

## PROJECTIVE NORMALITY OF ABELIAN VARIETIES

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ABSTRACT. We show that ample line bundles  $L$  on a  $g$ -dimensional simple abelian variety  $A$ , satisfying  $h^0(A, L) > 2^g \cdot g!$ , give projective normal embeddings, for all  $g \geq 1$ .

### 1. INTRODUCTION

Let  $A$  be an abelian variety of dimension  $g$  defined over the field of complex numbers and let  $L$  be an ample line bundle on  $A$ . Consider the associated rational map  $\phi_L : A \rightarrow \mathbb{P}^{d-1} = \mathbb{P}H^0(A, L)$ , where  $d = \dim H^0(A, L)$ . Suppose  $L = M^n$  for some ample line bundle  $M$  on  $A$ . Then Koizumi has shown that  $L$  gives a projectively normal embedding if  $n \geq 3$  (see [2]).

When  $n = 2$ , Ohbuchi (see [7]) has shown the following.

**Theorem 1.1.** *Suppose  $M$  is a symmetric ample line bundle on a  $g$ -dimensional abelian variety  $A$ . Then  $L = M^2$  gives a projectively normal embedding of  $A$  if and only if the origin  $0$  of  $A$  is not contained in  $Bs|M \otimes P_\alpha|$  for any  $\alpha \in \hat{A}_2 = \{\alpha \in \hat{A} : 2\alpha = 0\}$ , where  $\hat{A}$  is the dual abelian variety of  $A$ ,  $P$  is the Poincaré bundle on  $A \times \hat{A}$ ,  $P_\alpha = P|_{A \times \alpha}$  for  $\alpha \in \hat{A}$  and  $Bs|M \otimes P_\alpha|$  is the set of all base points of  $M \otimes P_\alpha$ .*

Suppose  $L \neq M^n$  for any ample line bundle  $M$  on  $A$  and  $n > 1$ . When  $g = 2$ , Lazarsfeld (see [4]) has shown that if  $\phi_L$  is birational onto its image, then  $\phi_L$  gives a projectively normal embedding, for  $d = 7, 9, 11$  and for  $d \geq 13$ . We showed that if the Neron Severi group  $NS(A)$  of  $A$  is  $\mathbb{Z}$ , generated by  $L$  and  $d \geq 7$ , then  $\phi_L$  gives a projectively normal embedding (see [1]).

In this article, we show

**Theorem 1.2.** *Suppose  $L$  is an ample line bundle on a  $g$ -dimensional simple abelian variety  $A$ . If  $d > 2^g \cdot g!$ , then  $L$  gives a projectively normal embedding, for all  $g \geq 1$ . (Here  $d = \dim H^0(A, L)$ ).*

We outline the proof of Theorem 1.2.

For a polarized abelian variety  $(A, L)$ , consider the multiplication maps

$$\rho_r : \text{Sym}^r H^0(A, L) \rightarrow H^0(A, L^r).$$

By definition,  $L$  gives a projectively normal embedding if  $\rho_r$  is surjective, for all  $r \geq 1$ . We first show that it suffices to show  $\rho_2$  is surjective. More precisely, we show that  $\rho_2$  surjective implies that the maps  $\rho_r$  are surjective, for  $r \geq 3$  (see Prop. 2.1).

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Received by the editors December 5, 2001 and, in revised form, October 20, 2002.  
2000 *Mathematics Subject Classification.* Primary 14C20, 14K05, 14K25, 14N05.

To prove the surjectivity of the map  $\rho_2$  we consider a finite isogeny  $A \rightarrow B = A/H$ , where  $H$  is a maximal isotropic subgroup of the fixed group  $K(L)$  of  $L$ . Then  $L$  descends down to a principal polarization  $M$  on  $B$ . Let  $\hat{H}$  denote the group of characters on  $H$ . By associating to a character  $\chi \in \hat{H}$  a degree 0 line bundle  $L_\chi$  on  $B$  one can identify  $\hat{H}$  as a subgroup of the dual abelian variety  $\text{Pic}^0(B)$  of  $B$ . The homomorphism  $\psi_M : B \rightarrow \text{Pic}^0(B), b \mapsto t_b^*M \otimes M^{-1}$  is an isomorphism and we denote  $H' = \psi_M^{-1}(\hat{H})$ .

We then show that the surjectivity of the map  $\rho_2$  is equivalent to showing that the subgroup  $H'$  of  $B$  generates the projective space  $\mathbb{P}H^0(B, M^2)$  and its translates  $\mathbb{P}H^0(t_\sigma^*M^2)$ , where  $\sigma \in B$  is such that  $\psi_M(2\sigma) = L_\chi, L_\chi \in \hat{H}$ , i.e., the images of points of  $H'$ , under the morphism  $B \xrightarrow{\phi_{t_\sigma^*M^2}} \mathbb{P}H^0(t_\sigma^*M^2) \simeq |t_\sigma^*M^2|, b \mapsto t_b^*\theta + t_{-b+2\sigma}^*\theta$  (due to Wirtinger), have their linear span as  $|t_\sigma^*M^2|$ . (Here we assume that  $M$  is symmetric and that  $\theta$  is the unique symmetric divisor in  $|M|$ .)

