.6. So, for simplicity, we restrict our consideration to the orders of differentiation covered there.

Theorem 3.4. *Let $p > 0$, $\alpha > -1$, $s \geq 0$ and assume $s < 1 + \frac{\alpha+1}{p}$.*

(a) *If $1 + \frac{\alpha+1}{p} - s > \varepsilon \geq 0$ and C_φ is bounded (compact, resp.) on $A^p_{\alpha,s+\varepsilon}$, then so is C_φ on $A^p_{\alpha,s}$.*

(b) *If $\varepsilon \geq 0$ and C_φ is bounded (compact, resp.) on $A^p_{\alpha,s}$, then so is C_φ on $A^p_{\alpha+\varepsilon,s}$.*

Proof. Let I be an arc in the unit circle, and let $\varphi(z) = w \in SI$. A standard argument using the Schwarz Lemma then tells us that $1-|z| \lesssim 1-|w| \leq |I|$, and so

$$\begin{aligned} N_{p,\alpha+\varepsilon+(1-s)p}(\varphi, w) &= \sum |\varphi'(z)|^{p-2}(1-|z|^2)^{\alpha+\varepsilon+(1-s)p} \\ &\lesssim |I|^\varepsilon N_{p,\alpha+(1-s)p}(\varphi, w), \end{aligned}$$

$w \in SI$. Hence

$$\mu^\varphi_{p,\alpha+\varepsilon+(1-s)p}(SI) \lesssim |I|^\varepsilon \mu^\varphi_{p,\alpha+(1-s)p}(SI),$$

and statement (b) is now an immediate consequence of Theorem 2.6. The proof of (a) is similar and will be omitted. □

We finish this section by giving the proof of Theorem 1.4 from the introduction, which we restate for convenience.

Theorem 3.5. *Let $p > 0$, $\alpha \geq -1$, $s \geq 0$ and assume $ps = \alpha + 2$.*

(a) *If $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is bounded, then $C_\varphi : \mathrm{VMOA} \to \mathrm{VMOA}$ is bounded.*
(b) *If φ is univalent and $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is compact, then $C_\varphi : \mathrm{VMOA} \to \mathrm{VMOA}$ is compact.*

In the proof below and elsewhere, we use the notation $\mathrm{dist}(a, \partial E)$ for the Euclidean distance between a point a and the boundary of a set E.

Proof. If C_φ is bounded on $A^p_{\alpha,s}$, then from (1.6) we have that $\varphi = C_\varphi z \in A^p_{\alpha,s} \subset \mathrm{VMOA}$. Also, it is easy to see that C_φ is bounded on VMOA if and only if $\varphi \in \mathrm{VMOA}$; see, for example, [Sm2]. This gives part (a).

For the proof of (b), we recall that when φ is univalent, C_φ is compact on VMOA if and only if

$$\lim_{|w|\to 1^-} \frac{\mathrm{dist}(w, \partial\varphi(\mathbf{D}))\chi_{\varphi(\mathbf{D})}(w)}{(1-|w|)} = 0; \tag{3.4}$$

see Theorem 4.1 in [Sm2]. Also, it is an easy consequence of the Koebe distortion theorem that if φ is univalent, then

$$(1-|z|^2)|\varphi'(z)| \approx \mathrm{dist}(\varphi(z), \partial\varphi(\mathbf{D})), \qquad z \in \mathbf{D}; \tag{3.5}$$

see Corollary 1.4 in [P].

First, consider the case $p > 1$ and $\alpha > -1$. With $ps = \alpha + 2 < p + \alpha + 1$, case (i) of Theorem 2.6 (b) tells us that C_φ is compact on $A^p_{\alpha,s}$ if and only if $\mu^\varphi_{p,p-2}(SI) = o(|I|^p)$ as $|I| \to 0$. We prove part (b) by showing that this fails when C_φ is not compact on VMOA. From (3.4), if C_φ is not compact on VMOA, then there is an $\varepsilon > 0$ and a sequence $\{w_n\} \subset \varphi(\mathbf{D})$ with $|w_n| \to 1$ and $\mathrm{dist}(w_n, \partial\varphi(\mathbf{D})) \geq \varepsilon(1-|w_n|)$. Let I_n be the arc of the unit circle with center $w_n/|w_n|$ and length $|I_n| = 2(1-|w_n|)$. Since φ is univalent,

$$\mu^\varphi_{p,p-2}(SI_n) = \int_{SI_n} \{|\varphi'(z)|(1-|z|^2)\}^{p-2} dA(w),$$

where $w = \varphi(z)$. From (3.5), $|\varphi'(z)|(1-|z|^2) \approx (1-|w_n|)$ for w in the disk with center w_n and radius $\varepsilon(1-|w_n|)/2$, which yields the lower bound

$$\mu^\varphi_{p,p-2}(SI_n) \gtrsim |I_n|^p.$$

Hence $\mu^\varphi_{p,p-2}(SI) \neq o(|I|^p)$, $|I| \to 0$, as desired.

Now, consider the general case $p > 0$, $\alpha \geq -1$ and suppose that $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is compact. With $ps = \alpha + 2$, choose q as in the proof of Theorem 3.2. That is, choose $q > p$ so large that $q > 1$ and put $\beta = sq - 2 > -1$. Then $\frac{\beta+2}{q} = \frac{\alpha+2}{p} = s$, and so $A^p_{\alpha,s} \subset A^q_{\beta,s}$ by (1.4). Thus, from Theorem 3.1 we see that $C_\varphi : A^q_{\beta,s} \to A^q_{\beta,s}$ is compact and thus so is $C_\varphi : \mathrm{VMOA} \to \mathrm{VMOA}$ by the result for the special case that we have proved above. The proof is complete. □

4. Composition Operators on $A^p_{\alpha,s}$

In this section we prove Theorems 1.1 and 1.2 from the introduction. For convenience, we divide these results into more easily managed pieces. As mentioned in the introduction, it is well known that every composition operator is bounded on A^p_β for all $p > 0$ and $\beta > -1$. Note that we have $A^p_{\alpha,s} \approx A^p_{\alpha-sp}$ by (1.1), when $sp < \alpha + 1$. Thus, it follows that every composition operator is bounded on $A^p_{\alpha,s}$ whenever $sp < \alpha + 1$. The next two theorems complete the description of the general situation, as stated in Theorem 1.1.

Theorem 4.1. *Let $p > 0$, $s \geq 0$ and $\alpha \geq -1$. If $s = \frac{\alpha+1}{p}$ and*

(a) *$p \geq 2$ or $\alpha = -1$, then every composition operator is bounded on $A^p_{\alpha,s}$;*
(b) *$p < 2$ and $\alpha > -1$, then some composition operators are not bounded on $A^p_{\alpha,s}$.*

Proof. If $\alpha = -1$, then $s = 0$ and so every composition operator is bounded on $A^p_{\alpha,s} = H^p$. If $\alpha > -1$, $p \geq 2$ and $ps = \alpha + 1$, then from (1.3) we have that $H^p \subset A^p_{\alpha,s}$. Hence part (a) follows from Theorem 1.3, since all composition operators are bounded on H^p.

Turning to the proof of (b), first note that $A^p_{\alpha,s} \approx A^p_{p-1,1}$ by (1.1), since $s = \frac{\alpha+1}{p}$. Also, $\varphi = C_\varphi z \in A^p_{p-1,1}$ is necessary for C_φ to be bounded on $A^p_{p-1,1}$. Thus it suffices to show that if $p < 2$ there is a bounded analytic function $F \notin A^p_{p-1,1}$. The case $p = 1$ of this statement is outlined in exercise 9(a) in Chapter VI of [G]. That construction can be modified to work for $p < 2$. For completeness, we sketch the argument.

Let $p < 2$ and consider the function

$$f(z) = \sum_{k=1}^{\infty} k^{-1/p} z^{2^k}.$$

Since the series for f is lacunary with square summable coefficients, it is known that $f \in$ BMOA. This is an easy consequence of BMOA being the dual of H^1 together with Paley's Inequality for the coefficients of an H^1 function ([D], p. 104), or see [Mi] for another approach to the proof. Next, it is easy to verify that if

$$z \in A_n = \{w \in \mathbf{D} : 1 - 2^{-n} \leq |w| < 1 - 2^{-n-1}\},$$

then $|f'(z)| \approx n^{-1/p} 2^n$. This leads to the approximation

$$\int_{A_n} |f'(z)|^p (1 - |z|^2)^{p-1} dA(z) \approx \frac{1}{n},$$

from which we see that $f \notin A^p_{p-1,1}$. This is not the required example, however, since f is not bounded. But since $f \in$ BMOA, there are bounded functions u_1 and u_2 on the unit circle such that $\mathrm{Re} f = u_1 + \tilde{u}_2$ where $\tilde{u}_2$ denotes the harmonic conjugate of u_2. Here, we are using the same notation for a boundary function and its harmonic extension. Then $|f'|^p \lesssim |\nabla u_1|^p + |\nabla u_2|^p$ by the Cauchy-Riemann equations, and it follows that there is a bounded real function u on the circle such that

$$\int_{\mathbf{D}} |\nabla u(z)|^p (1 - |z|^2)^{p-1} dA(z) = \infty.$$

Now let $F = \exp(u + i\tilde{u})$, so that F is a bounded analytic function satisfying $|F'| \approx |\nabla u|$. Thus $F \notin A^p_{p-1,1}$, and the proof is complete. $\square$

The proof of the next theorem, covering the case $s > \frac{\alpha+1}{p}$, requires two lemmas, which will also be used in the next section.

Lemma 4.2. *Let $p > 0$, $s \geq 0$, and $\alpha \geq -1$. Then the following inclusions hold*:

(a) $A^p_{\alpha,s} \subset A^p_{p+\alpha+\varepsilon,1+s}$ *for* $\varepsilon > 0$;

(b) $A^p_{p+\alpha-\varepsilon,1+s} \subset A^p_{\alpha,s}$ *for* $0 < \varepsilon < p$.

Moreover, both inclusions are bounded.

We remark in passing that, when $s = \varepsilon = 0$, $\alpha = -1$ and $p \geq 2$, the inclusion in (a) holds and this is just a restatement of the well-known Littlewood-Paley inequality. When $s = \varepsilon = 0$, $\alpha = -1$ and $1 < p \leq 2$, the inclusion in (b) holds and this is a restatement of the dual of the Littlewood-Paley inequality.

Proof. By definition of the holomorphic Sobolev spaces, it is sufficient to consider the case $s = 0$. First, consider the case $\alpha > -1$. Then, we have

$$A^p_\alpha \subset A^p_{\alpha+\varepsilon} \approx A^p_{p+\alpha+\varepsilon,1}, \qquad \varepsilon > 0$$

and

$$A^p_{p+\alpha-\varepsilon,1} \approx A^p_{\alpha,\varepsilon/p} \subset A^p_{\alpha+\varepsilon,\varepsilon/p} \approx A^p_\alpha, \qquad 0 < \varepsilon < p$$

where the equivalences are from (1.1) and the inclusions are clearly bounded.

Now, assume $\alpha = -1$. Let $f \in H^p$ and put

$$M^p_p(f,r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta, \qquad 0 < r < 1. \tag{4.1}$$

Then, for any $\delta > 0$, we have $M_p(f,r) = O\left((1-r)^{-\delta}\right)$, which implies $M_p(f',r) = O\left((1-r)^{-1-\delta}\right)$ (see Theorem 5.5 of [D]). Thus, for $\varepsilon > 0$, integration using polar coordinates shows that

$$\int_{\mathbf{D}} |f'(z)|^p (1-|z|^2)^{p-1+\varepsilon} dA(z) \lesssim \int_0^1 (1-r)^{-(1+\delta)p+p-1+\varepsilon} dr.$$

With δ small enough so that $p\delta < \varepsilon$, this integral is convergent, and so (a) holds. Now, assume $0 < \varepsilon < p$. Consider the case $p > 1$ first. Let p' be the conjugate exponent of p. By the fundamental theorem of calculus and Hölder's inequality, we have

$$\begin{aligned}\int_0^{2\pi} |f(e^{i\theta}) - f(0)|^p d\theta &\leq \int_0^{2\pi} \left\{\int_0^1 |f'(re^{i\theta})| dr\right\}^p d\theta \\ &\leq C\int_0^{2\pi} \left\{\int_0^1 |f'(re^{i\theta})|^p (1-r)^{(p-1-\varepsilon)} dr\right\} d\theta \\ &\approx \int_{\mathbf{D}} |f'(z)|^p (1-|z|^2)^{p-1-\varepsilon} dA(z),\end{aligned}$$

where

$$C = \left\{\int_0^1 (1-r)^{-p'(p-1-\varepsilon)/p} dr\right\}^{p/p'} < \infty.$$

It follows that

$$||f||^p_{H^p} \lesssim |f(0)|^p + \int_{\mathbf{D}} |f'(z)|^p (1-|z|^2)^{p-1-\varepsilon} dA(z),$$

and the same is true for $p = 1$ by a trivial modification of this argument. This proves (b) for $p \geq 1$ by Proposition 2.2. When $0 < p < 1$, we have the inclusions

$$A^p_{p-1-\varepsilon,1} \subset A^{p/(1-\varepsilon)}_{-1,0} = H^{p/(1-\varepsilon)} \subset H^p$$

where the first inclusion comes from (1.4). This completes the proof. □

We next show that certain inclusions between holomorphic Sobolev spaces are compact.