To see this, we show

**Proposition 1.3.** *Let  $\mathcal{L}$  be an ample line bundle on a simple abelian variety  $Z$  of dimension  $g$  and consider the associated rational map  $Z \xrightarrow{\phi_{\mathcal{L}}} \mathbb{P}H^0(\mathcal{L})$ . Then any finite subgroup  $G$  of  $Z$  of order strictly greater than  $h^0(\mathcal{L}) \cdot g!$ , generates the linear system  $\mathbb{P}H^0(\mathcal{L})$ . More precisely, the points  $\phi_{\mathcal{L}}(h)$  where  $h$  runs over all elements of  $G$  not in the base locus of  $\mathcal{L}$  span  $\mathbb{P}H^0(\mathcal{L})$  (see Prop. 3.4).*

We then apply Proposition 1.3 to  $\mathcal{L} = t_\sigma^*M^2$  to obtain bounds as asserted for a polarized abelian variety  $(A, L)$  in Theorem 1.2.

*Notation.* The varieties considered in this article are defined over the complex numbers.

Let  $\mathcal{L}$  be an ample line bundle on an abelian variety  $Z$  of dimension  $g$ .

1. The *fixed group* of  $\mathcal{L}$  is the group  $K(\mathcal{L}) = \{z \in Z : \mathcal{L} \simeq t_z^*\mathcal{L}\}, t_z : Z \rightarrow Z, x \mapsto z + x$ .
2. The *theta group* of  $\mathcal{L}$  is the group  $\mathcal{G}(\mathcal{L}) = \{(z, \phi) : \mathcal{L} \xrightarrow{\phi} t_z^*\mathcal{L}\}$ .
3. The *Weil form*  $e^{\mathcal{L}} : K(\mathcal{L}) \times K(\mathcal{L}) \rightarrow \mathbb{C}^*$  is the commutator map  $(x, y) \mapsto x'y'x'^{-1}y'^{-1}$ , for any lifts  $x', y' \in \mathcal{G}(\mathcal{L})$  of  $x, y \in K(\mathcal{L})$ .
4.  $h^0(\mathcal{L}) = \dim H^0(Z, \mathcal{L})$ .
5. If  $G$  is a finite subgroup of  $Z$ , then  $\text{Card}(G) = \text{order}(G)$ .

## 2. SURJECTIVITY OF THE MAPS $\rho_r, r \geq 3$

Suppose  $\mathcal{L}$  is an ample line bundle on a  $g$ -dimensional abelian variety  $A$ . Consider the multiplication maps

$$H^0(\mathcal{L})^{\otimes r} \xrightarrow{\rho_r} H^0(\mathcal{L}^r), \text{ for } r \geq 2.$$

The main result of this section is the following.

**Proposition 2.1.** *Suppose  $\mathcal{L}$  is an ample line bundle on an abelian variety  $A$ . If the multiplication map  $\rho_2$  is surjective, then  $\rho_r$  is surjective, for all  $r \geq 3$ .*

First, we recall

**Proposition 2.2.** *Suppose  $L$  and  $M$  are ample line bundles on an abelian variety  $A$ .*

1) *The multiplication map*

$$\sum_{\alpha \in U} H^0(L \otimes \alpha) \otimes H^0(M \otimes \alpha^{-1}) \longrightarrow H^0(L \otimes M)$$

is surjective, for any nonempty Zariski open subset  $U$  of  $\text{Pic}^0(A)$ .

2) *If the multiplication map  $H^0(L) \otimes H^0(M) \longrightarrow H^0(L \otimes M)$  is surjective, then the multiplication maps*

$$(a) H^0(L) \otimes H^0(M \otimes \alpha) \longrightarrow H^0(L \otimes M \otimes \alpha)$$

and

$$(b) H^0(L \otimes \alpha^{-1}) \otimes H^0(M \otimes \alpha) \longrightarrow H^0(L \otimes M)$$

are also surjective, for  $\alpha$  in some nonempty Zariski open subset  $U$  of  $\text{Pic}^0(A)$ .

*Proof.* 1) See [3], 7.3.3.

2) The proof is standard. □

*Proof of Proposition 2.1.* We prove by induction on  $r$ . Suppose the multiplication map  $\rho_r : H^0(\mathcal{L})^{\otimes r} \longrightarrow H^0(\mathcal{L}^r)$  is surjective, for some  $r \geq 2$ .