Lemma 4.3. *Let $p > 0$, $s \geq 0$, $\varepsilon > 0$ and $\alpha \geq -1$. Then the following inclusions are compact:*

$$A^p_{\alpha,s+\varepsilon} \subset A^p_{\alpha,s} \subset A^p_{\alpha+\varepsilon,s}.$$

Proof. We first consider the case $\alpha > -1$. Let $\{f_n\}$ be a bounded sequence in $A^p_{\alpha,s}$. To show that the inclusion $A^p_{\alpha,s} \subset A^p_{\alpha+\varepsilon,s}$ is compact, we must show that some subsequence of $\{f_n\}$ converges in $A^p_{\alpha+\varepsilon,s}$. It is well known that the inclusion $A^p_\alpha \subset A^p_{\alpha+\varepsilon}$ is compact; see, for example, [Sm1]. Thus there exists $g \in A^p_{\alpha+\varepsilon}$ and a subsequence of $\{f_n\}$ (which for convenience we continue to denote $\{f_n\}$) such that $\mathcal{R}^s f_n \to g$ in $A^p_{\alpha+\varepsilon}$. Now, choose $h \in H(\mathbf{D})$ such that $\mathcal{R}^s h = g$. It is then clear that $h \in A^p_{\alpha+\varepsilon,s}$ and $f_n \to h$ in $A^p_{\alpha+\varepsilon,s}$. This completes the proof that $A^p_{\alpha,s} \subset A^p_{\alpha+\varepsilon,s}$ is compact. Now, since we have by (1.1),

$$A^p_{\alpha,s+\varepsilon} \subset A^p_{\alpha+p\varepsilon,s+\varepsilon} \approx A^p_{\alpha,s}$$

and the first inclusion is compact, we conclude the compactness of the inclusion $A^p_{\alpha,s+\varepsilon} \subset A^p_{\alpha,s}$.

Now, assume $\alpha = -1$. Choose positive numbers δ_1, δ_2 such that $\delta_2 < \delta_1 < \min(p, p\varepsilon)$. Then, we have the relations

$$A^p_{-1,s+\varepsilon} \subset A^p_{p-1+p\varepsilon-\delta_1,1+s+\varepsilon} \approx A^p_{p-1-\delta_1,1+s} \subset A^p_{p-1-\delta_2,1+s} \subset A^p_{-1,s}.$$

The first and last inclusions are bounded from Lemma 4.2, the equivalence is from (1.1), while the remaining inclusion is compact by the previously established part of this lemma. Hence $A^p_{-1,s+\varepsilon} \subset A^p_{-1,s}$ is compact. For the compactness of $A^p_{-1,s} \subset A^p_{-1+\varepsilon,s}$ note that

$$A^p_{-1,s} \subset A^p_{p-1+\varepsilon/2,1+s} \approx A^p_{-1+\varepsilon/2,s} \subset A^p_{-1+\varepsilon,s}.$$

The first inclusion is bounded from Lemma 4.2, as the isomorphism from (1.1), while the remaining inclusion is compact by a previously established part of this lemma for $\alpha > -1$. Hence $A^p_{-1,s} \subset A^p_{-1+\varepsilon,s}$ is compact. The proof is complete. □

Theorem 4.4. *Let $p > 0$ and $\alpha \geq -1$. If $s > \frac{\alpha+1}{p}$, then some composition operators are not bounded on $A^p_{\alpha,s}$.*

Proof. Since $\varphi = C_\varphi z \in A^p_{\alpha,s}$ is necessary for C_φ to be bounded on $A^p_{\alpha,s}$, it suffices to show that if $s > \frac{\alpha+1}{p}$, then $H^\infty \setminus A^p_{\alpha,s} \neq \emptyset$, where H^∞ denotes the class of all bounded holomorphic functions on $\mathbf{D}$. Suppose to the contrary that $H^\infty \subset A^p_{\alpha,s}$. Then this inclusion map is continuous by the Closed Graph Theorem, while the inclusion map $A^p_{\alpha,s} \subset A^p_{\beta,s}$ was shown to be compact whenever $\alpha < \beta$ in Lemma 4.3. The hypothesized lower bound for s now allows us to choose $\delta \geq 1$ such that $\alpha + \delta < sp \leq \alpha + \delta + 1$. Moreover, we can choose $\varepsilon > 0$ so small that

$0 < \alpha + \delta + 1 + \varepsilon - sp < 1$, which gives $A^p_{\alpha+\delta-1+\varepsilon,s} \subset H^{p/(\alpha+\delta+1+\varepsilon-sp)}$ from (1.4). Consequently, we have a chain of inclusions

$$H^\infty \subset A^p_{\alpha,s} \subset A^p_{\alpha+\delta-1+\varepsilon,s} \subset H^{p/(\alpha+\delta+1+\varepsilon-sp)}.$$

Thus the inclusion $H^\infty \subset H^{p/(\alpha+\delta+1+\varepsilon-sp)}$ can be viewed as a product of a compact map and two bounded maps, and hence is compact. But $\{z^n\}$ is a bounded sequence in H^∞ for which no subsequence converges in $H^{p/(\alpha+\delta+1+\varepsilon-sp)}$. This contradicts the compactness of the inclusion map, and the proof is complete. □

The next theorem (also stated as Theorem 1.2 in the introduction) shows that the upper bounds for s in Theorems 4.1 and 4.4 can be extended when the symbol φ of the composition operator is of bounded valence.

Theorem 4.5. *Let $\alpha > -1$ and let φ be of bounded valence. Assume that either $0 \le s \le \frac{\alpha+2}{p}$ if $p \ge 2$, or $0 \le s < \frac{\alpha+1}{p} + \frac{1}{2}$ if $0 < p < 2$. Then C_φ is bounded on $A^p_{\alpha,s}$.*

Proof. We use the Carleson measure criteria from Theorem 2.6. We need to estimate

$$\mu^\varphi_{p,\alpha+(1-s)p}(SI) = \int_{SI} \sum_{\varphi(z)=w} |\varphi'(z)|^{p-2}(1-|z|^2)^{\alpha+(1-s)p} dA(w)$$

for arbitrary arcs $I \subset \partial\mathbf{D}$.

First, consider the case $p \ge 2$. By assumption we have $sp \le \alpha + 2$. Note that, since φ is of bounded valence, there is a uniformly bounded number of terms in the sum inside the integral above. Next, we set $w = \varphi(z)$ and use the Schwarz-Pick Lemma, which asserts that $|\varphi'(z)| \le (1-|w|^2)/(1-|z|^2)$, and then the elementary inequality $1-|z|^2 \le C(1-|w|^2)$ to get that

$$\begin{aligned}\mu^\varphi_{p,\alpha+(1-s)p}(SI) &\le \int_{SI} (1-|w|^2)^{p-2} \sum_{z\in\varphi^{-1}\{w\}} (1-|z|^2)^{\alpha+2-sp} dA(w)\\ &\lesssim \int_{SI} (1-|w|^2)^{\alpha+(1-s)p} dA(w)\\ &\lesssim |I|^{\alpha+2+(1-s)p},\end{aligned}$$

which from Theorem 2.6 is equivalent to boundedness of C_φ on $A^p_{\alpha,s}$. We note that the hypothesis $p \ge 2$ was used in getting the first inequality, and $sp \le \alpha + 2$ was used in the second inequality.

Now, consider the case $p < 2$. What we have now is $sp < \alpha + 1 + \frac{p}{2}$. By the area formula (Theorem 2.32 in [CM]), we have

$$\mu^\varphi_{p,\alpha+(1-s)p}(SI) = \int_{\varphi^{-1}(SI)} |\varphi'(z)|^p (1-|z|^2)^{\alpha+(1-s)p} dA(z).$$

Note that, since φ is of bounded valence, we have

$$\int_{\varphi^{-1}(SI)} |\varphi'(z)|^2 dA(z) = \int_{SI} \sum_{z\in\varphi^{-1}\{w\}} 1\, dA(w) \lesssim |I|^2$$

and thus

$$\mu^{\varphi}_{p,\alpha+(1-s)p}(SI) \leq \left(\int_{\varphi^{-1}(SI)} |\varphi'(z)|^2 dA(z)\right)^{p/2} \times \left(\int_{\varphi^{-1}(SI)} (1-|z|^2)^{2(\alpha+(1-s)p)/(2-p)} dA(z)\right)^{(2-p)/2}$$

$$\lesssim |I|^p \left(\int_{\varphi^{-1}(SI)} (1-|z|^2)^{2(\alpha+(1-s)p)/(2-p)} dA(z)\right)^{(2-p)/2} \tag{4.2}$$

where the first inequality is provided by Hölder's inequality. To estimate the integral above, recall that every composition operator C_φ is bounded on A^p_β, $\beta > -1$, and that this is equivalent to

$$\int_{\varphi^{-1}(SI)} (1-|z|^2)^{\beta} dA(z) \lesssim |I|^{\beta+2} \tag{4.3}$$

for all $I \subset \partial\mathbf{D}$; see section 4 in [MS]. Since, by hypothesis, $2(\alpha+(1-s)p)/(2-p) > -1$, we can combine the estimates in (4.2) and (4.3) to get that

$$\mu^{\varphi}_{p,\alpha+(1-s)p}(SI) \lesssim |I|^{p+\alpha+(1-s)p+(2-p)} = |I|^{\alpha+2+(1-s)p}.$$

From Theorem 2.6, this is equivalent to C_φ being bounded on $A^p_{\alpha,s}$. The proof is complete. □

Our final result in this section shows that the compactness of the inclusions in Lemma 4.3 extends to all composition operators with a certain restriction on s.

Theorem 4.6. *Let $p > 0$, $\alpha \geq -1$ and $\varepsilon > 0$. Assume that either (i) $\alpha > -1$, $0 \leq s < \frac{\alpha+1}{p}$ or (ii) $\alpha = -1$, $s = 0$ or (iii) $s = \frac{\alpha+1}{p}$, $p \geq 2$.*

(a) *$C_\varphi : A^p_{\alpha,s+\varepsilon} \to A^p_{\alpha,s}$ is compact.*
(b) *$C_\varphi : A^p_{\alpha-\varepsilon,s} \to A^p_{\alpha,s}$ is compact whenever $\alpha - \varepsilon \geq -1$.*

Proof. We may view the action of $C_\varphi : A^p_{\alpha,s+\varepsilon} \to A^p_{\alpha,s}$ as follows:

$$A^p_{\alpha,s+\varepsilon} \subset A^p_{\alpha,s} \xrightarrow{C_\varphi} A^p_{\alpha,s}.$$

Also, if $\alpha - \varepsilon \geq -1$, then we may view the action of $C_\varphi : A^p_{\alpha-\varepsilon,s} \to A^p_{\alpha,s}$ as follows:

$$A^p_{\alpha-\varepsilon,s} \subset A^p_{\alpha,s} \xrightarrow{C_\varphi} A^p_{\alpha,s}.$$

The inclusions above are both compact by Lemma 4.3, and every composition operator is bounded on $A^p_{\alpha,s}$ by Theorem 1.1. Thus, both operators are compact. The proof is complete. □

Carleson measure criteria for C_φ to be bounded or compact on H^p_1 are known; see Theorems 4.11 and 4.12 in [CM]. For s not an integer, characterizing when C_φ is bounded or compact on H^p_s seems much harder than the analogous problems on the Bergman spaces. The problem is that for the Hardy spaces, (1.1) does not provide isomorphisms with spaces defined using full derivatives. Thus, we are led to the following.

Problem. Characterize φ for which C_φ is bounded (compact) on H^p_s, $s > 0$.