Consider the composed multiplication map

$$H^0(\mathcal{L})^{\otimes r+1} \xrightarrow{Id \otimes \rho_r} H^0(\mathcal{L}) \otimes H^0(\mathcal{L}^r) \xrightarrow{\rho_{1,r}} H^0(\mathcal{L}^{r+1}).$$

To see the surjectivity of the map  $\rho_{r+1} = \rho_{1,r} \circ (Id \otimes \rho_r)$  we need to show that the map  $\rho_{1,r}$  is surjective.

Using Proposition 2.2 1), we can write

$$(*) \quad H^0(\mathcal{L}).H^0(\mathcal{L}^r) = \sum_{\alpha \in U} H^0(\mathcal{L}).H^0(\mathcal{L} \otimes \alpha^{-1}).H^0(\mathcal{L}^{r-1} \otimes \alpha)$$

for any nonempty Zariski open subset  $U$  of  $\text{Pic}^0(A)$ .

Since  $\rho_2$  is surjective, by Proposition 2.2 2) (a), there exists a nonempty Zariski open subset  $U'$  of  $\text{Pic}^0(A)$ , such that for  $\alpha^{-1} \in U'$ ,

$$(**) \quad H^0(\mathcal{L}).H^0(\mathcal{L} \otimes \alpha^{-1}) = H^0(\mathcal{L}^2 \otimes \alpha^{-1})$$

Now in (\*), using (\*\*) and again applying Proposition 2.2 1), we obtain

$$\begin{aligned} H^0(\mathcal{L}).H^0(\mathcal{L}^r) &= \sum_{\alpha^{-1} \in U'} H^0(\mathcal{L}^2 \otimes \alpha^{-1}).H^0(\mathcal{L}^{r-1} \otimes \alpha) \\ &= H^0(\mathcal{L}^{r+1}). \end{aligned}$$

□

### 3. SURJECTIVITY OF THE MAP $\rho_2$

Let  $Z$  be a  $g$ -dimensional abelian variety and let  $D$  be an ample divisor on  $Z$ . We denote  $M = \mathcal{O}(D)$  to be the ample line bundle on  $Z$ . Let  $G$  be a finite subgroup of  $Z$ . Consider the homomorphism  $\psi_M : Z \longrightarrow \text{Pic}^0(Z)$ ,  $z \mapsto t_z^*(M) \otimes M^{-1}$ . Let  $G' \subset \text{Pic}^0(Z)$  be the image of  $G$  under this homomorphism. Consider a finite subgroup  $J \subset \text{Pic}^0(Z)$  and containing the subgroup  $G'$ . Construct an étale cover  $\pi : X \longrightarrow Z$  corresponding to  $J$ , which is of degree equal to  $\text{Card}J$ . Let  $N = \mathcal{O}(\pi^{-1}D)$  be the ample line bundle on  $X$ .

Notice that if  $h \in G \cap K(M)$ , then  $t_h^*M \simeq M$ , and this implies that  $D + h$  is linearly equivalent to  $D$  on  $Z$ . If  $\psi_N : X \longrightarrow \text{Pic}^0(X)$  is the map  $x \mapsto t_x^*N \otimes N^{-1}$  and  $\hat{\pi} : \text{Pic}^0(Z) \longrightarrow \text{Pic}^0(X)$  is the dual of the map  $\pi$ , then since  $\hat{\pi}(J) =$

$\{0\}$ ,  $\pi^{-1}G \subset K(N) = \text{Ker}\psi_N$  (since  $\psi_N = \hat{\pi} \circ \psi_M \circ \pi$ ). This means that the divisors  $\pi^{-1}(D+h) \in |N|$ , for all  $h \in G$ .

Choose the subgroup  $J$  such that  $N$  is base point free. (In fact, if  $J$  contains the subgroup of 3-torsion points of  $\text{Pic}^0(Z)$  and  $G'$ , then, by the above discussion,  $X_{[3]} \subset K(N)$ , where  $X_{[3]}$  is the subgroup of 3-torsion points of  $X$ . This implies, by [3] 2.5.6, that  $N = K^3$ , for some ample line bundle  $K$  on  $X$  and by a theorem of Lefschetz (see [3], 4.5.1),  $N$  is very ample.)

We will use the following.