5. Composition with a Polygonal Map

Here, we find simple criteria for the compactness of the composition operators between holomorphic Sobolev spaces induced by polygonal maps.

Recall that $\operatorname{dist}(a, \partial E)$ denotes the Euclidean distance between a point a and the boundary of a set E.

Lemma 5.1. *Suppose that P is a polygon inscribed in the unit circle with a vertex at v and let $\pi\eta(v)$ be the vertex angle at v. Then, given a Riemann map φ of $\mathbf{D}$ onto P, there exists a neighborhood N_v of v such that*

$$|\varphi'(z)| \approx (1-|\varphi(z)|)^{1-1/\eta(v)},$$
$$(1-|z|) \approx (1-|\varphi(z)|)^{1/\eta(v)-1} \operatorname{dist}(\varphi(z), \partial P)$$

for all $z \in \varphi^{-1}(N_v)$.

Proof. Recall that φ extends to a homeomorphism of $\overline{\mathbf{D}}$ onto $\overline{P}$ (see, for example, Theorem 14.19 of [R2]). Assume $v = 1$ and $\varphi(1) = 1$ for simplicity. Also, let $\eta = \eta(v)$. Then, a reflection argument yields

$$1 - \varphi(z) = c(1-z)^{\eta} + O(|1-z|^{1+\eta})$$

for some constant $c \neq 0$ and for all z near 1. Thus, we have

$$|\varphi'(z)| \approx |1-z|^{\eta-1} \approx |1-\varphi(z)|^{1-1/\eta} \approx (1-|\varphi(z)|)^{1-1/\eta}$$

for z near 1. The last equivalence in the display above holds, because $\varphi(z)$ is contained in a nontangential region with the vertex at 1. This proves the first equivalence of the lemma. The second equivalence is now a consequence of the estimates

$$\frac{1}{4}(1-|z|^2)|\varphi'(z)| \leq \operatorname{dist}(\varphi(z), \partial P) \leq (1-|z|^2)|\varphi'(z)|,$$

which hold since φ is univalent; see Corollary 1.4 of [P]. The proof is complete. □

Lemma 5.2. *Let P be a polygon inscribed in the unit circle. Assume $b > -1$ and $a + b > -2$. Then, there exists a constant $C > 0$ such that*

$$\int_{P \cap SI} (1-|w|^2)^a \operatorname{dist}(w, \partial P)^b dA(w) \leq C|I|^{a+b+2}$$

for all arcs $I \subset \partial\mathbf{D}$.

Proof. Let us introduce a temporary notation. For an arc $I \subset \partial\mathbf{D}$ with center at $\zeta \in \partial\mathbf{D}$ and $|I| = 2\delta$, we let $S_\delta(\zeta) = SI$.

Assume that δ is sufficiently small and $P \cap S_\delta(\zeta) \neq \emptyset$. Then, there is a constant C_1, depending only on P, such that $S_\delta(\zeta) \subset S_{C_1\delta}(v)$ for some vertex v of P. Assume $v = 1$ for simplicity. Assuming that δ is sufficiently small so that $S_{C_1\delta}(1)$ contains no vertex of P other than 1, note that $1 - |w| \approx |1-w|$ for $w \in P \cap S_{C_1\delta}(1)$. Now,

we have

$$\begin{aligned}\int_{P\cap S_\delta(\zeta)} &(1-|w|^2)^a \operatorname{dist}(w,\partial P)^b dA(w)\\ &\le \int_{P\cap S_{C_1\delta}(1)} (1-|w|^2)^a \operatorname{dist}(w,\partial P)^b dA(w)\\ &\approx \int_{P\cap S_{C_1\delta}(1)} |1-w|^a \operatorname{dist}(w,\partial P)^b dA(w)\\ &\lesssim \int_0^\pi \int_0^{C_2\delta} r^a (r\sin\theta)^b r dr d\theta\\ &\approx \delta^{a+b+2}\end{aligned}$$

as asserted, where C_2 is a constant depending only on P. The estimate for large δ follows from the inequality

$$\int_P (1-|w|^2)^a \operatorname{dist}(w,\partial P)^b dA(w) < \infty,$$

which is clear from the argument above. The proof is complete. □

Recall that $D(z,1/2)$ denotes the pseudohyperbolic disk. Let $D(z)=D(z,1/2)$. In the following we let $dA_\alpha(z)=(1-|z|^2)^\alpha dA(z)$ for $\alpha>-1$. The following lemma is proved for $\alpha=0$ in [L2], and the same proof works for general α.

Lemma 5.3. *Let $\alpha>-1$ and μ be a positive finite Borel measure on $\mathbf{D}$. Assume $p>q>0$. Then, there is a constant C such that*

$$\left(\int_{\mathbf{D}} |f|^q d\mu\right)^{1/q} \le C\left(\int_{\mathbf{D}} |f|^p dA_\alpha\right)^{1/p}, \qquad f\in A^q_\alpha$$

if and only if $\tau\in L^{\frac{p}{p-q}}(A_\alpha)$ where $\tau(z)=\frac{\mu(D(z))}{A_\alpha(D(z))}$.

For a polygon P inscribed in the unit circle, recall that $\theta(P)$ denotes $1/\pi$ times the measure of the largest vertex angle of P.

Proposition 5.4. *Let $p_j>0$, $\alpha_j>-1$, $s_j\ge 0$ $(j=1,2)$ and assume*

$$s_1 < 1+\frac{\alpha_1+2}{p_1}-\frac{1}{p_2}, \qquad s_2<\frac{\alpha_2+1}{p_2}. \tag{5.1}$$

Let φ be a holomorphic function taking $\mathbf{D}$ into a polygon P inscribed in the unit circle. If

$$\theta(P) < \frac{p_1(\alpha_2+2-s_2p_2)}{p_2(\alpha_1+2-s_1p_1)}, \tag{5.2}$$

then $C_\varphi : A^{p_1}_{\alpha_1,s_1}\to A^{p_2}_{\alpha_2,s_2}$ is bounded.

Moreover, for functions φ of bounded valence, the second part of (5.1) can be replaced by the weaker condition that

$$s_2\le \frac{\alpha_2+2}{p_2} \text{ if } p_2\ge 2, \text{ or } s_2<\frac{\alpha_2+1}{p_2}+\frac{1}{2} \text{ if } 0<p_2<2. \tag{5.3}$$

In either case the equality can be allowed in (5.2) for $p_2\ge p_1$.

Proof. Let φ_0 be a Riemann mapping of $\mathbf{D}$ onto P and put $\psi = \varphi_0^{-1} \circ \varphi$. Then $\varphi = \varphi_0 \circ \psi$ and thus $C_\varphi = C_{\varphi_0 \circ \psi} = C_\psi C_{\varphi_0}$. Note that $C_\psi : A^{p_2}_{\alpha_2,s_2} \to A^{p_2}_{\alpha_2,s_2}$ is bounded by Theorem 1.1 or Theorem 1.2. This shows that we only need to prove the proposition for $\varphi = \varphi_0$. So, in the rest of the proof, we assume that φ is a Riemann map of $\mathbf{D}$ onto P. For simplicity, let $\beta_j = p_j + \alpha_j - s_j p_j$ and let $\gamma_j = 2 + \alpha_j - s_j p_j$ for $j = 1, 2$.

First, consider the case $p_2 \geq p_1$. By Theorem 2.6, we need to show that

$$\mu^{\varphi}_{p_2,\beta_2}(SI) = O(|I|^{(2+\alpha_1)p_2/p_1 + (1-s_1)p_2}) \tag{5.4}$$

for all arcs I. As in the proof of Lemma 5.2, we only need to consider I centered at a vertex of P for which $|I|$ is sufficiently small. Given such I, we have by Lemma 5.1 and Lemma 5.2,

$$\begin{aligned} \mu^{\varphi}_{p_2,\beta_2}(SI) &= \int_{SI\cap P} |\varphi'(\varphi^{-1}(w))|^{p_2-2}(1-|\varphi^{-1}(w)|^2)^{\beta_2} dA(w) \\ &\approx \int_{SI\cap P} \operatorname{dist}(w, \partial P)^{\beta_2}(1-|w|)^{(1/\theta-1)\gamma_2} dA(w) \\ &\lesssim |I|^{p_2+\gamma_2/\theta} \end{aligned} \tag{5.5}$$

where $\theta = \theta(P)$. In the last inequality we used the fact that $\beta_2 > -1$, $\gamma_2 \geq 0$ from (5.3) and thus

$$\beta_2 + \left(\frac{1}{\theta} - 1\right)\gamma_2 > -1. \tag{5.6}$$

Thus, we have (5.4) by (5.2) and (5.5). Also, the same proof works in case the equality holds in (5.2).

Next, consider the case $p_2 < p_1$. We may assume $\varphi(0) = 0$. Let $f \in A^{p_1}_{\alpha_1,s_1}$ be an arbitrary function such that $f(0) = 0$. Since $p_2 < p_1$, we have $\beta_1 > -1$ by the first part of (5.1). Thus, by (1.1) and Proposition 2.2, we have

$$||f||^{p_1}_{A^{p_1}_{\alpha_1,s_1}} \approx \int_{\mathbf{D}} |f'(w)|^{p_1}(1-|w|)^{\beta_1} dA(w).$$

Also, we have by (1.1), Proposition 2.2 and Lemma 5.1,

$$\begin{aligned} ||C_\varphi(f)||^{p_2}_{A^{p_2}_{\alpha_2,s_2}} &\approx ||C_\varphi(f)||^{p_2}_{A^{p_2}_{\beta_2,1}} \\ &\approx \int_{\mathbf{D}} |f'(\varphi(z))|^{p_2}|\varphi'(z)|^{p_2}(1-|z|^2)^{\beta_2} dA(z) \\ &= \int_P |f'(w)|^{p_2}|\varphi'(\varphi^{-1}(w))|^{p_2-2}(1-|\varphi^{-1}(w)|^2)^{\beta_2} dA(w) \\ &\approx \int_P |f'(w)|^{p_2} \operatorname{dist}(w, \partial P)^{\beta_2}(1-|w|)^{(1/\theta-1)\gamma_2} dA(w). \end{aligned}$$

Now, define measures

$$\begin{aligned} d\mu_1(w) &= (1-|w|)^{\beta_1} dA(w), \\ d\mu_2(w) &= \operatorname{dist}(w, \partial P)^{\beta_2}(1-|w|)^{(1/\theta-1)\gamma_2} \mathcal{X}_{\varphi(\mathbf{D})}(w) dA(w) \end{aligned}$$

and let

$$\tau(z) = \frac{\mu_2(D(z))}{\mu_1(D(z))}, \qquad z \in \mathbf{D}.$$

By Lemma 5.3, we need to show that $\tau \in L^p(\mu_1)$ where $p = p_1/(p_1 - p_2)$. Note that $\mu_2(D(z)) = 0$ if z is outside of some polygonal region Q. On the other hand, for $z \in Q$, we have

$$\mu_1(D(z)) \approx (1 - |z|^2)^{\beta_1+2},$$
$$\mu_2(D(z)) \approx (1 - |z|^2)^{\beta_2+(1/\theta-1)\gamma_2+2};$$

the first estimate is standard and the second one can be verified with (5.6) by modifying the proof of Lemma 5.2. Accordingly, we have

$$\tau(z) \approx (1 - |z|^2)^{\beta_2-\beta_1+(1/\theta-1)\gamma_2}\mathcal{X}_Q(z).$$

It follows that $\tau \in L^p(\mu_1)$ if and only if $p[\beta_2 - \beta_1 + (1/\theta - 1)\gamma_2] + \beta_1 > -2$, which turns out to be the same as (5.2). This completes the proof. □

Theorem 5.5. *Let $p_j > 0$, $\alpha_j \geq -1$, $s_j \geq 0$ $(j = 1, 2)$ and assume* (5.1) *holds. Let φ be a holomorphic function taking $\mathbf{D}$ into a polygon P inscribed in the unit circle. If* (5.2) *holds, then $C_\varphi : A^{p_1}_{\alpha_1,s_1} \to A^{p_2}_{\alpha_2,s_2}$ is compact.*

Moreover, for functions φ of bounded valence, the second part of (5.1) *can be replaced by the weaker condition* (5.3).

In Example 6.1 below, we show that (5.3) provides the sharp upper bound of $\frac{\alpha_2+2}{p_2}$ for s_2 when $p_2 \geq 2$. While we do not know whether it does the same when $p_2 < 2$, the upper bound for s_2 when $p_2 < 2$ cannot be extended to $\frac{\alpha_2+2}{p_2}$, as is shown by Example 6.2. Nevertheless, Example 6.4 shows the upper bound of $\theta(P)$ in (5.2) is sharp in either case.