**Lemma 3.1.** *Let  $V$  be a variety and  $\mathcal{V} \subset \text{Div}(V)$  be an irreducible family of effective Cartier divisors  $D_t$  on  $V$ . Suppose  $W = \bigcap_{t \in \mathcal{V}} D_t \subset V$  and is nonempty and  $r = \text{codim}(W)$ . Then there exist divisors  $D_j$ ,  $j = 1, 2, \dots, r$ , in  $\mathcal{V}$  that intersect properly and  $\dim W = \dim \bigcap_{i=1}^r D_i$ .*

*Proof.* We use induction on  $j$ . Let  $D_1, D_2, \dots, D_j$  ( $j < r$ ) be chosen in  $\mathcal{V}$  such that they intersect properly in  $V$ . Now write  $D_1 \cap D_2 \cap \dots \cap D_j = G_1 \cup G_2 \cup \dots \cup G_s$ , where  $G_1, \dots, G_s$  are irreducible components. Consider the closed subset  $\mathcal{W}_i \subset \mathcal{V}$  parametrizing divisors that contain  $G_i$  for  $i = 1, 2, \dots, s$ . (Note that  $\mathcal{W}_i \neq \mathcal{V}$ , otherwise  $G_i \subset W$ , which is not possible since  $\dim G_i > \dim W$ .) Let  $U$  be the complement of  $\bigcup_{i=1}^s \mathcal{W}_i$  in  $\mathcal{V}$ , which is nonempty since  $\mathcal{V}$  is irreducible. If  $D_{j+1} \in U$ , then  $D_1 \cap \dots \cap D_j \cap D_{j+1}$  has codimension  $j+1$  (communicated to us by A. Hirschowitz).  $\square$

*Remark 3.2.* Suppose  $D_1, D_2, \dots, D_r$  are linearly equivalent effective divisors on a variety  $V$ ,  $W = \bigcap_{i=1}^r D_i$  and is nonempty and  $r = \text{codim}(W)$ . If  $\mathbb{P}^k$  denotes the span of the points  $D_i$  in the linear system  $|D_1|$ , then  $W = \bigcap_{t \in \mathbb{P}^k} D_t$ . Hence, by Lemma 3.1, there are  $r$  divisors  $D_j \in \mathbb{P}^k$  that intersect properly.

With notation as above we have the following.

**Proposition 3.3.** *Let  $D$  be an ample divisor on a  $g$ -dimensional simple abelian variety  $Z$ . Let  $G$  be a finite subgroup of  $Z$  that is contained in  $D$ . Then  $\text{Card}(G) \leq D^g$  (which equals  $h^0(\mathcal{O}(D)) \cdot g!$ , by the Riemann-Roch Theorem).*

*Proof.* We prove this in several steps.

**Step 1:** We reduce to the case when the divisors  $D$  and  $D+h$ , for all  $h \in G$ , are linearly equivalent and  $\mathcal{O}(D)$  is base point free. Indeed, by the above discussion, choose a triple  $(X, N, \pi)$ , as above, corresponding to a subgroup  $J \subset \text{Pic}^0(Z)$  such that  $N$  is base point free and  $\psi_M(G) \subset J$ . This shows that the divisors  $\pi^{-1}D$  and  $\pi^{-1}(D+h)$ , for all  $h \in G$ , are linearly equivalent. Then we have a morphism  $\phi_N : X \rightarrow \mathbb{P}H^0(N)$ . Since  $\pi$  is a finite morphism of degree equal to  $\text{Card}(J)$ , by the projection formula, one sees that  $\deg(\pi^{-1}W) = \text{Card}(J) \cdot \deg(W)$ , for a subvariety  $W$  of  $Z$ . Since  $(\pi^{-1}D)^g = \text{Card}(J) \cdot D^g$ , if  $\text{Card}(\pi^{-1}G) \leq (\pi^{-1}D)^g$ , then  $\text{Card}(G) \leq D^g$ .

**Step 2:** We can now assume that  $D$  is an ample divisor on  $X$  and that  $G \subset D$  is a finite subgroup such that  $D$  is linearly equivalent to  $D+h$  for all  $h \in G$  and  $N = \mathcal{O}(D)$  is base point free. Let  $Y = \bigcap_{h \in G} D+h$  and  $s = \dim(Y)$ . By Lemma 3.2,  $Y \subset \bigcap_{j=1}^{g-s} D_j$  for some  $g-s$  divisors  $D_j \in |N|$  that intersect properly. Now  $\deg(Y) = [Y] \cdot [D^s]$  (here  $\deg(Y) = \deg(S)$ , where  $S \subset Y$  is of pure dimension  $s$ ). Since  $Y \subset \bigcap_{j=1}^{g-s} D_j$  we see that  $\deg(Y) \leq D^g$ . In particular, when  $s = 0$ , since  $G \subset Y$ , we get  $\text{Card}(G) \leq D^g$ .

**Step 3:** Suppose that  $s > 0$ . Let  $Y = Y_1 \cup Y_2 \cup \dots \cup Y_r$ , where  $Y_j, 1 \leq j \leq r$ , are the irreducible components of  $Y$  such that  $s = \dim Y_1 = \dim Y$ . Then  $\deg Y_1 \leq \deg Y$ . Since  $Y$  is  $G$ -invariant,  $\bigcup_{h \in G} Y_1 + h \subset Y$  and  $\sum_{h \in \frac{G}{G_{Y_1}}} \deg(Y_1 + h) \leq \deg Y$ , where  $G_{Y_1} = \{h \in G : Y_1 + h = Y_1\}$  is a subgroup of  $G$ . Hence we get the inequalities  $\text{Card}(\frac{G}{G_{Y_1}}) \cdot \deg Y_1 \leq \deg Y \leq D^g$ , i.e.,  $\text{Card}(G) \leq \frac{\text{Card}(G_{Y_1})}{\deg Y_1} \cdot D^g$ . To complete our proof, we need to show that  $\text{Card}(G_{Y_1}) \leq \deg Y_1$ .

**Step 4:** Now  $G_{Y_1} \subset \text{Stab}(Y_1) = \{a \in X : Y_1 + a = Y_1\}$ . Observe that  $\text{Stab}(Y_1) = \bigcap_{y \in Y_1} Y_1 - y$ . Now for a point  $y_0 \in Y_1$ ,  $\text{Stab}(Y_1) = (Y_1 - y_0) \bigcap_{y \in Y_1} Y_1 - y \subset (Y_1 - y_0) \bigcap_{h \in G, y \in Y_1} D + h - y$ . Let  $P = (Y_1 - y_0) \bigcap_{h \in G, y \in Y_1} D + h - y$ . We proceed to show that  $\deg(\text{Stab}(Y_1)) \leq \deg(P)$ . This will be true if  $\text{Stab}(Y_1)$  and  $P$  have the same dimension. Now, we have

$$\begin{aligned} \bigcap_{h \in G, y \in Y_1} D + h - y &= \bigcap_{y \in Y_1} Y - y \\ &= \bigcap_{y \in Y_1} ((Y_1 \cup Y_2 \cup \dots \cup Y_r) - y) \\ &= (\bigcap_{y \in Y_1} Y_1 - y) \cup (\bigcap_{y \in Y_1} Y_2 - y) \cup \dots \cup (\bigcap_{y \in Y_1} Y_r - y). \end{aligned}$$

(To see the above last equality: if  $x \in \bigcap_{y \in Y_1} (Y_1 \cup Y_2 \cup \dots \cup Y_r) - y$ , then  $x + y \in Y_1 \cup Y_2 \cup \dots \cup Y_r, \forall y \in Y_1$ . Via the translation map  $Y_1 \rightarrow Y_1 \cup Y_2 \cup \dots \cup Y_r, y \mapsto y + x$  and since  $Y_1$  is irreducible,  $x + y \in Y_j$ , for some  $j$  and for all  $y \in Y_1$ , i.e.,  $x \in \bigcap_{y \in Y_1} Y_j - y$  showing one way inclusion, the other inclusion being obvious.)

We now see that if  $j \neq 1$  and  $x \in \bigcap_{y \in Y_1} Y_j - y$ , then  $Y_1 + x \subset Y_j$ . If  $\dim Y_j < \dim Y_1$ , then this is absurd and so  $\bigcap_{y \in Y_1} Y_j - y$  is empty. If  $\dim Y_j \geq \dim Y_1$ , since  $Y_1$  is of maximal dimension in  $Y$ ,  $\dim Y_j = \dim Y_1$  and  $Y_1 + x = Y_j$ . This implies that  $\bigcap_{y \in Y_1} Y_j - y = \bigcap_{y \in Y_1} Y_1 + x - y = \text{Stab}(Y_1) + x$ . Hence  $\bigcap_{h \in G, y \in Y_1} D + h - y, P$  and  $\text{Stab}(Y_1)$  are of equal dimension, say equal to  $m$  and  $\deg(\text{Stab}(Y_1)) \leq \deg P$ .

**Step 5:** We proceed to show that  $\deg(P) \leq \deg(Y_1)$ . Consider the Poincaré line bundle  $\mathcal{P}$  on  $X \times \text{Pic}^0(X)$ . Let  $p_1$  and  $p_2$  denote the projections onto  $X$  and  $\text{Pic}^0(X)$  respectively from  $X \times \text{Pic}^0(X)$ . Consider the sheaf  $\mathcal{E} = p_{2*}(p_1^*N \otimes \mathcal{P})$  on  $\text{Pic}^0(X)$ . Since the vector spaces  $H^0(N \otimes \alpha)$  are of constant dimension for all  $\alpha \in \text{Pic}^0(X)$ , by Grauert's theorem,  $\mathcal{E}$  is a vector bundle on  $\text{Pic}^0(X)$ . Let  $\mathbb{P}(\mathcal{E})$  denote the associated projective bundle on  $\text{Pic}^0(X)$ . Consider the natural morphism  $p_2^*(\mathcal{E}) \rightarrow p_1^*N \otimes \mathcal{P}$ . This is surjective, since on any fibre  $X \times \alpha, (p_1^*N \otimes \mathcal{P})_\alpha \simeq N \otimes \alpha$  which is globally generated (since  $N$  is globally generated) and  $\mathcal{E}(\alpha) \simeq H^0(N \otimes \alpha)$ . Hence this defines a morphism  $\delta_N : X \times \text{Pic}^0(X) \rightarrow \mathbb{P}(\mathcal{E})$ . Let  $\mathbb{P}(\mathcal{E})^\vee$  denote the dual projective bundle over  $\text{Pic}^0(X)$ . In general, the parameter space  $\mathcal{V} \subset \mathbb{P}(\mathcal{E})^\vee$  of the family  $\{D + h - y\}_{h \in G, y \in Y_1}$  may not form an irreducible variety (unless  $G_{Y_1} = G$ ), but we construct an irreducible subvariety  $\mathcal{F} \subset \mathbb{P}(\mathcal{E})^\vee$  such that  $\mathcal{V} \subset \mathcal{F}$  and  $\bigcap_{h \in G, y \in Y_1} D + h - y = \bigcap_{t \in \mathcal{F}} D_t$ , where  $D_t$  denotes the divisor corresponding to  $t$  in  $\mathbb{P}(\mathcal{E})^\vee(**)$ .