Proof. Assume that (5.2) holds and choose $\varepsilon > 0$ sufficiently small so that (5.2) holds with $\alpha_1 + \varepsilon$ in place of α_1. By Lemma 4.3 we have $A^{p_1}_{\alpha_1,s_1} \subset A^{p_1}_{\alpha_1+\varepsilon,s_1}$ and the inclusion is compact. Thus, it is sufficient to show that $C_\varphi : A^{p_1}_{\alpha_1+\varepsilon,s_1} \to A^{p_2}_{\alpha_2,s_2}$ is bounded. In case $\alpha_2 > -1$, we see that $C_\varphi : A^{p_1}_{\alpha_1+\varepsilon,s_1} \to A^{p_2}_{\alpha_2,s_2}$ is bounded by Proposition 5.4.

So, assume $\alpha_2 = -1$. Note that with $\alpha_2 = -1$ there is no s_2 satisfying the second part of (5.1). Thus, we only need to be concerned about the case where φ is of bounded valence and (5.3) holds. First, consider the case $p_2 \leq 2$. In this case, we can view the action of C_φ as follows:

$$A^{p_1}_{\alpha_1+\varepsilon,s_1} \xrightarrow{C_\varphi} A^{p_2}_{p_2-1,s_2+1} \subset A^{p_2}_{-1,s_2}$$

where $C_\varphi : A^{p_1}_{\alpha_1+\varepsilon,s_1} \to A^{p_2}_{p_2-1,s_2+1}$ is bounded by Proposition 5.4 and the inclusion comes from (1.2). Therefore, $C_\varphi : A^{p_1}_{\alpha_1+\varepsilon,s_1} \to A^{p_2}_{\alpha_2,s_2}$ is bounded. Next, consider the case $p_2 > 2$. Choose $p_2' \in (2, p_2)$ and $\alpha_2' > -1$. Also, let $s_2' = \frac{\alpha_2'+2}{p_2'} + s_2 - \frac{1}{p_2}$. Then, we can view the action of C_φ as follows:

$$A^{p_1}_{\alpha_1+\varepsilon,s_1} \xrightarrow{C_\varphi} A^{p_2'}_{\alpha_2',s_2'} \subset A^{p_2}_{-1,s_2}$$

where $C_\varphi : A^{p_1}_{\alpha_1+\varepsilon,s_1} \to A^{p_2'}_{\alpha_2',s_2'}$ is bounded by Proposition 5.4 and the inclusion comes from (1.4). The proof is complete. □

Remarks. 1. As mentioned in the proof above, there is no s_2 satisfying the second part of (5.1) in case $\alpha_2 = -1$. Thus, we have no conclusion in Theorem 5.5 for general φ in case the target space is a Hardy-Sobolev space.

2. Note that the condition (5.2) holds vacuously if $\frac{\alpha_1+2}{p_1} - \frac{\alpha_2+2}{p_2} \leq s_1 - s_2$.

6. Examples

We now give several examples demonstrating that our theorems are sharp. For that purpose we introduce the so-called lens maps. For $0 < \eta < 1$ we denote by φ_η the function defined by

$$\varphi_\eta(z) = \frac{\sigma(z)^\eta - 1}{\sigma(z)^\eta + 1}, \qquad z \in \mathbf{D} \tag{6.1}$$

where $\sigma(z) = (1+z)/(1-z)$. Let $\varphi_\eta(\mathbf{D}) = L_\eta$. Then, φ_η is the Riemann map of $\mathbf{D}$ onto the subset L_η of $\mathbf{D}$ bounded by arcs of circles meeting at $z = \pm 1$ at an angle of $\eta\pi$, and fixing the points -1, 0, and 1. Because of the shape of the range L_η, such a map is called a "lens map". Note that L_η is contained in a polygon inscribed in the unit circle.

By a straightforward calculation, we have

$$1 - |\varphi(z)| \approx |1-z|^\eta, \qquad |\varphi'(z)| \approx |1-z|^{\eta-1} \tag{6.2}$$

for z near 1.

The first example shows that the upper bound $s \le \frac{\alpha+2}{p}$ in Theorem 1.2(a) is sharp. Also, this example shows that the upper bound $s_2 \le \frac{\alpha_2+2}{p_2}$ in Theorem 5.5 is sharp when $p_2 \ge 2$ and φ is of bounded valence.

Example 6.1. Let $p > 1$, $\alpha > -1$ and $\frac{\alpha+2}{p} < s < 1 + \frac{\alpha+1}{p}$. Then, there exists a lens map $\varphi_\eta \notin A^p_{\alpha,s}$. In particular, C_{φ_η} is not bounded on $A^p_{\alpha,s}$.

Proof. Choose $0 < \eta < 1$ sufficiently small so that $sp \ge \eta p + \alpha + 2$ and consider the corresponding lens map φ_η. Note that $A^p_{\alpha,s} \approx A^p_{\alpha+(1-s)p,1}$ by (1.1). Therefore, we have by Proposition 2.2 and (6.2),

$$\begin{aligned} ||\varphi_\eta||_{A^p_{\alpha,s}} &\approx \int_{\mathbf{D}} |\varphi_\eta'(z)|^p (1-|z|^2)^{\alpha+(1-s)p} dA(z) \\ &\approx \int_{\mathbf{D}} |1-z|^{(\eta-1)p} (1-|z|^2)^{\alpha+(1-s)p} dA(z). \end{aligned}$$

Note that $(\eta-1)p + \alpha + (1-s)p \le -2$, because $sp \ge \eta p + \alpha + 2$. Thus, an obvious estimate in an angle with vertex at 1 shows that the last integral above diverges, as desired. □

We do not know whether the upper bound $s < \frac{\alpha+1}{p} + \frac{1}{2}$ in Theorem 1.2(b) is sharp. However, the next example shows that the upper bound cannot be extended to $\frac{\alpha+2}{p}$ as in Theorem 1.2(a). Also, this is related to the assumption, $s_2 < \frac{\alpha_2+1}{p_2} + \frac{1}{2}$, in Theorem 5.5 when $p_2 < 2$.