**Step 6:** Construction of  $\mathcal{F}$ :

Consider the subspace  $T$  of  $H^0(X, N)$  spanned by sections  $s_h, h \in G$  such that the divisor of  $s_h$  is  $D + h$ . Consider the addition map  $m : X \times X \rightarrow X, (x, y) \mapsto x + y$ . Recall the skew-Pontryagin product of the sheaves  $\mathcal{O}_X$  and  $N, N \hat{*} \mathcal{O}_X = (p_1)_*(m^*N)$  (see [8], p. 653), where  $p_1$  (resp.  $p_2$ ) :  $X \times X \rightarrow X$

denotes the first (resp. second) projection. Then, by Grauert’s theorem,  $N^*\mathcal{O}_X$  forms a vector bundle on  $X$  with fibres  $(N^*\mathcal{O}_X)_x \simeq H^0(t_x^*N)$ . By [8], Remark 1.2,  $N^*\mathcal{O}_X \simeq N * \mathcal{O}_X$  where  $N * \mathcal{O}_X = m_*(p_1^*N)$  is the Pontryagin product and by [5], p. 161, there are isomorphisms  $\mathcal{O}_X \otimes H^0(X, N) \xrightarrow{f} N^*\mathcal{O}_X \simeq \psi_N^*\mathcal{E} \otimes N$  ( $\psi_N : X \rightarrow \text{Pic}^0(X)$  is the isogeny  $x \mapsto t_x^*N \otimes N^{-1}$ ). Consider the image  $F$  under  $f$  of the trivial subbundle  $\mathcal{O}_X \otimes T$  in  $N^*\mathcal{O}_X$ . Then the fibre of  $F$  at  $x \in X$  is the vector subspace of  $H^0(t_x^*N)$  spanned by the sections  $t_x^*s_h$  whose divisor is  $D+h-x$ , for  $h \in G$ . Now  $\mathbb{P}(F)$  is a projective subbundle of  $\mathbb{P}(\psi_N^*\mathcal{E} \otimes N) \simeq \mathbb{P}(\psi_N^*\mathcal{E})$  (since  $N$  is a line bundle). Since  $Y_1$  is irreducible, the projective bundle  $\mathbb{P}(F)$  restricted to  $Y_1$  is an irreducible subvariety, and let  $\mathcal{F}$  be the image of this irreducible variety in  $\mathbb{P}(\mathcal{E})$ . Hence  $\mathcal{F}$  is irreducible and, by construction, if  $R \in |F_y|, y \in Y_1$ , then  $\bigcap_{h \in G} D+h-y \subset R$  and  $\mathcal{F}$  satisfies property (\*\*).

**Step 7:** By Lemma 3.1, there exist divisors  $D_1, D_2, \dots, D_{g-m} \in \mathcal{F}$  such that  $\bigcap_{h \in G, y \in Y_1} D+h-y \subset D_1 \cap D_2 \cap \dots \cap D_{g-m}$ . Hence  $P \subset (Y_1 - y_0) \cap D_1 \cap D_2 \cap \dots \cap D_{g-m} \subset D_1 \cap D_2 \cap \dots \cap D_{g-m}$ . This implies that  $\deg(P) \leq \deg(Y_1 - y_0)$ , and by Step 2 and Step 4,  $\deg \text{Stab}(Y_1) \leq \deg Y_1 \leq D^g$  (since by Step 2,  $\deg(Y_1) \leq \deg(Y) \leq D^g$ ). Since  $X$  is simple,  $\text{Stab}(Y_1)$  is zero-dimensional and  $G_{Y_1} \subset \text{Stab}(Y_1)$  implies that  $\text{Card}(G_{Y_1}) \leq \deg(Y_1)$ . Hence by Step 3,  $\text{Card}(G) \leq D^g$ . This ends the proof. □

This is equivalent to the following.