Example 6.2. For each $p \in [1,2)$, there exist $\alpha > -1$, $0 \le s < 1$ with $s < \frac{\alpha+2}{p}$ and a univalent holomorphic self-map φ of $\mathbf{D}$ such that $\varphi \notin A^p_{\alpha,s}$. In particular, C_φ is not bounded on $A^p_{\alpha,s}$.

Proof. P. Jones and N. Makarov have shown (see Theorem D(2) in [JM]) that for any $p < 2$, there exist a univalent holomorphic self-map φ_p of $\mathbf{D}$ and a constant $c > 0$ such that the integral means of φ_p' satisfy

$$(1-r_n)^{p-1+c(2-p)^2} M_p^p(\varphi_p', r_n) \ge 1 \tag{6.3}$$

for some sequence $r_n \to 1$. Here, we are using the notation introduced in (4.1). Note that, for any $f \in A^p_\beta$, $\beta > -1$, we have

$$\int_0^1 M_p^p(f,r) r(1-r^2)^\beta dr < \infty,$$

and so

$$\begin{aligned}(1-r)^{\beta+1} M_p^p(f,r) &= (\beta+1) M_p^p(f,r) \int_r^1 (1-t)^\beta dt \\ &\le (\beta+1) \int_r^1 M_p^p(f,t)(1-t)^\beta dt = o(1)\end{aligned}$$

as $r \to 1$. This, together with (6.3), yields $\varphi_p' \notin A^p_{p-2+c(2-p)^2}$. In other words, we have $\varphi_p \notin A^p_{p-2+c(2-p)^2,1}$ by Proposition 2.2. Note that the hypothesis $p \ge 1$ is used here to assure that $p-2+c(2-p)^2 > -1$. Now, choose $s \in [0,1)$ such that $sp-2+c(2-p)^2 > -1$ and put $\alpha = sp-2+c(2-p)^2$. Then, we have $s < \frac{\alpha+2}{p}$. Also, since $A^p_{p-2+c(2-p)^2,1} \approx A^p_{\alpha,s}$ by (1.1), we have $\varphi_p \notin A^p_{\alpha,s}$. □

The next example shows that the lower bound $s > \frac{\alpha+2}{p}$ in Theorem 3.3 is sharp when $\alpha > -1$.

Example 6.3. Let $p > 1$, $\alpha > -1$ and put $s = \frac{\alpha+2}{p}$. Then, there exists a holomorphic self-map φ of $\mathbf{D}$ with $\varphi(1) = 1$ such that $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is bounded but φ does not have angular derivative at $z = 1$.

Proof. Let $\varphi = \varphi_\eta$ be any lens map. Note that $\alpha + (1-s)p > -1$. Thus, as in the proof of Proposition 5.4, we have

$$\mu_{p,\alpha+(1-s)p}(SI) \lesssim |I|^{p+(2+\alpha-sp)/\eta} = |I|^p,$$

so that $C_\varphi : A^p_{\alpha,s} \to A^p_{\alpha,s}$ is bounded by Theorem 2.6(i). Clearly, φ does not have an angular derivative at 1. □

The next example shows that the upper bound for $\theta(P)$ in Theorem 5.5 is sharp when $\alpha_1 > -1$.

Example 6.4. Let p_j, s_j, α_j be as in the hypotheses of Theorem 5.5. Assume $\alpha_1 > -1$ and

$$\frac{p_1(\alpha_2 + 2 - s_2 p_2)}{p_2(\alpha_1 + 2 - s_1 p_1)} < \eta < 1. \tag{6.4}$$

Then, $f \circ \varphi_\eta \notin A^{p_2}_{\alpha_2,s_2}$ for some $f \in A^{p_1}_{\alpha_1,s_1}$.

Proof. Let $\varphi = \varphi_\eta$. Choose $0 < a < 1$ such that $|\varphi(a)| \ge 1/2$. Also, by using (6.4), choose $\varepsilon > 0$ sufficiently small so that

$$\frac{(2+\alpha_2)/p_2 - s_2 + \varepsilon}{(2+\alpha_1)/p_1 - s_1} < \eta < 1. \tag{6.5}$$

Now, consider the test function $f_a(z) = \log(1 - \overline{\varphi(a)} z)$. Let $k \ge s_1$ be a positive integer. Then we have $A^{p_1}_{\alpha_1,s_1} \approx A^{p_1}_{\alpha_1+(k-s_1)p,k}$ by (1.1). Therefore, by Proposition

2.2, (2.3), (6.2) and (6.5),

$$\begin{aligned} ||f_a||_{A^{p_1}_{\alpha_1,s_1}} &\approx ||f_a||_{A^{p_1}_{\alpha_1+(k-s_1)p_1,k}} \\ &\approx (1-|\varphi(a)|^2)^{(2+\alpha_1)/p_1-s_1} \\ &\approx (1-a)^{[(2+\alpha_1)/p_1-s_1]\eta} \\ &\lesssim (1-a)^{[(2+\alpha_2)/p_2-s_2]+\varepsilon}. \end{aligned} \tag{6.6}$$

On the other hand, for $\alpha_2 > -1$, we have by (1.1) and (3.2),

$$\begin{aligned} ||f_a \circ \varphi||_{A^{p_2}_{\alpha_2,s_2}} &\approx ||f_a \circ \varphi||_{A^{p_2}_{\alpha_2+(1-s_2)p_2,1}} \\ &\gtrsim \frac{|\varphi'(a)|(1-a)^{(2+\alpha_2)/p_2+(1-s_2)}}{(1-|\varphi(a)|^2)} \end{aligned}$$

and thus by (6.2),

$$||f_a \circ \varphi||_{A^{p_2}_{\alpha_2,s_2}} \gtrsim (1-a)^{(2+\alpha_2)/p_2-s_2}. \tag{6.7}$$

This also holds for $\alpha_2 = -1$, because $A^{p_2}_{-1,s_2} \subset A^{2p_2}_{0,s_2}$ by (1.4). Consequently, we obtain from (6.7), (6.6) and Lemma 4.3 that

$$\frac{||f_a \circ \varphi||_{A^{p_2}_{\alpha_2,s_2}}}{||f_a||_{A^{p_1}_{\alpha_1,s_1}}} \gtrsim (1-a)^{-\varepsilon}.$$

Now, letting $a \to 1$, we see that C_φ does not take $A^{p_1}_{\alpha_1,s_1}$ into $A^{p_2}_{\alpha_2,s_2}$ by the Closed Graph Theorem. □

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