**Proposition 3.4.** *Let  $\mathcal{L}$  be an ample line bundle on a simple abelian variety  $Z$  and consider the associated rational map  $Z \xrightarrow{\phi_{\mathcal{L}}} \mathbb{P}H^0(\mathcal{L})$ . Then any finite subgroup  $G$  of  $Z$ , of order strictly greater than  $h^0(\mathcal{L}) \cdot g!$ , generates  $\mathbb{P}H^0(\mathcal{L})$ . More precisely, the points  $\phi_{\mathcal{L}}(g)$  where  $g$  runs over all elements of  $G$  not in the base locus of  $\mathcal{L}$  span  $\mathbb{P}H^0(\mathcal{L})$ .*

We recall the following result, which we will need in the proof of Theorem 1.2.

**Proposition 3.5** (Wirtinger). *Let  $(Z, \Theta)$  be a principally polarized abelian variety and  $\mathcal{L} = \mathcal{O}(\Theta)$  (here  $\Theta$  is assumed to be a symmetric divisor). There is a nondegenerate inner product  $R : H^0(\mathcal{L}^2) \otimes H^0(\mathcal{L}^2) \rightarrow \mathbb{C}$  (which is symmetric or skew-symmetric depending on whether the multiplicity of the zero element  $0$  on  $\Theta$ ,  $\text{mult}_0\Theta$ , is even or odd) such that if  $R$  induces the isomorphism  $R'$ ,*

$$\mathbb{P}(H^0(\mathcal{L}^2)) \simeq \mathbb{P}(H^0(\mathcal{L}^2)^*) = |2\Theta|,$$

then the composed morphism

$$Z \xrightarrow{\phi_{\mathcal{L}^2}} \mathbb{P}(H^0(\mathcal{L}^2)) \xrightarrow{R'} |2\Theta|$$

is the morphism

$$\phi : Z \rightarrow |2\Theta|, \quad x \mapsto \Theta_x + \Theta_{-x},$$

where  $\Theta_x$  is the translate of  $\Theta$  by  $x$  on  $Z$ .

*Proof.* See [6], Proposition, p. 335. □

*Proof of Theorem 1.2.* Consider a polarized simple abelian variety  $(A, L)$  of dimension  $g$  such that  $h^0(L) > 2^g \cdot g!$ .

Consider the multiplication map

$$H^0(L) \otimes H^0(L) \xrightarrow{\rho_2} H^0(L^2).$$

This map factors via

$$\text{Sym}^2 H^0(L) \xrightarrow{\rho_2} H^0(L^2).$$

Let  $H \subset K(L)$  be a maximal isotropic subgroup for the Weil form  $e^L$ . Consider the isogeny  $A \xrightarrow{\pi} B = \frac{A}{H}$ . Then  $L$  descends down to a principal polarization  $M$  on  $B$ . We may assume that  $M$  is symmetric, i.e.,  $M \simeq i^*M$ ,  $i(b) = -b, b \in B$ . Using the fact that  $\pi_*\mathcal{O}_A = \bigoplus_{\chi \in \hat{H}} L_\chi$ , where  $L_\chi$  denotes the degree 0 line bundle on  $B$  corresponding to the character  $\chi$  on  $H$ , by the projection formula,  $\pi_*L = \bigoplus_{\chi \in \hat{H}} M \otimes L_\chi$  and  $\pi_*L^2 = \bigoplus_{\chi \in \hat{H}} M^2 \otimes L_\chi$ . Hence we obtain the following decompositions:

$$H^0(L) = \bigoplus_{\chi \in \hat{H}} H^0(M \otimes L_\chi)H^0(L^2) = \bigoplus_{\chi \in \hat{H}} H^0(M^2 \otimes L_\chi).$$

Write  $\text{Sym}^2 H^0(L) = \sum_{\chi, \chi' \in \hat{H}} H^0(M \otimes L_{\chi'}) \cdot H^0(M \otimes L_{\chi \cdot \chi'^{-1}})$ . Consider the multiplication maps

$$\sum_{\chi' \in \hat{H}} H^0(M \otimes L_{\chi'}) \cdot H^0(M \otimes L_{\chi \cdot \chi'^{-1}}) \xrightarrow{\rho_\chi} H^0(M^2 \otimes L_\chi).$$

Since  $\rho_2 = \bigoplus_{\chi \in \hat{H}} \rho_\chi$ , it will suffice to show the surjectivity of  $\rho_\chi$  for each  $\chi \in \hat{H}$ .

Since the pair  $(B, M)$  is principally polarized, the homomorphism  $\psi_M : B \rightarrow \text{Pic}^0(B)$  is an isomorphism. Let  $H' = \psi_M^{-1}(\hat{H})$  and  $\theta \in |M|$  be the unique symmetric divisor.

**Case 1:** Suppose  $\chi$  is trivial.

We see that the surjectivity of the map  $\rho_{triv}$  is equivalent to showing that the reducible divisors  $\theta_h + \theta_{-h}$  generate the linear system  $|M^2|$ , for  $h \in H'$ . By Proposition 3.5, using the morphism  $\phi : B \rightarrow |M^2|$ , this is the same as saying that the image of the subgroup  $H'$  under the morphism  $\phi_{M^2}$  generates the projective space  $\mathbb{P}H^0(M^2)$ .

**Case 2:** Suppose  $\chi$  is nontrivial.

First, notice that if  $b \in B$ , then  $\psi_{M^2}(b) = \psi_M(2b)$ . Let  $\sigma \in B$  be such that  $\psi_{M^2}(\sigma) = L_\chi$ , i.e.,  $\psi_M(2\sigma) = L_\chi$ . Hence the map  $\rho_\chi$  is surjective if the reducible divisors  $\theta_h + \theta_{-h+2\sigma}$  span the linear system  $|t_\sigma^*M^2|$  for  $h \in H' = \psi_M^{-1}(\hat{H})$ . Now if  $b \in B$ , then  $\theta_b + \theta_{-b+2\sigma} = (\theta_\sigma)_{b-\sigma} + (\theta_\sigma)_{-b+\sigma}$ , which is the image of the divisor  $\theta_{b-\sigma} + \theta_{-b+\sigma}$  under the morphism  $|M^2| \rightarrow |t_\sigma^*M^2|$  given by the translation map  $B \xrightarrow{t_\sigma} B$ . Hence the morphism  $\phi_\sigma : A \rightarrow |t_\sigma^*M^2|$  is given as  $b \mapsto \theta_b + \theta_{-x+2\sigma}$ . This implies that  $\rho_\chi$  is surjective if and only if the points in  $\phi_\sigma(H')$  generate the linear system  $|t_\sigma^*M^2|$ .

Since the pair  $(A, L)$  is a simple polarized abelian variety with  $h^0(L) = \text{Card}(H') > 2^g \cdot g! = h^0(t_\sigma^*M^2) \cdot g!$ , by Proposition 3.4,  $\rho_\chi$  is surjective for all  $\chi \in \hat{H}$ . Hence, by Proposition 2.1, our proof is now complete.  $\square$

*Remark 3.6.* 1) Suppose  $g = 1$ . Then any line bundle of degree strictly greater than 2 on an elliptic curve gives a projectively normal embedding. Hence the bound is sharp.

2) Suppose  $g = 2$ . If  $L \simeq N^2$ , where  $N$  is an ample symmetric line bundle with  $h^0(N) = 2$  on an abelian surface  $A$ , then it follows that  $h^0(L) = 8$  (in terms of “type” of an ample line bundle,  $N$  is of type (1, 2) and hence  $L$  is of type (2, 4) and  $h^0(L) = 8$ ). By [3], 10.1.4,  $N$  has 4 base points, say  $x_1, x_2, x_3$  and  $x_4$ , which are 4-torsion points on  $A$  and, moreover,  $2x_i \in K(N) = \text{Ker } \psi_N$  where

$\psi_N : A \longrightarrow \text{Pic}^0(A)$ ,  $a \mapsto t_a^* N \otimes N^{-1}$ . Let  $\alpha_i = \psi_N(x_i)$ , for  $i = 1, 2, 3, 4$ . Now the points  $x_i$  are base points for  $N$ , for  $i = 1, 2, 3, 4$ , is equivalent to saying that the origin  $0 \in A$  is a base point for  $N \otimes \alpha_i$ , for  $i = 1, 2, 3, 4$ . Also  $2x_i \in K(N)$  implies that the points  $\alpha_i$  are 2-torsion points in  $\text{Pic}^0(A)$ . Hence by Ohbuchi's Theorem 1.1,  $L$  does not give a projectively normal embedding. So the bound is sharp.

3) Suppose  $g = 3$ . If  $L \simeq N^3$ , where  $N$  is a principal polarization on an abelian threefold  $A$ , then  $h^0(L) = 27$ . But by Koizumi's Theorem,  $L$  gives a projectively normal embedding. So the bound is not sharp in this case.

#### ACKNOWLEDGEMENTS

We thank M. Hindry and K. Paranjape for useful conversations and A. Beauville for pointing out a gap in Prop. 3.3 in an earlier version. We are grateful to A. Hirschowitz for his helpful suggestions incorporated here. This work was done at the University of Paris-6 and the University of Essen. Their hospitality and support from the French Ministry of Education, Research and Technology and DFG "Arithmetik und Geometrie" Essen, is gratefully acknowledged.

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